

INTERPOLATION OF LEBESGUE AND LORENTZ SPACES

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ABSTRACT. Given an operator T bounded between two pairs of function spaces, the problem of *interpolation* asks: what can we say about the boundedness of T between function spaces *interpolated* between our original spaces? We set the stage by defining the Lorentz function space $L^{p,q}(X)$ and present both the complex and real methods for interpolating bounds on linear and sub-linear operators.

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1. LEBESGUE SPACES

Let (X, μ) be a measure space and $1 \leq p \leq \infty$, the L^p SPACE, denoted $L^p(X)$, is the space of measurable functions $f : X \rightarrow \mathbb{C}$ such that the norm

$$\begin{aligned} \|f\|_{L^p} &:= \left(\int_X |f|^p d\mu \right)^{1/p}, & \text{when } p \neq \infty, \\ \|f\|_{L^\infty} &:= \operatorname{ess\,sup}_{x \in X} |f(x)|, & \text{when } p = \infty, \end{aligned}$$

is finite. The quantity above forms a monotone norm, that is, for $f, g \in L^p(X)$ and $\alpha \in \mathbb{C}$, it satisfies the following:

(a) Monotonicity, if $|f| \leq |g|$, then

$$\|f\|_{L^p} \leq \|g\|_{L^p}.$$

(b) Positive definiteness,

$$\|f\|_{L^p} \geq 0$$

with equality only if $f \equiv 0$.

(c) Absolute homogeneity,

$$\|\alpha f\|_{L^p} = |\alpha| \|f\|_{L^p}.$$

(d) The triangle inequality,

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

1.1. Complex interpolation. Let T be an operator mapping a subspace of measurable functions $Y \rightarrow \mathbb{C}$ to measurable functions $Y \rightarrow \mathbb{C}$, we say it is **LINEAR** if it satisfies

$$T(\alpha f + g) = \alpha T f + T g$$

for all $\alpha \in \mathbb{C}$ and $f, g : X \rightarrow \mathbb{C}$ in the domain of T . For $1 \leq p, q \leq \infty$, a linear operator is **STRONG-TYPE** (p, q) if it is bounded $T : L^p(X) \rightarrow L^q(Y)$, i.e. it satisfies the strong-type (p, q) inequality

$$\|T f\|_{L^q} \lesssim \|f\|_{L^p}, \quad \text{uniformly in } f \in L^p(X).$$

Proposition 1. *Let $T : L^p(X) \rightarrow L^q(Y)$ be a linear operator. Then the following are equivalent:*

- (a) T is strong-type (p, q) .
- (b) T is Lipschitz continuous.
- (c) T is continuous at the origin.

Proof. From linearity we see that (a) \implies (b) and (b) \implies (c) is trivial, so to close the chain of implications we show (c) \implies (a). Let $\varepsilon > 0$, then there exists $\delta > 0$ such that

$$\|Tf\|_{L^q} < \varepsilon, \quad \text{whenever } \|f\|_{L^p} \leq \delta.$$

For any $f \in L^p(X)$, we use homogeneity and apply the inequality above to the normalised function $\delta f / \|f\|_{L^p}$ to obtain the strong-type (p, q) inequality,

$$\|Tf\|_{L^q} = \frac{\|f\|_{L^p}}{\delta} \left\| T \left(\delta \frac{f}{\|f\|_{L^p}} \right) \right\|_{L^q} \leq \frac{\varepsilon}{\delta} \|f\|_{L^p},$$

completing the proof. \square

Remark. To avoid any philosophical malaise on whether an operator is well-defined on all L^p -functions, it is convenient to consider a linear operator acting on a sufficiently regular sub-class, namely test functions $C_c^\infty(\mathbb{R}^d)$ or Schwartz functions $\mathcal{S}(\mathbb{R}^d)$. Upon proving a strong-type (p, q) estimate for $p \neq \infty$ for such functions, we can extend the result by density and Lipschitz continuity to all L^p -functions.

Let $1 \leq p_0, p_1 \leq \infty$ and $0 \leq \theta \leq 1$, define $1 \leq p_\theta \leq \infty$ by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

The space $L^{p_\theta}(X)$ is known as an *interpolation space* between $L^{p_0}(X)$ and $L^{p_1}(X)$. More precisely, $L^{p_\theta}(X)$ “lives between” $L^{p_0}(X)$ and $L^{p_1}(X)$ in the sense that

$$L^{p_0}(X) \cap L^{p_1}(X) \subseteq L^{p_\theta}(X) \subseteq L^{p_0}(X) + L^{p_1}(X).$$

The first inclusion follows from Holder’s inequality, writing $f = f^{1-\theta} f^\theta$,

$$\|f\|_{L^{p_\theta}} \leq \|f\|_{L^{p_0}}^{1-\theta} \|f\|_{L^{p_1}}^\theta.$$

The second inclusion follows from decomposing $f = f \mathbb{1}_{|f|>1} + f \mathbb{1}_{|f|\leq 1}$, monotonicity of $|x| \mapsto |x|^p$, and noting that $p_0 < p_\theta < p_1$,

$$\|f \mathbb{1}_{|f|>1}\|_{L^{p_0}} \leq \|f\|_{L^{p_\theta}}, \quad \|f \mathbb{1}_{|f|\leq 1}\|_{L^{p_1}} \leq \|f\|_{L^{p_\theta}}.$$

Given $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and a linear operator $T : (L^{p_0} + L^{p_1})(X) \rightarrow (L^{q_0} + L^{q_1})(Y)$ of strong-type (p_0, q_0) and (p_1, q_1) , we see that T forms a map between the interpolation spaces. *Interpolation* is the problem of establishing the strong-type (p_θ, q_θ) inequality.

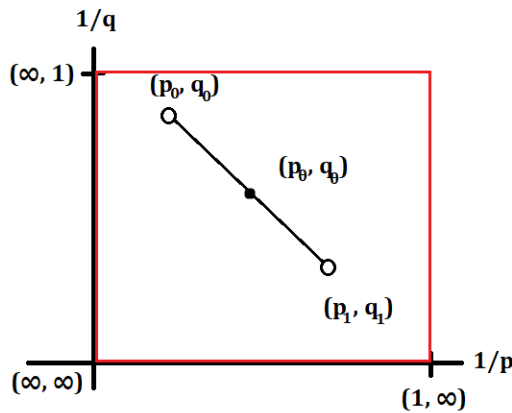


FIGURE 1. The *interpolation diagram*; given bounds for exponents (p_0, q_0) and (p_1, q_1) , interpolation furnishes bounds for exponents (p_θ, q_θ) on the intermediate line.

Theorem 2 (Riesz-Thorin interpolation). *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and $0 < \theta < 1$, and suppose (X, μ) and (Y, ν) are measure spaces, the latter σ -finite when $q_0 = q_1 = \infty$. If $T : (L^{p_0} + L^{p_1})(X) \rightarrow (L^{q_0} + L^{q_1})(Y)$ is a linear operator of strong-type (p_0, q_0) and (p_1, q_1) , i.e.*

$$\begin{aligned} \|Tf\|_{L^{q_0}} &\leq M_0 \|f\|_{L^{p_0}}, & \text{uniformly in } f \in L^{p_0}(X) \\ \|Tf\|_{L^{q_1}(Y)} &\leq M_1 \|f\|_{L^{p_1}}, & \text{uniformly in } f \in L^{p_1}(X) \end{aligned}$$

for some $0 < M_0, M_1 < \infty$. Then T satisfies the strong-type (p_θ, q_θ) inequality

$$\|Tf\|_{L^{q_\theta}} \leq M_0^{1-\theta} M_1^\theta \|f\|_{L^{p_\theta}}.$$

Proof. If $p_0 = p_1 = p_\theta$, the result follows from the log convexity of the L^p -norm. That is, by Holder's inequality,

$$\|g\|_{L^{q_\theta}}^{q_\theta} = \int_Y |g|^{(1-\theta)q_\theta} |g|^{\theta q_\theta} d\nu \leq \| |g|^{(1-\theta)q_\theta} \|_{L^{\frac{q_0}{(1-\theta)q_\theta}}} \| |g|^{\theta q_\theta} \|_{L^{\frac{q_1}{\theta q_\theta}}} = \|g\|_{L^{q_0}}^{(1-\theta)q_\theta} \|g\|_{L^{q_1}}^{\theta q_\theta}.$$

Then taking $Tf = g$ and applying the strong type (p_0, q_0) and (p_1, q_1) inequalities,

$$\|Tf\|_{L^{q_\theta}(Y)} \leq \|Tf\|_{L^{q_0}(Y)}^{1-\theta} \|Tf\|_{L^{q_1}(Y)}^\theta \leq M_0^{1-\theta} M_1^\theta \|f\|_{L^{p_\theta}(X)}.$$

