LAPLACE'S EQUATION

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Abstract. The prototypical example of an elliptic partial differential equation is *Poisson's equation*

$$\Delta u = f.$$

The equation is known as *Laplace's equation* when $f = 0$. The problem of solving the Poisson equation subject to boundary conditions $u_{|\partial\Omega} = \phi$ is known as the *Dirichlet problem*. We exposit four methods for solving the Dirichlet problem.

Contents

1. Fundamental solution

The fundamental solution of the Laplace operator is a distribution $K \in C_c^\infty(\mathbb{R}^d)^*$ such that

$$\Delta K = \delta_0.$$

From the perspective of electrostatics, we can view the fundamental solution as a potential field arising from the unit electric charge concentrated at the origin. In thermodynamics, the fundamental solution is a steady-state heat distribution given a unit heat source at the origin. We can construct solutions to Poisson's equation on $\mathbb{R}^d$ by convolving the fundamental solution with the source term; for any compactly supported distribution $f \in C^\infty(\mathbb{R}^d)^*$, we have

$$f = \delta_0 * f = \Delta K * f = \Delta(K * f).$$

The fundamental solution takes the form

$$K(x) := \begin{cases} \frac{1}{2\pi} \log \frac{1}{|x|}, & \text{if } d = 2, \\ \frac{1}{(d-2)A_d} |x|^{2-d}, & \text{if } d \geq 3, \end{cases}$$

where A_d is the surface area of the unit sphere in $\mathbb{R}^d$.

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1.1. **Derivation.** To motivate the construction of the fundamental solution, we remark that the Laplace operator is a homogeneous differential operator of order 2 invariant under rotations, while the Dirac delta is a homogeneous distribution of order $-n$. Thus we expect the fundamental solution to be homogeneous of order $2-n$ and spherically symmetric. Making the *ansatz* that the spherical derivatives vanish, K solves the equation

$$\frac{1}{r^{d-1}}\partial_r\left(r^{d-1}\partial_r K\right)=0$$

on $\mathbb{R}^d \setminus 0$. It follows that $r^{d-1}\partial_r u \equiv c_d$ for some constant depending on the dimension. Rearranging and integrating with respect to r, the fundamental solution takes the form

$$K(x) := \begin{cases} c_2 \log \frac{1}{|x|}, & \text{if } d=2, \\ c_d |x|^{2-d}, & \text{if } d \geq 3. \end{cases}$$

It remains to determine the value of the constant $c_d \in \mathbb{R}$. Let $\phi \in C_c^\infty(\mathbb{R}^d)$, then by dominated convergence theorem we can write

$$\phi(0) = \langle \phi, \Delta K\rangle = \langle \Delta\phi, K\rangle = c_d \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} K(x)\,\Delta\phi(x)\,dx.$$

Integrating by parts on the right, we obtain

$$\int_{|x|>\varepsilon} K(x)\,\Delta\phi(x)\,dx = \int_{|x|>\varepsilon}\phi(x)\Delta K(x)\,dx + \int_{|x|=\varepsilon}\partial_\nu\phi(x)u(x)\,dS - \int_{|x|=\varepsilon}\phi(x)\partial_\nu u(x)\,dS.$$

The first integral on the right vanishes by harmonicity of K away from the origin. The second integral vanishes when we pass the limit, since the sphere $|x|=\varepsilon$ has surface measure comparable to ε^{d-1}, so

$$\left|\int_{|x|=\varepsilon}\partial_\nu\phi(x)u(x)\,dS\right| \leq \int_{|x|=\varepsilon}\frac{|\partial_\nu\phi(x)|}{\varepsilon^{d-2}}\,dS \lesssim_{\phi,d} \varepsilon \xrightarrow{\varepsilon\to 0} 0.$$

The unit normal vector on the boundary of $|x|>\varepsilon$ is given by $\nu(x) = -x/|x|$, so the normal derivatives are exactly $\partial_\nu = -\partial_r$. It follows that the third integral on the right satisfies

$$-\int_{|x|=\varepsilon}\phi(x)\partial_\nu u(x)\,dS = \int_{|x|=\varepsilon}\phi(x)\frac{d-2}{\varepsilon^{d-1}}\,dS \xrightarrow{\varepsilon\to 0} (d-2)A_d\phi(0).$$

1.2. **Green's function.** The GREEN'S FUNCTION of the Laplace operator on a domain $\Omega \subseteq \mathbb{R}^d$ is a locally integrable $G : \overline{\Omega}\times\overline{\Omega} \to \overline{\mathbb{R}}$ smooth away from the diagonal $x=y$ and satisfying the Dirichlet problem

$$\begin{aligned} \Delta_y G(x,y) &= \delta_x(y), && y\in\Omega, \\ G(x,y) &= 0, && y \in \partial\Omega. \end{aligned}$$

Following a maximum principle argument, c.f. Section 2, we see that the Green's function is unique. We can view the Green's function as the analogue of the fundamental solution in the non-translation invariant case of the Dirichlet problem. In fact, existence is equivalent to the solvability of the Dirichlet problem

$$\begin{aligned} \Delta_y v(x,y) &= 0, && y\in\Omega, \\ v(x,y) &= K(y-x), && y \in \partial\Omega, \end{aligned}$$

as setting $G(x,y) := K(y-x) - v(x,y)$ furnishes the Green's function. Just as the fundamental solution gives rise to a representation of the solution to the Poisson equation on $\mathbb{R}^d$, we can write a solution to the Dirichlet problem in terms of the Green's function integrated against the source term f and the boundary terms ϕ.

Theorem 1 (Green's representation formula). *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded C^1-domain, and suppose $f:\Omega\to\mathbb{R}$ and $\phi:\partial\Omega\to\mathbb{R}$ are continuous. If $u\in C^2(\overline{\Omega})$ solves the Dirichlet problem*

$$\begin{aligned} \Delta u(x) &= f(x), && x\in\Omega, \\ u(x) &= \phi(x), && x \in \partial\Omega, \end{aligned}$$

then for $x\in\Omega$ it admits the representation

$$u(x) = \int_\Omega G(x,y)f(y)\,dy + \int_{\partial\Omega}\phi(y)\partial_\nu G(x,y)\,d\,\text{area}(y).$$

Proof. We want to apply integration by parts to the first integral on the right, however G admits a singularity at $x = y$. We can truncate the region of integration about the singularity, since by dominated convergence,

$$\int_\Omega G(x,y) f(y)\, dy = \lim_{\varepsilon \to 0} \int_{\Omega \setminus B_\varepsilon(x)} G(x,y) \Delta_y u(y)\, dy.$$

For $\varepsilon \ll 1$ such that $B_\varepsilon(x) \subseteq \Omega$, applying Green's second identity and properties of the Green's function gives

$$\begin{aligned}\int_{\Omega \setminus B_\varepsilon(x)} G(x,y)\Delta_y u(y)\, dy &= \int_{\Omega\setminus B_\varepsilon(x)} \Delta_y G(x,y)\, u(y)\, dy + \int_{\partial(\Omega \setminus B_\varepsilon(x))} \left(G(x,y)\partial_\nu u(y) - u(y)\partial_\nu G(x,y)\right) d\,\mathrm{area}(y) \\ &= -\int_{\partial\Omega} \phi(y)\partial_\nu G(x,y)\, d\,\mathrm{area}(y) - \int_{|x-y|=\varepsilon} \left(G(x,y)\partial_\nu u(y) - u(y)\partial_\nu G(x,y)\right) d\,\mathrm{area}(y).\end{aligned}$$

