Uniform Decay Estimates and the Lorentz Invariance of the Classical Wave Equation

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A long list of recent papers (e.g., [1]–[7]) have dealt with the problem of long-time or global existence to general classes of nonlinear wave equations subject to initial conditions which are small in a suitable sense. All of them have used as essential ingredient the uniform decay property of solutions to the classical wave equation

$$\Box u = 0, \tag{1}$$

where $\Box = \partial_t^2 - \partial_1^2 \cdots - \partial_n^2$ is the D'Alambertian of the $(n + 1)$-dimensional Minkowski space-time. More precisely, consider the standard initial value problem for (1)

$$u = 0,\ u_t = g(x) \quad \text{at} \quad t = 0 \tag{1a}$$

with g a smooth, compactly supported, function in $\mathbb{R}^n$. Then (see [2])

$$|u(t,x)| \leqq ct^{-(n-1)/2}\|g\|_{W^{[n/2],1}} \tag{1b}$$

for all $x \in \mathbb{R}^n$, $t > 0$ and $W^{s,1}$, $s \in N$, the classical Sobolev spaces in $\mathbb{R}^n$. We denote by $[n/2]$ the largest integer less than or equal to $\frac{1}{2}n$.

Often (see [2]), the estimate (1b) has been used together with the following L^2 estimate, which is a consequence of the energy identity for (1),

$$\|u'(t)\|_{L^2} \leqq \|g(t)\|_{L^2} \tag{1c}$$

for $t \geqq 0$, $u' = (u_t, u_1, \cdots, u_n)$ the space-time gradient of u and $\| \ \|_{L^2}$ the usual L^2 norm of $\mathbb{R}^n$.

In particular, interpolations between (1b) and (1c) have proved very successful in [4], [5]. The major drawback of (1b) however, which has made necessary the use of the above mentioned interpolation inequalities, is the presence on the right-hand side of (1b) of L^1 norms of derivatives of g. In fact the L^1 norm is a "bad" norm for wave equations in higher dimensions, more precisely the L^1 norm of solutions to (1) grows like $t^{(n-1)/2}$ as $t \to \infty$, which, when applied to nonlinear equations, counter-

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balances the decay rate gained in (1b). One could easily check that the results of [2] could be significantly improved and vastly simplified if one could replace in (1b) the $W^{[n/2],1}$ norm by some $W^{s,2}$-norm. The fact that this can actually be accomplished, by a suitable modification of the Sobolev spaces $W^{s,2}$, is our objective in this paper.

To illustrate the power of these new L^∞-L^2 decay estimates we shall apply them to prove the following result for nonlinear wave equations of the type

$$\Box u = F(u', u''), \tag{2}$$

where F is a smooth function of (u', u''), the first and second space-time derivatives of $u = u(t,x)$, $x \in \mathbb{R}^n$, vanishing together with its first derivatives for $(u', u'') = 0$. Consider the initial value problem

$$u = \varepsilon f(x), \qquad u_t = \varepsilon g(x) \qquad \text{at} \qquad t = 0, \tag{2a}$$

with $f, g \in C_0^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$ a small parameter, and define the life span $T_\infty = T_\infty(\varepsilon) = \sup \tau > 0$ for which there exists a C^∞-solution of (2), (2a) for all $x \in \mathbb{R}^n$, $0 \leqq t < \tau$. We have

THEOREM. (A) *If $n > 3$, there exists an ε_0, sufficiently small, depending on at most $2n + 8$ derivatives of F, f, g such that, for any $0 < \varepsilon \leqq \varepsilon_0$, $T_\infty(\varepsilon) = \infty$,* i.e., *all solutions, with sufficiently small initial data, remain smooth for all time.*

(B) *If $n = 3$, there exist constants ε_0,* A, *sufficiently small depending on at most 14 derivatives of F, f, g, such that for any $0 < \varepsilon \leqq \varepsilon_0$ we have*

$$T_\infty(\varepsilon) \geqq \exp\left\{A\,\frac{1}{\varepsilon}\right\}.$$

Part (A) of the theorem improves our previous global existence result for $n > 5$ (see [2] and also [4], [5]). Part (B) is precisely the almost-global existence result of [8].

