OSCILLATORY INTEGRALS

JASON ZHAO

ABSTRACT. An *oscillatory integral* takes the form

$$\int_{\mathbb{R}^d} e^{i\lambda\phi(x)} a(x)\, dx,$$

where $a : \mathbb{R}^d \to \mathbb{C}$ is the *amplitude*, $\phi : \mathbb{R}^d \to \mathbb{R}$ is the *phase*, and $\lambda \in \mathbb{R}$ is a parameter to measure the extent of oscillation. While we can naively control the integral by the L^1-norm of a, we can alternatively exploit the cancellation arising from the oscillating phase ϕ to compute precise asymptotic decay as $\lambda \to \pm\infty$.

CONTENTS

Formally, an OSCILLATORY INTEGRAL (of the first kind) is an integral of the form

$$\int_{\mathbb{R}^d} e^{i\lambda\phi(x)} a(x)\, dx$$

where $a : \mathbb{R}^d \to \mathbb{C}$ is known as the *amplitude*, $\phi : \mathbb{R}^d \to \mathbb{R}$ the *phase*, and $\lambda \in \mathbb{R}$ the *frequency*.

1. NON-STATIONARY PHASE

The integral exhibits oscillation and thus cancellation provided that the phase ϕ is non-stationary, i.e. ϕ does not admit critical points in the support of the amplitude a. Given oscillation, we can use integration-by-parts to exchange regularity of a for decay in λ.

Theorem 1 (Non-stationary phase on $\mathbb{R}$). *Let $a \in C_c^\infty(\mathbb{R})$, and suppose $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth phase such that ϕ' is non-zero on the support of a. For $N \in \mathbb{N}_0$, we have*

$$\left| \int_{\mathbb{R}} e^{i\lambda\phi(x)} a(x)\, dx \right| \lesssim_{N,\phi,a} \frac{1}{\lambda^N}$$

uniformly in $\lambda > 0$.

Proof. Consider the differential operator D and its formal adjoint D^* given by

$$Df := \frac{1}{i\lambda\phi'(x)} \frac{d}{dx}, \qquad D^* f = -\frac{d}{dx}\left(\frac{1}{i\lambda\phi'(x)} f\right).$$

It is clear that $De^{i\lambda\phi(x)} = e^{i\lambda\phi(x)}$, so integrating by parts,

$$\int_{\mathbb{R}} e^{i\lambda\phi(x)} a(x) dx = \int_{\mathbb{R}} D^N \left(e^{i\lambda\phi(x)}\right) a(x) dx = \int_{\mathbb{R}} e^{i\lambda\phi(x)} (D^*)^N a(x) dx.$$

Date: September 14, 2022.

Therefore by the triangle inequality and the product rule

$$\left|\int_{\mathbb{R}} e^{i\lambda\phi(x)}a(x)dx\right| \leq \lambda^{-N}\int_{\mathbb{R}}\left|\left(\frac{d}{dx}\frac{1}{\phi'(x)}\right)^N a(x)\right|dx$$
$$\lesssim \lambda^{-N}\sum_{k=0}^{N}\sum_{\beta+\alpha_1+\cdots+\alpha_k=N}\left\|\frac{\partial^\beta a(\partial^{\alpha_1}\phi'\cdots\partial^{\alpha_k}\phi')}{(\phi')^{N+k}}\right\|_{L^1} \lesssim_{N,\phi,a} \lambda^{-N}.$$

This completes the proof. □

Remark.

(a) The implicit constant depends only on the derivatives of a up to order N and the derivatives of ϕ up to order $N+1$.

(b) The prototypical example of non-stationary phase is the decay of the Fourier transform of a compactly supported function, which is the case where $\phi(x) = \pm x$.