We claim that the second term on the second line converges to $u(x)$. Indeed, writing $G(x,y) = K(y-x) - v(x,y)$, it follows from the triangle inequality and decay estimates on the fundamental solution that

$$\begin{aligned}\left| \int_{|x-y|=\varepsilon} G(x,y)\partial_\nu u(y)\, d\,\mathrm{area}(y)\right| &\le \varepsilon^{d-1} \sup_{|x-y|=\varepsilon} |\nabla u(y)| \sup_{|x-y|=\varepsilon} |G(x,y)| \\ &\lesssim \varepsilon^{d-1} \sup_{|x-y|=\varepsilon} |K(y-x)| + \varepsilon^{d-1} \sup_{|x-y|=\varepsilon} |v(x,y)| \xrightarrow{\varepsilon\to 0} 0.\end{aligned}$$

Furthermore,

$$\int_{|x-y|=\varepsilon} u(y)\partial_\nu G(x,y)\, d\,\mathrm{area}(y) = \int_{|x-y|=\varepsilon} u(y)\partial_\nu K(y-x)\, d\,\mathrm{area}(y) - \int_{|x-y|=\varepsilon} u(y)\partial_\nu v(x,y)\, d\,\mathrm{area}(y).$$

The second term on the right clearly vanishes by continuity of u and $\partial_\nu v$. To complete the proof of the claim, we need to show the first term converges to $u(x)$. Note the unit normal vector ν on the sphere $|y| = 1$ is exactly $y \in \mathbb{R}^d$. We compute

$$\partial_\nu K(y-x) = \frac{1}{A_d}|x-y|^{1-d} = \frac{1}{A_d}\varepsilon^{1-d} = \frac{1}{\mathrm{area}\, B_\varepsilon(x)},$$

for $y \in \partial B_\varepsilon(x)$ and $d \ge 3$; the case $d = 2$ is similar. We conclude

$$\int_{|x-y|=\varepsilon} u(y)\partial_\nu K(y-x)\, d\,\mathrm{area}(y) = \frac{1}{\mathrm{area}\, B_\varepsilon(x)} \int_{\partial B_\varepsilon(x)} u(y)\, d\,\mathrm{area}(y) \xrightarrow{\varepsilon \to 0} u(x)$$

completing the proof. □

Remark. The case $f \equiv 0$ corresponds to u harmonic. In view of Green's representation formula, we see that harmonic functions depend only on their boundary values, and, by smoothness of the Green's function away from the diagonal $x = y$, are smooth.

We are interested in construction Green's functions, and showing that the converse of Green's representation formula holds for harmonic functions, i.e. given continuous boundary values $\phi : \partial\Omega \to \mathbb{R}$ and vanishing source term $f \equiv 0$, the representation formula gives rise to a solution to the Dirichlet problem. To these ends, we consider domains Ω with symmetry which we can exploit by applying the *method of images*.

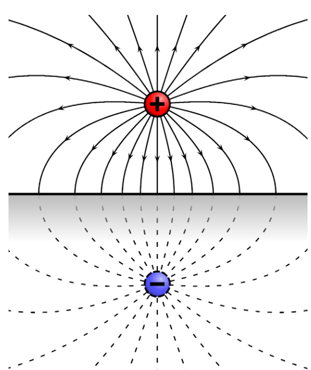

FIGURE 1. A dipole electric potential which vanishes at the boundary.

Recall that we can physically interpret the fundamental solution $y \mapsto K(y-x)$ as the potential arising from the unit electric charge concentrated at the pole $y = x$. We want the potential to vanish on the boundary, so "reflecting" the charge distribution across the boundary we obtain a dipole distribution such that the two potentials arising from the oppositely charged poles cancel out at the boundary. For example, consider the upper-half space

$$\mathbb{H} := \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : t > 0\}.$$

The distribution $(y,s) \mapsto K(y-x, s-t)$ is the potential of the positive unit charge at (x,t), while the distribution $(y,s) \mapsto -K(y-x, t-s)$ is the potential of the negative unit charge at $(x,-t)$. Thus the Green's function of the upper-half space is

$$G_{\mathbb{H}}(x,t,y,s) := K(y-x, s-t) - K(y-x, s+t).$$

The unit normal vector on the boundary $\partial\mathbb{H}$ is $\nu = (0,-1)$. The normal derivative on the boundary $s = 0$ is known as the Poisson kernel on the upper-half space, taking the form

$$P_t(x-y) := \partial_\nu G_{\mathbb{H}}(x,t,y,0) = \frac{2}{A_d} \frac{t}{(t^2 + |x-y|^2)^{\frac{d+1}{2}}}.$$

Observe that the Poisson kernel $\{P_t\}_t$ forms a spherically-symmetric approximation to the identity, so $P_t * \phi \to \phi$ pointwise as $t \to 0$ for continuous and bounded ϕ. Moreover, by construction $(x,t) \mapsto P_t(x-y)$ is harmonic on the upper-half space. This proves that convolving the boundary conditions with the Poisson kernel furnishes the solution to the Dirichlet problem.

Theorem 2 (Poisson integral formula for $\mathbb{H}$)**.** *Suppose $\phi : \partial\mathbb{H} \to \mathbb{R}$ is bounded and continuous, then $u : \mathbb{H} \to \mathbb{R}$ defined by the formula*

$$u(x) := \int_{\mathbb{R}^d} P_t(x-y)\phi(y)dy = (P_t * \phi)(x)$$

is the unique harmonic function extending continuously to the boundary such that $u_{|\partial\mathbb{H}} = \phi$.

Consider now the case of the ball $B \subseteq \mathbb{R}^d$ of radius $R > 0$ centered at the origin. Given $x \in \mathbb{R}^d$, define its reflection across the boundary of the ball by

$$\overline{x} := \frac{R^2}{|x|^2} x.$$

Following the construction of the Green's function for the upper-half space, we want to place a positive unit charge at $x \in B$ and a negative unit charge at $\overline{x} \in \mathbb{R}^d \setminus B$ such that the net potential vanishes on the boundary. The Green's function of the ball B is

$$G_B(x,y) := K(y-x) - K\left(\frac{|x|}{R}(y - \overline{x})\right).$$

The unit normal vector on the boundary ∂B is $\nu = y/R$. The normal derivative restricted to the boundary $|y| = R$ is known as the Poisson kernel on the ball, taking the form

$$P(x,y) := \partial_\nu G_B(x,y) = \frac{R^{d-2}(R^2 - |x|^2)}{|x-y|^d}.$$

As with the upper-half space, the Poisson kernel on the ball is harmonic, non-negative, and has unit mass. Integrating against boundary conditions furnishes the solution to the Dirichlet problem.

Theorem 3 (Poisson integral formula for ball)**.** *Let $B \subseteq \mathbb{R}^d$ be an open ball of radius $R > 0$ centered at the origin, and suppose $\phi : \partial B \to \mathbb{R}$ is continuous. Then $u : B \to \mathbb{R}$ defined by the formula*

$$u(x) := \frac{1}{\operatorname{area}\partial B} \int_{\partial B} P(x,y)\phi(y)d\operatorname{area}(y)$$

is the unique harmonic function extending continuously to the boundary such that $u_{|\partial B} = \phi$.