Thus we can assume $p_0 < p_1$; in particular, we avoid the endpoint cases and assume $1 < p_\theta < \infty$. We claim that it suffices to prove the result for simple functions with finite support, i.e. functions of the form

$$f = \sum_{k=1}^n a_k \mathbb{1}_{A_k}$$

for coefficients $a_k \in \mathbb{C}$ and disjoint finite measure sets $A_k \subseteq X$. We can find simple functions with finite measure support $\{f_n\}_n$ such that $f_n \rightarrow f$ pointwise and $|f_n| \leq |f|$. Assume $f \in L^{p_0}(X) \cap L^{p_1}(X)$, then by log convexity of the L^p -norm we also have $f \in L^{p_\theta}(X)$. By the L^p -dominated convergence theorem, $f_n \rightarrow f$ in L^{p_θ} and L^{p_0} . Observe

$$\begin{aligned} \|Tf\|_{L^{q_\theta}(Y)} &\leq \|T(f - f_n)\|_{L^{q_\theta}(Y)} + \|Tf_n\|_{L^{q_\theta}(Y)} \\ &\leq \|T(f - f_n)\|_{L^{q_0}(Y)}^{1-\theta} \|T(f - f_n)\|_{L^{q_1}(Y)}^\theta + M_0^{1-\theta} M_1^\theta \|f_n\|_{L^{p_\theta}(X)} \\ &\leq M_0^{1-\theta} M_1^\theta \left(\|f - f_n\|_{L^{p_0}(X)}^{1-\theta} \|f - f_n\|_{L^{p_1}(X)}^\theta + \|f - f_n\|_{L^{p_\theta}(X)} + \|f\|_{L^{p_\theta}(X)} \right). \end{aligned}$$

The first line follows from the triangle inequality and linearity of T , the second follows from log convexity of the L^q -norm and Riesz-Thorin for simple functions, the third follows from the strong type (p_0, q_0) and (p_1, q_1) inequalities and the L^{p_θ} -triangle inequality. Since $|f_n| \leq |f|$, we know $\|f - f_n\|_{L^{p_1}(X)} \leq 2\|f\|_{L^{p_1}(X)}$. This allows us to pass the limit $n \rightarrow \infty$ on the right to obtain

$$\|Tf\|_{L^{q_\theta}(Y)} \leq M_0^{1-\theta} M_1^\theta \|f\|_{L^{p_\theta}(X)},$$

proving the claim.

To prove Riesz-Thorin for simple functions, we argue by duality. For $1 \leq q \leq \infty$, then

$$\|Tf\|_{L^{q_\theta}(Y)} = \sup_{\|g\|_{L^{q'_\theta}(Y)}=1} \left| \int_Y (Tf)g d\nu \right|.$$

Suppose one of $q_0, q_1 \neq 1$, then $1 \leq q'_\theta < \infty$. By density we can take the supremum over simple functions. Fix simple functions with finite measure support f and g , and set

$$F(s) := \int_Y T \left(|f|^{(1-s)\frac{p_\theta}{p_0} + s\frac{p_\theta}{p_1}} \operatorname{sgn}(f) \right) |g|^{(1-s)\frac{q'_\theta}{q_0} + s\frac{q'_\theta}{q_1}} \operatorname{sgn}(g) d\nu,$$

with the convention $q'_\theta/q_0 = q'_\theta/q_1 = 1$ in the endpoint case $q_0 = q_1 = q_\theta = \infty$. Since we assumed f and g were simple functions and T is linear, the integrand is entire for each fixed $y \in Y$ and satisfies $|F(s)| \lesssim e^{c|z|}$. From the finite measure support assumption, we can apply Fubini-Tonelli and Morera's theorem to conclude F is entire. Hence, the conditions for Hadamard's three lines theorem have been satisfied.

If $q_0 = q_1 = q_\theta = 1$, density of simple functions fails, so we consider instead any $g \in L^\infty(Y)$. Nevertheless, the conditions of Hadamard's theorem continue to be satisfied, since the exponent of $|g|$ reduces to a constant and

$$F(s) = \int_Y T \left(|f|^{(1-s)\frac{p_\theta}{p_0} + s\frac{p_\theta}{p_1}} \operatorname{sgn}(f) \right) g d\nu.$$

By choice of exponents and noting $f = |f| \operatorname{sgn}(f)$,

$$F(\theta) = \int_Y (Tf)g \, dv.$$

Applying Holder's inequality, the strong type (p_0, q_0) inequality, and recalling $|a^{it}| = 1$ for any $a, t \in \mathbb{R}$, we have

$$\begin{aligned} |F(it)| &\leq \left\| T \left(|f|^{(1-it)\frac{p_\theta}{p_0} + it\frac{p_\theta}{p_1}} \operatorname{sgn}(f) \right) \right\|_{L^{q_0}(Y)} \left\| |g|^{(1-it)\frac{q'_\theta}{q_0} + it\frac{q'_\theta}{q_1}} \operatorname{sgn}(g) \right\|_{L^{q'_0}(Y)} \\ &\leq M_0 \left\| |f|^{\frac{p_\theta}{p_0}} \right\|_{L^{p_0}(Y)} \left\| |g|^{\frac{q'_\theta}{q_0}} \right\|_{L^{q'_0}(Y)} \leq M_0 \|f\|_{L^{p_\theta}(X)}^{\frac{p_\theta}{p_0}} \|g\|_{L^{q'_\theta}(Y)}^{\frac{q'_\theta}{q_0}}, \end{aligned}$$

and similarly, applying instead the strong type (p_1, q_1) inequality,

$$\begin{aligned} |F(1+it)| &\leq \left\| T \left(|f|^{-it\frac{p_\theta}{p_0} + (1+it)\frac{p_\theta}{p_1}} \operatorname{sgn}(f) \right) \right\|_{L^{q_1}(Y)} \left\| |g|^{-it\frac{q'_\theta}{q_0} + (1+it)\frac{q'_\theta}{q_1}} \operatorname{sgn}(g) \right\|_{L^{q'_1}(Y)} \\ &\leq M_0 \left\| |f|^{\frac{p_\theta}{p_1}} \right\|_{L^{p_1}(Y)} \left\| |g|^{\frac{q'_\theta}{q_1}} \right\|_{L^{q'_1}(Y)} \leq M_0 \|f\|_{L^{p_\theta}(X)}^{\frac{p_\theta}{p_1}} \|g\|_{L^{q'_\theta}(Y)}^{\frac{q'_\theta}{q_1}}. \end{aligned}$$

Hadamard's three lines theorem furnishes the inequality $|F(\theta)| \leq \sup_t |F(it)|^{1-\theta} \sup_t |F(1+it)|^\theta$. Collecting this inequality with the previous two inequalities and comparing exponents, we obtain

$$\left| \int_Y (Tf)g \, dv \right| \leq M_0^{1-\theta} M_1^\theta \|f\|_{L^{p_\theta}(X)} \|g\|_{L^{q'_\theta}(Y)}.$$

By duality, this finishes the proof. \square

2. LORENTZ SPACES

Given a measure space (X, μ) , the two basic quantitative notions of "size" of a function $f : X \rightarrow \mathbb{C}$ are the "height" in the range and "width" in the domain. The L^p -norm primarily quantified control over the former; to quantify control over both notions, we introduce for $1 \leq p, q \leq \infty$ the LORENTZ SPACE $L^{p,q}(X)$, the space of measurable functions $f : X \rightarrow \mathbb{C}$ for which

$$\begin{aligned} \|f\|_{L^{p,q}}^* &:= p^{1/q} \left\| \lambda \mu(\{x \in X : |f(x)| > \lambda\})^{1/p} \right\|_{L^q((0, \infty), \frac{d\lambda}{\lambda})} \\ &= p^{1/q} \left(\int_0^\infty \lambda^q \mu(\{x \in X : |f(x)| \geq \lambda\})^{q/p} \frac{d\lambda}{\lambda} \right)^{1/q} \end{aligned}$$

is finite. The quantity above forms a monotone quasi-norm, that is, for $f, g \in L^{p,q}(X)$ and $\alpha \in \mathbb{C}$, it satisfies the following properties:

(a) Monotonicity, if $|f| \leq |g|$ then

$$\|f\|_{L^{p,q}}^* \leq \|g\|_{L^{p,q}}^*.$$

(b) Positive definiteness,

$$\|f\|_{L^{p,q}}^* \geq 0,$$

with equality only if $f \equiv 0$; this is clear.