(c) If the amplitude a is not compactly supported, then the best decay is λ^{-1}. For example,

$$\int_a^b e^{i\lambda x}dx = \frac{e^{i\lambda b}-e^{i\lambda a}}{i\lambda}.$$

Theorem 2 (Non-stationary phase on $\mathbb{R}^d$). *Let $a \in C_c^\infty(\mathbb{R}^d)$, and suppose $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth phase such that $\nabla\phi$ is non-zero on the support of a. For $N \in \mathbb{N}_0$, we have*

$$\left|\int_{\mathbb{R}} e^{i\lambda\phi(x)}a(x)\,dx\right| \lesssim_{N,\phi,a} \frac{1}{\lambda^N}$$

uniformly in $\lambda > 0$.

Proof. Consider the differential operator D and its formal adjoint D^* given by

$$Df := \frac{\nabla\phi(x)}{i\lambda|\nabla\phi(x)|^2}\cdot\nabla f, \qquad D^*f = -\nabla\cdot\left(\frac{\nabla\phi(x)}{i\lambda|\nabla\phi(x)|^2}f\right).$$

As in the one-dimensional case, we see that $De^{i\lambda\phi(x)} = e^{i\lambda\phi(x)}$, so integrating by parts

$$\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}a(x)\,dx = \int_{\mathbb{R}^d} D\left(e^{i\lambda\phi(x)}\right)a(x)\,dx = \int_{\mathbb{R}^d} e^{i\lambda\phi(x)}D^*a(x)\,dx = -\frac{1}{i\lambda}\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}\nabla\cdot\left(\frac{a\nabla\phi}{|\nabla\phi|^2}\right)(x)\,dx.$$

The triangle inequality furnishes the result for $N = 1$. Note that $\nabla\cdot(a\nabla\phi/|\nabla\phi|^2) \in C_c^\infty(\mathbb{R}^d)$, so we iterate to obtain the estimate for all $N \in \mathbb{N}_0$. □

Remark. The implicit constant obtained from this proof becomes frighteningly complicated. However, one class of functions which behaves well under the operator D^* is the *symbol class*

$$\partial_x^\alpha a(x) \lesssim_\alpha \langle x\rangle^{m-|\alpha|}$$

of order $m \in \mathbb{R}$. It is an exercise in the product rule to verify that the symbol class is closed under multiplication and division by symbols satisfying $|a| \gtrsim 1$. If a and $\nabla\phi$ are in the symbol class for $m = 0$ and $|\nabla\phi| \gtrsim 1$, then D^* forms a *pseudo-differential operator* of order 1 sending symbols of order m to order $m-1$.

2. Scaling

To illustrate the principle of scaling, consider the one-dimensional oscillatory integral

$$I_{J,\phi}(\lambda) := \int_J e^{i\lambda\phi(x)}\,dx$$

where $J \subseteq \mathbb{R}$ is a finite interval and $\phi : J \to \mathbb{R}$ is a smooth phase function. Then

$$I_{L(J),\phi\circ L^{-1}}(\lambda) = |\det(L)|I_{J,\phi}(\lambda)$$

for any invertible affine transformation $L : \mathbb{R} \to \mathbb{R}$.

Theorem 3 (van der Corput lemma on $\mathbb{R}$). *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth phase such that $|\partial_x^k\phi| \geq 1$ for some $k \in \mathbb{N}$; in the case $k = 1$, we assume further that $\partial_x\phi$ is monotone. Then*

$$\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \lesssim_k \lambda^{-1/k}$$

uniformly in $\lambda > 0$, the phase ϕ, and the interval $[a,b]$.