1.3. C^∞ **elliptic regularity.** When solving a linear partial differential equation distributionally, *a priori* we do not know whether the solution exhibits any regularity in the strong sense. In the case of the Laplace operator, we have *elliptic regularity*, the property that regularity is not "lost" when solving the Poisson equation. The classic example is Weyl's lemma: if the Laplacian of a distribution is smooth, then the distribution is also smooth.

Theorem 4 (Weyl's lemma)**.** *Let $u \in C_c^\infty(\Omega)^*$ be a distributional solution to the Poisson equation*

$$\Delta u = f$$

for $f \in C^\infty(\Omega)$. Then u is smooth.

Proof. Fix $x \in \Omega$ and suppose $B_{5\varepsilon}(x) \subseteq \Omega$. Choose a cut-off $\chi \in C_c^\infty(\Omega)$ such that $\chi \equiv 1$ on the ball $B_{4\varepsilon}(x)$. Smoothness is a local property, so it suffices to show χu is smooth at x. Observe that χu defines a compactly supported distribution on $\mathbb{R}^d$, so we can write

$$\chi u = \delta_0 * (\chi u) = \Delta K * (\chi u) = K * \Delta(\chi u).$$

Choose another cut-off $\eta \in C_c^\infty(\Omega)$ supported on $B_{3\varepsilon}(x)$ and satisfies $\eta \equiv 1$ on the ball $B_{2\varepsilon}(x)$. By construction, $\eta\Delta(\chi u) = \eta\Delta u = \eta f$, so we can write

$$\chi u = \delta_0 * (\chi u) = \Delta K * (\chi u) = K * \Delta(\chi u) = K * (\eta f) + K * (1-\eta)\Delta(\chi u).$$

Since $\eta f \in C_c^\infty(\mathbb{R}^d)$, the first term on the right is smooth. We claim that the second term on the right is smooth at x, which would complete the proof. Choose the final cut-off $\psi \in C_c^\infty(\mathbb{R}^d)$ supported in $|x| < \varepsilon$ such that $\psi \equiv 1$ in a neighborhood of the origin, then

$$K * (1-\eta)\Delta(\chi u) = (\psi K) * (1-\eta)\Delta(\chi u) + (1-\psi)K * (1-\eta)\Delta(\chi u).$$

By construction, $(1-\psi)K \in C^\infty(\mathbb{R}^d)$, so the second term on the right is smooth. On the other hand, the first term on the right vanishes in a neighborhood of x. Recall the support of the convolution is the sum of the supports,

$$\operatorname{supp}(\psi K) * (1-\eta)\Delta(\chi u) \subseteq \operatorname{supp}(\psi K) + \operatorname{supp}(1-\eta)\Delta(\chi u) \subseteq \{x : |x| \le \varepsilon\} + \{x : |x-y| > 2\varepsilon\}.$$

This proves that the convolution vanishes in $B_\varepsilon(x)$, completing the proof. □

Remark. This proof relied only on the fact that the fundamental solution K was smooth on $\mathbb{R}^d \setminus 0$. That is, if $P(D)$ is a constant coefficient linear partial differential operator with fundamental solution smooth away from the origin, then it is HYPOELLIPTIC, i.e. any distribution satisfying $P(D)u \in C^\infty(\Omega)$ must also be smooth. In fact, hypoellipticity is equivalent to the fundamental solution being smooth away from the origin. The heat operator $\partial_t - \Delta$ is an example of a non-elliptic operator which is hypoelliptic.

2. Maximum principle

The *strong maximum principle* is the property that a continuous function $u : \Omega \to \mathbb{R}$ cannot achieve a maximum on a bounded domain $\Omega \subseteq \mathbb{R}^d$. The *weak maximum principle* follows as a direct corollary, stating that if u extends continuously to the boundary, then it achieves its maximum on the boundary. As a motivating example, the class of C^2-convex functions, i.e. those satisfying $\nabla^2 u \ge 0$, obeys the maximum principle. Such functions also obey the differential inequality $\Delta u \ge 0$; this weaker condition turns out to be sufficient for the maximum principle. We say that an upper semi-continuous function $u : \Omega \to \overline{\mathbb{R}}$ is DISTRIBUTIONALLY SUB-HARMONIC if

$$\langle \Delta\phi, u\rangle \ge 0$$

for all non-negative test function $\phi \in C_c^\infty(\Omega)$. If $u \in C^2(\Omega)$ and

$$\Delta u \ge 0,$$

then we say u is CLASSICALLY SUB-HARMONIC. Integrating by parts shows that classical sub-harmonicity implies distributional sub-harmonicity.

2.1. **Mean value property.**

Our proof of the maximum principle will rely on the following mean value characterisation of sub-harmonic functions;

Theorem 5 (Sub-mean value property)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be open, and suppose $u : \Omega \to \overline{\mathbb{R}}$ is sub-harmonic. Then for any closed ball $B \subseteq \Omega$ is a ball centered at $x_0 \in B$ we have*

$$u(x_0) \le \frac{1}{\operatorname{vol} B}\int_B u(y)\,dy,$$
$$u(x_0) \le \frac{1}{\operatorname{area} \partial B}\int_{\partial B} u(y)\,d\operatorname{area}.$$

Conversely, if u is continuous and satisfies the inequalities above for all $x_0 \in \Omega$ and sufficiently small balls $B \subseteq \Omega$ centered at x_0, then u is sub-harmonic.

Proof. The first inequality follows from the second by converting to spherical coordinates, so we aim towards the latter. It suffices to prove the result assuming smoothness by replacing u with the convolution smoothing $u * \phi_\varepsilon$ where $\phi \in C_c^\infty(|x| \le 1)$ is non-negative and $\int \phi = 1$. Indeed, $u * \phi_\varepsilon \to u$ uniformly on B and, since u is distributionally sub-harmonic, we have

$$\Delta(u * \phi_\varepsilon)(x) = \int_\Omega \Delta\phi_\varepsilon(x-y)\,u(y)dy \ge 0,$$

i.e. $u * \phi_\varepsilon$ is classically sub-harmonic. Assume then u is smooth, we argue by a monotonicity formula, defining

$$\Phi(r) := \frac{1}{\operatorname{area} \partial B_r(x_0)} \int_{\partial B_r(x_0)} u(y)\, d\operatorname{area}(y) = \frac{1}{\operatorname{area} \partial B_1(0)} \int_{\partial B_1(0)} u(x_0 + ry)\, d\operatorname{area}(y).$$

To conclude the sub-mean value property, it would suffice to show Φ is non-decreasing in r since $\Phi(r) \to u(x_0)$ as $r \to 0$ by continuity of u. Differentiating, applying the divergence theorem and sub-harmonicity, we obtain

$$\Phi'(r) = \frac{1}{\operatorname{area} \partial B_1(0)} \int_{\partial B_1(0)} y \cdot \nabla u(x_0 + ry)\, d\operatorname{area}(y) = \frac{1}{\operatorname{area} \partial B_1(0)} \int_{B_1(0)} \Delta u(x_0 + ry)\, dy \geq 0,$$

as desired.