(c) Absolute homogeneity,

$$\|\alpha f\|_{L^{p,q}}^* = |\alpha| \|f\|_{L^{p,q}}^*.$$

(d) A quasi-triangle inequality,

$$\|f + g\|_{L^{p,q}}^* \leq 2\|f\|_{L^{p,\infty}}^* + 2\|g\|_{L^{p,q}}^*.$$

By convention we set $L^{\infty,\infty}(X) := L^\infty(X)$, and in general when $q = \infty$, the Lorentz space $L^{p,\infty}(X)$ is known as the WEAK L^p SPACE, endowed with the norm

$$\|f\|_{L^{p,\infty}}^* := \sup_{\lambda > 0} \lambda \mu(\{x \in X : |f(x)| > \lambda\})^{1/p}.$$

Remark. The Lorentz space coincides with the usual Lebesgue space when $p = q$, that is, $L^{p,p}(X) = L^p(X)$. This follows from the *layered cake representation*, writing using the fundamental theorem of calculus

$$|f(x)|^p = \int_0^{|f(x)|} p\lambda^{p-1} d\lambda = p \int_0^\infty \lambda^{p-1} \mathbb{1}_{[0,|f(x)|]}(\lambda) d\lambda = p \int_0^\infty \lambda^p \mathbb{1}_{|f|>\lambda}(x) \frac{d\lambda}{\lambda}.$$

The representation takes its name from writing $|f(x)|^p$ as the sum of contributions from the “layers” λ below $|f(x)|$. Integrating and applying Fubini’s theorem,

$$\|f\|_{L^p}^p = \int_X |f(x)|^p d\mu = p \int_X \int_0^\infty \lambda^p \mathbb{1}_{|f|>\lambda}(x) \frac{d\lambda}{\lambda} = p \int_0^\infty \lambda^p \mu(\{x \in X : |f(x)| > \lambda\}) \frac{d\lambda}{\lambda} = (\|f\|_{L^{p,p}}^*)^p.$$

2.1. Weak L^p space. As a primer for studying the general $L^{p,q}$ -spaces, we first consider the weak L^p -spaces. The prototypical example of a function in weak L^p -space however not in L^p -space is

$$f(x) := |x|^{-d/p}.$$

In fact, one can think of every weak L^p -function as dominated pointwise by a rearrangement of $|x|^{-d/p}$. Several classical inequalities can be reformulated in terms of weak L^p -space, such as the Hardy-Littlewood maximal inequality and, in probability,

Lemma 3 (Chebyshev’s inequality). *Let $f \in L^p(X)$ and $\lambda > 0$, then*

$$\mu(\{x \in X : |f(x)| > \lambda\}) \leq \frac{1}{\lambda^p} \int_{|f|>\lambda} |f|^p d\mu.$$

Moreover, $L^p(X) \hookrightarrow L^{p,\infty}(X)$ via the inequality

$$\|f\|_{L^{p,\infty}}^* \leq \|f\|_{L^p}.$$

Proof. We have

$$\mu(\{x \in X : |f(x)| > \lambda\}) = \frac{1}{\lambda^p} \int_{|f|>\lambda} \lambda^p d\mu \leq \frac{1}{\lambda^p} \int_{|f|>\lambda} |f|^p d\mu.$$

This proves Chebyshev’s inequality. Rearranging and taking the p -th root gives

$$\lambda \mu(\{x \in X : |f(x)| > \lambda\})^{1/p} \leq \left(\int_X |f|^p d\mu \right)^{1/p}.$$

Taking the supremum over $\lambda > 0$ on the left furnishes the continuous embedding $L^p(X) \hookrightarrow L^{p,\infty}(X)$. \square

Remark. The case $p = 1$ of Chebyshev’s inequality is also referred to as Markov’s inequality.

Theorem 4 (Weak L^p -duality). *Let $1 < p < \infty$ and $f \in L^{p,\infty}(X)$, then*

$$\|f\|_{L^{p,\infty}} := \sup_{0 < \mu(A) < \infty} \mu(A)^{-1/p'} \left| \int_X f \mathbb{1}_A d\mu \right|$$

defines a norm satisfying

$$\|f\|_{L^{p,\infty}}^* \leq \|f\|_{L^{p,\infty}} \leq \frac{1}{p'} \|f\|_{L^{p,\infty}}^*.$$

Proof. It is clear from definition that $\|\cdot\|_{L^{p,\infty}}$ defines a norm, so it remains to show that it is comparable to the quasi-norm $\|\cdot\|_{L^{p,\infty}}^*$. Decomposing into real, imaginary, positive and negative components, we can assume without loss of generality $f \geq 0$. Since $f \in L^{p,\infty}(X)$, we know that the super-level sets $|f| > \lambda$ have finite measure, so rearranging Markov’s inequality we obtain

$$\lambda \mu(f > \lambda)^{1/p} \leq \mu(f > \lambda)^{-1/p'} \int_{f>\lambda} f d\mu \leq \|f\|_{L^{p,\infty}}.$$

Taking the supremum with respect to $\lambda > 0$ on the left gives the desired lower bound on $\|f\|_{L^{p,\infty}}$. For the upper bound, we apply the layered cake representation and the definition of the weak L^p -quasinorm to write

$$\begin{aligned} \int_A f d\mu &= \int_0^\infty \mu(x \in A : f > \lambda) d\lambda \leq \int_0^\infty \min\{\mu(f > \lambda), \mu(A)\} \\ &\leq \int_0^\infty \min\{\lambda^{-p} \|f\|_{L^{p,\infty}}^*, \mu(A)\} d\lambda. \end{aligned}$$

We split the integral on the right, remarking that $\lambda^{-p} \|f\|_{L^{p,\infty}}^* \leq \mu(A)$ if and only if $\|f\|_{L^{p,\infty}}^* \mu(A)^{-1/p} \leq \lambda$, so by the fundamental theorem of calculus

$$\begin{aligned} \int_0^\infty \min\{\lambda^{-p} \|f\|_{L^{p,\infty}}^*, \mu(A)\} d\lambda &= \int_0^{\|f\|_{L^{p,\infty}}^* \mu(A)^{-1/p}} \mu(A) d\lambda + \int_{\|f\|_{L^{p,\infty}}^* \mu(A)^{-1/p}}^\infty \lambda^{-p} \|f\|_{L^{p,\infty}}^* d\lambda \\ &= \|f\|_{L^{p,\infty}}^* \mu(A)^{1/p'} + \frac{1}{p-1} \|f\|_{L^{p,\infty}}^* \mu(A)^{1/p'} = \frac{1}{p'} \|f\|_{L^{p,\infty}}^* \mu(A)^{1/p'}. \end{aligned}$$

Rearranging and taking the supremum over $0 < \mu(A) < \infty$ furnishes the desired upper bound. \square

Remark. The argument fails in the case $p = 1$; in fact, the weak L^1 -space cannot be normed provided that the measure μ is non-zero and non-atomic. Consider for example the Lebesgue measure on \mathbb{R} , and assume towards a contradiction that there exists a norm $\|\cdot\|$ such that $\|\cdot\| \sim \|\cdot\|_{L^{1,\infty}}^*$. Define $f_n \in L^{1,\infty}(\mathbb{R})$ by

$$f_n(x) = \frac{1}{|x - n|}. \quad (*)$$

We compute the quasinorms,

$$\|f_n\|_{L^{1,\infty}}^* = \sup_{\lambda > 0} \lambda \mu(|x| < 1/\lambda) = 2.$$

On the other hand, recall that the harmonic series grows logarithmically, so for $x \in [0, N]$ and $N \gg 1$ we have the pointwise lower bound

$$\log N \lesssim \sum_{n=1}^N f_n.$$

Collecting our results and applying the triangle inequality, we obtain

$$N \log N \lesssim \left\| \sum_{n=1}^N f_n \right\|_{L^{1,\infty}}^* \sim \left\| \sum_{n=1}^N f_n \right\| \leq \sum_{n=1}^N \|f_n\| \lesssim \sum_{n=1}^N \|f_n\|_{L^{1,\infty}}^* \sim N,$$

a contradiction.

2.2. Characterisation of $L^{p,q}$. The $L^{p,q}$ -quasinorm can at first glance appears rather mysterious and cumbersome to work with directly, so it will be both enlightening and convenient to introduce characterisations of $L^{p,q}$ -functions in terms of simple functions. Much like the L^p -norm, the $L^{p,q}$ -quasinorm of a step function is

$$\|H \mathbb{1}_E\|_{L^{p,q}}^* \sim H \mu(E)^{1/p}.$$

The difference lies in how these quantities are summed in the case of simple functions. For $H_n > 0$ distinct heights and $\{E_n\}_n$ a disjoint family of measurable sets, the L^p -norm of the corresponding simple function is the ℓ_n^p -sum of step function L^p -norms

$$\left\| \sum_{n \in \mathbb{Z}} H_n \mathbb{1}_{E_n} \right\|_{L^p} = \left\| \|H_n \mathbb{1}_{E_n}\|_{L^p} \right\|_{\ell_n^p} = \|H_n \mu(E_n)^{1/p}\|_{\ell_n^p}.$$

The $L^{p,q}$ -norm is instead comparable to the ℓ^q -sum of step function L^p -norms. For a generic function $f \in L^{p,q}(X)$, we can decompose and approximate by simple functions in two fashions: vertically into step functions of height $H_n \sim 2^n$ and some width $\mu(E_n)$, or horizontally into step functions of width $\mu(E_n) \sim 2^n$ and some heights H_n .

