MAXIMAL FUNCTIONS

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ABSTRACT. Given a family of linear operators $\{T_t\}_{t>0}$ and $f \in L^p(\mathbb{R}^d)$, we are interested in finding the conditions under which the convergence $T_t f \to f$ holds pointwise almost everywhere. Typically one knows that pointwise convergence holds for a dense subclass of "nice" functions, such as test functions $f \in C_c^\infty(\mathbb{R}^d)$ or Schwartz functions $f \in \mathcal{S}(\mathbb{R}^d)$. We would like to pass this result to the limit uniformly in t to extend it to $f \in L^p(\mathbb{R}^d)$. This motivates the analysis of the maximal operator

$$T^* f(x) := \sup_t |T_t f(x)|,$$

and the study of its boundedness on $L^p(\mathbb{R}^d)$. Applications include the ergodic theorem (dynamical systems), the classical Dirichlet problem (PDE), and pointwise a.e. convergence of Fourier series (classical Fourier analysis).

CONTENTS

1. APPROXIMATION TO THE IDENTITY

Let $\phi \in L^1(\mathbb{R}^d)$ such that $\int \phi = 1$, then define

$$\phi_t(x) := \frac{1}{t^d}\phi(x/t).$$

We say that the family $\{\phi_t\}_{t>0}$ forms an APPROXIMATION OF THE IDENTITY. The name comes from the convergence of the family to the Dirac measure at the origin δ_0 in the sense of tempered distributions: for any $f \in \mathcal{S}$, it follows from a change of variables and the dominated convergence theorem that

$$\lim_{t\to 0}\langle \phi_t, f\rangle = \lim_{t\to 0}\int_{\mathbb{R}^d} \frac{1}{t^d}\phi(t/x)f(x)\,dx = \lim_{t\to 0}\int_{\mathbb{R}^d}\phi(x)f(tx)\,dx = g(0) = \langle \delta_0, g\rangle.$$

Since the Dirac measure furnishes the identity with respect to the convolution operation, we further have

$$\lim_{t\to 0}(\phi_t * f)(x) = f(x)$$

for any $f \in \mathcal{S}$ and $x \in \mathbb{R}^d$. We will consider the following question: in what sense does the convergence $\phi_t * f \to f$ hold, and under what conditions? In particular, we are interested in pointwise convergence and convergence in $L^p(\mathbb{R}^d)$. The latter is immediate;

Theorem 1. *Let $1 \leq p < \infty$ and suppose $\{\phi_t\}_t$ forms an approximation of the identity, then*

$$\lim_{t\to 0}||\phi_t * f - f||_{L^p} = 0$$

for $f \in L^p(\mathbb{R}^d)$. The convergence holds in the endpoint case $p = \infty$ when $f \in C_0(\mathbb{R}^d)$.

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Proof. Using $\int \phi = 1$ and a change of variables, we can write

$$(\phi_t * f)(x) - f(x) = \int_{\mathbb{R}^n} \phi(y)(f(x-ty) - f(x))\, dy.$$

Taking the L^p-norm and applying Minkowski's integral inequality to the right, we obtain

$$||\phi_t * f - f||_{L^p} \leq \int_{\mathbb{R}^n} |\phi(y)|\, ||f(x-ty) - f(x)||_{L^p_x}\, dy.$$

We need to exploit smallness of the L^1-norm of ϕ outside a large radius along with smallness of the L^p-norm for $|ty| \ll 1$. For the latter, it follows from the dominated convergence theorem in the case $1 \leq p < \infty$ or uniform continuity in the case $p = \infty$ that for every $\varepsilon > 0$ we can choose $\delta > 0$ such that

$$||f(x+h) - f(x)||_{L^p_x} < \frac{\varepsilon}{2||\phi||_{L^1}}$$

whenever $|h| < \delta$. To control the former, choose $t \ll 1$ such that

$$\int_{|y|>\delta/t} |\phi(y)|\, dy < \frac{\varepsilon}{4||f||_{L^p}}.$$

We conclude

$$\begin{aligned} ||\phi_t * f - f||_{L^p} &\leq \left(\int_{|y| \geq \delta/t} + \int_{|y|<\delta/t} \right) |\phi(y)|\, ||f(x-ty) - f(x)||_{L^p_x}\, dy \\ &\leq 2||f||_{L^p} \int_{|y|>\delta/t} |\phi(y)|\, dy + \frac{\varepsilon}{2||\phi||_{L^1}} \int_{|y|<\delta/t} |\phi(y)|\, dy < \varepsilon. \end{aligned}$$

□

As a consequence, we know that there exists a sequence $t_k \to 0$ such that

$$\lim_{k\to\infty} (\phi_{t_k} * f)(x) = f(x)$$

for a.e. $x \in \mathbb{R}^d$. Hence if the limit of $\phi_t * f$ exists pointwise, then it must equal f a.e. However, $f \in L^p(\mathbb{R}^d)$ is far from sufficient for establishing pointwise convergence. We do however know that the result holds Schwartz functions $f \in \mathcal{S}$, which form a dense subspace of $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$ and $C_0(\mathbb{R}^d)$ for the endpoint $p = \infty$. Therefore, it suffices to show that the set of functions $f \in L^p(\mathbb{R}^d)$ such that $\phi_t * f \to f$ pointwise a.e. forms a closed subspace of $L^p(\mathbb{R}^d)$. It turns out that this is closely related to establishing a weak-type (p, q) bound for the corresponding MAXIMAL OPERATOR defined as

$$M_\phi f(x) := \sup_{t>0} |(\phi_t * f)(x)|.$$

More generally, we have the following theorem:

Theorem 2. *Let (X, μ) be a measure space and $\{T_t\}_{t>0}$ be a family of linear operators on $L^p(X, \mu)$. Suppose that the corresponding* MAXIMAL OPERATOR

$$T^* f(x) := \sup_{t>0} |T_t f(x)|$$

is weak-type (p, q) for exponents $1 \leq p \leq \infty$ and $1 \leq q < \infty$, then the set

$$\{f \in L^p(X, \mu) : \lim_{t\to 0} T_t f(x) = f(x) \text{ a.e.}\}$$

is closed in $L^p(X, \mu)$.

Proof. Let $\{f_n\}_n$ be a sequence of functions converging to $f \in L^p(X, \mu)$ in norm and such that $T_t f_n \to f_n$ pointwise a.e., we want to show that $T_t f \to f$ pointwise a.e. For $x \in X$ such that $T_t f_n(x) \to f_n(x)$, it follows from the triangle inequality that

$$\limsup_{t\to 0} |T_t f(x) - f(x)| \leq \limsup_{t\to 0} |T_t(f - f_n)(x) - (f - f_n)(x)| \leq T^*(f - f_n)(x) + |(f - f_n)(x)|.$$