Conversely, suppose that u satisfies the sub-mean value property. Then for any $\varepsilon \ll 1$ such that $B_\varepsilon(x) \subseteq \Omega$, we have the inequality

$$0 \leq \int_{|y| \leq \varepsilon} (u(x-y) - u(x))\, dy.$$

Let $\phi \in C_c^\infty(\Omega)$ be a non-negative test function supported on $K \subseteq \Omega$, and denote K_ε the ε-neighborhood of K, in particular $K_\varepsilon \subseteq \Omega$ for $\varepsilon \ll 1$. Integrating the inequality above against ϕ and applying Fubini's theorem gives

$$0 \leq \frac{1}{\varepsilon^{2+d}} \int_\Omega \int_{|y| \leq \varepsilon} (u(x-y) - u(x))\, \phi(x)\, dy dx = \int_{K_\varepsilon} u(x) \left(\frac{1}{\varepsilon^{2+d}} \int_{|y| \leq \varepsilon} (\phi(x-y) - \phi(x))\, dy \right) dx.$$

Consider the Taylor expansion of our test function $\phi(x-y) - \phi(x) = -\sum_j \partial_j \phi(x) y_j + \frac{1}{2} \sum_{i,j} \partial_i \partial_j \phi(x) y_i y_j + O(|y|^3)$, observing that by symmetry the integral over the first order terms vanish, the integral over the second order terms vanish off the diagonal $i \neq j$, and the integral over the third order term is controlled by ε, i.e.

$$\int_{|y| \leq \varepsilon} y_j\, dy = 0, \qquad \frac{1}{\varepsilon^{2+d}} \int_{|y| \leq \varepsilon} y_i y_j\, dy = c_d \delta_{ij}, \qquad \frac{1}{\varepsilon^{2+d}} \int_{|y| \leq \varepsilon} O(|y|^3)\, dy = O(\varepsilon)$$

for some constant $c_d > 0$. Collecting our results and taking $\varepsilon \to 0$, we conclude

$$0 \leq \frac{c_d}{2} \int_K u(x) \Delta \phi(x)\, dx,$$

i.e. u is distributionally sub-harmonic. □

Corollary 6. *The class of sub-harmonic functions form a convex hull, that is, if $u, v : \Omega \to \mathbb{R}$ are sub-harmonic, then $\max\{u, v\} : \Omega \to \mathbb{R}$ is sub-harmonic.*

Proof. It suffices to show $\max\{u, v\}$ satisfies the sub-mean value property. Applying the sub-mean value property to u and v, we obtain

$$\begin{aligned} u(x_0) &\leq \frac{1}{\operatorname{vol} B} \int_B u(y)\, dy \leq \frac{1}{\operatorname{vol} B} \int_B \max\{u(y), v(y)\}\, dy \\ v(x_0) &\leq \frac{1}{\operatorname{vol} B} \int_B v(y)\, dy \leq \frac{1}{\operatorname{vol} B} \int_B \max\{u(y), v(y)\}\, dy \end{aligned}$$

as desired. □

Corollary 7 (Mean value property)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be open, and suppose $u : \Omega \to \overline{\mathbb{R}}$ is harmonic. Then for any ball $B \subseteq \Omega$ is a ball centered at $x_0 \in B$ we have*

$$\begin{aligned} u(x_0) &= \frac{1}{\operatorname{vol} B} \int_B u(y)\, dy, \\ u(x_0) &= \frac{1}{\operatorname{area} \partial B} \int_B u(y)\, d\operatorname{area}(y). \end{aligned}$$

Conversely, if u is continuous and satisfies the equalities above for all $x_0 \in \Omega$ and sufficiently small balls $B \subseteq \Omega$ centered at x_0, then u is harmonic.

Proof. If u is harmonic, then the proof of the sub-mean value property continues to hold replacing u with $-u$, which furnishes equalities in place of the inequalities. □

2.2. C^ω **elliptic regularity.** One can view the mean value property as an instance of elliptic regularity. If u is harmonic, then by commuting differentiation with the Laplacian, we see that $\partial_j u$ is also harmonic. Applying the mean value theorem and integration by parts gives control of a derivative by u itself. Iterating furnishes control over all derivatives, more precisely,

Theorem 8 (Cauchy estimates)**.** *Let $B \subseteq \mathbb{R}^d$ be the ball of radius $R > 0$ centered at $x_0 \in \mathbb{R}^d$, and suppose $u : B \to \mathbb{R}$ is harmonic and extends continuously to the boundary. Then*

$$|\partial^\alpha u(x_0)| \leq \left(\frac{d|\alpha|}{R}\right)^{|\alpha|} \sup_B |u|.$$

In particular, u is real analytic.

Proof. Commuting differentiation with the Laplacian, observe that $\partial^\alpha u$ is harmonic. We argue inductively; by the mean value property and the divergence theorem,

$$\partial_j u(x_0) = \frac{1}{\operatorname{vol} B} \int_B \partial_j u \, dy = \frac{1}{\operatorname{vol} B} \int_B \operatorname{div}(u\mathbf{e}_j) \, dy = \frac{1}{\operatorname{vol} B} \int_{\partial B} u\mathbf{e}_j \cdot \nu \, d \operatorname{area}.$$

It follows from $\operatorname{area} \partial B_r(x) / \operatorname{vol} B_r(x) = d/r$ and the triangle inequality that

$$|\partial_j u(x_0)| \leq \frac{d}{R} \sup_B |u|.$$

This proves the result for $|\alpha| = 1$. Set $m = |\alpha|$ and $\alpha_1, \dots, \alpha_m$ be a decreasing set of multi-indices $\alpha_j < \alpha_{j+1}$ such that $|\alpha_j| = j$ and $\alpha_m = \alpha$. We apply the $|\alpha| = 1$ case to control α_{j+1}-derivatives by α_j-derivatives on balls of radii $R/|\alpha|$. Iterating, we obtain

$$|\partial^\alpha u(x_0)| \leq \left(\frac{d|\alpha|}{R}\right) \sup_{B_{R/|\alpha|}(x_0)} |\partial^{\alpha_{m-1}} u| \leq \left(\frac{d|\alpha|}{R}\right)^2 \sup_{B_{2R/|\alpha|}(x_0)} |\partial^{\alpha_{m-2}} u| \leq \dots \leq \left(\frac{d|\alpha|}{R}\right)^{|\alpha|} \sup_{B_R(x_0)} |u|$$

as desired. □

Lemma 9. *Let $\{u_n\}_n$ be a sequence of harmonic functions on a domain $\Omega \subseteq \mathbb{R}^d$ converging uniformly on compact sets to u. Then u is harmonic.*

Proof. The mean value property is preserved under the limit, so u also satisfies the mean value property,

$$u(x_0) = \lim_{n\to\infty} u_n(x_0) = \lim_{n\to\infty} \frac{1}{\operatorname{vol} B} \int_B u_n(y) \, dy = \frac{1}{\operatorname{vol} B} \int_B u(y) \, dy,$$

for all $x_0 \in \Omega$ and closed balls $B \subseteq \Omega$ centered at x_0. □

Theorem 10 (Montel's theorem)**.** *A family of harmonic functions $\mathcal{F}$ on a domain $\Omega \subseteq \mathbb{R}^d$ is pre-compact, i.e. every sequence has a sub-sequence converging uniformly on every compact $K \subseteq \Omega$, if and only if $\mathcal{F}$ is uniformly bounded.*

Proof. We claim that $\mathcal{F}$ is equicontinuous on every compact $K \subseteq \Omega$. Choose $R > 0$ such that $B_{2R}(x) \subseteq \Omega$ for every $x \in K$, then by the first-order Cauchy estimate

$$|\nabla u(x)| \leq \frac{d}{R} \sup_B |u| \lesssim 1$$

uniformly in $u \in \mathcal{F}$ and $x \in K$. In particular, this holds for a compact neighborhood $K_\varepsilon \subseteq \Omega$, so it follows from the mean value theorem that u is Lipschitz continuous on K. By Arzela-Ascoli, for every sequence $\{u_n\}_n \subseteq \mathcal{F}$ there exists a sub-sequence converging uniformly on K.