Proposition 5 (Vertically dyadic layer cake decomposition). *Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$, and suppose that $f \in L^{p,q}(X)$. Decomposing*

$$f = \sum_{m \in \mathbb{Z}} f \mathbb{1}_{2^m \leq |f| < 2^{m+1}} =: \sum_{m \in \mathbb{Z}} f_m,$$

then

$$\|f\|_{L^{p,q}}^* \sim_{p,q} \left\| \|f_m\|_{L^p} \right\|_{\ell_m^q}.$$

In particular, $L^{p,q_1}(X) \hookrightarrow L^{p,q_2}(X)$ whenever $q_1 \leq q_2$.

Proof. By construction,

$$|f| \sim \sum_{m \in \mathbb{Z}} 2^m \mathbb{1}_{2^m \leq |f| < 2^{m+1}}$$

pointwise, so by monotonicity we can consider without loss of generality functions of the form $f = \sum_m 2^m \mathbb{1}_{E_m}$ where $\{E_m\}_m$ is a disjoint family of measurable sets. For such functions, the result takes the form

$$\|f\|_{L^{p,q}}^* \sim_{p,q} \left\| 2^m \mu(E_m)^{1/p} \right\|_{\ell_m^q}.$$

We compute

$$\begin{aligned} (\|f\|_{L^{p,q}}^*)^q &= p \sum_{m \in \mathbb{Z}} \int_{2^{m-1}}^{2^m} \lambda^q \mu(|f| > \lambda)^{q/p} \frac{d\lambda}{\lambda} = p \sum_{m \in \mathbb{Z}} \int_{2^{m-1}}^{2^m} \lambda^q \left(\sum_{n \geq m} \mu(E_n) \right)^{q/p} \frac{d\lambda}{\lambda} \\ &\sim_{p,q} \sum_{m \in \mathbb{Z}} 2^{mq} \left(\sum_{n \geq m} \mu(E_n) \right)^{q/p} \sim \left\| 2^m \left(\sum_{n \geq m} \mu(E_n) \right)^{1/p} \right\|_{\ell_m^q}^q. \end{aligned}$$

Clearly

$$\left\| 2^m \mu(E_m)^{1/p} \right\|_{\ell_m^q} \leq \left\| 2^m \left(\sum_{n \geq m} \mu(E_n) \right)^{1/p} \right\|_{\ell_m^q}.$$

For the converse inequality, we appeal to a change of indices $n = m + k$ and the triangle inequality,

$$\begin{aligned} \left\| 2^m \left(\sum_{n \geq m} \mu(E_n) \right)^{1/p} \right\|_{\ell_m^q} &\lesssim \left\| 2^n \sum_{n \geq m} \mu(E_n)^{1/p} \right\|_{\ell_m^q} \sim \left\| 2^m \sum_{k \geq 0} \mu(E_{m+k})^{1/p} \right\|_{\ell_m^q} \\ &\leq \sum_{k \geq 0} 2^{-k} \left\| 2^{m+k} \mu(E_{m+k})^{1/p} \right\|_{\ell_m^q} \sim \|2^m \mu(E_m)^{1/p}\|_{\ell_m^q}. \end{aligned}$$

This completes the proof. \square

Corollary 6 (Density of simple functions). *For $1 \leq p, q < \infty$ the space of simple function is dense in $L^{p,q}(X)$.*

Proof. Fix $f \in L^{p,q}(X)$ and $\varepsilon > 0$. Performing a vertically dyadic layer cake decomposition, there exists $n_\varepsilon \in \mathbb{N}$ sufficiently large such that

$$\left\| \|f_m\|_{L^p} \right\|_{\ell_{|m| \geq n_\varepsilon}^q} \ll \varepsilon.$$

Fix $N \gg 1$ to be chosen later, define

$$E_{m,k} := \{x \in X : \frac{k}{2^N} \leq |f_m(x)| \leq \frac{k+1}{2^N}\}$$

for $2^{m+N} \leq k \leq 2^{m+N+1} - 1$. Define the simple functions

$$\phi_m := \sum_{k=2^{m+N}}^{2^{m+N+1}} \frac{k}{2^N} \mathbb{1}_{E_{m,k}}, \quad \phi := \sum_{|m| \leq n_\varepsilon} \phi_m.$$

We claim that $\|f - \phi\|_{L^{p,q}}^* \lesssim \varepsilon$. By the triangle inequality and choice of n_ε we have

$$\|f - \phi\|_{L^{p,q}}^* \leq 2 \left(\left\| \sum_{|m| \leq n_\varepsilon} f_m - \phi_m \right\|_{L^{p,q}}^* + \left\| \sum_{|m| > n_\varepsilon} f_m \right\|_{L^{p,q}}^* \right) \leq 2 \left(\left\| \sum_{|m| \leq n_\varepsilon} f_m - \phi_m \right\|_{L^{p,q}}^* + \varepsilon \right).$$

It remains to control the first term on the right. By the triangle inequality and Holder's inequality,

$$\begin{aligned} \left\| \sum_{|m| \leq n_\varepsilon} f_m - \phi_m \right\|_{L^{p,q}}^* &\leq 2^{2n_\varepsilon+1} \sum_{|m| \leq n_\varepsilon} \|f_m - \phi_m\|_{L^{p,q}}^* \lesssim 2^{2n} \sum_{|m| \leq n_\varepsilon} \left\| 2^k \mu(2^k \leq |f_m - \phi_m| < 2^{k+1})^{1/p} \right\|_{\ell_k^q} \\ &\lesssim 2^{2n_\varepsilon} \sum_{|m| \leq n_\varepsilon} \left\| 2^k \mu(2^m \leq |f| < 2^{m+1})^{1/p} \right\|_{\ell_k^q} \lesssim 2^{2n} \sum_{|m| \leq n_\varepsilon} 2^{-N} \mu(2^m \leq |f| < 2^{m+1})^{1/p} \\ &\lesssim 2^{2n_\varepsilon} \sum_{|m| \leq n_\varepsilon} 2^{-N} \frac{\|f_m\|_{L^p}}{2^m} \lesssim 2^{2n-N} \left\| \|f_m\|_{L^p} \right\|_{\ell_m^q} \left\| 2^{-m} \right\|_{\ell_{|m| \leq n_\varepsilon}^q} \lesssim \frac{2^{3n}}{2^N} \|f\|_{L^{p,q}}^*. \end{aligned}$$

Choosing $N \gg 1$ completes the proof. \square

Proposition 7 (Horizontally dyadic layer cake decomposition). *Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$, and suppose that $f \in L^{p,q}(X)$. Decomposing,*

$$f = \sum_{m \in \mathbb{Z}} f \mathbb{1}_{H_{m+1} \leq |f| < H_m} =: \sum_{m \in \mathbb{Z}} f_m,$$

where $H_m := \inf\{\lambda > 0 : \mu(|f| > \lambda) \leq 2^{m-1}\}$, then

$$\|f\|_{L^{p,q}}^* \sim \|H_m 2^{m/p}\|_{\ell_m^q}.$$

Proof. It is clear that $H_m \rightarrow \|f\|_{L^\infty}$ as $m \rightarrow -\infty$ and, since $f \in L^{p,q}(X)$, we also have $H_m \rightarrow 0$ as $m \rightarrow \infty$. Furthermore we have

$$2^{m-1} < \mu(|f| > \lambda) \leq 2^m.$$

whenever $H_{m+1} \leq \lambda < H_m$. For $q \neq \infty$, we compute

$$\begin{aligned} (\|f\|_{L^{p,q}}^*)^q &= p \int_0^\infty \lambda^q \mu(|f| > \lambda)^{q/p} \frac{d\lambda}{\lambda} = p \sum_{m \in \mathbb{Z}} \int_{H_{m+1}}^{H_m} \lambda^q \mu(|f| > \lambda)^{q/p} \frac{d\lambda}{\lambda} \\ &\sim \sum_{m \in \mathbb{Z}} \int_{H_{m+1}}^{H_m} \lambda^q 2^{mq/p} \frac{d\lambda}{\lambda} \sim \sum_{m \in \mathbb{Z}} 2^{mq/p} (H_m^q - H_{m+1}^q) \sim \sum_{m \in \mathbb{Z}} 2^{mq/p} H_m^q = \|2^{\frac{m}{p}} H_m\|_{\ell_m^q}^q. \end{aligned}$$

For the case $q = \infty$ we have

$$\begin{aligned} \|f\|_{L^{p,\infty}}^* &= \sup_{\lambda > 0} \lambda |\{x : |f(x)| > \lambda\}|^{\frac{1}{p}} = \sup_{m \in \mathbb{Z}} \sup_{H_{m+1} \leq \lambda < H_m} \lambda |\{x : |f(x)| > \lambda\}|^{\frac{1}{p}} \\ &\sim \sup_{m \in \mathbb{Z}} H_m 2^{\frac{m}{p}} = \|H_m 2^{\frac{m}{p}}\|_{\ell_m^\infty}, \end{aligned}$$

completing the proof. \square

2.3. Duality. Analogous to the L^p -space setting, the continuous duals of non-endpoint $L^{p,q}$ -spaces are represented by the dual exponent $L^{p',q'}$ -spaces. The dual characterisation thereby furnishes a norm on $L^{p,q}(X)$ and moreover a Banach space structure. To this end, we will need the following preliminary lemma, which states that the triangle inequality, up to a uniform constant, continues to hold for quasi-norms provided the component quasi-norms are on different dyadic scales.