The L^∞-L^2 decay estimates, which we hinted at above, depend on the invariance properties of $\Box$ and are otherwise *a priori*. Some of these invariance properties were used in [9] to derive the weighted L^∞ and L^1 estimates which replaced (1b) in the proof of the almost-global existence theorem of [8]. They are reminiscent of the local energy decay estimates of C. Morawetz [10] and of the conformal method of D. Christodoulou and C. Bruhat (see [11]).

Consider the Minkowski space $\mathbb{R} \times \mathbb{R}^n$ with coordinates $x_0 = t$, $x = (x_1, \cdots x_n)$, the Lorentz metric

$$\eta = (\eta_{ab})_{a,b=0,1,\cdots,n} = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{3}$$

and the D'Alembertian

(3a) $$\Box = -\eta^{ab}\partial_a\partial_b = \partial_0^2 - \partial_1^2 \cdots - \partial_n^2, \qquad a,b = 0, 1, \cdots, n,$$

with $$\partial_0 = -\frac{\partial}{\partial t} \text{ and } \partial_i = \frac{\partial}{\partial x_i}, \qquad i = 1, \cdots, n.$$

Consider the generators of the Lorentz group, the first-order operators,

(3b) $$\Omega_{ab} = x_a\partial_b - x_b\partial_a, \qquad a, b = 0, 1, \cdots, n.$$

Thus,

$$\Omega_{ij} = x_i\partial_j - x_j\partial_i, \qquad i, j = 1, \cdots, n,$$

$$\Omega_{0i} = t\partial_i + x_i\partial_t, \qquad i = 1, \cdots, n.$$

Define also

(3c) $$L_0 = \eta^{ab}x_a\partial_b = t\partial_t + x_1\partial_1 + \cdots + x_n\partial_n.$$

For convenience we shall write $L_i = \Omega_{0i}$, $i = 1, \cdots, n$.

We have the crucial properties

(3d) $$[\Omega_{ab},\Box] = 0, \qquad a, b = 0, 1, \cdots, n,$$

and

(3e) $$[L_0,\Box] = -2\Box.$$

Also, for every $a, b, c, d, = 0, 1, \cdots, n$,

(3f) $$[\Omega_{ab},\Omega_{cd}] = \eta_{bc}\Omega_{ad} + \eta_{ad}\Omega_{bc} - \eta_{bd}\Omega_{ac} - \eta_{ac}\Omega_{bd},$$
$$[L_0,\Omega_{ab}] = 0.$$

Moreover,

(3g) $$[\Omega_{ab},\partial_c] = \eta_{bc}\partial_a - \eta_{ac}\partial_b, \qquad a, b, c, = 0, 1, \cdots, n,$$
$$[L_0,\partial_a] = -\partial_a, \qquad a = 0, 1, \cdots, n.$$

To check these relations note that $\partial_a x_b = \eta_{ab}$ for $a, b = 0, 1, \cdots, n$.

In particular each of the following families of first-order operators generates Lie-algebras, i.e., their $\mathbb{R}$-linear span is a Lie-algebra:

(3h) $$\Omega = (\Omega_{ij})_{1\leq i<j\leq n},$$

(3i) $$\overline{\Omega} = (\Omega_{ab})_{0\leq a<b\leq n},$$

(3j) $$\Gamma = (L_0,\overline{\Omega},\partial),$$

with $\partial = (\partial_a)_{0 \leqq a \leqq n}$. To each of these families we can associate generalized Sobolev norms by the following procedure. Let $A = (A_i)_{1 \leqq i \leqq \sigma}$ be one of these families, $u = u(t,x)$ a smooth function of $t,x \in \mathbb{R}^n$ decaying sufficiently rapidly at spatial infinity for each fixed t. Define

(3k) $$\|u(t)\|^2_{A,k} = \sum_{|\alpha| \leqq k} \|A^\alpha u(t)\|^2_{L^2(\mathbb{R}^n)},$$

where A^α is the product operator $\Pi^\sigma_{i=1}(A_i^{\alpha_i})$ and α a σ-index, $|\alpha| = \Sigma^\sigma_{i=1} \alpha_i$.