It follows that

$$\begin{aligned} \mu\{x \in X : \limsup_{t\to 0} |T_t f(x) - f(x)| > \lambda\} &\leq \mu\{x \in X : T^*(f - f_n)(x) > \lambda/2\} + \mu\{x \in X : |(f - f_n)(x)| > \lambda/2\} \\ &\leq \left(\frac{2C}{\lambda} ||f - f_n||_{L^p} \right)^q + \left(\frac{2}{\lambda} ||f - f_n||_{L^p} \right)^p \overset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

where in the first inequality we use sub-additivity of the measure μ, and in the second inequality we apply the weak-type (p,q)-inequality and Chebyshev's inequality. As this holds for all $\lambda > 0$, we can write

$$\mu\{x \in X : \limsup_{t\to 0} |T_t f(x) - f(x)| > 0\} \leq \sum_{k=1}^{\infty} \mu\{x \in \mathbb{R}^d : \limsup_{t\to 0} |T_t f(x) - f(x)| > 1/k\} = 0,$$

which shows $T_t f(x) \to f(x)$ for a.e. $x \in \mathbb{R}^d$, as desired. □

1.1. **One-sided maximal function.** Let $f \in L^1_{\text{loc}}(\mathbb{R})$, then the ONE-SIDED HARDY-LITTLEWOOD MAXIMAL FUNCTION of f is defined by

$$Mf(x) := \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(t)| dt.$$

The operator M is known as the one-sided Hardy-Littlewood maximal operator. It is clear that the operator is sub-linear, and it follows from the dominated convergence theorem that the averaging operator

$$A_h f(x) := \frac{1}{h} \int_x^{x+h} |f(t)| dt$$

is continuous. As the maximal operator is defined pointwise as the supremum of the averaging operators indexed by $h > 0$, it follows that Mf is lower semi-continuous. Thus we can view the maximal operator as a "smoothing" operator, allowing us to make quantitative comparisons between pointwise values of a generic function. Furthermore it controls the approximation to the identity formed by the step function $\phi := \mathbb{1}_{[0,1]}$, so in view of Theorem 2, we aim to show a weak-type (p,q)-inequality to conclude the classical Lebesgue differentiation theorem;

Theorem 3 (Lebesgue differentiation theorem)**.** *Let $f \in L^1_{loc}(\mathbb{R})$, then*

$$f(x) = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt$$

for a.e. $x \in \mathbb{R}$.

Points where the Lebesgue differentiation theorem hold are known as LEBESGUE POINTS. Dimensional analysis shows that no weak-type (p,q) inequality can hold in the off-diagonal case $p \neq q$. Indeed, for $\alpha > 0$, set $f_\alpha(x) := f(x/\alpha)$. Then by a change of variables,

$$Mf_\alpha(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(t/\alpha)|\, dt = \sup_{h>0} \frac{\alpha}{h} \int_{x/\alpha}^{x/\alpha+h/\alpha} |f(t)|\, dt = \sup_{h>0} \frac{1}{h} \int_{x/\alpha}^{x/\alpha+h} |f(t)|\, dt = Mf(x/\alpha)$$

and

$$||f_\alpha||_{L^p} = \left(\int_{\mathbb{R}} f(t/\alpha)\, dt\right)^{1/p} = \alpha^{1/p} ||f||_{L^p}.$$

Therefore a weak-type (p,q) inequality applied to f_α gives

$$\alpha^{1/q} ||Mf||_{L^{q,\infty}} \lesssim \alpha^{1/p} ||f||_{L^p},$$

where the bound can only hold uniformly provided that $p = q$. We will give the original proof of the weak-type $(1,1)$ inequality due to F. Riesz, relying on a precursor of the Calderon-Zygmund decomposition;

Lemma 4 (Rising sun lemma)**.** *Let $F : [a,b] \to \mathbb{R}$ be continuous and let $A \subseteq (a,b)$ be the set of $x \in (a,b)$ such that there exists $y \in (x,b)$ satisfying $F(y) > F(x)$. Then there exists an at most countable collection of disjoint open intervals $\{(a_k, b_k)\}_k$ such that*

$$A = \bigcup_k (a_k, b_k), \qquad F(a_k) \leq F(b_k).$$

Proof. It is clear by definition and continuity that A is open, so, assuming it is non-empty, we know that it takes the form of an at most countable disjoint union of open intervals. It remains to verify $F(a_k) \leq F(b_k)$. Assume otherwise, i.e. $F(a_k) > F(b_k)$. By the extreme value theorem, we can find $x \in [a_k, b_k)$ satisfying

$$F(x) = \max_{y\in[a_k,b_k]} F(y).$$

If $x > a_k$, then by definition of A we know $F(y) > F(x)$ for some $y \in (x,b)$. It follows that

$$F(b_k) < F(a_k) \leq F(x) < f(y).$$

We claim that $y \in (b_k, b)$, which by definition implies $b_k \in A$, a contradiction. Indeed, if the claim fails, then $y \in [a_k, b_k]$, contradicting the choice of x as maximising f on $[a_k, b_k]$. This completes the proof. □

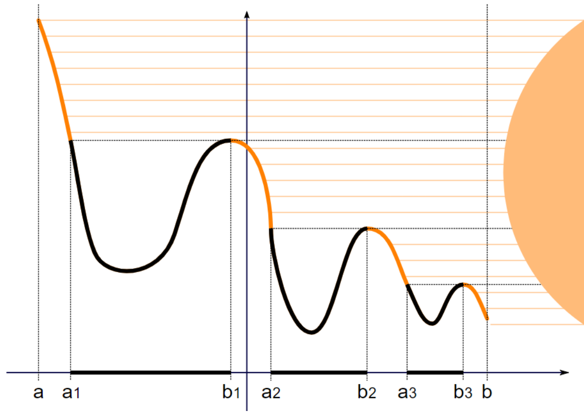


FIGURE 1. The points b_k form the "peaks" of the "hills" casting "shadows" on the region (a_k, b_k) as the "sun rises".

Theorem 5 (One-sided Hardy-Littlewood maximal inequality). *Let $f \in L^1(\mathbb{R})$, then*

$$|\{x \in \mathbb{R} : Mf(x) > \lambda\}| \leq \frac{1}{\lambda} \int_{Mf > \lambda} |f(t)| \, dt.$$

In particular, the maximal operator is weak-type $(1,1)$ and strong-type (p,p) for $1 < p \leq \infty$.

Proof. The strong-type (∞, ∞) inequality is clear, so Marcinkiewicz interpolation furnishes the strong-type (p,p) inequalities for $1 < p \leq \infty$ provided we show the weak-type $(1,1)$-inequality. We argue using the rising sun lemma, letting $F : \mathbb{R} \to \mathbb{R}$ be the continuous function

$$F(x) := \int_{-\infty}^{x} |f(t)| \, dt - \lambda x.$$

Observe that $Mf(x) > \lambda$ if and only if there exists $h > 0$ such that $F(x+h) - F(x) > 0$. This allows us to write the super-level set $Mf(x) > \lambda$ as an ascending union of sets $A_k \subseteq [-k, k]$ given by

$$\begin{aligned} \{x \in \mathbb{R} : Mf(x) > \lambda\} &= \{x \in \mathbb{R} : F(y) > F(x) \text{ for some } y \in (x, \infty)\} \\ &= \bigcup_{k \in \mathbb{N}} \{x \in [-k,k] : F(y) > F(x) \text{ for some } y \in (x,k)\} =: \bigcup_{k \in \mathbb{N}} A_k. \end{aligned}$$

The rising sun lemma yields disjoint open intervals $\{(a_i, b_i)\}_i$ such that

$$A_k = \bigcup_i (a_i, b_i), \qquad F(a_i) \leq F(b_i).$$

It follows from σ-additivity and the construction of F that

$$\begin{aligned} \int_{A_k} |f(t)| \, dt = \sum_i \int_{a_i}^{b_i} |f(t)| \, dt &= \sum_i \lambda(b_i - a_i) + F(b_i) - F(a_i) \\ &\geq \sum_i \lambda(b_i - a_i) = \lambda |A_k|. \end{aligned}$$

Taking $k \to \infty$, monotone convergence furnishes the maximal inequality. $\square$

Remark. A strong-type $(1,1)$ inequality is impossible; suppose without loss of generality that $f \in L^1_{\text{loc}}(\mathbb{R})$ has non-zero mass in $[-1, 0]$, then for $x < -1$ we have the pointwise bound

$$Mf(x) \geq \frac{1}{|x|} \int_x^0 |f(t)| dt \geq \frac{1}{|x|} \int_{-1}^0 |f(t)| dt \gtrsim_f \frac{1}{|x|}.$$

Following the proof of Theorem 2, we obtain as a consequence the Lebesgue differentiation theorem. Making appropriate modifications, we can in fact prove a slightly stronger statement;

Corollary 6. *Let $f \in L^1_{loc}(\mathbb{R})$, then*

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| \, dt = 0$$

for a.e. $x \in \mathbb{R}$.