There exists a compact exhaustion $K_n \subseteq \Omega$ of the domain, i.e. $\bigcup_m K_m = \Omega$. We inductively extract sub-sequences $\{u_{n,m}\}_n$ converging uniformly on K_m. The diagonal sequence $\{u_{n,n}\}_n$ converges uniformly on every K_n and moreover, since they form an exhaustion of Ω, every compact $K \subseteq \Omega$. □

2.3. **Maximum principles.** We are now ready to establish the maximum principle for sub-harmonic functions and its consequences.

Theorem 11 (Strong maximum principle)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and connected and $u : \Omega \to \mathbb{R}$ be sub-harmonic. If u achieves its maximum, i.e. there exists $x_0 \in \Omega$ such that*

$$u(x_0) = \sup u(\Omega),$$

then u is constant.

Proof. We know from upper semi-continuity that the level set $u = \sup u(\Omega)$ is closed, so it suffices by connectivity to show it is also open. Let $B \subseteq \Omega$ be an open ball centered at x_0, then by the sub-mean value property and $u(x_0) \geq u(y)$ we have

$$0 \geq \int_B (u(x_0) - u(y))\, dy = \int_B |u(x_0) - u(y)|\, dy.$$

It follows from semi-continuity that $u \equiv u(x_0)$ on the ball B, as desired. □

Corollary 12 (Weak maximum principle). *Let $\Omega \subseteq \mathbb{R}^d$ be open and $u : \Omega \to \mathbb{R}$ be sub-harmonic. If u is bounded and extends continuously to the boundary, then it achieves its maximum on the boundary, i.e.*

$$\sup u(\partial\Omega) = \sup u(\Omega).$$

Corollary 13 (Comparison principle). *Let $\Omega \subseteq \mathbb{R}^d$ be open, and suppose $u, h : \Omega \to \mathbb{R}$ are sub-harmonic and harmonic respectively extending continuously to the boundary. If $u \leq h$ on the boundary ∂B, then $u \leq h$ on the entire domain Ω.*

Proof. The difference $u - h$ is sub-harmonic on Ω extending continuously to the boundary, so it obeys the weak maximum principle, i.e. the maximum is on the boundary. Since $u - h \leq 0$ on the boundary $\partial\Omega$, we have $u - h \leq 0$ on the entire domain Ω. □

Theorem 14 (Lindelof maximum principle). *Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded, and suppose $u : \Omega \to \overline{\mathbb{R}}$ be sub-harmonic. Then*

$$\sup u(\Omega) = \sup_{x \in \partial\Omega} \limsup_{y \to x} u(y).$$

Furthermore, if u is bounded above and $F \subseteq \partial\Omega$ is finite, then

$$\sup u(\Omega) = \sup_{x \in \partial\Omega \setminus F} \limsup_{y \to x} u(y).$$

Proof. Since Ω is bounded, we can choose a maximising sequence $\{y_n\}_n \subseteq \Omega$ converging to $x \in \overline{\Omega}$. If $x \in \partial\Omega$, we have proven

$$\sup u(\Omega) = \sup_{x \in \partial\Omega} \limsup_{y \to x} u(y).$$

Otherwise $x \in \Omega$, which by the strong maximum principle implies $u \equiv \sup u(\Omega)$ on the connected component containing x, and again the equality above holds.

Suppose now u is bounded above. For $\varepsilon > 0$, define

$$v_\varepsilon(y) := u(y) - \varepsilon \sum_{x \in F} K(y - x),$$

where K is the fundamental solution. Note that $K(y - x) \to \infty$ as $y \to x$, so since u is bounded above we have $v_\varepsilon(y) \to -\infty$ as $y \to x$. It follows that

$$\sup v_\varepsilon(\Omega) \leq \sup_{x \in \partial\Omega} \limsup_{y \to x} v_\varepsilon(y) \leq \sup_{x \in \partial\Omega \setminus F} \limsup_{y \to x} v_\varepsilon(y).$$

In the case $d = 2$, we write

$$\sup_{x \in \partial\Omega \setminus F} \limsup_{y \to x} v_\varepsilon(y) \leq \sup_{x \in \partial\Omega \setminus F} \limsup_{y \to x} u(y) + \varepsilon \log(1 + \operatorname{diam} \Omega).$$

In the case $d = 3$, using $K \geq 0$ we write

$$\sup_{x \in \partial\Omega \setminus F} \limsup_{y \to x} v_\varepsilon(y) \leq \sup_{x \in \partial\Omega \setminus F} \limsup_{y \to x} u(y).$$

As $v_\varepsilon \to u$ pointwise taking $\varepsilon \to 0$, we conclude the result. □

Remark. The first Lindelof maximum principle fails when the domain is not bounded; let $\mathbb{H} \subseteq \mathbb{R}^2$ be the upper-half space $y > 0$ and define $u : \mathbb{H} \to \mathbb{R}$ by $u(x, y) := y$. Clearly the maximum is not achieved on the boundary $y = 0$. The second Lindelof maximum principle fails when u is not bounded above; let $\mathbb{D} \subseteq \mathbb{C}$ be the unit disc and define $u : \mathbb{D} \setminus 0 \to \mathbb{R}$ by $u(z) := -\log|z|$. Then the result fails taking $F = \{0\}$ and noting u vanishes on $\partial\mathbb{D}$.

Theorem 15 (Harnack's inequality). *Let $\Omega \subseteq \mathbb{R}^d$ be open and connected, and suppose $u : \Omega \to [0, \infty)$ is a non-negative harmonic function. Then for any $U \Subset \Omega$ which is open and connected, there exists a constant $C(\Omega, U) > 0$ such that*

$$\sup u(U) \leq C(\Omega, U) \inf(U).$$

In particular, if $u(x) = 0$ for some $x \in \Omega$, then $u \equiv 0$.