Remark. Any two different orderings of the operators A will produce equivalent norms in (3k).

According to (3k) we shall denote by $\| \|_\Omega$, $\| \|_{\overline{\Omega}}$, $\| \|_\Gamma$ the norms generated by the families Ω, $\overline{\Omega}$, Γ.

Given $\bar{x} = (x_0,x) \in \mathbb{R}^{n+1}$, introduce the Minkowski distance $[\bar{x}]^2 = -\eta^{ab}x_a x_b = t^2 - |x|^2$, where $| \, |$ denotes the usual Euclidean distance. Using the operators L_0, Ω_{ab} we can express $\square$ in the following form:

(3l) $$\square = \frac{1}{[\bar{x}]^2}[L_0^2 + (n-1)L_0 - \Delta_{H^n}],$$

where Δ_{H^n} is the Laplace-Beltrami operator of the n-dimensional hyperboloid H^n, $[\bar{\omega}]^2 = 1$, $\bar{\omega} = (\omega_0,\omega) \in \mathbb{R} \times \mathbb{R}^n$. In fact,

(3m) $$\Delta_{H^n} = \sum_{i=1}^n L_i - \Delta_{S^{n-1}},$$

with $\Delta_{S^{n-1}} = \Sigma_{1 \leqq i < j \leqq n} \Omega^2_{ij}$ the Laplace-Beltrami operator of the sphere $S^{n-1} = \{x \in \mathbb{R}^n / |x| = 1\}$.

Given $\bar{x} = (t,x) \in \mathbb{R} \times \mathbb{R}^n$ a time-like vector, i.e., $t > |x|$, we introduce the pseudospherical coordinates

(3n) $$t = x_0 = \rho \cosh \theta_0, \qquad x = \rho \sinh \theta_0 \cdot \xi,$$

with $\rho^2 = t^2 - |x|^2$, $\theta_0 \in \mathbb{R}$, and $\xi \in \mathbb{R}^n$, $|\xi| = 1$, $\xi \in \mathbb{R}^n$. Note that

(3o) $$L_0 = \rho \, \partial\rho, \qquad \sum_{i=1}^n \frac{x_i}{|x|} L_i = \partial_{\theta_0} = r \, \partial_t + t \, \partial_r,$$

with $r = |x|$, $\partial_r = \Sigma^n_{i=1}(x_i/|x|)\partial_i$. We shall also denote by L_r the operator $\Sigma^n_{i=1}(x_i/|x|)L_i$.

The following lemmas are immediate applications of the standard Sobolev inequalities on S^{n-1} and H^n.

LEMMA 1. *Let $u = u(\xi)$ be a smooth function on S^{n-1}. Then, for every $\xi \in S^{n-1}$,*

$$|u(\xi)| \leqq C_n \left(\sum_{|\alpha| \leqq [(n-1)/2]+1} \|\Omega^\alpha u\|^2_{L^2(S^{n-1})} \right)^{1/2},$$

where Ω^α denotes the product of the operators $\Omega_{ij}^{\alpha_{ij}}$, $1 \leqq i < j \leqq n$, in a fixed given order, e.g. *the lexicographic one, and $\alpha = (\alpha_{ij})_{1 \leqq i<j \leqq n}$ are $\frac{1}{2}n(n-1)$ indices with $|\alpha| = \Sigma_{1 \leqq i<j \leqq n} \alpha_{ij}$. Also, $[\frac{1}{2}(n-1)]$ denotes the largest integer smaller than or equal to $\frac{1}{2}(n-1)$.*

