DISTRIBUTION THEORY

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ABSTRACT. Distributions

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1. DISTRIBUTIONS

The theory of distributions arose from the study of linear differential equations

1.1. **Test functions.** A TEST FUNCTION $\phi : \Omega \to \mathbb{C}$ is a smooth function with compact support. Denote $C_c^\infty(\Omega)$ the space of test functions

The space of test functions, denoted by $C_c^\infty(\Omega)$, is endowed with a family of semi-norms

$$||\phi||_{m,K} := \sup_{x\in K} |\nabla^m \phi(x)|$$

where $K \subseteq \Omega$ is compact. This induces a topology in which a sequence $\{\phi_n\}_n \subseteq C_c^\infty(\Omega)$ converges to $\phi \in C_c^\infty(\Omega)$ if and only if there exists a compact set $K \subseteq \Omega$ such that $\operatorname{supp} \phi_n \subseteq K$ for all n and

$$\partial^\alpha \phi_n \overset{n\to\infty}{\longrightarrow} \partial^\alpha \phi$$

uniformly for all multi-indices α.

Proposition 1. *Let $T : C_c^\infty(\Omega) \to X$ be a linear map from the space of test functions into a Frechet space. Then the following are equivalent:*

(a) T is continuous.
(b) T is sequentially continuous.
(c) $T : C_c^\infty(K) \to X$ is continuous for each compact $K \Subset \Omega$.

1.2. **Distributions.** A DISTRIBUTION is a continuous linear functional $T : C_c^\infty(\Omega) \to \mathbb{C}$ denoted by

$$\phi \mapsto \langle \phi, T \rangle.$$

The space of distributions, denoted by $C_c^\infty(\Omega)^*$, is endowed with the weak-$*$ topology. In this topology, a sequence $\{T_n\}_n \subseteq C_c^\infty(\Omega)^*$ converges to $T \in C_c^\infty(\Omega)^*$ if and only if

$$\langle \phi, T_n \rangle \overset{n\to\infty}{\longrightarrow} \langle \phi, T \rangle$$

for all $\phi \in C_c^\infty(\Omega)$. It furthermore forms a vector space over $\mathbb{C}$ with respect to the conjugate complex structure, that is, scalar multiplication of a distribution $T \in C_c^\infty(\Omega)^*$ by $\lambda \in \mathbb{C}$ is defined as

$$\langle \phi, \lambda T \rangle := \overline{\lambda} \langle \phi, T \rangle.$$

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A distribution $T \in C_c^\infty(\Omega)^*$ is of ORDER k if k is the smallest non-negative integer such that T is continuous with respect to the $C^k(\Omega)$-topology.

Example. Some examples

- Locally integrable functions $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ form distributions $\langle -, u\rangle : C_c^\infty \to \mathbb{C}$ via the L^2-inner product
$$\langle \phi, u\rangle := \int_{\mathbb{R}^d} \phi \overline{u}\, dx.$$
- Radon measures μ form distributions $\langle -, \mu\rangle : C_c^\infty(\mathbb{R}^d) \to \mathbb{C}$ via the integral of test functions
$$\langle \phi, \mu\rangle := \int_{\mathbb{R}^d} \phi \, d\overline{\mu}.$$

Proposition 2. *A linear functional $T : C_c^\infty(\Omega) \to \mathbb{C}$ is a distribution if and only if for every compact $K \subseteq \Omega$ there exists $k \geq 0$ such that*
$$|\langle \phi, T\rangle| \lesssim ||f||_{C^k(K)}$$
for all $f \in C_c^\infty(K)$.

Theorem 3. *Let $\{T_n\}_n \subseteq C_c^\infty(\Omega)^*$ be a sequence of distributions such that $\{\langle \phi, T_n\rangle\}_n \subseteq \mathbb{R}$ is a convergent sequence for all $\phi \in C_c^\infty(\Omega)$. Then*
$$\langle \phi, T\rangle := \lim_{n\to\infty} \langle \phi, T_n\rangle$$
defines a distribution $T \in C_c^\infty(\Omega)^*$. Moreover, for every compact $K \Subset \Omega$, there exists $N \in \mathbb{N}$ such that
$$|\langle \phi, T\rangle| \lesssim \sum_{|\alpha|\leq N} \sup_{x\in K} |\partial^\alpha \phi(x)|$$
and $\langle \phi_n, T_n\rangle \to \langle \phi, T\rangle$ whenever $\phi_n \to \phi$.