Proof. The result holds for test functions $C_c^\infty(\mathbb{R})$ by application of uniform continuity. In the general case, taking suitable cutoffs allows us to assume without loss of generality $f \in L^1(\mathbb{R})$. Let $\{f_n\}_n \subseteq C_c^\infty(\mathbb{R})$ converge to f in $L^1(\mathbb{R})$ and pointwise a.e. By the triangle inequality,

$$\limsup_{h\to 0} \frac{1}{h}\int_x^{x+h} |f(t)-f(x)|\,dt \le M(f-f_n)(x) + |f_n(x)-f(x)|.$$

It follows that

$$\begin{aligned}|\{x\in\mathbb{R} : \limsup_{h\to 0} \frac{1}{h}\int_x^{x+h} |f(t)-f(x)|\,dt > \lambda\}| &\le |\{M(f-f_n) > \frac{\lambda}{2}\}| + |\{|f_n - f| > \frac{\lambda}{2}\}| \\ &\le \frac{2}{\lambda}||f-f_n||_{L^1} + \frac{2}{\lambda}||f-f_n||_{L^1} \overset{n\to\infty}{\longrightarrow} 0\end{aligned}$$

where in the first inequality we use sub-additivity of the Lebesgue measure, in the second inequality we apply the weak-type $(1,1)$-inequality and Markov's inequality. As this holds for all $\lambda > 0$, we can write

$$|\{x\in\mathbb{R} : \limsup_{h\to 0} \int_x^{x+h} |f(t)-f(x)|\,dt > 0\}| \le \sum_{k=1}^{\infty} |\{x\in\mathbb{R} : \limsup_{h\to 0}\int_x^{x+h} |f(t)-f(x)|\,dt > 1/k\}| = 0$$

which completes the proof. ☐

Remark. Because the proof relies on a density argument, it offers no quantitative rate for the speed of convergence. Indeed, the convergence can be arbitrarily slow.

1.2. **Maximal function on $\mathbb{R}^d$.** Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, the HARDY-LITTLEWOOD MAXIMAL FUNCTION of f is defined by

$$Mf(x) := \sup_{r>0} \frac{1}{B_r(x)}\int_{B_r(x)} |f(y)|dy.$$

The study of this operator is almost completely analogous to the one-sided operator, however we require a different proof of the weak-type $(1,1)$ which does not rely on the available order structure when $d = 1$. We replace the rising sun lemma with a covering argument and the scaling property of Lebesgue measure.

Lemma 7 (Vitali-Wiener covering lemma)**.** *Given a finite collection of balls $\{B_{r_j}(x_j)\}_{j\in J}$, there exists a sub-collection $I \subseteq J$ of pair-wise disjoint balls such that*

$$\bigcup_{j\in J} B_{r_j}(x_j) \subseteq \bigcup_{i\in I} B_{3r_i}(x_i).$$

Proof. We construct the sub-collection by running the following algorithm:

(a) Add the ball of largest radius to the sub-collection.
(b) Discard all balls intersecting the sub-collection.
(c) If no balls remain, then we are done. Otherwise, we iterate the algorithm.

By construction, the sub-collection consists of pair-wise disjoint balls. Let $B_{r_i}(x_i)$ be a ball added in step (a) and $B_{r_j}(x_j)$ be a ball discarded in step (b). We chose r_i as the maximum radius of all balls at that point in the procedure, so it follows that $r_j \le r_i$. Hence by the triangle inequality

$$B_{r_j}(x_j) \subseteq B_{3r_i}(x_i).$$

A ball must be either added or discarded, so the algorithm is exhaustive. ☐

Theorem 8 (Hardy-Littlewood maximal inequality)**.** *The Hardy-Littlewood maximal operator M is weak-type $(1,1)$ and strong-type (p,p) for $1 < p \le \infty$.*

Proof. The strong-type (∞,∞) inequality is clear, so Marcinkiewicz interpolation furnishes the strong-type (p,p) inequalities for $1 < p \le \infty$ provided we show the weak-type $(1,1)$ inequality. Let $K \subseteq \mathbb{R}^d$ be a compact subset satisfying

$$K \subseteq \{x\in\mathbb{R}^d : Mf(x) > \lambda\}.$$

We claim that

$$|K| \lesssim_d \frac{1}{\lambda}||f||_{L^1}.$$

The weak-type $(1,1)$ inequality follows immediately from inner regularity of the Lebesgue measure and monotone convergence. By construction, for every $x \in K$ there exists $r(x) > 0$ such that

$$|B_{r(x)}(x)| \leq \frac{1}{\lambda} \int_{B_{r(x)}(x)} |f(y)| \, dy.$$

The collection $\{B_{r(x)}(x)\}_x$ forms a cover of K, so we use compactness to extract a finite sub-cover, and the Vitali-Wiener covering lemma to extract a sub-collection of disjoint balls B_j satisfying $K \subseteq \bigcup_j 3B_j$. Thus

$$|K| \leq \sum_j |3B_j| = 3^d \sum_j |B_j| = \frac{3^d}{\lambda} \sum_j \int_{B_j} |f(y)| \, dy \leq \frac{3^d}{\lambda} ||f||_{L^1},$$

proving the claim, as desired. □

We finish this section with an answer to our original question of a.e. pointwise convergence when convolving against an approximation to the identity.

Theorem 9. *Suppose $\{\phi_t\}_t$ forms an approximation to the identity such that $|\phi| \leq \psi$ for radially decreasing $\psi \in L^1(\mathbb{R}^d)$, then the maximal operator M_ϕ is weak-type $(1,1)$ and strong-type (p,p) for $1 < p \leq \infty$. Furthermore*

$$\lim_{t \to 0} (\phi_t * f)(x) = f(x)$$

for a.e. $x \in \mathbb{R}^d$ when $f \in L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$ and $f \in C_0(\mathbb{R}^d)$ in the endpoint case $p = \infty$.