Proof. We induct on k; consider the case $k = 1$, then integrating by parts gives

$$\int_a^b e^{i\lambda\phi(x)}dx = \left[\frac{e^{i\lambda\phi(b)}}{i\lambda\partial_x\phi(b)} - \frac{e^{i\lambda\phi(a)}}{i\lambda\partial_x\phi(a)}\right] - \int_a^b e^{i\lambda\phi(x)}\frac{d}{dx}\left[\frac{1}{i\lambda\partial_x\phi(x)}\right]dx.$$

As $|\partial_x\phi| \geq 1$, the first term on the right is bounded by $2/\lambda$. To bound the second term by $2/\lambda$, we want to apply the fundamental theorem of calculus. Applying the triangle inequality removes the phase term,

$$\left|\int_a^b e^{i\lambda\phi(x)}\frac{d}{dx}\left[\frac{1}{i\lambda\partial_x\phi(x)}\right]dx\right| \leq \frac{1}{\lambda}\int_a^b \left|\frac{d}{dx}\left[\frac{1}{\partial_x\phi(x)}\right]dx\right|.$$

We know $1/\partial_x\phi$ is monotone, so its derivative is either non-negative or non-positive, allowing us to "reverse" the triangle inequality and take the absolute values out of the integral,

$$\frac{1}{\lambda}\left|\int_a^b \frac{d}{dx}\left[\frac{1}{\partial_x\phi(x)}\right]dx\right| = \frac{1}{\lambda}\left|\frac{1}{\partial_x\phi(b)} - \frac{1}{\partial_x\phi(a)}\right| \leq \frac{2}{\lambda}.$$

This proves the lemma for $k = 1$. Assume for induction the claim holds for $k \geq 1$ and $|\partial_x^{(k+1)}\phi| \geq 1$. Without loss of generality, assume $\partial_x^{k+1}\phi \geq 1$; in particular, $\partial_x^k\phi$ is strictly increasing on $[a, b]$, so there exists at most one point $c \in [a, b]$ such that $\partial_x^k\phi(c) = 0$.

Consider the case where c exists; as $\partial_x^{k+1}\phi \geq 1$, we know

$$|\partial_x^k\phi(x)| \geq \delta \qquad \text{whenever } |x - c| > \delta$$

for any choice of $\delta > 0$. Rescaling, it follows that

$$\partial_y^k(\phi(\delta^{-1/k}y)) = \delta^{-1}\partial_x^k\phi(\delta^{-1/k}y) \geq 1 \qquad \text{whenever } y \in [\delta^{1/k}a, \delta^{1/k}(c-\delta)] \cup [\delta^{1/k}(c+\delta), \delta^{1/k}b].$$

Hence we can apply the induction hypothesis on the rescaled function $y \mapsto \phi(\delta^{-1/k}y)$,

$$\left|\left(\int_a^{c-\delta} + \int_{c+\delta}^b\right)e^{i\lambda\phi(x)}dx\right| = \left|\left(\int_{\delta^{1/k}a}^{\delta^{1/k}(c-\delta)} + \int_{\delta^{1/k}(c+\delta)}^{\delta^{1/k}b}\right)e^{i\lambda\phi(\delta^{-1/k}y)}\delta^{1/k}dy\right| \lesssim \delta^{-1/k}\lambda^{-1/k},$$

and similarly for the integral on $[c + \delta, b]$. On the other hand, we estimate naively

$$\left|\int_{c-\delta}^{c+\delta} e^{i\lambda\phi(x)}\right| \leq 2\delta.$$

Choosing $\delta = \lambda^{-1/(k+1)}$, we obtain

$$\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \lesssim \delta^{-1/k}\lambda^{-1/k} + 2\delta \sim \lambda^{-\frac{1}{k+1}}.$$

This completes the proof for this case.

Consider the case where $\partial_x^k\phi(x) \neq 0$ for all $x \in [a, b]$, e.g. without loss of generality $\partial_x^k\phi > 0$. As $\partial_x^{k+1}\phi \geq 1$, we know that

$$\partial_x^k(x) \geq \delta \qquad \text{whenever } x \in [a + \delta, b].$$

Following the argument from the previous case, we have

$$\left|\int_{a+\delta}^b e^{i\lambda\phi(x)}dx\right| \lesssim \lambda^{-1/k}\delta^{-1/k}$$

and the naive estimate

$$\left|\int_a^{a+\delta} e^{i\lambda\phi(x)}dx\right| \leq \delta.$$

Choosing $\delta = \lambda^{1/(k+1)}$, we conclude the result. $\square$

Remark. Monotonicity of $\partial_x\phi$ is necessary in the case $k = 1$; writing $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we can bound the integral from below,

$$\left|\int_a^b e^{i\phi(x)}dx\right| \geq \left|\int_a^b \cos(\phi(x))dx\right|.$$