Proof. It suffices to show

$$u(a) \leq C(\Omega, U)u(b)$$

for any $a, b \in U$. Since U has compact closure in Ω, there exists $r > 0$ such that $B_{4r}(x) \subseteq \Omega$ for all $x \in U$, and a finite cover of Ω by the balls $B_r(x)$. We claim that Harnack's inequality holds on balls,

$$\sup u(B_r(x)) \leq 3^d \inf u(B_r(x)).$$

Fix $a, b \in B_r(x)$, then by the mean value property

$$\begin{aligned} u(a) &= \frac{1}{\operatorname{vol} B_r(a)} \int_{B_r(a)} u(y)\, dy \\ &\leq \frac{1}{\operatorname{vol} B_r(a)} \int_{B_{3r}(b)} u(y)\, dy = \frac{\operatorname{vol} B_{3r}(b)}{\operatorname{vol} B_r(a)} u(b) = 3^d u(b), \end{aligned}$$

where the inequality follows from non-negativity of u and $B_r(a) \subseteq B_{3r}(b) \subseteq \Omega$. This proves the claim. To extend Harnack's inequality to U, we remark that any two points $a, b \in U$ can be connected by a continuous path covered by the balls $B_r(x)$ in the finite cover. Iterating Harnack's inequality on these balls along the path, we obtain

$$u(a) \leq (3^d)^{\#\text{ of balls}} u(b),$$

completing the proof. ☐

Remark. The constant $C(\Omega, U)$ does not depend on the choice of harmonic function u. Thus Harnack's inequality states that positive harmonic functions are controlled solely by the geometry of their domain.

Corollary 16 (Harnack convergence theorem)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and connected, and suppose $u_n : \Omega \to \mathbb{R}$ is an increasing sequence of harmonic functions such that $\sup_n u_n(x_0) < \infty$ for some $x_0 \in \Omega$. Then $\{u_n\}_n$ converges uniformly on compact sets to a harmonic function.*

Proof. Without loss of generality, let $K \subseteq \Omega$ be compact such that $x_0 \in \Omega$. Since the sequence is increasing, $u_n - u_m \geq 0$ for all $n \geq m$. By Harnack's inequality and monotone convergence,

$$0 < \sup_K (u_n - u_m) \lesssim \inf_K (u_n - u_m) \leq u_n(x_0) - u_m(x_0) \overset{n,m\to\infty}{\longrightarrow} 0.$$

This proves $\{u_n\}_n$ converges uniformly to some $u : \Omega \to \mathbb{R}$ on every compact K. In particular, each u_n satisfies the mean value property, so passing to the limit, it follows that u also satisfies the mean value property and is therefore harmonic. ☐

2.4. **Perron's method.** Consider the Dirichlet problem

$$\begin{aligned} \Delta u &= 0, \\ u_{|\partial\Omega} &= \phi \end{aligned}$$

on an open bounded domain $\Omega \subseteq \mathbb{R}^d$ for continuous boundary values $\phi : \partial\Omega \to \mathbb{R}$. We construct a suitable candidate for the solution to the Dirichlet problem via the maximum principle. A continuous sub-harmonic function $v : \Omega \to \mathbb{R}$ is a SUB-SOLUTION if

$$\limsup_{y\to x} v(y) \leq \phi(x)$$

for all $x \in \partial\Omega$. The PERRON SOLUTION $u : \Omega \to \mathbb{R}$ is defined as the pointwise maximum over all sub-solutions,

$$u(x) := \sup\{v(x) : v \text{ is a sub-solution}\}.$$

We claim u solves the Dirichlet problem for sufficiently regular domains. Note first that u is well-defined, i.e. sub-solutions exist and u is bounded. Indeed, the boundary $\partial\Omega$ is compact and so the boundary values ϕ are bounded. The constant function equal to the lower bound is a sub-solution, and from the Lindelof maximum principle u is bounded above by

$$\sup v(\Omega) = \sup_{x\in\partial\Omega} \limsup_{y\to x} v(y) \leq \sup \phi(\partial\Omega)$$

whenever v is a sub-solution. We now turn to showing the Perron solution is harmonic;

Lemma 17. *Let $\Omega \subseteq \mathbb{R}^d$ be open, and suppose $u : \Omega \to \mathbb{R}$ is continuous and sub-harmonic. If $h : \overline{B} \to \mathbb{R}$ is a harmonic function on the ball $\overline{B} \subseteq \Omega$ such that $u_{|\partial B} = h_{|\partial B}$, then the* HARMONIC LIFT $u_B : \Omega \to \mathbb{R}$ *defined by*

$$u_B(x) := \begin{cases} u(x), & \text{if } x \notin B, \\ h(x), & \text{if } x \in \overline{B} \end{cases}$$

is sub-harmonic and $u \leq u_B$.

Proof. From the comparison principle we see that $u \leq u_B$. Fix a closed ball $D \subseteq \Omega$ centered at $x_0 \in \Omega$ and let $g : D \to \mathbb{R}$ be the harmonic function agreeing with u_B on the boundary. We claim $u_B \leq g$ on the ball D; it would follow that u_B satisfies the sub-mean value property,

$$u_B(x_0) \leq g(x_0) = \frac{1}{\operatorname{area} \partial D} \int_{\partial D} u_B(y) d \operatorname{area}(y),$$

i.e. u_B is sub-harmonic. Since $u \leq u_B \leq g$ on the boundary ∂D, the comparison principle implies $u \leq g$ on the entire ball D. This proves $u_B \leq g$ on $D \setminus B$. It remains to consider the region $D \cap B$. Note $g - h$ is harmonic, so it achieves its maximum over $D \cap B$ on the boundary, which consists of two components $\partial D \cap B$ and $\partial B \cap D$. On the former $g = u_B = h$, on the latter, $h = u$, so we are done. □

Theorem 18. *Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded, and suppose $\phi : \partial\Omega \to \mathbb{R}$ is continuous. Then the Perron solution to the corresponding Dirichlet problem is harmonic.*

Proof. Fix $x_0 \in \Omega$ and choose a sequence of sub-solutions $\{v_n\}_n$ such that $v_n(x_0) \to u(x_0)$. We can assume the sequence is increasing and bounded below by $\inf \phi$ by replacing with the pointwise maximum

$$x \mapsto \max\{v_1(x), \dots, v_n(x), \inf \phi(\partial\Omega)\},$$

as sub-solutions are closed under pointwise maximums. Furthermore, the harmonic lift $(v_n)_B$ on a ball $B \subseteq \Omega$ centered at x_0 is a sub-solution since it is sub-harmonic and the boundary values remain unchanged. Set

$$v(x) := \lim_{n\to\infty} (v_n)_B(x).$$

By Harnack's convergence theorem, v is harmonic in B. Moreover, $v(x_0) = u(x_0)$ by the comparison principle,

$$u(x_0) \geq (v_n)_B(x_0) \geq v_n(x_0) \overset{n\to\infty}{\longrightarrow} u(x_0).$$

Showing $u \equiv v$ on B completes the proof. By construction, $v \leq u$, so assume towards a contradiction there exists $x_1 \in B$ such that $v(x_1) < u(x_1)$. Repeating the beginning of the proof, we can find a bounded non-decreasing sequence of sub-solutions $\{w_n\}_n$ such that $w_n(x_1) \to u(x_1)$. Define

$$w(x) := \lim_{n\to\infty} (\max\{v_n, w_n\})_B(x).$$

By Harnack's convergence theorem, w is harmonic in B. Moreover, $v(x_0) = w(x_0)$ by the comparison principle,

$$v(x_0) \leq w(x_0) \leq u(x_0) = v(x_0).$$

Note $v - w$ is a non-positive harmonic function on B vanishing at x_0, so by the weak maximum principle $v \equiv w$ on B. However $w(x_1) = u(x_1)$ and, by assumption, $v(x_1) \neq u(x_1)$, a contradiction. □