Lemma 8. Let $\|\cdot\|$ denote a quasinorm on a topological vector space X . Let $f_1, \dots, f_N \in X$ satisfy

$$\|f_n\| \leq 2^{-\varepsilon n}$$

for some $\varepsilon > 0$. Then

$$\left\| \sum_{n=1}^N f_n \right\| \lesssim_\varepsilon 1,$$

where the implicit constant is independent of N .

Proof. There exists a constant $C > 1$ such that for functions f and g the quasinorm satisfies the following quasi-triangle inequality

$$\|f + g\| \leq C\|f\| + C\|g\|.$$

Let $\eta > 0$ such that $C = 2^\eta$. We consider first the case where $\varepsilon > \eta$, then

$$\begin{aligned} \left\| \sum_{n=1}^N f_n \right\| &\leq C\|f_1\| + C \left\| \sum_{n=2}^N f_n \right\| \leq \dots \leq C\|f_1\| + \dots + C^N \|f_N\| \\ &\leq \sum_{n=1}^N C^n \|f_n\| \leq \sum_{n=1}^N 2^{(\eta-\varepsilon)n} \leq \frac{1}{1-2^{\eta-\varepsilon}} \lesssim_\varepsilon 1. \end{aligned}$$

In the first line we apply the quasi-triangle inequality N -times and the second line we note that we obtain a convergent geometric series. If $\varepsilon \leq \eta$, choose $M_\varepsilon \in \mathbb{N}$ such that $\varepsilon M_\varepsilon > \eta$. For notational convenience, let $d \in \mathbb{N}$ such that $dM_\varepsilon > N$ and set $f_{N+1} = \dots = f_{dM_\varepsilon} = 0$. Define

$$g_k = \sum_{n=kM_\varepsilon+1}^{(k+1)M_\varepsilon} f_n$$

for $k = 0, \dots, d-1$. Trivially $\sum_n f_n = \sum_k g_k$, so to reduce to the previous case it remains to prove analogous bounds $\|g_k\| \lesssim_\varepsilon 2^{-\varepsilon M_\varepsilon k}$ where the implicit constant is uniform in k . We apply the quasi-triangle inequality M_ε -times to obtain the desired inequality for each k ,

$$\begin{aligned} \left\| \sum_{n=kM_\varepsilon+1}^{(k+1)M_\varepsilon} f_n \right\| &\leq C\|f_{kM_\varepsilon+1}\| + \dots + C^{M_\varepsilon} \|f_{(k+1)M_\varepsilon}\| \\ &\leq C^{M_\varepsilon} (2^{-\varepsilon(kM_\varepsilon+1)} + \dots + 2^{-\varepsilon(k+1)M_\varepsilon}) \leq C^{M_\varepsilon} M_\varepsilon 2^{-\varepsilon k M_\varepsilon}. \end{aligned}$$

Arguing as we did in the first case, we conclude

$$\left\| \sum_{n=1}^N f_n \right\| = \left\| \sum_{k=0}^{d-1} g_k \right\| \leq \frac{C^{M_\varepsilon} M_\varepsilon}{1 - 2^{\eta - \varepsilon M_\varepsilon}} \lesssim_\varepsilon 1.$$

This completes the proof. \square

Theorem 9 ($L^{p,q}$ -Holder's inequality). *Let $1 \leq p, p_1, p_2 < \infty$ and $1 \leq q, q_1, q_2 \leq \infty$ such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then

$$\|fg\|_{L^{p,q}}^* \lesssim \|f\|_{L^{p_1,q_1}}^* \|g\|_{L^{p_2,q_2}}^*.$$

Proof. By absolute homogeneity we can renormalize, replacing f and g with $f/\|f\|_{L^{p_1,q_1}}^*$ and $g/\|g\|_{L^{p_2,q_2}}^*$. It suffices then to prove the case where $\|f\|_{L^{p_1,q_1}}^* = \|g\|_{L^{p_2,q_2}}^* = 1$. Set

$$\begin{aligned} H_{n,f} &= \inf\{\lambda > 0 : |\{x : |f(x)| > \lambda\}| \leq 2^{n-1}\}, \\ H_{n,g} &= \inf\{\lambda > 0 : |\{x : |g(x)| > \lambda\}| \leq 2^{n-1}\}, \\ E_{n,f} &= \{x : H_{n+1,g} \leq |f(x)| < H_{n,f}\}, \\ E_{n,g} &= \{x : H_{n+1,g} \leq |g(x)| < H_{n,g}\}, \end{aligned}$$

where $\{E_{n,f}\}_n$ and $\{E_{n,g}\}_n$ are families of disjoint measurable sets. The horizontal layered cake decomposition furnishes the estimates

$$\|H_{n,f} 2^{\frac{n}{p_1}}\|_{\ell^{q_1}(\mathbb{Z})} \sim \|H_{n,g} 2^{\frac{n}{p_2}}\|_{\ell^{q_2}(\mathbb{Z})} \sim 1.$$

By construction,

$$|fg| \leq \sum_k \sum_n H_{n,f} H_{n+k,g} \mathbf{1}_{E_{n,f} \cap E_{n+k,g}}.$$

From monotonicity of the quasi-norm and the quasi-triangle inequality, it remains to show

$$\left\| \sum_{k \geq 1} \sum_n H_{n,f} H_{n+k,g} \mathbf{1}_{E_{n,f} \cap E_{n+k,g}} \right\|_{L^{p,q}}^* \leq \left\| \sum_{k \leq 0} \sum_n H_{n,f} H_{n+k,g} \mathbf{1}_{E_{n,f} \cap E_{n+k,g}} \right\|_{L^{p,q}}^* \lesssim 1.$$

We prove the former; the latter is symmetric. In the proof of the horizontal dyadic layered cake decomposition, we showed that $\mu(E_{n,f}) \sim \mu(E_{n,g}) \sim 2^n$. In particular, $\mu(E_{n,f} \cap E_{n+k,g}) \lesssim 2^n$ uniformly in n and k , so by Holder's inequality in $\ell_n^q(\mathbb{Z})$ we obtain

$$\begin{aligned} \left\| \sum_n H_{n,f} H_{n+k,g} \mathbf{1}_{E_{n,f} \cap E_{n+k,g}} \right\|_{L^{p,q}}^* &\lesssim \|H_{n,f} H_{n+k,g} 2^{\frac{n}{p}}\|_{\ell_n^q(\mathbb{Z})} = 2^{-\frac{k}{p_2}} \|H_{n,f} 2^{\frac{n}{p_1}} H_{n+k,g} 2^{\frac{n+k}{p_2}}\|_{\ell_n^q(\mathbb{Z})} \\ &\lesssim 2^{-\frac{k}{p_2}} \|H_{n,f} 2^{\frac{n}{p_1}}\|_{\ell_n^{q_1}(\mathbb{Z})} \|H_{n,f} 2^{\frac{n}{p_2}}\|_{\ell_n^{q_2}(\mathbb{Z})} \lesssim 2^{-\frac{k}{p_2}}. \end{aligned}$$

Note the second line we make the change of variables $n+k \mapsto n$. Summing in $k \geq 1$ and applying the previous lemma, we conclude the desired inequality. A remark on the symmetric proof for the sum over terms $k \leq 0$, we simply replace $n+k$ with $n-k$ and $2^{-k/p_2}$ with $2^{k/p_2}$. \square

For brevity we denote the dyadic numbers $2^{\mathbb{Z}}$, that is, the numbers of the form 2^n for $n \in \mathbb{Z}$. In a similar spirit to Lemma 8, we can show that the q -th power of a convergent sum of dyadics is comparable to the sum of the q -th powers and the maximum of the q -th powers for $1 \leq q < \infty$. This result will be convenient algebraically,

Lemma 10. *Let $1 \leq q < \infty$ and suppose $\mathcal{F} \subseteq 2^{\mathbb{Z}}$ is finite. Then*

$$\sum_{N \in \mathcal{F}} N^q \sim \left(\sum_{N \in \mathcal{F}} N \right)^q \sim \max_{N \in \mathcal{F}} N^q.$$

Proof. We have

$$\sum_{N \in \mathcal{F}} N^q \leq \left(\sum_{N \in \mathcal{F}} N \right)^q \leq \left(\max_{N \in \mathcal{F}} 2N \right)^q \leq 2^q \sum_{N \in \mathcal{F}} N^q.$$

\square

Theorem 11 ($L^{p,q}$ -duality). *Let $1 < p < \infty$ and $1 \leq q \leq \infty$, and suppose $f \in L^{p,q}(X)$, then*

$$\|f\|_{L^{p,q}}^* \sim_{p,q} \sup_{\|g\|_{L^{p',q'}}^* \leq 1} \left| \int_X f \bar{g} d\mu \right|.$$

Furthermore, if $q \neq \infty$, then $L^{p,q}(X)$ forms a Banach space whose continuous dual is $L^{p',q'}(X)$.