Proof. □

1.3. **Adjoint operators.** Given a continuous linear operator $A : C_c^\infty(\Omega) \to C_c^\infty(\Omega)$ on the space of the test functions, there exists a unique continuous linear ADJOINT OPERATOR $A^* : C_c^\infty(\Omega)^* \to C_c^\infty(\Omega)^*$ on the space of distributions satisfying
$$\langle \phi, A^* T\rangle = \langle A\phi, T\rangle$$
for all $\phi \in C_c^\infty(\Omega)$ and $T \in C_c^\infty(\Omega)^*$.

- Multiplication by a smooth function $\psi \in C^\infty(\Omega)$ is self-adjoint,
$$\langle \phi, \psi T\rangle := \langle \psi\phi, T\rangle.$$
- Differentiation is skew-adjoint. Proceeding inductively,
$$\langle \phi, \partial^\alpha T\rangle := (-1)^{|\alpha|} \langle \partial^\alpha \phi, T\rangle.$$
- Scaling
$$\mathrm{Dil}_\lambda$$
- CONVOLUTION WITH A TEST FUNCTION: by Fubini's theorem and a change of variables, convolution by $\psi \in C_c^\infty(\mathbb{R}^d)$ is adjoint to convolution by $\tilde{\psi}(x) := \psi(-x)$, so we define for $u \in \mathcal{D}'(\mathbb{R}^d)$
$$(\psi * u)(\phi) := u(\tilde{\psi} * \phi).$$
- SCALING THE DOMAIN: for $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\lambda > 0$ define $S_\lambda u(x) = u(\lambda x)$. By a change of variables, the adjoint is $S_\lambda^* \phi(x) = \lambda^{-n} \phi(x/\lambda)$; hence we define
$$(S_\lambda u)(\phi) = u(S_\lambda^* \phi).$$
We say u is HOMOGENEOUS OF ORDER α if
$$u(S_\lambda^* \phi) = \lambda^\alpha f(\phi)$$
for all $\lambda > 0$ and $\phi \in C_c^\infty(\Omega)$.

The SUPPORT OF A DISTRIBUTION u, denoted $\operatorname{supp} u$, is defined as the complement of the largest open set $U \subseteq \Omega$ such that $u(\phi) = 0$ for all $u \in C_c^\infty(U)$. The space of COMPACTLY SUPPORTED DISTRIBUTIONS on Ω is denoted $\mathcal{E}'(\Omega)$. We use this notation because this space can be viewed as the continuous dual of $\mathcal{E}(\Omega) := C^\infty(\Omega)$ by truncating smooth functions outside the support of u. Let $\chi \in C_c^\infty(\Omega)$ such that $\chi \equiv 1$ on U, then

$$u(\phi) := u(\chi\phi)$$

for $\phi \in C^\infty(\Omega)$. This is consistent with the definition on test functions $\phi \in C_c^\infty(\Omega)$. Indeed, $u((1-\chi)\phi) = 0$ since $\operatorname{supp}(1-\chi)\phi \subseteq U$, so by linearity

$$u(\phi) = u(\chi\phi + (1-\chi)\phi) = u(\chi\phi) + u((1-\chi)\phi) = u(\chi\phi).$$

Example.
We can view $L^1_{\text{loc}}(\Omega) \subseteq \mathcal{D}'(\Omega)$ by identifying a locally integrable function with its action on test functions. Notice the topologies are compatible, and so this inclusion is continuous.
The DIRAC MASS at $x \in \Omega$ is defined as

$$\delta_x(\phi) := \phi(x),$$

and has $\operatorname{supp} \delta_x = \{x\}$. The Dirac mass at $x = 0$ is homogeneous of order $-n$. We can moreover write δ_0 as the distributional limit of functions L^1_{loc}. For example, in one dimension $n\mathbb{1}_{[0,1/n]} \to \delta_0$ as $n \to \infty$.
The HEAVISIDE FUNCTION $H : \mathbb{R} \to \mathbb{R}$ is defined

$$H(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

This is an asymmetric function which is homogeneous of order zero. The derivative of the Heaviside function is the Dirac mass at zero, $H' = \delta_0$.