3. ENERGY METHOD

The methods discussed thus far do not have robust generalisations outside the realm of constant-coefficient linear partial differential equations. We will instead appeal to functional analysis, where the natural function spaces to consider are the Sobolev spaces $W^{1,p}(\Omega)$ for $1 < p < \infty$, in which case the boundary conditions need to be taken in the sense of traces. Suppose then $\Omega \subseteq \mathbb{R}^d$ is a C^1-domain, and let $f \in W^{-1,p}(\Omega)$ and $\phi \in W^{1-1/p,p}(\partial\Omega)$, then the Dirichlet problem takes the form

$$\begin{aligned} \Delta u &= f, && \text{on } \Omega, \\ u &= \phi, && \text{on } \partial\Omega. \end{aligned}$$

We refer to solutions $u \in W^{1,p}(\Omega)$ as WEAK SOLUTIONS. In view of Sobolev embedding, weak solutions are in fact classical solutions provided sufficient regularity or integrability.

3.1. **Existence and uniqueness.** We can reduce to solving the Dirichlet problem with homogeneous boundary conditions by choosing an extension of the boundary values $g \in W^{1,p}(\Omega)$, that is, $g_{|\partial\Omega} = \phi$. Indeed, suppose $v \in W_0^{1,p}(\Omega)$ solves

$$\begin{aligned} \Delta v &= f + \Delta g, &\quad &\text{on } \Omega, \\ v &= 0, & &\text{on } \partial\Omega, \end{aligned}$$

then $u = v - g$ solves the original Dirichlet problem. The space $W_0^{1,p}(\Omega)$ is reflexive and dual to the negative Sobolev space $W^{-1,p}(\Omega) = (W_0^{1,p}(\Omega))^*$. It follows by duality and self-adjointness of the Laplace operator that uniqueness of a solution to the Dirichlet problem implies existence of a solution. More precisely, we record the following general lemma:

Lemma 19 (Existence-uniqueness duality). *Let $P : X \to Y$ be a linear operator between Banach spaces, and denote $P^* : Y^* \to X^*$ its adjoint, then*

- *uniqueness furnishes existence for the dual problem, i.e. if $||u||_X \lesssim ||Pu||_Y$, then $\operatorname{Im} P^* = X^*$,*
- *existence furnishes uniqueness for the dual problem, i.e. if $\operatorname{Im} P = Y$, then $||v||_{Y^*} \lesssim ||P^* v||_{X^*}$.*

In particular, if X is reflexive, then the a priori estimate furnishes existence and uniqueness for the problem $Pu = f$.

Theorem 20 (Energy estimate). *Let $\Omega \subseteq \mathbb{R}^d$ be a C^1-domain, and suppose $f \in W^{-1,p}(\Omega)$. Then a solution $u \in W_0^{1,p}(\Omega)$ to the Dirichlet problem*

$$\begin{aligned} \Delta u &= f, &\quad &\text{on } \Omega, \\ u &= 0, & &\text{on } \partial\Omega, \end{aligned}$$

satisfies the energy estimate

$$||u||_{W_0^{1,p}} \lesssim ||f||_{W^{-1,p}}.$$

In particular, there exists a unique weak solution to the Dirichlet problem.

Proof. Choose a sequence $\{u_n\}_n \subseteq C_c^\infty(\Omega)$ such that $u_n \to u$ in $W_0^{1,p}(\Omega)$. Moreover, by duality $\Delta u_n \to f$ in $W^{-1,p}(\Omega)$. The operator $\partial_j / |\nabla|$ is a Mikhlin multiplier, so it is bounded on $L^p(\mathbb{R}^d)$ and thus

$$||\partial_j u_n||_{L^p(\mathbb{R}^d)} \lesssim |||\nabla| u_n||_{L^p(\mathbb{R}^d)} \sim ||\Delta u_n||_{W^{-1,p}(\Omega)}.$$

Summing in j, passing to the limit and applying the Poincare inequality on the left, we conclude

$$||u||_{W_0^{1,p}} \lesssim ||f||_{W^{-1,p}}.$$

This completes the proof. ☐

Remark. It is illustrative to consider the case $p = 2$ which can be proven using elementary tools. Integrating the equation against u and integrating by parts, we can write

$$\int_\Omega uf \, dx = \int_\Omega u \Delta u \, dx = -\int_\Omega |\nabla u|^2 \, dx.$$

Applying duality to the left and the Poincare inequality to the right, we obtain

$$||u||^2_{H_0^1} \sim ||u||^2_{\dot{H}^1} \leq ||u||_{H_0^1} ||f||_{H^{-1}}.$$

3.2. $W^{k,p}$ **elliptic regularity.** The energy estimate *a priori* only furnishes solutions to the Dirichlet problem in the weak sense. Ideally, we would like to know if u admits higher order weak derivatives, which by Sobolev embedding would furnish strong derivatives and imply the solution is in fact a classical solution. We appeal to elliptic regularity, claiming that if $f \in W^{k,p}(\Omega)$, then solving the Poisson equation

$$\Delta u = f$$

for $u \in W^{1,p}(\Omega)$ does not "lose" regularity in the sense that any solution satisfies $u \in W^{k+2,p}(\Omega)$. This would follow from an argument similar to the proof of the energy estimate provided we knew *a priori* our solution had the desired regularity. We instead replace the derivative operators with the difference quotient

$$D_i^h u(x) := \frac{u(x + h\mathbf{e}_i) - u(x)}{h}.$$

Theorem 21 ($W^{k,p}_{\text{loc}}$ elliptic regularity)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be a domain. For $f \in W^{k,p}(\Omega)$, suppose that $u \in W^{1,p}(\Omega)$ is a solution to Poisson's equation*

$$\Delta u = f.$$

Then $u \in W^{k+2,p}_{loc}(\Omega)$ and for each $V \Subset \Omega$ we have the estimate

$$||u||_{W^{k+2,p}(V)} \lesssim_V ||f||_{W^{k,p}(\Omega)} + ||u||_{W^{k,p}(\Omega)}.$$

Proof. We argue inductively, considering first the case $k = 0$. Let $V \Subset W \Subset \Omega$, and choose a non-negative cut-off $\chi \in C_c^\infty(W)$ satisfying $\chi \equiv 1$ on V. The operator $D_i^h \langle \nabla \rangle / (\Delta + 1)$ is a Mikhlin multiplier uniformly in h, so it is bounded on $L^p(\mathbb{R}^d)$. In particular,

$$\begin{aligned} ||D_i^h \langle \nabla \rangle (\chi u)||_{L^p(\mathbb{R}^d)} &\lesssim ||(\Delta + 1)(\chi u)||_{L^p(\mathbb{R}^d)} \\ &\leq ||\Delta(\chi u)||_{L^p(\mathbb{R}^d)} + ||\chi u||_{L^p(\mathbb{R}^d)} \lesssim_\chi ||f||_{L^p(\Omega)} + ||u||_{L^p(\Omega)} \end{aligned}$$

uniformly in h. It follows that $\langle \nabla \rangle (\chi u) \in W^{1,p}(W)$ and

$$\sum_{|\alpha|=1,2} ||\partial^\alpha u||_{L^p(V)} \lesssim \sum_i ||\partial_i \langle \nabla \rangle (\chi u)||_{L^p(W)} \lesssim ||f||_{L^p(\Omega)} + ||u||_{L^p(\Omega)}.$$