We can construct a phase function satisfying $|\phi'(x)| \gg 1$ whenever $\cos(\phi(x)) < 0$ and $|\phi'(x)| \ll 1$ whenever $\cos(\phi(x)) > 0$. It would follow that

$$|\{x : \cos(\phi(x)) > 0\}| \gg |\{x : \cos(\phi(x)) < 0\}|$$

and thus

$$\left|\int_a^b \cos(\phi(x))dx\right| \sim \int_a^b \cos(\phi(x))dx \overset{b-a\to\infty}{\longrightarrow} \infty.$$

Corollary 4. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth phase such that $|\partial_x^k \phi| \geq 1$ for some $k \in \mathbb{N}$; in the case $k = 1$, we assume further that $\partial_x \phi$ is monotone. For amplitude functions $\psi : \mathbb{R} \to \mathbb{C}$ of bounded variation,*

$$\left|\int_a^b e^{i\lambda\phi(x)}\psi(x)dx\right| \lesssim_k \lambda^{-1/k}\left(|\psi(b)| + ||\psi||_{\mathrm{BV}}\right).$$

Proof. Integrating by parts in the Riemann-Stieltjes sense, we can write

$$\int_a^b e^{i\lambda\phi(x)}a(x)dx = \int_a^b \psi(x)\frac{d}{dx}\left(\int_a^x e^{i\lambda\phi(y)}dy\right)dx = \psi(b)\int_a^b e^{i\lambda\phi(y)}dy - \int_a^b\left(\int_a^x e^{i\lambda\phi(y)}dy\right)da.$$

By the triangle inequality and the van der Corput lemma, it follows that

$$\left|\int_a^b e^{i\lambda\phi(x)}\psi(x)dx\right| \lesssim \lambda^{-1/k}\left(|\psi(b)| + ||\psi||_{\mathrm{BV}}\right)$$

as desired. □

Theorem 5 (van der Corput lemma on $\mathbb{R}^d$)**.** *Let $\psi \in C_c^\infty(\mathbb{R}^d)$, and suppose $\phi : \mathbb{R}^d \to \mathbb{R}$ be a smooth phase such that $|\partial^\alpha \phi| \geq 1$ for some $|\alpha| > 0$ throughout the support on ψ. Then*

$$\left|\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}\psi(x)\,dx\right| \lesssim_{k,\phi} \lambda^{-d/k}(||\psi||_{L^\infty} + ||\nabla\psi||_{L^1})$$

3. Stationary phase

It follows from the principle of non-stationary phase that determining the asymptotics of an oscillatory integral reduces to localising the integral to a neighborhood of the stationary set $\nabla\phi = 0$. In the case where the critical points are isolated, the *principle of stationary phase* roughly states that

$$\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}a(x)\,dx \sim \sum_{\nabla\phi(x_0)=0} e^{i\lambda\phi(x_0)}a(x_0)|\{x \approx x_0 : \phi(x) = \phi(x_0) + O(\lambda^{-1})\}|.$$

The size of the region where the phase is close to stationary depends on the order to which ϕ vanishes at the critical point. The model cases are the monomial phases $\phi(x) = |x|^k$ which vanish up to order $k-1$, in which case we expect the asymptotic expansion

$$\int_{\mathbb{R}^d} e^{i\lambda|x|^k}a(x)\,dx \sim a(0)\lambda^{-\frac{d}{k}} + O(\lambda^{-\frac{d}{k}-1}),$$

which is consistent with the van der Corput lemma.

