INTEGRAL OPERATORS

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ABSTRACT. This article surveys the study of linear operators taking the form

$$Tf(y) := \int_{\mathbb{R}^d} K(x,y) f(x)\, dx$$

where $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is known as the *kernel* of the *integral operator* T. A fundamental problem in harmonic analysis is determining the boundedness of the operator T between function spaces given certain conditions on the kernel K. This has applications in establishing the Sobolev embedding inequalities and energy estimates for linear PDE.

CONTENTS

1. SCHUR'S TEST

Let (X,μ) and (Y,ν) be measure spaces, and let $K : X \times Y \to \mathbb{C}$ be a measurable function. Formally, an INTEGRAL OPERATOR is a linear operator of the form

$$Tf(y) := \int_X K(x,y) f(x)\, d\mu(x)$$

mapping a function $f : X \to \mathbb{C}$ to a function $Tf : Y \to \mathbb{C}$. The function K is known as the KERNEL of the integral operator T. A priori, the integral on the right is not well-defined, motivating the introduction various integrability conditions on K, which upon appealing to Minkowski's integral inequality or Holder's inequality we see that the integral defining $Tf(y)$ converges absolutely for almost every $y \in Y$. Furthermore, we can show that T forms a bounded operator between Lebesgue spaces.

1.1. Strong-type integrability conditions.

Assuming uniform L^1-integrability conditions on $K(x,y)$ in x and y, we can show that T satisfies a strong-type $(1,1)$ and (∞,∞) inequality, which by complex interpolation *a la* Riesz-Thorin would furnish a strong-type (p,p) inequality. This is the classical statement of Schur's test, which is the particular case on the diagonal of the general Schur's test stated below:

Theorem 1 (Strong-type Schur's test). *Suppose that $K : X \times Y \to \mathbb{C}$ obeys the bounds*

$$||K(x,y)||_{L_y^{q_0}(Y)} \le B_0 \qquad \text{uniformly for a.e. } x \in X,$$
$$||K(x,y)||_{L_x^{p_1'}(X)} \le B_1 \qquad \text{uniformly for a.e. } y \in Y,$$

for some constants $B_0, B_1 > 0$ and exponents $1 \le p_1, q_0 \le \infty$. Setting $p_0 := 1$ and $q_1 := \infty$, define the exponents $1 \le p_\theta, q_\theta \le \infty$ for $0 < \theta < 1$ by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

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Then the integral operator T satisfies the strong-type (p_θ, q_θ) inequality

$$||Tf||_{L^{q_\theta}(Y)} \leq B_0^\theta B_1^{1-\theta} ||f||_{L^{p_\theta}(X)}.$$

Proof. We argue by complex interpolation. The strong-type $(1, q_0)$ inequality follows from the triangle inequality and Minkowski's integral inequality,

$$||Tf||_{L^{q_0}(Y)} \leq \left|\left|\int_X |K(x,y)|\,|f(x)|\,d\mu(x)d\nu(y)\right|\right|_{L_y^{q_0}(Y)} \leq \int_X ||K(x,y)||_{L_y^{q_0}(Y)} |f(x)|\,d\mu(x) \leq B_0 ||f||_{L^1(X)}.$$

The strong-type (p_1, ∞) inequality follows from Holder's inequality

$$||Tf||_{L^\infty(Y)} \leq \sup_{y\in Y} ||K(x,y)||_{L_x^{p_1'}(X)} ||f||_{L^{p_1}(X)} \leq B_1 ||f||_{L^{p_1}(X)}.$$

We conclude the desired strong-type (p_θ, q_θ) inequality for $0 < \theta < 1$ via Riesz-Thorin interpolation. $\square$

Remark. Note that we did not exploit the sign of the kernel K anywhere in the proof of Schur's test, which suggests that Schur's test is ill-equipped for dealing with kernels exhibiting oscillation, such as the Fourier transform which has kernel $K(x,y) := e^{2\pi i x\cdot y}$, or cancellation, such as the Riesz transform which has kernel $K(x,y) := \frac{x_i - y_i}{|x-y|^{d+1}}$.