Proof. The maximal operator M_ϕ inherits the weak-type $(1,1)$ and strong-type (p,p) inequalities from the Hardy-Littlewood maximal operator provided that we verify the pointwise bound

$$M_\phi f(x) \leq ||\psi||_{L^1} Mf(x).$$

The pointwise convergence result would then follow from Theorem 2. Fix $t > 0$ and observe that $|\phi_t * f| \leq \psi_t * |f|$. We can find a sequence of non-negative radially decreasing simple functions $\phi_k := \sum_j a_{j,k} \mathbb{1}_{B_{j,k}}$, where $B_{j,k}$ are balls centered at the origin and $a_{j,k} > 0$, increasing pointwise to ϕ_t as $k \to \infty$. Observe that

$$||\phi_k||_{L^1} = \sum_j a_{j,k} |B_{j,k}|, \qquad \lim_{k \to \infty} ||\phi_k||_{L^1} = ||\psi_t||_{L^1} = ||\psi||_{L^1}.$$

Moreover, by definition of the maximal function, $(\mathbb{1}_{B_{j,k}} * |f|)(x) \leq |B_{j,k}| Mf(x)$. Collecting our results, we conclude from monotone convergence that

$$|(\phi_t * f)(x)| \leq \lim_{k \to \infty} \sum_j a_{j,k} (\mathbb{1}_{B_{j,k}} * |f|)(x) \leq ||\psi||_{L^1} Mf(x).$$

Since t was arbitrary, this completes the proof. □

Remark. As an application to partial differential equations, define the Poisson kernel by

$$P_t(x) := \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}}.$$

One can show that $u(t,x) := (P_t * f)(x)$ solves the Laplace's equation $\Delta u = 0$ on the upper-half plane $\mathbb{R}^{d+1}_+$, with, by the theorem, the boundary data $u(0,x) = f(x)$ satisfied a.e. whenever $f \in L^p(\mathbb{R}^d)$. Similarly, define the Gauss-Weierstrass kernel by

$$W_t(x) := t^{-n} e^{\pi |x|^2 / t^2}.$$

One can show that $u(t,x) := (W_t * f)(x)$ solves the heat equation $\partial_t u - \Delta u = 0$ with, by the theorem, the initial data $u(0,x) = f(x)$ satisfied a.e. whenever $f \in L^p(\mathbb{R}^d)$.

2. Weighted maximal inequalities

It is of interest to characterise the non-negative Borel measures $d\mu$ such that the maximal operator M satisfies a strong-type (p,p) inequality with respect to $d\mu$, that is,

$$||Mf||_{L^p(d\mu)} \lesssim_{p,d} ||f||_{L^p(d\mu)}$$

for some $1 < p < \infty$. The proof of the Hardy-Littlewood maximal inequality relied on the scaling property of the Lebesgue measure $|\alpha E| = \alpha^d |E|$ for any measurable $E \subseteq \mathbb{R}^d$ and scalar $\alpha > 0$, however it would have sufficed to use the weaker "doubling" property $|2B| \lesssim |B|$ for any ball $B \subseteq \mathbb{R}^d$. More generally, we say that a Radon measure μ on $\mathbb{R}^d$ is a DOUBLING MEASURE if

$$\sup_{x \in \mathbb{R}^d,\, r>0} \frac{\mu(B_{2r}(x))}{\mu(B_r(x))} < \infty.$$

Define the maximal operator with respect to μ by

$$M_\mu f(x) := \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)|\, d\mu(y).$$

Then replicating the proof of the Hardy-Littlewood maximal inequality furnishes an analogous result for M_μ,

Theorem 10. *Let μ be a doubling measure on $\mathbb{R}^d$. The maximal operator M_μ is weak-type $(1,1)$ and strong-type (p,p) for $1 < p \leq \infty$ with respect to μ, i.e.*

$$||M_\mu f||_{L^{1,\infty}(d\mu)} \lesssim ||f||_{L^1(d\mu)}, \qquad ||M_\mu f||_{L^p(d\mu)} \lesssim ||f||_{L^p(d\mu)}.$$

This shows that $Mf \lesssim Mf_\mu$ is a sufficient condition for the usual maximal operator M satisfying a weak-type $(1,1)$ inequality and strong-type (p,p) inequality with respect to μ. Furthermore, we can restrict our attention to measures of the form $d\mu = \omega dx$ for some non-negative locally integrable $\omega : \mathbb{R}^d \to [0,\infty)$.

Theorem 11. *Let μ be a non-negative Borel measure on $\mathbb{R}^d$ and $1 \leq p < \infty$. If the maximal operator satisfies the weak-type (p,p) inequality with respect to μ, that is,*

$$\lambda \mu(\{x \in \mathbb{R}^d : Mf(x) > \lambda\})^{1/p} \lesssim ||f||_{L^p(d\mu)}$$

then $d\mu \ll dx$.

Proof. Let $K \subseteq \mathbb{R}^d$ be compact such that $|K| = 0$, we want to show that $\mu(K) = 0$. Define

$$U_n := \{x \in \mathbb{R}^d : \operatorname{dist}(x,K) < 1/n\}.$$

These are nested open neighborhoods of K satisfying $\bigcup_n U_n = K$. Setting $f_n := \mathbb{1}_{U_n \setminus K}$, by the dominated convergence theorem

$$\int_{\mathbb{R}^d} |f_n|^p d\mu \overset{n\to\infty}{\longrightarrow} 0.$$

Let $x \in K$, then since K has Lebesgue measure zero,

$$Mf_n(x) \geq \frac{1}{|B_{1/n}(x)|} \int_{B_{1/n}(x)} \mathbb{1}_{U_n/K}(y)\, dy = \frac{1}{|B_{1/n}(x)|} \int_{\mathbb{R}^d \setminus K} \mathbb{1}_{B_{1/n}(x)}(y)\, dy = \frac{1}{|B_{1/n}(x)|} \int_{\mathbb{R}^d} \mathbb{1}_{B_{1/n}(x)}(y)\, dy = 1.$$

It follows from the weak-type (p,p) inequality that

$$\mu(K) \leq \mu(\{x \in \mathbb{R}^d : Mf_n(x) > 1/2\}) \lesssim \frac{\int_{\mathbb{R}^d} |f_n|^p\, d\mu}{(1/2)^p} \overset{n\to\infty}{\longrightarrow} 0.$$

This completes the proof. $\square$

2.1. A_1 **condition.** We say that $\omega : \mathbb{R}^d \to [0, \infty)$ is a WEIGHT if it is non-negative, locally integrable, and $\omega \not\equiv 0$. We abuse notation by writing ω for the associated measure

$$\omega(E) := \int_E \omega(y)\, dy.$$

The weight satisfies the A_1 CONDITION, writing $\omega \in A_1$, if

$$M\omega(x) \lesssim \omega(x)$$

for a.e. $x \in \mathbb{R}^d$.

Proposition 12. *Let $\omega : \mathbb{R}^d \to [0, \infty)$ be a weight. The following are equivalent:*

(a) $\omega \in A_1$.