1.4. **Localisation.** While distributions generally do not admit pointwise values, they are nonetheless characterised by their local behaviour. To make this statement precise, we construct a partition of unity

Theorem 4. *Let $\Omega_k \subseteq \mathbb{R}^d$ be a collection of open subsets, and suppose $u_k \in C_c^\infty(\Omega_k)^*$ is a family of distributions satisfying the compatability condition*

$$(u_k)_{|\Omega_j \cap \Omega_k} = (u_j)_{|\Omega_j \cap \Omega_k}.$$

Then there exists a unique distribution $u \in C_c^\infty(\bigcup_k \Omega_k)^*$ such that $u_{|\Omega_k} = u_k$.

Proof. We first establish uniqueness. Since distributions are linear, it suffices to show that if $u_{|\Omega_k} \equiv 0$ for all k, then $u \equiv 0$.

Choose a partition of unity $\chi_k \in C_c^\infty(\mathbb{R}^d)$ subordinate to Ω_k and the functional $u : C_c^\infty(\bigcup_k \Omega_k) \to \mathbb{C}$ by

$$\langle \phi, u \rangle := \sum_k \langle \chi_k \phi, u_k \rangle.$$

□

Theorem 5. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution supported in $\{0\} \subseteq \mathbb{R}^n$. Prove that there exists $N \in \mathbb{N}$ and coefficients $c_\alpha \in \mathbb{C}$ such that*

$$u = \sum_{|\alpha| \leq N} c_\alpha \partial^\alpha \delta.$$

Proof. We claim that there exists $N \in \mathbb{N}$ such that

$$u(\phi) \leq N \sum_{|\alpha| \leq N, |\beta| \leq N} ||x^\alpha \partial^\beta \phi||_\infty$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. Assume towards contradiction otherwise, then for all $N \in \mathbb{N}$ there exists $\phi_N \in \mathcal{S}(\mathbb{R}^n)$ such that

$$|u(\phi_N)| > N \sum_{|\alpha| \leq N, |\beta| \leq N} ||x^\alpha \partial^\beta \phi_N||_\infty.$$

Set

$$\psi_N = \left(N \sum_{|\alpha| \leq N, |\beta| \leq N} ||x^\alpha \partial^\beta \phi_N||_\infty \right)^{-1} \phi_N.$$

By construction, $\psi_N \in \mathcal{S}(\mathbb{R}^n)$ satisfying $|u(\psi_N)| \geq 1$ and

$$||x^\alpha \partial^\beta \psi_N||_\infty \leq \frac{1}{N}$$

for all $|\alpha|, |\beta| \leq N$. However, the inequality above implies $\psi_N \to 0$ in $\mathcal{S}(\mathbb{R}^n)$, a contradiction.

It remains to show that if $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\partial^\alpha \psi(0) = 0$ for $|\alpha| \leq N$, then $u(\psi) = 0$. We can find a bump function $\eta \in C_c^\infty(\mathbb{R}^n)$ satisfying $\eta \equiv 1$ on $|x| \leq 1$ and $\eta \equiv 0$ on $|x| \geq 2$. Set

$$\eta_\varepsilon(x) := \eta(x/\varepsilon), \qquad ||\partial^\alpha \eta_\varepsilon||_\infty = \varepsilon^{-|\alpha|} ||\partial^\alpha \eta||_\infty.$$

Since u is supported on $\{0\}$,

$$u(\phi) = u((1-\eta_\varepsilon)\phi) + u(\eta_\varepsilon \phi) = u(\eta_\varepsilon \phi).$$

Observe that $|x^\alpha| \lesssim 1$ for $|\alpha| \leq N$ on bounded sets, particularly $\operatorname{supp} \eta_\varepsilon \subseteq \{|x| \leq 2\varepsilon\}$. It follows from this observation, the initial claim, and the product rule that

$$|u(\phi)| \leq N \sum_{|\alpha| \leq N, |\beta| \leq N} ||x^\alpha \partial^\beta (\eta_\varepsilon \phi)||_\infty \lesssim_N \sum_{|\beta| \leq N} \sup_{|x| \leq 2\varepsilon} |\partial^\beta (\eta_\varepsilon \phi)(x)| \lesssim_\eta \sum_{|\beta| \leq N} \varepsilon^{|\beta| - N} \sup_{|x| \leq 2\varepsilon} |\partial^\beta \phi(x)|.$$

