HODGE THEORY

JASON ZHAO

Contents

1. Preliminaries

1.1. **Linear algebra.** Let $(V, \langle \cdot, \cdot \rangle)$ be a real inner product space of dimension n with a choice of orientation $e_1 \wedge \cdots \wedge e_n$. The Hodge star operator $\star : \bigwedge^k V \to \bigwedge^{n-k} V$

$$\star(e_{i_1} \wedge \cdots \wedge e_{i_k}) = (-1)^{\sigma} e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}.$$

Lemma 1. $\star\star = (-1)^{n(n-k)}$.

1.2. **Manifolds.** We endow the space of k-forms with the inner product

$$(\alpha, \beta) := \int_M \alpha \wedge \star\beta = \int_M \langle \alpha, \beta \rangle d \operatorname{vol}$$

It is easy to see that this is positive definite,

$$(\omega, \omega) = \int_M \langle \omega, \omega \rangle d \operatorname{vol}$$

is non-negative and zero if and only if $\omega = 0$ by continuity. For symmetry, observe that $\langle \omega, \eta \rangle = \langle \star\omega, \star\eta \rangle$. Therefore

$$(\omega, \eta) = \int_M \langle \omega, \eta \rangle d \operatorname{vol} = \int_M \langle \star\omega, \star\eta \rangle d \operatorname{vol}$$

2. Hodge decomposition theorem

Denote $\mathcal{H}^k(M)$ the space of harmonic k forms. Then

Theorem 2 (Hodge decomposition theorem)**.** *Let M be a compact, oriented Riemannian manifold of dimension n. Then the space of differential forms admits the following orthogonal decompositions*

$$\begin{aligned} \Omega^k(M) &= \operatorname{im} \Delta \oplus \ker \Delta \\ &= \operatorname{im} d\delta \oplus \operatorname{im} \delta d \oplus \ker \Delta \\ &= d(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M)) \oplus \mathcal{H}^k(M). \end{aligned}$$

Proof. Choose an orthonormal basis □

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