This proves the base case. Assume the result holds for k, then following the same multiplier argument and applying the induction hypothesis we obtain

$$\begin{aligned} ||D_i^h \langle \nabla \rangle^{k+1} (\chi u)||_{L^p(\mathbb{R}^d)} &\lesssim ||(\Delta + 1) \langle \nabla \rangle^k (\chi u)||_{L^p(\mathbb{R}^d)} \\ &\leq ||\Delta(\chi u)||_{W^{k,p}(\mathbb{R}^d)} + ||\chi u||_{W^{k,p}(\mathbb{R}^d)} \lesssim_\chi ||f||_{L^p(\Omega)} \lesssim_\chi ||f||_{W^{k,p}(\Omega)} + ||u||_{L^p(\Omega)} \end{aligned}$$

uniformly in h. It follows that $\langle \nabla \rangle^{k+1} (\chi u) \in W^{1,p}(W)$ and

$$\sum_{1 \leq |\alpha| \leq k+2} ||\partial^\alpha u||_{L^p(V)} \lesssim \sum_i ||\partial_i \langle \nabla \rangle^{k+1} (\chi u)||_{L^p(W)} \lesssim ||f||_{W^{k,p}(\Omega)} + ||u||_{L^p(\Omega)}.$$

This completes the proof. $\square$

Remark. By Sobolev embedding, it follows from $k > d/2$ that $f \in C_{\text{loc}}(\Omega)$ and $u \in C^2_{\text{loc}}(\Omega)$, so u is classical solution to Poisson's equation.

4. Dirichlet's principle

For $f \in H^{-1}(\Omega)$, define the Dirichlet energy $E : H^1_0(\Omega) \to \overline{\mathbb{R}}$ by

$$E[u] := \frac{1}{2} ||u||_{\dot{H}^1} + \langle u, f \rangle = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \int_\Omega u f \, dx.$$

Dirichlet's principle states that minimising the energy is equivalent to solving the Dirichlet problem with homogeneous boundary conditions. This is an example of *variational calculus*, and we say that the Poisson equation arises as the *Euler-Lagrange equation* of the Lagrangian $L(x, u, \nabla u) = |\nabla u|^2 + uf$.

Theorem 22 (Dirichlet's principle)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be a C^1-domain, and suppose $f \in H^{-1}(\Omega)$. Then $u \in H^1_0(\Omega)$ solves the Dirichlet problem*

$$\begin{aligned} \Delta u &= f, \qquad \text{on } \Omega, \\ u &= 0, \qquad \text{on } \partial\Omega, \end{aligned}$$

if and only if it minimises the Dirichlet energy,

$$E[u] = \min_{v \in H^1_0(\Omega)} E[v].$$

Proof. Suppose u is a minimiser, then for any test function $\phi \in C_c^\infty(\Omega)$ define $e : \mathbb{R} \to \mathbb{R}$ by

$$e(t) := E[u + t\phi].$$

By construction, e is minimised at $t = 0$, so writing $|\nabla(u + tv)|^2 = |\nabla u|^2 + 2t \nabla u \cdot \nabla \phi + t^2 |\nabla \phi|^2$, differentiating under the integral sign, and integrating by parts, we obtain

$$0 = \frac{d}{dt}\Big|_{t=0} e = \int_\Omega \nabla u \cdot \nabla \phi \, dx + \int_\Omega f \phi \, dx = \int_\Omega (f - \Delta u) \, \phi \, dx.$$

Since ϕ was arbitrary, we conclude $\Delta u = f$ in the sense of distributions, thereby solving the Dirichlet problem.

Suppose u is a solution to the Dirichlet problem, and let $v \in H_0^1(\Omega)$, then testing $0 = \Delta u - f$ against $u - v$ and integrating by parts gives

$$
\begin{aligned}
0 = \int_\Omega (\Delta u - f)(u-v)\,dx &= \int_\Omega f(u-v)\,dx - \int_\Omega \nabla u \cdot \nabla(u-v)\,dx \\
&= \int_\Omega uf\,dx - \int_\Omega vf\,dx - \int_\Omega |\nabla u|^2\,dx + \int_\Omega \nabla u \cdot \nabla v\,dx.
\end{aligned}
$$

By Cauchy-Schwartz and the arithmetic-geometric mean inequality, $\nabla u \cdot \nabla v \leq \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla v|^2$. Rearranging above and applying this inequality, we obtain

$$
\int_\Omega |\nabla u|^2\,dx - \int_\Omega uf\,dx \leq \tfrac{1}{2}\int_\Omega |\nabla u|^2\,dx + \tfrac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega vf\,dx.
$$

Rearranging gives $E[u] \leq E[v]$. Since v was arbitrary, we conclude u is a minimiser of the Dirichlet energy. □

Theorem 23 (Existence and uniqueness of a minimiser)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be a C^1-domain, and suppose $f \in H^{-1}(\Omega)$. Then there exists a unique solution $u \in H_0^1(\Omega)$ to the Dirichlet energy $E : H_0^1(\Omega) \to \overline{\mathbb{R}}$.*

Proof. Uniqueness follows from strict convexity of the energy. Indeed, E is the sum of a strictly convex functional $u \mapsto ||u||_{\dot{H}^1}^2$ and a linear functional $u \mapsto \langle u, f\rangle$, so if $u, v \in H_0^1(\Omega)$ are distinct minimisers, then

$$
E[\tfrac{1}{2}u + \tfrac{1}{2}v] < \frac{E[u] + E[v]}{2} = \min_{w \in H_0^1} E[w],
$$

a contradiction.

Existence follows from coercivity and weak lower semi-continuity of the energy on the weakly compact space $H_0^1(\Omega)$. We first show coercivity; by duality and the Poincare inequality, the energy is bounded below by

$$
E[u] \geq \frac{1}{2}||u||_{\dot{H}^1}^2 - ||u||_{H_0^1}||f||_{H^{-1}} \gtrsim ||u||_{H_0^1}^2 - ||u||_{H_0^1}||f||_{H^{-1}}.
$$

It follows that $E[u] \to \infty$ whenever $||u||_{H_0^1} \to \infty$. In particular, a minimising sequence $\{u_k\}_k \subseteq H_0^1(\Omega)$ of the energy must be bounded. We can therefore pass to a sub-sequence such that $u_k \rightharpoonup u$ for some $u \in H_0^1(\Omega)$. By showing weak lower semi-continuity, that is,

$$
E[u] \leq \liminf_{k\to\infty} E[u_k],
$$

we can conclude u is the minimiser. By the arithmetic-geometric mean inequality, $|y|^2 \geq |x|^2 + 2x\cdot(y-x)$ for all $x, y \in \mathbb{R}^d$. Hence

$$
E[u_n] \geq E[u] + \int_\Omega \nabla u \cdot (\nabla u_n - \nabla u)\,dx + \int_\Omega (u - u_n) f\,dx \overset{n\to\infty}{\longrightarrow} E[u],
$$

where by construction $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ and $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\Omega)$. This completes the proof. □