LEMMA 2. *Let u be a smooth, compactly supported function in the future-directed hyperboloid $H^n_+ = \{\overline{\omega} = (\omega_0, \omega) \in H^n / \omega_0 > 0\}$. Then, for every $\overline{\omega} \in H^n_+$,*

$$|u(\overline{\omega})| \leqq C_n \left(\sum_{|\beta| \leqq [n/2]+1} \|\overline{\Omega}^\beta u\|^2_{L^2(H^n_+)} \right)^{1/2},$$

where $\overline{\Omega}^\beta = \Pi_{0 \leqq a<b \leqq n} \Omega_{ab}^{\beta_{ab}}$ in a fixed given order, and $\beta = (\beta_{ab})_{0 \leqq a<b \leqq n}$ are $\frac{1}{2}n(n+1)$ indices, $|\beta| = \Sigma_{0 \leqq a<b \leqq n} \beta_{ab}$.

We shall prove now three propositions which, together, constitute the heart of our proof of the theorem. They are all consequences of Lemmas 1 and 2.

PROPOSITION 1. *Let u be a smooth function in $\mathbb{R}^n$, compactly supported or vanishing sufficiently rapidly at infinity. We have, for $x \in \mathbb{R}^n$, $|x| \neq 0$,*

$$|u(x)| \leqq C_n \left(\frac{1}{|x|} \right)^{(n-1)/2} \|u\|^{1/2}_{\Omega,[(n-1)/2]+1} \cdot \|\partial_r u\|^{1/2}_{\Omega,[(n-1)/2]+1}, \tag{4}$$

$\partial_r = \Sigma_{i=1}^n (x_i/|x|)\, \partial_i$ and $\| \ \|$ denotes the usual L^2-norm in $\mathbb{R}^n$.

Proof: Introduce polar coordinates, $x = r\xi$, $r = |x|$, $\xi \in S^{n-1}$. Thus,

$$u^2(r\xi) \leqq 2 \frac{1}{r^{n-1}} \int_r^\infty |u\ \partial_r u(\lambda\xi)| \lambda^{n-1}\, d\lambda.$$

Hence,

$$\int_{|\xi|=1} u^2(r\xi)\, dS_\xi \leqq 2 \frac{1}{r^{n-1}} \|u\|_{L^2(\mathbb{R}^n)} \|\partial_r u\|_{L^2(\mathbb{R}^n)}$$

and, similarly

$$\sum_{|\alpha| \leqq k} \left(\int_{S^{n-1}} |\Omega^\alpha u(r\xi)|^2\, dS_\xi \right)^{1/2} \leqq \left(\frac{1}{|x|} \right)^{(n-1)/2} \|u\|^{1/2}_{\Omega,k} \cdot \|\partial_r u\|^{1/2}_{\Omega,k},$$

$k = [(n-1)/2] + 1$, $\partial_r = \Sigma_{i=1}^n (x_i/|x|)\, \partial_i$, which combined with Lemma 1 proves the desired inequality.

PROPOSITION 2. *Let $u = u(t,x)$ be a smooth function in $\mathbb{R} \times \mathbb{R}^n$ compactly supported in $\mathbb{R}^n$, or vanishing sufficiently fast at infinity, for any fixed $t \geqq 0$. Then, for any $\bar{x} = (t,x)$, $t \geqq 2|x| > 0$,*

$$|u(t,x)|^2 \leqq C_n(t^2 - |x|^2)^{-n/2} \sup_{0\leqq s\leqq 2\rho} \|u(s)\|_{\bar{\Omega},[n/2]+1} \tag{5}$$
$$\cdot (\|u(s)\|_{\bar{\Omega},[n/2]+1} + \|L_0u(s)\|_{\bar{\Omega},[n/2]+1}),$$

where $\rho^2 = t^2 - |x|^2$.