3.1. **Fresnel phase.**

To motivate our approach, consider the quadratic case $\phi(x) = |x|^2$. In this case, the oscillating factor takes the form of a complex Gaussian. Recall the Fourier transform of a Gaussian

$$\int_{\mathbb{R}^d} e^{-\pi z|x|^2}e^{-2\pi i x\cdot\xi}\,dx = z^{-d/2}e^{-\pi|\xi|^2/z},$$

for $\operatorname{Re} z > 0$. Passing $z \to -i\lambda/\pi$ in the sense of tempered distributions, it follows from Plancharel's theorem and, formally, taking the Taylor expansion of the exponential,

$$\begin{aligned}\int_{\mathbb{R}^d} e^{i\lambda|x|^2}a(x)\,dx &= (-i\lambda/\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-i\pi^2|\xi|^2/\lambda}\widehat{a}(\xi)\,d\xi \\ &= (-i\lambda/\pi)^{-d/2}\sum_{n=0}^\infty \int_{\mathbb{R}^d}(-i\pi^2|\xi|^2\lambda^{-1})^n\frac{1}{n!}\widehat{a}(\xi)\,d\xi \\ &= (i\pi)^{\frac{d}{2}}\sum_{n=0}^\infty \lambda^{-n-\frac{d}{2}}\frac{i^n}{n!}(\Delta^n a)(0)\end{aligned}$$

for $a \in C_c^\infty(\mathbb{R}^d)$. While some care needs to be taken when dealing with the convergence of the sum, nonetheless we expect the full asymptotic expansion to take the above form.

3.2. **Quadratic phase.** With this primer at hand, we now consider the case of non-degenerate quadratic phases $\phi(x) = x \cdot Qx$ where $Q \in \mathbb{R}^{d\times d}$ is a non-degenerate symmetric matrix. An analogous formula for the Fourier transform of the corresponding "Gaussian" holds, however our approach will need to be more subtle to deal with negative eigenvalues. Writing

$$\int_{\mathbb{R}^d} e^{i\lambda x\cdot Qx} a(x)\, dx = \int_{\mathbb{R}^d} e^{x\cdot(i\lambda Q - I)x} e^{|x|^2} a(x)\, dx,$$

we take the Fourier transform of $e^{x\cdot(i\lambda Q-I)x} \in \mathcal{S}(\mathbb{R}^d)$ and $e^{|x|^2}a(x) \in C_c^\infty(\mathbb{R}^d)$. To compute the former, we argue by complex analysis; given symmetric matrices $A, B \in \mathbb{R}^{d\times d}$ with A positive definite, we define the square root of $Z := A + iB$ via analytic continuation along the map $s \mapsto A + isB$ for $s \in [0,1]$.

Lemma 6. *Let $A, B \in \mathbb{R}^{d\times d}$ be symmetric matrices with A positive definite. Denoting $Z = A + iB$, we have*

$$\int_{\mathbb{R}^d} e^{-\pi x\cdot Zx} dx = \sqrt{\det(Z^{-1})}.$$

Proof. We first prove the case $B = 0$, and conclude the general case via analytic continuation. Let $\sqrt{A} \in \mathbb{R}^{d\times d}$ be the positive-definite square root of A, then making the change of variables $y = \sqrt{A}x$ gives

$$\int_{\mathbb{R}^d} e^{-\pi x\cdot Ax} dx = \frac{1}{\det\sqrt{A}} \int_{\mathbb{R}^d} e^{-\pi|y|^2} dy = \sqrt{\det(A^{-1})}.$$

For the general case, define $Z(s) := A + isB$ and consider the analytic map

$$\Phi(z) := \int_{\mathbb{R}^d} e^{-\pi x\cdot Z(s)x} dx.$$

Observe $\operatorname{Re} Z(s) = A - (\operatorname{Im} s)B$ is positive definite on the strip

$$|\operatorname{Im} s| < \frac{1}{||B||\ ||A^{-1}||},$$

where $||\cdot||$ is the usual matrix operator norm induced by the Euclidean metric. It follows from the first case that $\Phi(s) = \sqrt{\det(Z(s)^{-1})}$ on the strip whenever $\operatorname{Re} s = 0$. By the uniqueness theorem, this result extends globally to the entire strip, particularly when $s = 1$ which is exactly the desired identity. □

