Qualifying Exam Syllabus

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13 October 2022, 10 AM – 1 PM, Evans 961

Committee: Richard Bamler (Exam Chair), Sung-Jin Oh (Advisor), Ruixiang Zhang, Sunčica Čanić (Department of Academic Senate Representative).

1 Major topic: Partial differential equations (Analysis)

References: Hormander, *The analysis of linear partial differential operators I*; Evans, *Partial differential equations*; Tao, *Nonlinear dispersive equations: local and global analysis.*

- **Distributions** (Hormander I–IV)
  Test functions, distributions, tempered distributions, differentiation, multiplication by smooth function, convolution, approximations, fundamental solutions.
- **Four important PDE** (Evans 2)
  Energy methods, fundamental solutions, Laplace's equation (mean value property, maximum principle, Harnack's inequality), heat equation (maximum principle, smoothing), wave equation (finite speed of propagation, Huygen's principle).
- **Sobolev spaces** (Evans 5 and Tao Appendix A)
  Weak derivatives, Holder spaces, Sobolev spaces, approximations, extensions, traces, Sobolev embedding, compact embedding, Poincare inequality.
- **Second-order elliptic equations** (Evans 6)
  Energy method, existence and uniqueness, Fredholm alternative, local and global H^k-elliptic regularity, maximum principles.
- **Calculus of variations** (Evans 8.1, 8.2, 8.4, 8.6)
  First variation, Euler-Lagrange equations, existence of minimisers, constrained minimisers, Lagrange multipliers, conservation laws.
- **Dispersive equations** (Tao 2.1–2.6, 3.1–3.6)
  Linear and non-linear Schrodinger and wave equations, notions of solutions, dispersive estimates, Strichartz estimates, monotonicity formulae, $X^{s,b}$ spaces, local existence theory, conservation laws, global existence theory, scattering theory.

2 Major topic: Harmonic analysis (Analysis)

References: Stein, *Harmonic Analysis*; Duoandikoetxea, *Fourier Analysis*; Grafakos, *Classical Harmonic Analysis*; Tao, *Nonlinear dispersive equations: local and global analysis.*

- **Function spaces and interpolation** (Grafakos 1.1–1.4)
  L^p-spaces, convolution, Young's inequality, approximations to the identity, Riesz-Thorin interpolation, Lorentz spaces, duality, Marcinkiewicz interpolation, Schur's test.

- **Fourier transform** (Duoandikoetxea 1.1–1.9)
  Fourier coefficients and series, criteria for pointwise convergence, summability methods and L^p-convergence of Fourier series, Fourier transform, Schwartz functions, Fourier inversion, Plancharel theorem, Riemann-Lebesgue lemma, Hausdorff-Young inequality.

- **Maximal functions** (Duoandikoetxea 2.1–2.7, Stein II.1, V.1–V.3)
  Approximation to the identity, Wiener-Vitali covering lemma, rising sun lemma, Hardy-Littlewood maximal function, Lebesgue differentiation theorem, doubling measures, A_p weights, weighted maximal inequality, vector-valued maximal inequality.

- **Singular integrals** (Duoandikoetxea 5.1–5.4)
  Calderon-Zygmund decomposition, convolution kernels, general Calderon-Zygmund kernels, truncated integrals and principal values, singular integrals, Fourier multipliers, Mikhlin multipliers.

- **Littlewood-Paley theory** (Tao Appendix A)
  Littlewood-Paley projections, Bernstein and Sobolev-Bernstein inequalities, Hardy-Littlewood-Sobolev inequality, Gagliardo-Nirenberg inequality, Sobolev embedding, square function, characterisations of Holder and Sobolev spaces, fractional product rule, fractional chain rule.

- **Oscillatory integrals** (Stein VIII.1–VIII.2)
  Oscillatory integrals of the first kind, non-stationary phase, van der Corput lemma, stationary phase, dispersive estimates.

3 Minor topic: Smooth manifolds (Geometry)

References: Lee, *Introduction to Smooth Manifolds*; Warner, *Foundations of Differentiable Manifolds and Lie Groups*

- **Manifolds** (Lee 1–6)
  Smooth structures, smooth functions, partitions of unity, sub-manifolds, immersions, submersions, embeddings, Sard's theorem, Whitney's embedding theorem.

- **Vector bundles** (Lee 8–11)
  Vector fields, Lie brackets, flows, Lie derivatives, time-dependent vector fields, tangent and cotangent bundles, differentials.

- **Differential forms and integration** (Lee 12, 14–16)
  Tensors, wedge products, k-forms, pullbacks, exterior derivative, closed and exact forms, Poincare's lemma, orientation, manifolds with boundary, integration of forms, Stokes' theorem.

- **de Rham cohomology** (Lee 17)
  de Rham groups, compactly supported de Rham groups, homotopy invariance, degree theory, Mayer-Vietoris sequences, Poincare duality.

- **Hodge theory** (Warner 6)
  Laplace-Beltrami operator, harmonic forms, elliptic regularity, Hodge decomposition.