(b) For all balls $B \subseteq \mathbb{R}^d$ and a.e. $x \in B$, we have

$$\frac{1}{|B|} \int_B \omega(y)\, dy \lesssim \omega(x).$$

(c) For all balls $B \subseteq \mathbb{R}^d$ and measurable $f \geq 0$, we have

$$\frac{1}{|B|} \int_B f(y)\, dy \lesssim \frac{1}{\omega(B)} \int_B f(y)\, \omega(y)\, dy.$$

Proof. (a) $\implies$ (b). Fix a ball $B \subseteq \mathbb{R}^d$ of radius $r > 0$, By the A_1 condition, we can choose a generic $x \in B$ such that $M\omega(x) \lesssim \omega(x)$. It follows that

$$\frac{1}{|B|} \int_B \omega(y)\, dy \leq \frac{1}{|B|} \int_{B_{2r}(x)} \omega(y)\, dy \leq \frac{|B_{2r}(x)|}{|B|} M\omega(x) \lesssim 2^d \omega(x).$$

(b) $\implies$ (c). Fix a ball $B \subseteq \mathbb{R}^d$ and measurable $f \geq 0$. We can write

$$\begin{aligned}\frac{1}{|B|} \int_B f(y)\, dy &= \frac{1}{\omega(B)} \int_B f(y) \left(\frac{1}{|B|} \int_B \omega(z)\, dz \right) dy \\ &\lesssim \frac{1}{\omega(B)} \int_B f(y)\, \omega(y)\, dy.\end{aligned}$$

(c) $\implies$ (a). Let $x \in \mathbb{R}^d$ be a Lebesgue point of ω. Fix $0 < r < R$, letting $B := B_R(x)$ and $f := \mathbb{1}_{B_r(x)}$ in (c) gives

$$\frac{|B_r(x)|}{|B_R(x)|} \lesssim \frac{\omega(B_r(x))}{\omega(B_R(x))}. \tag{*}$$

Rearranging, we obtain

$$\frac{1}{|B_R(x)|} \int_{B_R(x)} \omega(y)\, dy \lesssim \frac{1}{|B_r(x)|} \int_{B_r(x)} \omega(y)\, dy \xrightarrow{r \to 0} \omega(x).$$

Taking the supremum over R on the left, we conclude $M\omega(x) \lesssim \omega(x)$ for every Lebesgue point $x \in \mathbb{R}^d$, completing the proof. ☐

Remark. The characterisation (b) implies that $\widetilde{M}\omega(x) \lesssim \omega(x)$, where $\widetilde{M}$ is the UNCENTERED MAXIMAL FUNCTION

$$\widetilde{M}f(x) := \sup_{B \ni x \text{ ball}} \int_B |f(y)|\, dy.$$

The inequality (*) implies that ω is a doubling measure, while (c) implies $Mf \lesssim M_\omega f$. It follows from Theorem 10 that the maximal operator M is weak-type $(1, 1)$ and strong-type (p, p) for $1 < p \leq \infty$ with respect to the measure ω. In fact, the A_1 condition characterises exactly the measures for which the weighted weak-type $(1, 1)$ inequality holds.

Theorem 13. *Let $\omega : \mathbb{R}^d \to [0, \infty)$ be a weight. Then the maximal operator M is weak-type $(1, 1)$ with respect to ω, i.e.*

$$\omega(\{x \in \mathbb{R}^d : Mf(x) > \lambda\}) \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^d} |f(x)| \omega(x)\, dx$$

if and only if $\omega \in A_1$.

Proof. The converse was shown in the remark above, so it remains to establish the forward implication. We will aim towards the characterisation (c) in Proposition 12 of the A_1 condition. For any $x \in B_r(x_0)$, we have $B_r(x_0) \subseteq B_{2r}(x)$ and $2^d|B_r(x_0)| = |B_{2r}(x)|$. Thus

$$\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} f(y)\,dy \le \frac{2^d}{|B_{2r}(x)|}\int_{B_{2r}(x)} f(y)\,dy$$

for any $f \ge 0$ measurable. It follows that

$$\frac{2^{-d-1}}{|B_r(x_0)|}\int_{B_r(x_0)} f(y)\,dy < Mf(x).$$

Suppose f is supported in $B_r(x_0)$, then taking λ equal to the left-hand side in the weak-type $(1,1)$-inequality and rearranging, we obtain

$$\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} f(y)\,dy \lesssim \frac{1}{\omega(B_r(x_0))}\int_{B_r(x_0)} f(x)\omega(x)\,dx,$$

as desired. □

2.2. A_p **condition.** Let $1 < p < \infty$, we say that a weight $\omega : \mathbb{R}^d \to [0,\infty)$ satisfies the A_p CONDITION, writing $\omega \in A_p$, if

$$\sup_{B\subseteq\mathbb{R}^d \text{ balls}} \left(\frac{1}{|B|}\int_B \omega(y)\,dy\right)\left(\frac{1}{|B|}\int_B \omega(y)^{-p'/p}dy\right)^{p/p'} < \infty.$$

We note for convenience that the above condition is equivalent to

$$\sup_{B\subseteq\mathbb{R}^d \text{ balls}} \frac{\omega(B)}{|B|^p}||\omega^{-\frac{1}{p-1}}||_{L^1(B)}^{p-1} < \infty.$$

We record some easy properties of the A_p class,

(a) The A_p condition is invariant under translation $\omega(x) \mapsto \omega(x-x_0)$, scalar multiplication $\omega(x) \mapsto \lambda\omega(x)$, and rescaling $\omega(x) \mapsto \omega(\lambda x)$.
(b) The A_p class is increasing in p, that is, $A_p \subseteq A_q$ whenever $1 \le p < q < \infty$; this follows from Holder's inequality,

$$\begin{aligned}||\omega^{-\frac{1}{q-1}}||_{L^1(B)}^{q-1} &\le \left(||\omega^{-\frac{1}{q-1}}||_{L^{\frac{q-1}{p-1}}(B)}|B|^{1-\frac{p-1}{q-1}}\right)^{q-1}\\ &\le ||\omega^{-\frac{1}{p-1}}||_{L^{p-1}(B)}^{p-1}|B|^{q-p}.\end{aligned}$$

(c) We have $\omega \in A_p$ if and only if $\omega^{-p'/p} \in A_{p'}$.

Proposition 14. *Let $\omega : \mathbb{R}^d \to [0,\infty)$ be a weight and $1 < p < \infty$. The following are equivalent:*

(a) $\omega \in A_p$.
(b) For balls $B \subseteq \mathbb{R}^d$ and measurable $f \ge 0$, we have

$$\left(\frac{1}{|B|}\int_B f(y)\,dy\right)^p \lesssim \frac{1}{\omega(B)}\int_B f(y)^p\omega(y)\,dy.$$

Proof. (a) $\implies$ (b). Fix a ball $B \subseteq \mathbb{R}^d$. By Holder's inequality

$$\frac{1}{|B|}\int_B f(y)\,dy \le \frac{1}{|B|}\left(\int_B f(y)^p\omega(y)\,dy\right)^{1/p}\left(\int_B \omega(y)^{-p'/p}\,dy\right)^{1/p'}.$$

Thus by the A_p condition

$$\left(\frac{1}{|B|}\int_B f(y)\,dy\right)^p \le \frac{1}{|B|^p}\left(\int_B f(y)^p\omega(y)\,dy\right)\left(\int_B \omega(y)^{-p'/p}\,dy\right)^{p/p'} \lesssim \frac{1}{\omega(B)}\int_B f(y)^p\omega(y)\,dy.$$

(b) $\implies$ (a). Fix $\varepsilon > 0$ and set $f := (\omega+\varepsilon)^{-\frac{1}{p-1}}$. Then

$$\begin{aligned}\frac{\omega(B)}{|B|^p}\left(\int_B (\omega+\varepsilon)^{-\frac{1}{p-1}}\,dy\right)^p &\lesssim \int_B (\omega+\varepsilon)^{-\frac{p}{p-1}}\omega\,dy \\ &\lesssim \int_B (\omega+\varepsilon)^{-\frac{1}{p-1}}\,dy.\end{aligned}$$

