# VECTOR FIELDS

JASON ZHAO

Contents



## 1. Vector fields

Given a smooth manifold $M$, a vector field is a smooth assignment of a tangent vector $X_p \in T_pM$ to every point $p \in M$, i.e. $X$ is a section of the tangent bundle. We denote the space of vector fields $\mathfrak{X}(M)$. In local coordinates $x = (x^1, \ldots, x^n)$ on $U \subseteq M$, vector fields take the form

$$X_p = X^i(p) \frac{\partial}{\partial x^i}\Big|_p$$

where $X^i : U \to \mathbb{R}$ are smooth component functions of $X$ in the given coordinates.

### 1.1. Frames.

### 1.2. Integral curves.

An integral curve $\gamma : I \to M$ of a vector field $X \in \mathfrak{X}(M)$ is a smooth curve which flows along the field, i.e.

$$\frac{d\gamma}{dt}(t) = X_{\gamma(t)}.$$

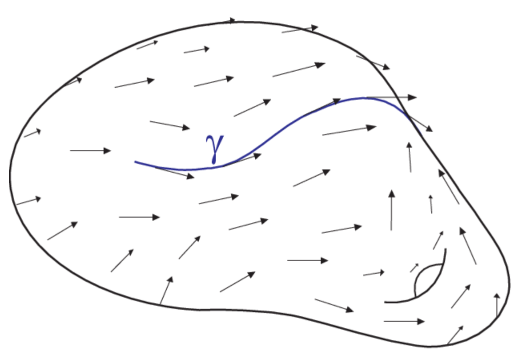


Figure 1. An integral curve $\gamma$ of a vector field $X$. The *velocity* of the curve at a point $\gamma(t)$ is given exactly by the tangent vector $X_{\gamma(t)}$.

*Date*: October 3, 2022.

**Theorem 1** (Fundamental theorem of ODE). *Let $X \in \mathfrak{X}(M)$ be a vector field and $p \in M$, then there exists an open neighborhood $U \subseteq M$ of $p$ and a smooth map $\Phi : U \times (-\varepsilon, \varepsilon) \to M$ such that, setting*

$$\gamma_x(t) = \phi_t(x) = \Phi(x,t),$$

*the curves $\gamma_x$ are the unique integral curves of $X$ with initial data $\gamma(0) = x$, and the flows $\phi_t$ are a 1-parameter group of diffeomorphisms, i.e. $\phi_t \circ \phi_s = \phi_{t+s}$ and $\phi_t^{-1} = \phi_{-t}$.*

*Remark.* If $M$ is compact without boundary, then there exists a unique global flow $\Phi : M \times \mathbb{R} \to M$, which we construct by constructing local flows on a finite sub-cover and piecing together by uniqueness to obtain a flow map $\Phi : M \times (-\varepsilon, \varepsilon) \to M$. Iterating furnishes a flow for all time.

A counter-example to the existence of a global flow in the non-compact case is $M = \mathbb{R} \setminus 0$ and $X = \frac{\partial}{\partial x}$. Viewed on $\mathbb{R}$, the flow $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$\Phi(x,t) = x + t,$$

which does not restrict to a global flow $\mathbb{R} \setminus 0$ for any time interval $(-\varepsilon, \varepsilon)$.

**Corollary 2.** *Let $X \in \mathfrak{X}(M)$ be a vector field and suppose $X(p) \neq 0$ at some point $p \in M$. Then there exist local coordinates in a neighborhood of $p$ such that $X$ is a constant vector field.*

*Proof.* Working in local coordinates $x : U \to \mathbb{R}^n$, suppose that the coordinate function satisfies $X^j(p) \neq 0$. Let $\Sigma \subseteq U$ be the hypersurface given by $x^j = 0$, then restricting the flow $\Phi : (-\varepsilon, \varepsilon) \times \Sigma \to M$ furnishes a local diffeomorphism. By the inverse function theorem, we obtain coordinates such that

$$X = \frac{\partial}{\partial y^j}.$$

□

## 2. Lie algebra

2.1. **Lie bracket.** Viewing vector fields $X, Y \in \mathfrak{X}(M)$ as first-order differential operators, i.e. derivations, the COMMUTATOR of $X$ and $Y$,

$$[X,Y] := XY - YX$$

measures the degree to which $X$ and $Y$ fail to commute. The commutator forms another first-order differential operator $[X,Y] \in \mathfrak{X}(M)$, which in local coordinates $x$ takes the form

$$[X,Y] = \left( X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

In view of the following properties,

- anti-commutativity, $[X,Y] = -[Y,X]$,
- bi-linearity, $[aX + bY, Z] = a[X,Z] + b[Y,Z]$,
- the Jacobi identity, $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$,

furnishes a LIE ALGEBRA structure to the space of vector fields, in which case the commutator is referred to as the LIE BRACKET. One can view the Jacobi identity as describing precisely how associativity fails.