We want to show the right-hand side vanishes taking $\varepsilon \to 0$. For the highest order terms $|\beta| = N$, it follows from uniform continuity that

$$\lim_{\varepsilon \to 0} \sup_{|x| \leq 2\varepsilon} |\partial^\beta \phi(x)| = \partial^\beta \phi(0) = 0.$$

For the lower order terms $|\beta| < N$, we apply the Taylor estimate, noting $\partial^\alpha \phi(0) = 0$ for $|\alpha| < N$,

$$\varepsilon^{N-|\beta|} \sup_{|x| \leq 2\varepsilon} |\partial^\beta \phi(x)| \leq \varepsilon^{N-|\beta|} \sum_{|\alpha| = N - |\beta|} \frac{1}{(N-|\beta|)!} \sup_{|x| < 2\varepsilon} |x^\alpha \partial^{\beta+\alpha} \phi(x)| \sim_{N,\beta} \sum_{|\alpha| = N} \sup_{|x| < 2\varepsilon} |\partial^\alpha \phi(x)|.$$

Again, since the order N derivatives are uniformly continuous and vanish at the origin, the right-hand side vanishes as $\varepsilon \to 0$, as desired.

By Taylor's theorem, we can write $\phi \in \mathcal{S}'(\mathbb{R}^n)$ as

$$\phi(x) = \sum_{|\alpha| \leq N} \frac{\partial^\alpha \phi(0)}{\alpha!} x^\alpha + \psi(x)$$

where $\psi \in C^\infty(\mathbb{R}^n)$. Then, since $x^\alpha \eta \in \mathcal{S}(\mathbb{R}^n)$ and $\eta \psi \in C_c^\infty(\mathbb{R}^n)$ satisfies $\partial^\alpha(\eta\psi)(0) = \partial^\alpha \psi(0) = 0$ for $|\alpha| \leq N$, it follows from the previous result that

$$u(\phi) = u((1-\eta)\phi) + u(\eta\phi) = \sum_{|\alpha| \leq N} \frac{u(x^\alpha \eta)}{\alpha!} \partial^\alpha \phi(0).$$

This completes the proof. □

2. Convolution

Integration, interpreted as an averaging, acts as a smoothing operator. For example, we know from Lebesgue differentiation theorem that the local averaging operator

$$(\mathbb{1}_{[0,1]} * f)(x) := \int_x^{x+1} f(t)\, dt$$

is continuous and differentiable a.e. when $f \in L^1_{\text{loc}}(\mathbb{R})$. Thus integration allows us to compare pointwise values of an approximation of f, a task ill-posed for generic locally integrable functions which are only defined pointwise up to a.e. modification. To generalise the averaging operator, given $f, g : \mathbb{R}^d \to \mathbb{C}$, we formally define the CONVOLUTION of f against ϕ via the integral

$$(\phi * f)(x) := \int_{\mathbb{R}^d} \phi(x-y) f(y)\, dy.$$

Thus $(\phi * f)(x)$ is the "average" of f against the weight $\phi(x-y)dy$ centered at x.

2.1. **Convolution of test functions.**

Proposition 6. *Let $f, g, h : \mathbb{R}^d \to \mathbb{C}$ be sufficiently regular, then the following properties hold:*

(a) convolution is commutative
$$f * g = g * f,$$
(b) convolution is associative,
$$(f * g) * h = f * (g * h),$$
(c) the support of the convolution is contained in the algebraic sum of the supports,
$$\operatorname{supp}(f * g) \subseteq \operatorname{supp} f + \operatorname{supp} g,$$
(d) the derivatives of the convolution are given by
$$\partial^\alpha (f * g) = (\partial^\alpha f) * g = f * (\partial^\alpha g),$$
(e) if $f \in C^\infty(\mathbb{R}^d)$, then $f * g \in C^\infty(\mathbb{R}^d)$.