Rearranging,

$$\frac{\omega(B)}{|B|^p}\left(\int_B (\omega+\varepsilon)^{-\frac{1}{p-1}}\,dy\right)^{p-1} \lesssim 1.$$

By monotone convergence, letting $\varepsilon \to 0$ we conclude $\omega \in A_p$. □

Remark. Letting $f := \mathbb{1}_{B_r(x)}$ and $B := B_{2r}(x)$ in (b) gives

$$\left(\frac{|B_r(x)|}{|B_{2r}(x)|}\right)^p \lesssim \frac{\omega(B_r(x))}{\omega(B_{2r}(x))},$$

which implies A_p weights are doubling measures. Furthermore, (b) implies $|Mf|^p \lesssim M_\omega(|f|^p)$, so it follows from Theorem 10 that the maximal operator M is weak-type (p,p). If we were to show that an A_p weight is an A_q weight for some $1 < q < p$, then M would furthermore satisfy a strong-type (p,p) inequality by interpolating between the weak-type (q,q) inequality and the trivial strong-type (∞,∞) inequality. This follows from a reverse Holder's inequality.

Lemma 15 (Reverse Holder's inequality)**.** *Let $\omega \in A_p$, then there exists $r > 0$ such that*

$$\left(\frac{1}{|B|}\int_B \omega(y)^r\,dy\right)^{1/r} \lesssim \frac{1}{|B|}\int_B \omega(y)\,dy$$

uniformly over all balls $B \subseteq \mathbb{R}^d$.

Theorem 16. *Let $\omega : \mathbb{R}^d \to [0,\infty)$ be a weight and $1 \le p < \infty$. The maximal operator M is strong-type (p,p) with respect to ω if and only if $\omega \in A_p$.*

Proof. The proof of the forward implication is analogous to that of Theorem 13. To prove the converse, by Marcinkiewicz interpolation it suffices to show that $\omega \in A_q$ for some $1 < q < p$. Recall that $\omega \in A_p$ if and only if $\omega^{-p'/p} \in A_{p'}$. Applying the reverse Holder inequality to the latter, there exists $r > 1$ such that

$$\left(\frac{1}{|B|}\int_B \omega(y)^{-p'r/p}\,dy\right)^{1/r} \lesssim \frac{1}{|B|}\int_B \omega(y)^{-p'/p}\,dy.$$

Recall $\omega \in A_p$ if and only if

$$\frac{\omega(B)}{|B|}\left(\frac{1}{|B|}\int_B \omega(y)^{-p'/p}dy\right)^{p/p'} \lesssim 1.$$

Combining with the reverse Holder inequality, we obtain

$$\left(\frac{\omega(B)}{|B|}\right)^{p'/p}\left(\frac{1}{|B|}\int_B \omega(y)^{-p'r/p}\,dy\right)^{1/r} \lesssim 1.$$

Since $r > 1$, there exists $1 < q < p$ such that $p'r/p = q'/q$, i.e. $q - 1 = (p-1)/r$. Rewriting the exponents above in terms of q and raising the exponents on both sides by p/p', we obtain

$$\frac{\omega(B)}{|B|}\left(\frac{1}{|B|}\int_B \omega(y)^{-q'/q}\,dy\right)^{q/q'} \lesssim 1,$$

i.e. $\omega \in A_q$. This completes the proof. □

3. Vector-valued maximal function

The weak-type $(1,1)$ inequality and strong-type (p,p) inequalities extend to functions $f : \mathbb{R}^d \to \ell^q(\mathbb{N})$ taking values in the sequence spaces for $1 < q \leq \infty$. We denote the norms

$$|f(x)| := ||f(x)||_{\ell^q_n}, \qquad ||f||_{L^p} := \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{1/p}.$$

The vector-valued maximal function of f is defined by

$$\overline{M}_q f(x) := ||Mf_n(x)||_{\ell^q_n}.$$

Theorem 17 (Vector-valued maximal inequality). *Let $1 < p, q < \infty$ and $f : \mathbb{R}^d \to \ell^q(\mathbb{N})$, then the vector-valued maximal operator $\overline{M}_q$ satisfies the weak-type $(1,1)$ inequality,*

$$|\{x \in \mathbb{R}^d : \overline{M}_q(x) > \lambda\}| \lesssim_{d,p} \frac{1}{\lambda} ||f||_{L^1},$$

and the strong-type (p,p) inequality,

$$||\overline{M}_q f||_{L^p} \lesssim_{d,p} ||f||_{L^p}.$$

Remark.

- Both the weak-type $(1,1)$ and strong-type (p,p) inequalities fail in the case $q = 1$. Fix $N \in \mathbb{N}$, we divide the unit interval into sub-intervals of length $1/N$,
$$[0,1] = [0,1/N] \cup \cdots \cup [(N-1)/N, 1] =: I_1 \cup \cdots \cup I_N.$$
Define $f_N : \mathbb{R} \to \ell^1_n(\mathbb{N})$ by
$$f_N := (\mathbb{1}_{I_1}, \ldots, \mathbb{1}_{I_N}, 0, \ldots).$$
Then $|f_N| = \mathbb{1}_{[0,1]}$ and $||f_N||_{L^p} = 1$. On the other hand, for any $x \in [0,1]$, observe $I_n \subseteq [x - n/N, x + n/N]$ and $I_n \subseteq [x - N/2, x + N/2]$. Arguing combinatorially, we can bound below $\overline{M}_1 f_N$ pointwise by
$$\begin{aligned} \overline{M}_1 f_N(x) &= \sum_{n=1}^{N} \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} \mathbb{1}_{I_n}(y) \, dy \\ &\gtrsim \sum_{n=1}^{\lfloor N/2 \rfloor} \frac{1}{n/N} |I_n| \gtrsim \sum_{n=1}^{\lfloor N/2 \rfloor} \frac{1}{n} \gtrsim \log N \end{aligned}$$
for any $x \in [0,1]$. This shows that $||\overline{M}_1 f_N||_{L^{p,\infty}} \gtrsim \log N$ for $N \gg 1$.
- The strong-type (∞,∞) bound fails dramatically for all $1 < q < \infty$ in that there exists a bounded function $f : \mathbb{R} \to \ell^q(\mathbb{N})$ such that $\overline{M}_q f \equiv \infty$. Define
$$f := (\mathbb{1}_{[2^{n-1}, 2^n]})_{n \in \mathbb{N}}.$$
Then $|f(x)| = \mathbb{1}_{[0,\infty)}$ and $||f||_{L^\infty} = 1$. On the other hand, observe that $[2^{n-1}, 2^n] \subseteq [x - 2^{n+1}, x + 2^{n+1}]$ for any $|x| \leq 2^n$. We can therefore bound below the maximal function pointwise by
$$M\mathbb{1}_{[2^{n-1},2^n]}(x) \geq \frac{1}{2^{n+2}} \int_{[x-2^{n+1}, x+2^{n+1}]} \mathbb{1}_{[2^{n-1},2^n]}(y) \, dy \geq \frac{1}{8}$$
for any $|x| \leq 2^n$. Hence
$$\overline{M}_q f(x) = \left(\sum_{n \,:\, 2^n \geq |x|} \left| M\mathbb{1}_{[2^{n-1},2^n]}(x) \right|^q \right)^{1/q} \geq \left(\sum_{n \,:\, 2^n \geq |x|} \frac{1}{8^q} \right)^{1/q} = \infty.$$