Proof. The right-hand side is controlled by the left-hand side by Holder's inequality, we will show the converse. By absolute homogeneity, we can assume without loss of generality $\|f\|_{L^{p,q}}^* = 1$, and using the vertically dyadic layered cake decomposition we can assume f takes the form

$$f = \sum_{n \in \mathbb{Z}} 2^n \mathbb{1}_{F_n},$$

for $\{F_n\}_n$ a family of disjoint measurable sets. Then

$$1 = \left(\|f\|_{L^{p,q}}^* \right)^q = \|2^n \mu(F_n)^{1/p}\|_{\ell_n^q}^q = \sum_{n \in \mathbb{Z}} 2^{nq} \mu(F_n)^{q/p}.$$

We want to construct $g \in L^{p',q'}(X)$ such that $\int_X f \bar{g} d\mu \sim 1$ and $\|g\|_{L^{p',q'}}^* \lesssim 1$. To this end, set

$$g := \sum_{n \in \mathbb{Z}} \left(2^n \mu(F_n)^{1/p} \right)^{q-1} \mu(F_n)^{-1/p'} \mathbb{1}_{F_n} = \sum_{n \in \mathbb{Z}} 2^{n(q-1)} \mu(F_n)^{(q-p)/p} \mathbb{1}_{F_n}.$$

Then

$$\int_X f \bar{g} d\mu = \sum_{n \in \mathbb{Z}} \left(2^n \mu(F_n)^{1/p} \right)^{q-1} 2^n \mu(F_n)^{1/p} = \sum_{n \in \mathbb{Z}} \left(2^n \mu(F_n)^{1/p} \right)^q = (\|f\|_{L^{p,q}}^*)^q = 1.$$

Denoting $S_N \subseteq \mathbb{Z}$ the collection of integers $n \in \mathbb{Z}$ satisfying $2^{n(q-1)} \mu(F_n)^{(q-p)/p} \sim N$, we apply the vertical layered cake decomposition and Lemma 10 to estimate the $L^{p',q'}$ -norm of g by

$$\begin{aligned} \left(\|g\|_{L^{p',q'}}^* \right)^{q'} &\sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'} \left(\sum_{n \in S_N} \mu(F_n) \right)^{q'/p'} \sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'} \left(\sum_{n \in S_N} N^{\frac{p}{q-p}} 2^{-n(q-1)\frac{p}{q-p}} \right)^{q'/p'} \\ &\sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'(1+\frac{p}{p'}\frac{1}{q-p})} \sum_{n \in S_N} N^{\frac{p}{q-p}} 2^{-n(q-1)q'\frac{p}{q-p}\frac{p'}{p}} \sim \sum_{N \in 2^{\mathbb{Z}}} \left(2^{n(q-1)} \mu(F_n)^{\frac{q-p}{p}} \right) 2^{-nq\frac{p-1}{q-p}} \\ &\sim \sum_{n \in \mathbb{Z}} \mu(F_n)^{\frac{q'}{p}(q-1)} 2^{n(q-1)q'\frac{q-1}{q-p}-nq\frac{p-1}{q-p}} \sim \sum_{n \in \mathbb{Z}} 2^{nq} \mu(F_n)^{q/p} \sim \|2^n \mu(F_n)^{1/p}\|_{\ell_n^q}^{q/q'} \sim 1, \end{aligned}$$

as desired.

To show duality for $1 < p < \infty$ and $1 \leq q < \infty$, consider $\ell : L^{p,q}(X) \rightarrow \mathbb{C}$ a bounded linear functional, we want to show that there exists $g \in L^{p',q'}(X)$ such that

$$\ell(f) = \int_X f \bar{g} d\mu.$$

Observe that $E \mapsto \ell(\mathbb{1}_E)$ forms a complex measure absolutely continuous with respect to μ . Thus by Radon-Nikodym, there exists $g \in L_{\text{loc}}^1(X)$ such that

$$\ell(\mathbb{1}_E) = \int_X \mathbb{1}_E \bar{g} d\mu.$$

This extends by linearity and density replacing $\mathbb{1}_E$ with any $f \in L^{p,q}(X)$. By the dual characterisation of the $L^{p,q}$ -norm and boundedness of ℓ , we conclude $g \in L^{p',q'}(X)$, completing the proof. \square

Remark. Modifying the example in the case $q = \infty$ (*), one can show that $L^{1,q}(X)$ cannot admit a norm for all $1 < q < \infty$.

2.4. Real interpolation. Let T be an operator mapping a subspace of measurable functions $X \rightarrow \mathbb{C}$ to measurable functions $Y \rightarrow \mathbb{C}$, we say it is SUB-LINEAR if it satisfies

- absolute homogeneity, $|T(\alpha f)| = |\alpha| |Tf|$,
- sub-linearity, $|T(f+g)| \leq |Tf| + |Tg|$,

for all $\alpha \in \mathbb{C}$ and $f, g : X \rightarrow \mathbb{C}$ in the domain of T . For $1 \leq p, q \leq \infty$, a sub-linear operator T is

- STRONG-TYPE (p, q) if it is bounded $T : L^p(X) \rightarrow L^q(Y)$, i.e. it satisfies the strong-type (p, q) inequality

$$\|Tf\|_{L^q} \lesssim \|f\|_{L^p}, \quad \text{uniformly in } f \in L^p(X)$$

- **WEAK-TYPE** (p, q) if it is bounded $T : L^p(X) \rightarrow L^{q,\infty}(Y)$, i.e. it satisfies the weak-type (p, q) inequality

$$\|Tf\|_{L^{q,\infty}}^* \lesssim \|f\|_{L^p}, \quad \text{uniformly in } f \in L^p(X),$$

- **RESTRICTED WEAK-TYPE** (p, q) for $q \neq \infty$ if the weak-type (p, q) inequality holds for step functions $f = \mathbb{1}_F$.

Remark. Strong-type implies weak-type, while weak-type implies restricted weak-type, however the converses generally fail. For example, the Hardy-Littlewood maximal function is weak-type $(1, 1)$ however not strong-type $(1, 1)$. The operator

$$Tf(x) := |x|^{-d/q} \int_{\mathbb{R}^d} f(y) |y|^{-d/p'} dy$$

is restricted weak-type (p, q) however not weak-type (p, q) .

A typical technique to prove boundedness of a linear operator is to show the desired inequality $\|Tf\| \lesssim \|f\|$ for an elementary class of functions, such as simple functions or test functions. In the case of linear operators, boundedness implies Lipschitz continuity $\|Tf - Tg\| \lesssim \|f - g\|$, so we can extend the inequality by density to the entire space. This fails for sub-linear operators, so we will instead take a more technical approach in the spirit of Lemma 8.

Lemma 12. *Let $f : X \rightarrow [0, \infty)$ be measurable, then there exists sequence of measurable simple functions $f_k = \sum_n 2^n \mathbb{1}_{F_{k,n}}$ such that $f_k \leq 2^{1-k} f$ and f can be decomposed pointwise as*

$$f = \sum_{k \in \mathbb{N}} f_k.$$

Furthermore, if $\|\cdot\|$ is a monotonic norm, then

$$\|f\| \sim \sum_{k \in \mathbb{N}} \|f_k\|.$$

We conclude that if T is a sub-linear operator bounded on simple functions, then T is bounded.

Proof. For fixed $x \in X$, consider the binary expansion of $f(x) \in [0, \infty)$. We construct recursively

$$f_k(x) := 2^{n_k(x)}, \quad n_k(x) := \sup\{n \in \mathbb{Z} : 2^n \leq f(x) - \sum_{i=1}^{k-1} f_i(x)\},$$

arguing inductively gives $f_k \leq 2^{1-k} f$. Informally, $f_k(x)$ corresponds to the k -th non-zero entry in the binary expansion of $f(x)$. Under this interpretation, it is clear that $f = \sum_k f_k$ pointwise, f_k is measurable since the map from $y \in [0, \infty)$ to the k -th non-zero entry in its binary expansion is continuous, and f_k are simple,

$$f_k := \sum_{n \in \mathbb{Z}} 2^n \mathbb{1}_{F_{k,n}}, \quad F_{k,n} = \{x \in X : n_k(x) = n\}.$$

It follows from the triangle inequality and pointwise bound on f_k that

$$\|f\| \leq \sum_{k \in \mathbb{N}} \|f_k\| \leq \sum_{k \in \mathbb{N}} 2^{1-k} \|f\| \lesssim \|f\|.$$

Suppose T is sub-linear and bounded on simple functions, then by sub-linearity and the triangle inequality

$$\|Tf\| \leq \sum_{n \in \mathbb{N}} \|Tf_k\| \lesssim \sum_{n \in \mathbb{N}} \|f_k\| \sim \|f\|$$

completing the proof. \square

Proposition 13 (Characterisations of restricted weak-type (p, q)). *Let $1 < p, q < \infty$ and suppose T is a sub-linear operator. Then the following are equivalent:*

- (a) *T is of restricted weak-type (p, q) , i.e.*

$$\|T\mathbb{1}_F\|_{L^{q,\infty}}^* \lesssim \mu(F)^{1/p}$$

uniformly in $F \subseteq X$ measurable.