Proof: Introduce pseudospherical coordinates $\bar{x} = \rho\bar{\omega}$, $\rho^2 = t^2 - |x|^2$ and $[\bar{\omega}]^2 = 1$. Since $|x| \leqq \frac{1}{2}t$ we have $|\theta_0| \leqq \frac{1}{2}\log 3$ in (5). Thus,

$$\rho^{n+1}u^2(\rho\bar{\omega}) \leqq (n+1)\int_0^\rho [u^2(\lambda\bar{\omega}) + |L_0u(\lambda\bar{\omega})|\,|u(\lambda\bar{\omega})|]\lambda^n\,d\lambda.$$

Now integrate this expression on the set $\Sigma = \{\bar{\omega} \in H^n_+/|\omega| \leqq \frac{1}{2}\omega_0\}$ and note that $\lambda^n\,d\lambda\,dS_{\bar{\omega}}$, with $dS_{\bar{\omega}}$ the area element of H^n, is precisely the area element $dt\,dx$ of $\mathbb{R} \times \mathbb{R}^n$. Hence,

$$\rho^{n+1}\int_\Sigma u^2(\rho\bar{\omega})\,dS_{\bar{\omega}} \leqq (n+1)\iint_{\substack{0<s^2-|y|^2\leqq\rho^2\\|y|\leqq s/2}} [|u(s,y)|^2 + |u(s,y)|\,|L_0u(s,y)|]\,ds\,dy$$
$$\leqq (n+1)\int_0^{2\rho} ds\int_{|y|\leqq s/2} dy\,[|u(s,y)|^2 + |u(s,y)|\,|L_0u(s,y)|]$$
$$\leqq 2(n+1)\rho\sup_{0\leqq s\leqq 2\rho}[\|u(s)\|^2_{L^2(\mathbb{R}^n)} + \|u(s)\|_{L^2(\mathbb{R}^n)}\|L_0u(s)\|_{L^2(\mathbb{R}^n)}]$$

and similarly, since L_0 commutes with $\bar{\Omega}^\beta$,

$$\rho^{n+1}\sum_{|\beta|\leqq k}\int_\Sigma |\bar{\Omega}^\beta u(\rho\bar{\omega})|^2\,dS_{\bar{\omega}}$$
$$\leqq 2(n+1)\rho\sup_{0\leqq s\leqq 2\rho}\|u(s)\|_{\bar{\Omega},k}(\|u(s)\|_{\bar{\Omega},k} + \|L_0u(s)\|_{\bar{\Omega},k})$$

which, together with the appropriate localized version of Lemma 2, proves (5).

Remark. We can avoid the presence of L_0 in (5) by losing some decay. Indeed, we also have,

$$|u(t,x)|^2 \leqq c_1(t^2 - |x|^2)^{-(n-1)/2}\sup_{0\leqq s\leqq 2\rho}\|u(s)\|_{\bar{\Omega},[n/2]+1} \tag{5'}$$
$$\cdot (\|u(s)\|_{\bar{\Omega},[n/2]+1} + \|u'(s)\|_{\bar{\Omega},[n/2]+1}),$$

with u' the vector $(u_0, u_1, \cdots, u_n)$, $u_a = \partial_a u$, $a = 0, \cdots, n$.

Combining (5) with the estimate (4) applied to $u(t,x)$ for every fixed t we conclude that

$$(6) \qquad |u(t,x)| \leqq Ct^{-(n-1)/2}[\sup_{0 \leqq s \leqq 2\rho} (\|u(s)\|^2_{\bar{\Omega},[n/2]+1} + \|u'(s)\|^2_{\bar{\Omega},[n/2]+1})]^{1/2}$$

or

$$(6a) \quad |u'(x,t)| \leqq Ct^{-(n-1)/2}[\sup_{0 \leqq s \leqq 2\rho} \|u'(s)\|^2_{\bar{\Omega},[n/2]+1} + \sup_{0 \leqq s \leqq 2\rho} \|u''(s)\|^2_{\bar{\Omega},[n/2]+1}]^{1/2}$$

uniformly for $x \in \mathbb{R}^n$, $t > 0$.