**Proposition 3.**

## 3. Lie derivative

The LIE DERIVATIVE is the derivative of a tensor field along the flow $\phi_t : M \to M$ of a vector field $X \in \mathfrak{X}(M)$. The flows define local diffeomorphisms, so the pullback $(\phi_t)^*$ lifts to covariant tensor algebra homomorphism

$$\begin{array}{ccc} T^*_{\phi_t(p)}M & \xrightarrow{(\phi_t)^*} & T^*_pM \\ \big\downarrow & & \big\downarrow \\ T(T^*_{\phi_t(p)}M) & \xrightarrow{(\phi_t)^*} & T(T^*_pM) \end{array}$$

while the pushforward $(\phi_{-t})_*$ lifts to a contravariant tensor algebra homomorphism

$$
\begin{array}{ccc}
T_{\phi_{-t}(p)}M & \xrightarrow{(\phi_{-t})_*} & T_pM \\
\big\downarrow & & \big\downarrow \\
T(T_{\phi_{-t}(p)}M) & \xrightarrow{(\phi_{-t})_*} & T(T_pM)
\end{array}
$$

The Lie derivative of a section $A \otimes B \in \bigotimes^k T^*M \otimes \bigotimes^l TM$ is defined as

$$\mathcal{L}_X(A \otimes B) = \frac{d}{dt}\Big|_{t=0}(\phi_t)^* A \otimes \frac{d}{dt}\Big|_{t=0}(\phi_{-t})_* B.$$

The Lie derivative agrees with the exterior derivative on scalar functions in that if $f \in C^\infty(M)$ then

$$\mathcal{L}_X f = \frac{d}{dt}\Big|_{t=0} f(\phi_t) = df(X) = Xf.$$

3.1. **Differential forms.** The Lie derivative of a differential form $\omega \in \Omega^k(M)$ along a vector field $X \in \mathfrak{X}(M)$ with flow $\phi_t : M \to M$ is given by

$$\mathcal{L}_X \omega = \frac{d}{dt}\Big|_{t=0} \phi_t^* \omega.$$

We have already seen that the Lie derivative agrees with the exterior derivative on 0-forms. The relationship between the two notions of a derivative extends to general $k$-forms via *Cartan's magic formula*.

**Lemma 4.** *Let $X \in \mathfrak{X}(M)$, then the Lie derivative $\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$ commutes with the exterior derivative, i.e.*

$$\mathcal{L}_X d\omega = d\mathcal{L}_X \omega,$$

*and satisfies the product rule,*

$$\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X \beta).$$

*Proof.* The Lie derivative and exterior derivative commute since pullback and differentiating with respect to time commute with the exterior derivative. We verify the product rule by computation,

$$\mathcal{L}_X(\alpha \wedge \beta) = \frac{d}{dt}\Big|_{t=0} \phi_t^*(\alpha \wedge \beta) = \frac{d}{dt}\Big|_{t=0} (\phi_t^* \alpha \wedge \phi_t^* \beta) = (\mathcal{L}_X \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X \beta).$$

□

**Theorem 5** (Cartan's magic formula)**.** *Let $\omega \in \Omega^k(M)$ be a k-form and suppose $X \in \mathfrak{X}(M)$ is a vector field, then*

$$\mathcal{L}_X \omega = \iota_X d\omega + d\iota_X \omega.$$

*Proof.* The operator $\iota_X d + d\iota_X$ agrees with the Lie derivative $\mathcal{L}_X$ on smooth functions, commutes with the exterior derivative and satisfies the product rule. We prove the result for 1-forms and induct on degree. Let $\omega \in \Omega^1(M)$, which in coordinates takes the form

$$\omega = f_i dx^i.$$

Then applying the three properties, we obtain

$$\mathcal{L}_X \omega = \mathcal{L}_X(f_i dx^i) = \mathcal{L}_X f_i dx^i + f_i \mathcal{L}_X dx^i = (\iota_X d + d\iota_X) f_i dx^i + f_i d\mathcal{L}_X x^i = (\iota_X d + d\iota_X)\omega.$$

Suppose $k \geq 2$ and Cartan's formula holds on $\Omega^j(M)$ for $j < k$, then writing $\omega \in \Omega^k(M)$ as $\omega = \alpha \wedge \beta$ for some $\alpha \in \Omega^j(M)$ and $\beta \in \Omega^l(M)$ with $0 < j, l < k$ we have

$$\mathcal{L}_X \omega = (\mathcal{L}_X \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X \beta) = (\iota_X d + d\iota_X)\omega.$$

This completes the proof. □

**Proposition 6.** *Let $\omega \in \Omega^k(M)$ be a differential k-form and suppose $X, X_1, \ldots, X_k \in \mathfrak{X}(M)$ are vector fields, then*

$$\mathcal{L}_X(\omega(X_1, \ldots, X_k)) = (\mathcal{L}_X \omega)(X_1, \ldots, X_k) + \sum_{i=1}^k \omega(X_1, \ldots, \mathcal{L}_X X_i, \ldots, X_k).$$

*Proof.* By Cartan's formula,

$$(\mathcal{L}_X \omega)(X_1, \ldots, X_k) = (\iota_X d + d\iota_X)\omega(X_1, \ldots, X_k)$$

□

### 3.2. **Vector fields.**

**Proposition 7.** *Let $M$ be a manifold and $X, Y \in \mathfrak{X}(M)$, then*

$$\mathcal{L}_X Y = [X, Y].$$

*Proof.* We compute

$$\begin{aligned}
X(Yf) &= \mathcal{L}_X(Yf) \\
&= \mathcal{L}_X(df(Y)) \\
&= (\mathcal{L}_X df)(Y) + df(\mathcal{L}_X Y) && \text{Cartan's formula} \\
&= (d\mathcal{L}_X f)Y + (\mathcal{L}_X Y)(f) && \mathcal{L}_X \text{ commutes with } d \\
&= d(X(f))Y + (\mathcal{L}_X Y)(f) \\
&= Y(Xf) + (\mathcal{L}_X Y)(f).
\end{aligned}$$

Rearranging gives the result. ☐

**Proposition 8.** *Let $\omega \in \Omega^k(M)$ and $X_1, \ldots, X_{k+1} \in \mathfrak{X}(M)$, then*

$$\begin{aligned}
d\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k} &(-1)^{i+1} X_i(\omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1})) \\
&+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}).
\end{aligned}$$