- (b) *T satisfies the inequality*

$$\int_Y |T\mathbb{1}_F| \mathbb{1}_E dv \lesssim \mu(F)^{1/p} \nu(E)^{1/q'}$$

uniformly in $F \subseteq X$ and $E \subseteq Y$ measurable.

(c) T forms a bounded operator $T : L^{p,1}(X) \rightarrow L^{q,\infty}(Y)$, i.e.

$$\|Tf\|_{L^{q,\infty}}^* \lesssim \|f\|_{L^{p,1}}^*$$

uniformly in $f \in L^{p,1}(X)$.

Proof. (a) \implies (b) using Holder's inequality,

$$\int_Y |T\mathbb{1}_F| \mathbb{1}_E d\nu \leq \|T\mathbb{1}_F\|_{L^{1,1}}^* \lesssim \|T\mathbb{1}_F\|_{L^{q,\infty}}^* \|\mathbb{1}_E\|_{L^{q',1}} \lesssim \mu(F)^{1/p} \nu(E)^{1/q'}.$$

(c) \implies (a) follows from taking $f = \mathbb{1}_F$,

$$\|T\mathbb{1}_F\|_{L^{q,\infty}}^* \lesssim \|\mathbb{1}_F\|_{L^{p,1}}^* \sim \mu(F)^{1/p}.$$

(b) \implies (c) using Lemma 12, the dual characterisation of the $L^{p,q}$ -norm and density of simple functions, we can take a vertically dyadic layered cake decomposition and assume $f \in L^{p,1}(X)$ and $g \in L^{q',1}(Y)$ take the form

$$f = \sum_{n \in \mathbb{Z}} 2^n \mathbb{1}_{F_n}, \quad g = \sum_{m \in \mathbb{Z}} 2^m \mathbb{1}_{E_m}$$

for $\{F_n\}_n$ and $\{E_m\}_m$ families of disjoint measurable sets in X and Y respectively. Suppose $\|g\|_{L^{q',1}}^* \lesssim 1$, then it follows from (b) that

$$\begin{aligned} \left| \int_X T f \bar{g} d\nu \right| &\lesssim \sum_{m,n \in \mathbb{Z}} 2^{m+n} \int_Y |T\mathbb{1}_{F_n}| \mathbb{1}_{E_m} d\nu \\ &\lesssim \sum_{m,n \in \mathbb{Z}} 2^{m+n} \mu(F_n)^{1/p} \nu(E_m)^{1/q'} \leq \|2^m \nu(E_m)^{1/q'}\|_{\ell_m^1} \|2^n \mu(F_n)^{1/p}\|_{\ell_n^1} \lesssim \|f\|_{L^{p,q}}^*. \end{aligned}$$

This completes the proof. \square

Remark. If $p = q = 1$, the equivalence fails; in this case, (c) states that T is weak type $(1,1)$. For the non-equivalence of (a) and (c), Hagelstein and Jones [HJ05] exhibited an operator which is restricted weak-type $(1,1)$ however not weak-type $(1,1)$. For an example of the non-equivalence of (b) and (c), consider

$$(Tf)(x) := |x|^{-d} \int_{\mathbb{R}^d} f(y) dy,$$

which is weak-type $(1,1)$ however does not satisfy (b).

Theorem 14 (Hunt interpolation [Hun64]). *Let $1 \leq p_0 \neq p_1 \leq \infty$ and $1 \leq q_0 \neq q_1 \leq \infty$, and suppose T is a sub-linear operator which is bounded $L^{p_0,1}(X) \rightarrow L^{q_0,\infty}(Y)$ and $L^{p_1,1}(X) \rightarrow L^{q_1,\infty}(Y)$, i.e.*

$$\begin{aligned} \|Tf\|_{L^{q_0,\infty}}^* &\lesssim \|f\|_{L^{p_0,1}}^*, & \text{uniformly in } f \in L^{p_0,1}(X) \\ \|Tf\|_{L^{q_1,\infty}}^* &\lesssim \|f\|_{L^{p_1,1}}^*, & \text{uniformly in } f \in L^{p_1,1}(X). \end{aligned}$$

Then for all $1 \leq r \leq \infty$ and $0 < \theta < 1$ the operator $T : L^{p_{\theta},r}(X) \rightarrow L^{q_{\theta},r}(Y)$ is bounded, i.e.

$$\|Tf\|_{L^{q_{\theta},r}}^* \lesssim \|f\|_{L^{p_{\theta},r}}^*.$$

Proof. We first reduce the endpoint cases to the non-endpoint cases. From the restriction $p_0 \neq p_1$ and $q_0 \neq q_1$, the interpolated exponents (p_{θ}, q_{θ}) must lie in the interior of the interpolation square, so it suffices to interpolate between non-endpoint exponents $(p_{\theta_0}, q_{\theta_0})$ and $(p_{\theta_1}, q_{\theta_1})$ for some $0 < \theta_0 < \theta < \theta_1 < 1$. Thus by Proposition 13 we need to show the restricted weak-type (p_{θ}, q_{θ}) estimate holds for all $0 < \theta < 1$. Indeed,

$$\begin{aligned} \|T\mathbb{1}_F\|_{L^{q_{\theta},\infty}}^* &= \sup_{\lambda > 0} \lambda \nu(|T\mathbb{1}_F| > \lambda)^{1/q_{\theta}} = \sup_{\lambda > 0} \left(\lambda \nu(|T\mathbb{1}_F| > \lambda)^{1/q_0} \right)^{1-\theta} \left(\lambda \nu(|T\mathbb{1}_F| > \lambda)^{1/q_1} \right)^{\theta} \\ &\lesssim (\|T\mathbb{1}_F\|_{L^{q_0,\infty}}^*)^{1-\theta} (\|T\mathbb{1}_F\|_{L^{q_1,\infty}}^*)^{\theta} \lesssim \mu(F)^{1/q_{\theta}}. \end{aligned}$$

Assume then $1 < p_0 \neq p_1 < \infty$ and $1 < q_0 \neq q_1 < \infty$. In this regime, we can use Lemma 12, the dual characterisation of the $L^{p,q}$ -norm, and density of simple functions to take without loss of generality $f \in L^{p_{\theta},r}(X)$ and $g \in L^{q'_{\theta},r'}(Y)$ of the form

$$f = \sum_{n \in \mathbb{Z}} 2^n \mathbb{1}_{F_n}, \quad g = \sum_{m \in \mathbb{Z}} 2^m \mathbb{1}_{E_m}$$

for $\{F_n\}_n$ and $\{E_m\}_m$ families of disjoint measurable sets in X and Y respectively. Using sub-linearity and the previous remarks, we need to estimate the quantity

$$\|T\|_{L^{q_\theta,\infty}}^* \sim \sup \left\{ \left| \int_Y T f \bar{g} d\nu \right| : \|g\|_{L^{q'_\theta,r}}^* \lesssim 1 \right\} \lesssim \left\{ \sum_{m,n \in \mathbb{Z}} \int_Y |T \mathbb{1}_{F_n}| \mathbb{1}_{E_m} d\nu : \|2^m \nu(E_m)^{1/q'_\theta}\|_{\ell_m^{r'}} \lesssim 1 \right\}.$$