In particular, if u is a solution of (1), with compactly supported initial data, we have, according to the energy identity,

$$(6b) \qquad \|(\bar{\Omega}^{\beta}u)'(s)\|_{L^2} \leqq \|(\bar{\Omega}^{\beta}u)'(0)\|_{L^2}$$

for any $s \geqq 0$ and any $\frac{1}{2}n(n+1)$-index β. On the other hand, according to the commutation properties of Ω and ∂ we have, for some large constant M_k,

$$(6c) \qquad \frac{1}{M_k}\|u'(s)\|_{\bar{\Omega},k} \leqq \sum_{|\beta| \leqq k} \|(\bar{\Omega}^{\beta}u)'(s)\|_{L^2} \leqq M_k\|u'(s)\|_{\bar{\Omega},k}.$$

Hence,

$$(6d) \qquad \|u'(s)\|_{\bar{\Omega},k} \leqq M_k^2\|u'(0)\|_{\bar{\Omega},k} = M_k^2\|u'(s)\|_{\bar{\Omega},k}|_{s=0}$$

and combining this with (6a) we derive the following global, L^{∞}-L^2 decay estimate:

$$(6e) \qquad |u'(t,x)| \leqq C_n t^{-(n-1)/2}\|u'(0)\|_{\bar{\Omega},[n/2]+1},$$

for every $x \in \mathbb{R}^n$, $t > 0$.

Though estimates (6), (6a) are perfectly suitable for the proof of the theorem they have the undesirable feature of depending on possible future times of the L^2-norm on the right-hand side. We can avoid this by proceeding as follows. First, note, according to the definition of L_0, L_i, $i = 1, \cdots, n$, that

$$(7a) \qquad \partial_r = \frac{1}{t^2 - r^2}(tL_r - rL_0),$$

with $\partial_r = \Sigma_{i=1}^3 (x_i/|x|)\partial_i$, $L_r = \Sigma_{i=1}^3 (x_i/|x|)L_i$. Hence, denoting by Lu the vector $(L_0u, L_1u, \cdots, L_nu)$, we have

$$|\partial_r u(t,x)| \leqq c\frac{1}{|t-r|}|Lu(t,x)|,$$

with $r = |x| \neq 0,t$. More generally,

$$(7b) \qquad |\partial_r^k u(t,x)| \leqq C_k \frac{1}{|t-r|^k}\sum_{|\alpha| \leqq k}|L^{\alpha}u(t,x)|, \qquad r,\ t > 0,\ r = |x| \neq t,$$

for every $k > 0$, $r,t > 0$, $r = |x| \neq t$ and C_k a constant which depends on k.

For fixed $\xi \in \mathbb{R}^n$, $|\xi| = 1$, take $v(t,r) = v(t,r,\xi) = (t-r)^k u(t,r,\xi)$. Clearly, (7c) $$v(t,r) = \frac{1}{(k-1)!}\int_r^t (\lambda - r)^{k-1}\frac{d^k}{d\lambda^k}v(t,\lambda)\,d\lambda,$$

for all $0 \leqq r < t$.

Making use of (7b) we have

$$\left|\frac{d^k}{d\lambda^k}v(t,\lambda)\right| \leqq C_k M_k(t,\lambda,\xi),$$

with

$$M_k = \sum_{|\alpha|\leqq k}|L^\alpha u(t,\lambda,\xi)|.$$

Therefore,

$$\begin{aligned}|v(t,r,\xi)| &\leqq C_k\frac{1}{(k-1)!}\int_r^t(\lambda - r)^{k-1}M_k(t,\lambda,\xi)\,d\lambda\\ &\leqq C_k I_k(t,r)\left(\int_r^t \lambda^{n-1}M_k^2(t,\lambda,\xi)\,d\lambda\right)^{1/2},\end{aligned}$$

where

(7d) $$I_k(t,r) = \left(\int_r^t (\lambda - r)^{2k-2}\lambda^{-n+1}\,d\lambda\right)^{1/2}.$$

Integrating with respect to ξ, $|\xi| = 1$, we derive

$$\left(\int_{|\xi|=1}|u(t,r,\xi)|^2\,dS_\xi\right)^{1/2} \leqq C_k I_k(t,r)\frac{1}{(t-r)^k}\|u(t)\|_{L,k},$$

or, applying Lemma 1 once more,

(7e) $$|u(t,x)| \leqq C_k I_k(t,r)\frac{1}{(t-r)^k}\|u(t)\|_{\bar{\Omega},k+1+[(n-1)/2]},$$

for all $0 \leqq |x| = r < t$.