The case $q = \infty$ follows from the usual scalar-valued maximal inequality since

$$\overline{M}_\infty f = ||Mf_n||_{\ell^\infty_n} \leq M||f_n||_{\ell^\infty_n} = M|f|.$$

As remarked, the strong-type (∞,∞) inequality fails in the case $1 < q < \infty$, and so we need to take a more subtle approach paralleling the proof of boundedness for Calderon-Zygmund operators. Interchanging the sum and integral and applying the scalar-valued maximal inequality gives

$$||\overline{M}_p f||_{L^p}^p = \int_{\mathbb{R}^d} \sum_{n \in \mathbb{N}} |Mf_n(x)|^p \, dx \lesssim \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^d} |f_n(x)|^p \, dx = ||f||_{L^p}^p.$$

This furnishes the strong-type (p,p) inequality for the diagonal $p=q$. The case $1<p<q$ reduces by Marcinkiewicz interpolation to showing the weak-type $(1,1)$ inequality, which we will prove via a Calderon-Zygmund decomposition. The case $q<p<\infty$ follows from duality and a weighted maximal inequality.

3.1. **The case $p \leq q$.** We prove the weak-type $(1,1)$-inequality via the higher dimensional successor of the rising sun lemma, relying crucially on the dyadic structure of $\mathbb{R}^d$:

Lemma 18 (Calderon-Zygmund decomposition). *Let $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$, there exists a decomposition $f = g + b$, where g is the "good" part and b is the "bad" part, such that*

(a) $|g| \leq \lambda$ a.e.,

(b) $b = f\mathbb{1}_{\bigcup_k Q_k}$, where $\{Q_k\}_k$ is a collection of cubes with pair-wise disjoint interiors satisfying

$$\lambda < \frac{1}{|Q_k|}\int_{Q_k} |b(y)|dy \leq 2^d\lambda.$$

Proof. Since $f \in L^1(\mathbb{R}^d)$, we can sub-divide $\mathbb{R}^d$ into dyadic cubes $Q \subseteq \mathbb{R}^d$ satisfying

$$\frac{1}{|Q|}\int_Q |f(y)|dy \leq \lambda.$$

We run the following algorithm: fixing one such cube Q, we sub-divide it into 2^d congruent dyadic cubes. Consider one of these smaller cubes $Q' \subseteq Q$, if it satisfies

$$\frac{1}{|Q'|}\int_{Q'} |f(y)|dy > \lambda \tag{*}$$

then we stop the algorithm and add Q' to the collection of cubes in the support of b. Such a cube satisfies

$$\lambda < \frac{1}{|Q'|}\int_{Q'} |f(y)|dy \leq \frac{2^d}{|Q|}\int_Q |f(y)|dy \leq 2^d\lambda.$$

If Q' does not satisfy (*), we continue the algorithm, further sub-dividing Q' into 2^d congruent dyadic cubes and examining each one. It remains only to check $|g| \leq \lambda$ a.e. Suppose $x \notin \bigcup_k Q_k$, then by construction the average of $|f|$ is bounded by λ for any dyadic cube containing x. Let $x \in Q$, we can find a radius $r_Q > 0$ such that $B_{r_Q}(x) \subseteq Q$ and $|B_{r_Q}(x)| \sim |Q|$. Since Lebesgue points are generic, we conclude

$$|f(x)| = \lim_{r\to 0}\frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)|\,dy \lesssim \lim_{x\in Q,\,\operatorname{diam} Q\to 0}\frac{1}{|Q|}\int_Q |f(y)|\,dy < \lambda$$

for a.e. x in the support of g. □

Proof of Theorem 17 for $p \leq q$. It suffices by the strong-type (q,q) inequality and Marcinkiewicz interpolation to establish the weak-type $(1,1)$ inequality. For $f \in L^1(\mathbb{R}^d)$, we perform a Calderon-Zygmund decomposition $f = g + b$ at the level $\lambda > 0$. By the triangle inequality,

$$|\{x \in \mathbb{R}^d : \overline{M}_q f(x) > \lambda\}| \leq |\{x \in \mathbb{R}^d : \overline{M}_q g(x) > \lambda/2\}| + |\{x \in \mathbb{R}^d : \overline{M}_q b(x) > \lambda/2\}|.$$

We claim that the contributions from the "good" and "bad" parts given on the right are controlled by $||f||_{L^1}/\lambda$, which would complete the proof. It follows from Chebyshev's inequality, the strong-type (q,q) inequality, and the "good" inequality $|g| \leq \lambda$ a.e. that the "good" part satisfies

$$|\{x \in \mathbb{R}^d : \overline{M}_q g(x) > \lambda/2\}| \leq \frac{||\overline{M}_q g||_{L^q}^q}{(\lambda/2)^q} \lesssim \frac{||g||_{L^q}^q}{\lambda^q} \leq \frac{||g||_{L^1}}{\lambda} \leq \frac{||f||_{L^1}}{\lambda}.$$

The result above continues to hold replacing the "good" part by the average of the "bad" part on every cube,

$$b_n^{\text{ave}} := \sum_k \mathbb{1}_{Q_k}\frac{1}{|Q_k|}\int_{Q_k} |b_n(y)|\,dy.$$

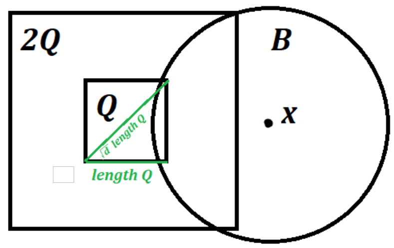
Indeed, by construction of b, we have the "good" inequality for b^{ave},

$$|b^{\text{ave}}(x)| \leq \frac{1}{|Q_k|}\int_{Q_k} |b(y)|\,dy \leq 2^d\lambda.$$

We claim that $\overline{M}_q b \lesssim \overline{M}_q b^{\text{ave}}$ whenever $x \notin \bigcup_k 2Q_k$; it would follow that

$$
\begin{aligned}
|\{x \in \mathbb{R}^d : \overline{M}_q b(x) > \lambda/2\}| &\leq \sum_k |2Q_k| + |\{x \notin \bigcup_k 2Q_k : \overline{M}_q b(x) > \lambda/2\}| \\
&\leq \frac{2^d}{\lambda} \sum_k \int_{Q_k} |b(y)|\, dy + |\{x \notin \bigcup_k 2Q_k : \overline{M}_q b^{\text{ave}}(x) \gtrsim \lambda\}| \lesssim \frac{||f||_{L^1}}{\lambda},
\end{aligned}
$$

completing the proof. Fix $x \notin \bigcup_k 2Q_k$ and choose $r > 0$ such that $B_r(x)$ intersects Q_k for some k. It follows that $2r >$ length Q_k, so $Q_k \subseteq B_{r+\text{length}\, Q_k \sqrt{d}}(x) \subseteq B_{r(1+2\sqrt{d})}(x)$.