From Proposition 13, the restricted weak-type inequalities imply

$$\begin{aligned} \int_Y |T \mathbb{1}_{F_n}| \mathbb{1}_{E_m} d\nu &\lesssim 2^{n+m} \min\{\mu(F_n)^{1/p_0} \nu(E_m)^{1/q'_\theta}, \mu(F_n)^{1/p_1} \nu(E_m)^{1/q'_1}\} \\ &\lesssim 2^{n+m} N^{\frac{1}{p_0}} M^{\frac{1}{q'_\theta}} \min\{N^{\frac{1}{p_0}-\frac{1}{p_\theta}} M^{\frac{1}{q'_\theta}-\frac{1}{q_\theta}}, N^{\frac{1}{p_1}-\frac{1}{p_\theta}} M^{\frac{1}{q'_1}-\frac{1}{q_\theta}}\} =: 2^{n+m} N^{\frac{1}{p_\theta}} M^{\frac{1}{q'_\theta}} A(N, M), \end{aligned}$$

where for algebraic convenience we replace the measures $\mu(F_n)$ and $\nu(E_m)$ with dyadics $N, M \in 2^{\mathbb{Z}}$ satisfying $N \sim \mu(F_n)$ and $M \sim \nu(E_m)$. We claim that

$$\sum_{N \in 2^{\mathbb{Z}}} A(N, M) \sim \sum_{M \in 2^{\mathbb{Z}}} A(N, M) \sim 1$$

uniformly in $N, M \in 2^{\mathbb{Z}}$. This would complete the proof, as summing in n and m , applying Holder's inequality in $N, M \in 2^{\mathbb{Z}}$, and Lemma 10, we can conclude

$$\begin{aligned} \sum_{m,n \in \mathbb{Z}} \int_Y |T \mathbb{1}_{F_n}| \mathbb{1}_{E_m} d\nu &\lesssim \sum_{N, M \in 2^{\mathbb{Z}}} A(N, M) \left(\sum_{\substack{n \in \mathbb{Z} \\ \mu(F_n) \sim N}} 2^n N^{\frac{1}{p_\theta}} \right) \left(\sum_{\substack{m \in \mathbb{Z} \\ \nu(E_m) \sim M}} 2^m M^{\frac{1}{p_\theta}} \right) \\ &\lesssim \left(\sum_{N, M \in 2^{\mathbb{Z}}} A(N, M) \left(\sum_{\substack{n \in \mathbb{Z} \\ \mu(F_n) \sim N}} 2^n N^{\frac{1}{p_\theta}} \right)^r \right)^{1/r} \left(\sum_{N, M \in 2^{\mathbb{Z}}} A(N, M) \left(\sum_{\substack{m \in \mathbb{Z} \\ \nu(E_m) \sim M}} 2^m M^{\frac{1}{p_\theta}} \right)^{r'} \right)^{1/r'} \\ &\lesssim \left(\sum_{N \in 2^{\mathbb{Z}}} \left(\sum_{\substack{n \in \mathbb{Z} \\ \mu(F_n) \sim N}} 2^n \mu(F_n)^{\frac{1}{p_\theta}} \right)^r \right)^{1/r} \left(\sum_{M \in 2^{\mathbb{Z}}} \left(\sum_{\substack{m \in \mathbb{Z} \\ \nu(E_m) \sim M}} 2^m \nu(E_m)^{\frac{1}{p_\theta}} \right)^{r'} \right)^{1/r'} \\ &\lesssim \left(\sum_{n \in \mathbb{Z}} 2^{nr} \mu(F_n)^{\frac{r}{p_\theta}} \right)^{1/r} \left(\sum_{m \in \mathbb{Z}} 2^{mr'} \nu(E_m)^{\frac{r'}{p_\theta}} \right)^{1/r'} \\ &\lesssim \left\| 2^n \mu(F_n)^{1/p_\theta} \right\|_{\ell_n^r} \left\| 2^m \nu(E_m)^{1/q'_\theta} \right\|_{\ell_m^{r'}} \lesssim \|f\|_{L^{p_\theta,r}}^*. \end{aligned}$$

We prove the claim for summing in N uniformly in M ; the other case is symmetric. Write

$$N^{\frac{1}{p_0}-\frac{1}{p_\theta}} M^{\frac{1}{q'_\theta}-\frac{1}{q_\theta}} = \left(N^{\frac{1}{p_1}-\frac{1}{p_0}} M^{\frac{1}{q'_1}-\frac{1}{q'_\theta}} \right)^{-\theta}, \quad N^{\frac{1}{p_1}-\frac{1}{p_\theta}} M^{\frac{1}{q'_1}-\frac{1}{q_\theta}} = \left(N^{\frac{1}{p_1}-\frac{1}{p_0}} M^{\frac{1}{q'_1}-\frac{1}{q'_\theta}} \right)^{1-\theta}.$$

Fixing M , the transition for the minimum defining $A(N, M)$ occurs at $N = N_0$ for which the two quantities above are equal. Assuming without loss of generality that $p_0 < p_\theta < p_1$, splitting the regimes of summation with respect to the transition, and summing dyadically,

$$\begin{aligned} \sum_{N \in 2^{\mathbb{Z}}} A(N, M) &= \sum_{N \geq N_0} \left(N^{\frac{1}{p_1}-\frac{1}{p_0}} M^{\frac{1}{q'_1}-\frac{1}{q'_\theta}} \right)^{-\theta} + \sum_{N < N_0} \left(N^{\frac{1}{p_1}-\frac{1}{p_0}} M^{\frac{1}{q'_1}-\frac{1}{q'_\theta}} \right)^{1-\theta} \\ &= \left(N_0^{\frac{1}{p_1}-\frac{1}{p_0}} M^{\frac{1}{q'_1}-\frac{1}{q'_\theta}} \right)^{-\theta} + \left(N_0^{\frac{1}{p_1}-\frac{1}{p_0}} M^{\frac{1}{q'_1}-\frac{1}{q'_\theta}} \right)^{1-\theta} = 2, \end{aligned}$$

as $B^{-\theta} = B^{1-\theta}$ if and only if $B = 1$. □

Corollary 15 (Marcinkiewicz interpolation). *Let $1 \leq p_0 \leq p_1 < \infty$ and $1 \leq q_1 \neq q_2 \leq \infty$ such that $p_i \leq q_i$, and suppose T is a sub-linear operator of weak-type (p_0, q_0) and (p_1, q_1) , i.e.*

$$\begin{aligned} \|Tf\|_{L^{q_0,\infty}}^* &\lesssim \|f\|_{L^{p_0}}^*, & \text{uniformly in } f \in L^{p_0}(X), \\ \|Tf\|_{L^{q_1,\infty}}^* &\lesssim \|f\|_{L^{p_1}}^*, & \text{uniformly in } f \in L^{p_1}(X). \end{aligned}$$

Then T satisfies a strong-type (p_θ, q_θ) inequality, i.e.

$$\|Tf\|_{L^{q_\theta}} \lesssim \|f\|_{L^{p_\theta}}.$$

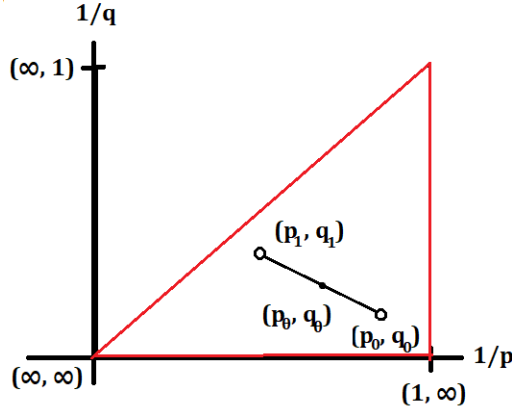


FIGURE 2. The *interpolation diagram*; the restriction $p_0 \leq q_0$ and $p_1 \leq q_1$ guarantees that $p_\theta \leq q_\theta$, so the intermediate line lies on the lower triangle.

Proof. Suppose $p_0 < p_1$, then recalling weak-type implies restricted weak-type, we apply Hunt interpolation for $r = q_\theta$ and use the embedding $L^{p_\theta, q_\theta}(X) \subseteq L^{p_\theta, p_\theta}(X)$ which holds since $p_\theta \leq q_\theta$,

$$\|Tf\|_{L^{q_\theta}} = \|Tf\|_{L^{q_\theta, q_\theta}}^* \lesssim \|f\|_{L^{p_\theta, q_\theta}}^* \lesssim \|f\|_{L^{p_\theta, p_\theta}}^* = \|f\|_{L^{p_\theta}}.$$

Assume now $p := p_0 = p_\theta = p_1$, we have that T is weak-type (p, q_0) and (p, q_1) , i.e.

$$\begin{aligned} \nu(|Tf| > \lambda) &\lesssim \left(\frac{\|f\|_{L^p}}{\lambda} \right)^{q_0}, \\ \nu(|Tf| > \lambda) &\lesssim \left(\frac{\|f\|_{L^p}}{\lambda} \right)^{q_1}. \end{aligned}$$

Taking a layered cake decomposition, and assume without loss of generality $q_0 < q_1$, then

$$\begin{aligned} \|Tf\|_{L^{q_\theta}}^{q_\theta} &\sim \int_0^\infty \lambda^{q_\theta} \nu(|Tf| > \lambda) \frac{d\lambda}{\lambda} \lesssim \int_0^\infty \lambda^{q_\theta} \min \left\{ \left(\frac{\|f\|_{L^p}}{\lambda} \right)^{q_0}, \left(\frac{\|f\|_{L^p}}{\lambda} \right)^{q_1} \right\} \frac{d\lambda}{\lambda} \\ &\lesssim \int_0^{\|f\|_{L^p}} \lambda^{q_\theta} \left(\frac{\|f\|_{L^p}}{\lambda} \right)^{q_0} \frac{d\lambda}{\lambda} + \int_{\|f\|_{L^p}}^\infty \lambda^{q_\theta} \left(\frac{\|f\|_{L^p}}{\lambda} \right)^{q_1} \frac{d\lambda}{\lambda} \lesssim \|f\|_{L^p}^{q_\theta}. \end{aligned}$$

This completes the proof. □

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