Proof. (a) follows from a change of variables, (b) follows from Fubini's theorem, (d) follows from differentiating under the integral, (e) follows from (d). To show (c), suppose $x \in \mathbb{R}^d$ such that $(f * g)(x) \neq 0$. Writing
$$(f * g)(x) = \int_{\mathbb{R}^d} f(y) g(x - y) dy,$$
note the integral is non-zero only if there exists $y \in \operatorname{supp} f$ and $x - y \in \operatorname{supp} g$. □

Proposition 7. *The space of test functions $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$ and $C^k(\Omega)$ for $k \in \mathbb{N}$.*

Proof. Let $f \in L^p(\mathbb{R}^d)$, we want to find $\{f_\varepsilon\}_\varepsilon \subseteq C_c^\infty(\mathbb{R}^d)$ such that $f_\varepsilon \to f$ in L^p-norm. By dominated convergence and the triangle inequality we can assume without loss of generality that f is compactly supported. Fix a test function $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\int \phi = 1$ and $\phi \equiv 0$ on $|x| > 1$, set
$$\phi_\varepsilon(x) := \frac{1}{\varepsilon^d} \phi(x/\varepsilon),$$
then $\int \phi_\varepsilon = 1$ and $\phi_\varepsilon \equiv 0$ on $|x| > \varepsilon$. Moreover, $f * \phi_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ and satisfies
$$f(x) - (f * \phi_\varepsilon)(x) = \int_{\mathbb{R}^d} (f(x) - f(x - y)) \phi_\varepsilon(y)\, dy = \int_{|z| \leq 1} (f(x) - f(x - \varepsilon z)) \phi(z)\, dz.$$
Then by Minkowski's integral inequality and dominated convergence theorem
$$||f - f * \phi_\varepsilon||_{L^p} \leq \int_{|z| \leq 1} ||f(x) - f(x - \varepsilon z)||_{L^p_x} \phi(z)\, dz \xrightarrow{\varepsilon \to 0} 0.$$
Similarly, we can write
$$\partial^\alpha f(x) - \partial^\alpha (f * \phi_\varepsilon)(x) = \int_{|z| \leq 1} (\partial^\alpha f(x) - \partial^\alpha f(x - \varepsilon z)) \phi(z)\, dz.$$
By continuity and compactness, $\partial^\alpha f(x) - \partial^\alpha f(x - \varepsilon z) \to 0$ uniformly as $\varepsilon \to 0$ for $|z| \leq 1$ and x in the compact support of f. This allows us to pass the limit uniformly under the integral sign. □

2.2. **Convolution of distributions.**

Proposition 8. *Let $T \in C_c^\infty(\mathbb{R}^d)^*$ and $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$. Then*

(a) the convolution defines a smooth function $T * \psi \in C^\infty(\mathbb{R}^d)$ satisfying
$$(T * \psi)(x) = T(\operatorname{Trans}_x \tilde{\psi}),$$
(b) the derivatives of the convolution are given by
$$\partial^\alpha (T * \psi) = (\partial^\alpha T) * \psi = T * (\partial^\alpha \psi).$$
(c) the support of the convolution is contained in the algebraic sum of the supports,
$$\operatorname{supp}(T * \psi) \subseteq \operatorname{supp} u + \operatorname{supp} \psi,$$
(d) convolution is associative,
$$(T * \psi) * \phi = T * (\psi * \phi).$$

Proof.

(a) We first prove that $x \mapsto T(\operatorname{Trans}_x \tilde{\psi})$ is smooth. Denote $e_j \in \mathbb{R}^d$ an elementary basis vector. The convergence $(\operatorname{Trans}_{x+he_j} \psi - \operatorname{Trans}_x \psi)/h \to \partial_j \psi$ as $h \to 0$ holds in the sense of test functions, so

$$\lim_{h\to 0} \frac{(T*\psi)(x+he_j) - (T*\psi)(x)}{h} = \lim_{h\to 0} u\left(\frac{\operatorname{Trans}_{x+he_j} \tilde{\psi} - \operatorname{Trans}_x \tilde{\psi}}{h}\right) = u(\partial_j \operatorname{Trans}_x \tilde{\psi}).$$