Since b is supported in $\bigcup_k Q_k$, we can write

$$
\begin{aligned}
Mb_n(x) &= \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |b_n(y)|\, dy = \sup_{r>0} \frac{1}{|B_r(x)|} \sum_k \int_{Q_k \cap B_r(x)} |b_n(y)|\, dy \\
&\lesssim \sup_{r>0} \frac{1}{|B_{r(1+2\sqrt{d})}(x)|} \int_{B_{r(1+2\sqrt{d})}(x)} \sum_k \mathbb{1}_{Q_k}(z) \left(\frac{1}{|Q_k|} \int_{Q_k} |b_n(y)|\, dy \right) dz \leq \overline{M} b_n^{\text{ave}}(x),
\end{aligned}
$$

for every n. This proves the claim, concluding the proof. □

3.2. **The case** $p \geq q$**.** To complete the analogy with Calderon-Zygmund operators, we need a self-adjointness-type result for the maximal operator. This takes the form of a weighted maximal inequality.

Theorem 19 (Weighted maximal inequality)**.** *Let $\omega : \mathbb{R}^d \to [0, \infty)$ be a weight and $1 < p \leq \infty$, then the maximal operator M satisfies the bounds*

$$
||Mf||_{L^{1,\infty}(\omega dx)} \lesssim ||f||_{L^1(M\omega dx)}, \qquad ||Mf||_{L^p(\omega dx)} \lesssim ||f||_{L^p(M\omega dx)}
$$

for scalar-valued $f : \mathbb{R}^d \to \mathbb{C}$.

Proof. Appealing to Marcinkiewicz interpolation, the strong-type (p, p) inequality holds provided we show the weak-type $(1, 1)$ inequality and strong-type (∞, ∞) inequality. The latter follows from the usual strong-type (∞, ∞) inequality, remarking that $\omega dx \ll dx$ and $M\omega dx \ll dx$ and $dx \ll M\omega dx$.

It remains to show the weak-type $(1, 1)$ inequality. Let $K \subseteq \mathbb{R}^d$ be a compact subset satisfying

$$
K \subseteq \{x \in \mathbb{R}^d : Mf(x) > \lambda\}.
$$

We claim that

$$
\omega(K) \lesssim \frac{1}{\lambda} ||f||_{L^1(M\omega dx)}.
$$

The weak-type $(1, 1)$ inequality follows immediately from inner regularity of ωdx and monotone convergence. By construction, for every $x \in K$ there exists $r(x) > 0$ such that

$$
|B_{r(x)}(x)| \leq \frac{1}{\lambda} \int_{B_{r(x)}(x)} |f(y)|\, dy. \tag{*}
$$

The collection $\{B_{r(x)}(x)\}_x$ forms a cover of K, so we use compactness to extract a finite sub-cover, and the Vitali-Wiener covering lemma to extract a sub-collection of disjoint balls $B_{r_j}(x_j)$ satisfying $K \subseteq \bigcup_j B_{3r_j}(x_j)$. In place of the scaling property of the Lebesgue measure, we control the measure on the scaled balls with respect to ω by the maximal function of ω. Note $B_{3r_j}(x_j) \subseteq B_{4r_j}(z)$ any $z \in B_{r_j}(x_j)$, so

$$
\omega(B_{3r_j}(x_j)) \leq \int_{B_{4r_j}(z)} \omega(y)\, dy \leq |B_{4r_j}(z)|\, M\omega(z).
$$

Integrating both sides against $|f|$ on the ball $B_{r_j}(x_j)$, we obtain

$$\begin{aligned}\omega(B_{3r_j}(x))\int_{B_{r_j}(x_j)}|f(y)|\,dy &\leq |B_{4r_j}(z)|\int_{B_{r_j}(x_j)}|f(z)|\,M\omega(z)\,dz\\ &\lesssim_d \frac{1}{\lambda}\left(\int_{B_{r_j}(x_j)}|f(y)|\,dy\right)\left(\int_{B_{r_j}(x_j)}|f(z)|\,M\omega(z)\,dz\right),\end{aligned}$$

where the second inequality follows from $|B_{4r_j}(z)|\sim|B_{r_j}(x_j)|$ and (*). We conclude

$$\omega(K)\leq\sum_j\omega(B_{3r_j}(x_j))\lesssim_d\frac{1}{\lambda}\sum_j\int_{B_{r_j}(x_j)}|f(z)|\,M\omega(z)\,dz\leq\frac{1}{\lambda}||f||_{L^1(M\omega dz)}$$

as desired. □

Remark. If $\omega\equiv 1$ then $M\omega\equiv 1$ and we recover the classical Hardy-Littlewood maximal inequality. For the theorem to be non-vacuous, we need the maximal function $M\omega$ to be finite a.e. This occurs precisely when

$$\sup_{r\gg 1}\frac{1}{r^d}\int_{|y|\leq r}\omega(y)\,dy\lesssim 1.$$

Assume the above holds, then $M\omega(x_0)<\infty$ for any Lebesgue point for ω. Indeed, the averages of ω on small balls are controlled by Lebesgue differentiation theorem, while averages on large balls are controlled by the condition above

$$\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}\omega(y)\,dy\leq\frac{1}{|B_r(x_0)|}\int_{|y|\leq|x_0|+r}\omega(y)\,dy\lesssim\frac{|B_{|x_0|+r}(0)|}{|B_r(x_0)|}\frac{1}{(|x_0|+r)^d}\int_{|y|\leq|x_0|+r}\omega(y)\,dy\lesssim 1$$

for $r\gg 1$. This shows that $M\omega(x_0)<\infty$. Conversely, suppose $M\omega(x_0)<\infty$, then

$$\frac{1}{r^d}\int_{|y|\leq r}\omega(y)\,dy\leq\frac{1}{r^d}\int_{B_{|x_0|+r}(x_0)}\omega(y)\,dy\lesssim\frac{(|x_0|+r)^d}{r^d}(M\omega)(x_0)\lesssim 1$$

for $r\geq|x_0|$ uniformly.

Proof of Theorem 17 for $p\geq q$. Observe that $1<(p/q)'\leq\infty$. By duality we can write

$$\begin{aligned}||\overline{M}_q f||_{L^p}^q=|||\overline{M}_q f|^q||_{L^{p/q}} &= \sup_{||\omega||_{L^{(p/q)'}}\leq 1}\int_{\mathbb{R}^d}|\overline{M}_q f(x)|^q\omega(x)\,dx=\sup_{||\omega||_{L^{(p/q)'}}\leq 1}\sum_{n\in\mathbb{N}}\int_{\mathbb{R}^d}|Mf_n(x)|^q\omega(x)\,dx\\ &\lesssim\sup_{||\omega||_{L^{(p/q)'}}\leq 1}\sum_{n\in\mathbb{N}}\int_{\mathbb{R}^d}|f_n(x)|^q M\omega(x)\,dx=\sup_{||\omega||_{L^{(p/q)'}}=1}\int_{\mathbb{R}^d}|f(x)|^q M\omega(x)\,dx\\ &\lesssim\sup_{||\omega||_{L^{(p/q)'}}\leq 1}|||f|^q||_{L^{p/q}}||M\omega||_{L^{(p/q)'}}\\ &\lesssim\sup_{||\omega||_{L^{(p/q)'}}\leq 1}|||f|^q||_{L^{p/q}}||\omega||_{L^{(p/q)'}}\leq||f||_{L^p}^q,\end{aligned}$$

where the first inequality follows from the weighted maximal inequality, the second follows from Holder's inequality, and the third follows from the strong-type $((p/q)',(p/q)')$ bound for the maximal operator. □