For n odd we pick $k = \frac{1}{2}(n+1)$ in which case $I_k \leqq (t-r)^{1/2}$. For n even we take $k = \frac{1}{2}(n+2)$ and have $I_k \leqq (1/\sqrt{2})(t^2 - r^2)^{1/2}$. In both cases, (7e) yields

(7f) $$|u(t,x)| \leqq C_n t^{-n/2}\|u(t)\|_{\bar{\Omega},n+2},$$

provided that $|x| \leqq \frac{1}{2}t$.

If we now combine this with the estimate for $|x| > \frac{1}{2}t$ which follows from Proposition 1, we conclude the proof of (i) of the following statement (the estimate (ii) is an immediate consequence of (6)).

PROPOSITION 3. *Let $u = u(t,x)$ be a smooth function of $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ compactly supported in $\mathbb{R}^n$, or vanishing sufficiently fast at infinity, for each fixed $t \geqq 0$. Then, for any $t \geqq 0$, $x \in \mathbb{R}^n$, we have*

(i) $|u(t,x)| \leqq C_n(1 + t)^{-(n-1)/2}\|u(t)\|_{\Gamma,(n+2)}$,

(ii) $|u(t,x)| \leqq C_n(1 + t)^{-(n-1)/2} \sup_{0 \leqq s \leqq 2t} \|u(s)\|_{\Gamma,[n/2]+2}$.

Remark. The decay rates of Proposition 3 can be improved in the interior of the light cone. Indeed, for $|x| \leqq \frac{1}{2}t$, we have

$$|u'(t,x)| \leqq C \frac{1}{t} |Lu(t,x)|. \tag{8a}$$

We now indicate the main steps in our new proof of the theorem. This uses, as main ingredient, the estimate (i) of Proposition 3. For simplicity we assume that the nonlinear term $F(u',u'')$ in (2) is linear in u'', i.e.,

$$F(u',u'') = \sum_{a,i} f^{ai}(u')\, \partial^2_{ai} + \sum_{a} f^a(u')\, \partial_a u, \tag{9}$$

where f^{ai}, f^a are smooth functions of u' with $f^{ai}(0) = f^a(0) = 0$ for all $a = 0, 1, 2, 3$, $i = 1, 2, 3$.

Moreover we can assume that

$$\sum_{a,i} |f^{ai}(u')| \leqq \tfrac{1}{2}, \tag{9a}$$

for any u', $|u'| \leqq 1$.

Step 1 (Generalized energy estimates). For every integer $N \geqq 0$, there exists a constant $C_N > 0$, depending only on F, with the following property:

Whenever $u(t,x)$ is a C^∞ solution of (2) for $x \in \mathbb{R}^3$, $0 \leqq t \leqq T_0$,

$$\|u'(t)\|_N \leqq C_N \|u'(0)\|_N \exp\left\{ C_N \int_0^t |u'(s)|_{1+N/2}\, ds \right\}, \tag{9b}$$

provided that $u(x,0)$, $u_t(x,0)$ have compact support, and

$$|u'(t)|_{N/2} \leqq 1 \qquad \text{for} \qquad 0 \leqq t \leqq T_0, \tag{9c}$$

with $N/2$ the smallest integer greater than $N/2$ and,

$$\|u(t)\|_N = \|u(t)\|_{\Gamma,N} \tag{9d}$$

introduced in (3k),

$$|u(t)|_N = \sum_{|\gamma| \leqq N} |\Gamma^\gamma u(t)|_{L^\infty}. \tag{9e}$$

The proof of Step 1 is almost the same as that of Theorem B* in [1] where, in that case, we did not use the operators L_0, L_1, L_2, L_3. The only modification, responsible for the somewhat unsatisfactory presence of the norm $| \ |_{1+N/2}$ in (9b) instead of $| \ |_1$, is due to our lack of sharp interpolation inequalities for the norms (9d) similar to those we used in [1], Lemmas 2, 3. In fact those lemmas are replaced here with the following cruder, but completely elementary, estimates:

LEMMA 3. *Let $u(t,x)$, $v(t,x)$ be C^∞ functions supported in x for any fixed t. Then, for any 11 component-index α, $|\alpha| = k$ and any $t \geqq 0$, we have*

$$\|\Gamma^\alpha(uv)(t)\| \leqq C(|u(t)|_{k/2}\|v(t)\|_k + |v(t)|_{k/2} \cdot \|u(t)\|_k).$$

LEMMA 4. *Let $v = (v_1, \cdots, v_m)$ be C^∞ functions in x,t, compactly supported in x for any fixed t, and assume that $f(v)$ is uniformly bounded, together with its derivatives of order up to k, on $|v| = |v_1| + \cdots + |v_m| \leqq 1$. Then, given any 11 component-index α, $|\alpha| = k$, we have*

$$\|\Gamma^\alpha f(v(t))\| \leqq C\|v(t)\|_k$$

whenever $|v(t)|_{k/2} \leqq 1$.

Step 2 (L^∞ estimates). This is now an immediate application of Proposition 3, (i) combined with (9b). We define, for any $t > 0$,

$$M_t(u) = \sup_{1 \leqq s \leqq t} (1 + s)^{-(n-1)/2} |u'(s)|_{N_0}, \tag{10a}$$

with N_0 a fixed integer to be chosen below. According to Proposition 3 (i), we have

$$M_t(u) \leqq C \sup_{0 \leqq s \leqq t} \|u'(s)\|_{\Gamma, N_0+n+2}. \tag{10b}$$

Step 3 (Proof of the theorem). Making use of (9b), (10b) we infer that

$$M_t(u) \leqq C_{N_0}\|u'(0)\|_{\Gamma, N_0+n+2} \cdot \exp\left\{C_{N_0}\left(\int_0^t |u'(s)|_{1+(N_0+n+2)/2}\, ds\right)\right\}, \tag{10c}$$

with C_{N_0} a constant depending on N_0 and n. If $N_0 \geqq n + 6$, then $1 + \frac{1}{2}(N_0 + n + 4) \leqq N_0$. Thus fixing N_0 by

$$N_0 = n + 6, \tag{10d}$$

and making use of (10a), (10c) and the choice of the initial data (2a), we infer that, for any $T > 0$,

$$M_t(u) \leqq C \in \exp\{CM_t(u)\} \tag{10e}$$

for $n > 3$ or

$$M_t(u) \leqq C \in \exp\{c \log(1 + t) M_t(u)\} \tag{10e$'$}$$

for $n = 3$, where C is a constant depending only on n.

The proof of part (A) of the theorem continues now as follows. Choose ε sufficiently small in (2a) such that

$$\exp\{CM_0(u)\} < 2, \tag{10f}$$

with C the constant in (10e). Define $T_0 = \sup\{0 \leqq t < \infty / M_t(u) < \log 2/c\}$. If $T_0 = \infty$, the theorem follows immediately by standard application of the local existence theorem. If $T_0 < \infty$, we have to conclude that $M_{T_0} = \log 2/C$. On the other hand, from (10e)

$$M_t(u) \leqq 2C\varepsilon < \frac{\log 2}{C} \tag{10g}$$

for all $0 \leqq t < T_0$, provided that

$$\varepsilon < \frac{\log 2}{4c^2}. \tag{10h}$$

Thus choosing ε_0 sufficiently small so that (10f), (10h) are simultaneously verified, we reach a contradiction. The proof of part (B) of the theorem follows in the same vein from (10e′) (see also [8]).

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