Proposition 7. *Let $A, B \in \mathbb{R}^{d\times d}$ be symmetric matrices with A positive-definite. Denoting $Z = A + iB$, we have*

$$\int_{\mathbb{R}^d} e^{-\pi x\cdot Zx} e^{-2\pi i\xi\cdot x} dx = e^{-\pi\xi\cdot Z^{-1}\xi} \sqrt{\det(Z^{-1})}.$$

Proof. For brevity denote the left-hand and right-hand sides by $L(\xi)$ and $R(\xi)$ respectively. We claim that

$$\nabla L(\xi) = -2\pi Z^{-1}\xi L(\xi), \qquad \nabla R(\xi) = -2\pi Z^{-1}\xi R(\xi).$$

The previous lemma furnishes $L(0) = R(0)$, so L and R satisfy an ordinary differential equation with the same initial data. We conclude from uniqueness the desired equality $L \equiv R$ from the claim. To prove the claim, recall

$$\frac{1}{2}\nabla(x\cdot Ax) = Ax$$

for any symmetric matrix A. The claim therefore follows for R by the chain rule. To prove the claim for L, we integrate by parts

$$\begin{aligned}
\nabla_\xi L(\xi) &= -2\pi i \int_{\mathbb{R}^d} x e^{-\pi x\cdot Zx} e^{i\xi x} dx \\
&= iZ^{-1} \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \left(\nabla_x e^{-\pi x\cdot Zx}\right) dx \\
&= -iZ^{-1} \int_{\mathbb{R}^d} \left(\nabla_x e^{2\pi i x\cdot\xi}\right) e^{-\pi x\cdot Zx} dx \\
&= -2\pi Z^{-1}\xi \int_{\mathbb{R}^d} e^{-\pi x\cdot Zx} e^{2\pi i x\cdot\xi} dx = -Z^{-1}\xi L(\xi).
\end{aligned}$$

This completes the proof of the claim and thereby the lemma. □

Theorem 8 (Asymptotics for quadratic phase). *Let $a \in C_c^\infty(\mathbb{R}^d)$, and suppose $Q \in \mathbb{R}^{d\times d}$ is a non-degenerate symmetric matrix, then*

$$\int_{\mathbb{R}^d} e^{-i\lambda x\cdot Qx} a(x)\,dx = \sum_{n=0}^{N} c_n \lambda^{-n-\frac{d}{2}} + O_{N,a,d}(\lambda^{-N-\frac{d}{2}-1})$$

for coefficients

$$c_n := \left(\frac{i\pi}{\det Q}\right)^{\frac{d}{2}} \frac{i^n}{n!}(\Delta_Q^n a)(0), \qquad \Delta_Q = \sum_{j,k=1}^{d} Q^{jk}\partial_j\partial_k.$$

Proof. By diagonalising via an orthonormal change of variables, we can assume without loss of generality that the phase takes the form $x \cdot Qx = \mu_1|x_1|^2 + \cdots + \mu_d|x_d|^2$, where $\mu_j \in \mathbb{R}$ are the eigenvalues of Q. Denote $b(x) := e^{|x|^2}a(x)$, we use Plancharel's theorem and Proposition 7 to write

$$\int_{\mathbb{R}^d} e^{i\lambda x\cdot Qx} a(x)\,dx = \int_{\mathbb{R}^d} e^{x\cdot(i\lambda Q - I)x} e^{|x|^2} a(x)\,dx = \int_{\mathbb{R}^d} \prod_{j=1}^{d} e^{(i\lambda\mu_j - 1)|x_j|^2} e^{|x|^2} a(x)\,dx$$

$$= \pi^{d/2} \prod_{j=1}^{d} (1 - i\lambda\mu_j)^{-1/2} \int_{\mathbb{R}^d} \prod_{j=1}^{d} e^{-\pi^2|\xi_j|^2/(i\lambda\mu_j - 1)}\, \widehat{b}(\xi)\,d\xi.$$