Arguing inductively gives the result. Then, since the Riemann sums converge in the sense of test functions,

$$\begin{aligned}\langle T(\operatorname{Trans}_x \tilde{\psi}), \phi\rangle &= \int_{\mathbb{R}^d} T(\operatorname{Trans}_x \tilde{\psi})\phi(x)dx = \lim_{h\to 0} \sum_{k\in\mathbb{Z}^d} T(\operatorname{Trans}_{kh} \tilde{\psi})\phi(kh)h^n \\ &= \lim_{h\to 0} T\left(y \mapsto \sum_{k\in\mathbb{Z}^n} \operatorname{Trans}_{kh} \tilde{\psi}(y)\phi(kh)h^n\right) = T\left(\int_{\mathbb{R}^d} \tilde{\psi}(y-x)\phi(x)dx\right) = T(\tilde{\psi}*\phi).\end{aligned}$$

(b) In the proof of a., we showed

$$\partial^\alpha(u*\psi) = T(\partial^\alpha \operatorname{Trans}_x \tilde{\psi}) = \partial^\alpha T(\operatorname{Trans}_x \tilde{\psi}) = (\partial^\alpha T)*\psi.$$

We can also write

$$\partial^\alpha(T*\psi) = T(\partial^\alpha \operatorname{Trans}_x \tilde{\psi}) = u(\operatorname{Trans}_x \widetilde{\partial^\alpha\psi}) = u*(\partial^\alpha\psi).$$

(c) Observe that $(T*\psi)(x) \neq 0$ only if $x - y \in \operatorname{supp}\psi$ for some $y \in \operatorname{supp} T$. Thus $x \in \operatorname{supp} T + \operatorname{supp}\phi$.

(d) By convergence of the Riemann sums in the sense of test functions,

$$\begin{aligned}(T*(\psi*\phi))(x) &= \lim_{h\to 0} u\left(\sum_{k\in\mathbb{Z}^d} \operatorname{Trans}_x \widetilde{\operatorname{Trans}_{kh}} \phi h^d \psi(kh)\right) \\ &= \lim_{h\to 0} \sum_{k\in\mathbb{Z}^n} (T*\psi)(x-kh)h^d\phi(kh) = \int_{\mathbb{R}^d} (T*\psi)(x-y)\phi(y)dy = ((T*\psi)*\phi)(x).\end{aligned}$$

□

2.3. **Convolution of compactly supported distributions.** Let $u, v, \phi \in C_c^\infty(\mathbb{R}^d)$, then by a change of variables $z = x - y$ and Fubini's theorem, we can write

$$(u*v)(\phi) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(y)v(x-y)\phi(x)\,dx\,dy = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(y)v(z)\phi(z+y)\,dy\,dz = u(v(\operatorname{Trans}_y \phi)).$$

The expression on the right continues to be well-defined for distributions u and v given that at least one is compactly supported. One important example is when we convolve a distribution with the Dirac mass at zero,

$$(u*\delta_0)(\phi) = u(\delta_0(\operatorname{Trans}_y \phi)) = u(\phi).$$

We can therefore view the space of compactly supported distributions $\mathcal{E}'(\mathbb{R}^d)$ as a commutative algebra with respect to the convolution operator, where the identity element with respect to convolution is given by δ_0.

2.4. **Fundamental solutions.** A LINEAR PARTIAL DIFFERENTIAL OPERATOR OF ORDER k takes the form

$$P(x,\partial) = \sum_{|\alpha|\le k} c_\alpha(x)\partial^\alpha$$

for coefficients $c_\alpha \in C^\infty(\Omega)$. We say that a distribution $u \in \mathcal{D}'(\Omega)$ is a FUNDAMENTAL SOLUTION if

$$P(x,\partial)u = \delta_0.$$

Consider now an operator with constant coefficients $P(x,\partial) \equiv P(\partial)$ defined on global distributions $\mathcal{D}'(\mathbb{R}^d)$. If we instead replaced the Dirac mass with a generic compactly supported distribution $f \in \mathcal{E}'(\mathbb{R}^d)$, we can obtain a solution for the corresponding differential equation by convolving the fundamental solution with f;

$$f = \delta_0 * f = P(\partial)u * f = P(\partial)(u*f).$$