□

3.3. **Non-degenerate phase.** We generalise the previous discussion by considering phases $\phi : \mathbb{R}^d \to \mathbb{R}$ with a unique critical point $x_0 \in \mathbb{R}^d$ which is non-degenerate, i.e. $\det \nabla^2\phi(x_0) \neq 0$. A fact unique to phases which vanishes only up to first order is the *Morse lemma*, which states that any such phase admits local coordinates under which it takes the form of a quadratic form.

Lemma 9 (Morse lemma). *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a smooth function and $x_0 \in \mathbb{R}^d$ a non-degenerate isolated critical point of ϕ. Then there exists a local diffeomorphism $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\nabla\Phi(x_0) = I$ and*

$$\phi(x) = \phi(x_0) + \frac{1}{2}\Phi(x)\cdot\nabla^2\phi(x_0)\Phi(x).$$

Theorem 10 (Asympotic expansion for non-degenerate phase). *Let $a \in C_c^\infty(\mathbb{R}^d)$, and suppose $\phi : \mathbb{R}^d \to \mathbb{R}$ is a smooth phase admitting a unique non-degenerate critical point $x_0 \in \mathbb{R}^d$ in the support of a, then*

$$\int_{\mathbb{R}^d} e^{i\lambda\phi(x)} a(x)\,dx = \sum_{n=0}^{N} c_n \lambda^{-n-\frac{d}{2}} e^{i\lambda\phi(x_0)} + O_{N,a,d}(\lambda^{-N-\frac{d}{2}-1})$$

for some coefficients c_n depending only only finitely many derivatives of a and ϕ at x_0. In particular,

$$c_0 := \frac{(2\pi)^{d/2} e^{i\frac{\pi}{4}\,\mathrm{sgn}\,\nabla^2\phi(x_0)}}{|\det \nabla^2\phi(x_0)|^{1/2}} a(x_0).$$

Proof. If a vanishes in a neighborhood of x_0, the claim follows from the principle of non-stationary phase, so localising we can assume that the change of variables $y = \Phi(x)$ from the Morse lemma is global on the support of a. Applying the change of variables,

$$\int_{\mathbb{R}^d} e^{i\lambda\phi(x)} a(x)\,dx = e^{i\lambda\phi(x_0)} \int_{\mathbb{R}^d} e^{\frac{1}{2}i\lambda y\cdot\nabla^2\phi(x_0)y} (a\circ\Phi^{-1})(y) |\det \nabla\Phi^{-1}(y)|\,dy.$$

Noting $(a\circ\Phi^{-1})(0) = a(x_0)$ and $\det\nabla\Phi^{-1}(0) = 1$, we see that Theorem 8 completes the proof. □

Theorem 11 (Asymptotic expansion for finite-order vanishing phase). *Let $a \in C_c^\infty(\mathbb{R})$, and suppose $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth phase admitting a unique critical point $x_0 \in \mathbb{R}$ in the support of a. If $\phi(x_0)$ vanishes up to order $k-1$, that is $\phi^{(1)}(x_0) = \cdots = \phi^{(k-1)}(x_0) = 0$ and $\phi^{(k)}(x_0) \neq 0$, then*

$$\int_{\mathbb{R}} e^{i\lambda\phi(x)} a(x)\,dx = \sum_{n=0}^{N} c_n \lambda^{-n/k} e^{i\lambda\phi(x_0)} + O_{N,a,\phi,k}(\lambda^{-(N+1)/k})$$

for some coefficients c_n depending only on finitely many derivatives of a and ϕ at x_0. In particular,

$$c_0 \sim |\phi^{(k)}(x_0)|^{-1/k} |a(x_0)|.$$