Corollary 2 (Strong-type Schur's test, diagonal)**.** *Suppose that $K : X \times Y \to \mathbb{C}$ obeys the bounds*

$$\int_X |K(x,y)| d\mu(x) \leq A \qquad \text{uniformly for a.e. } y \in Y,$$
$$\int_Y |K(x,y)| d\nu(y) \leq B \qquad \text{uniformly for a.e. } x \in X,$$

for some constants $A, B > 0$. Then for $1 \leq p \leq \infty$ the integral operator T satisfies the strong-type (p,p) inequality

$$||Tf||_{L^p(Y)} \leq A^{1/p'} B^{1/p} ||f||_{L^p(X)}.$$

Corollary 3 (Young's convolution inequality)**.** *Let $1 \leq p, q, r \leq \infty$ satisfy*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

Then

$$||f * g||_{L^r(\mathbb{R}^d)} \leq ||f||_{L^p(\mathbb{R}^d)} ||g||_{L^q(\mathbb{R}^d)}.$$

Proof. The result follows from the strong-type Schur's test for kernel $K(x,y) := g(x-y)$ where $g \in L^q(\mathbb{R}^d)$. Since the L^q-norm is translation invariant, we have

$$||K(x,y)||_{L_x^q} = ||K(x,y)||_{L_y^q} = ||g||_{L^q}.$$

Working through the exponent numerology, we conclude Young's convolution inequality. $\square$

1.2. **Weak-type integrability conditions.** If we replace the strong Lebesgue integrability conditions in the strong-type Schur's test by weak Lebesgue integrability conditions, we can use real interpolation to formulate a weak-type analogue of Schur's test:

Theorem 4 (Weak-type Schur's test)**.** *Suppose that $K : X \times Y \to \mathbb{C}$ obeys the bounds*

$$||K(x,y)||_{L^{q_0,\infty}(Y)} \leq B_0 \qquad \text{uniformly for a.e. } x \in X,$$
$$||K(x,y)||_{L^{p_1',\infty}(X)} \leq B_1 \qquad \text{uniformly for a.e. } y \in Y,$$

for some constants $B_0, B_1 > 0$ and exponents $1 < p_1, q_0 < \infty$. Setting $p_0 := 1$ and $q_1 := \infty$, define the exponents $1 < p_\theta, q_\theta < \infty$ for $0 < \theta < 1$ by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then the integral operator T satisfies the strong-type (p_θ, q_θ) inequality

$$||Tf||_{L^{q_\theta}(Y)} \lesssim_{p_1, q_0, \theta} B_0^\theta B_1^{1-\theta} ||f||_{L^{p_\theta}(X)}.$$

Proof. We argue by real interpolation. The restricted weak-type $(1, q_0)$ and (p_1, ∞) inequalities follow from the triangle inequality and Fubini-Tonelli,

$$\int_Y |T\mathbb{1}_E(y)|\,\mathbb{1}_F(y)\,d\nu(y) \le \int_F \left(\int_E |K(x,y)|\,d\mu(x)\right) d\nu(y) \le B_1\mu(E)^{1/p_1}\nu(F),$$
$$\int_Y |T\mathbb{1}_E(y)|\,\mathbb{1}_F(y)\,d\nu(y) \le \int_E \left(\int_F |K(x,y)|\,d\nu(y)\right) d\mu(x) \le B_0\nu(F)^{1/q_0'}\mu(E),$$

for all measurable $E \subseteq X$ and $F \subseteq Y$. We conclude the desired strong-type (p_θ, q_θ) inequality for $0 < \theta < 1$ via Marcinkiewicz interpolation. ☐

Remark. Note that we needed to exclude the endpoints $p_1, q_0 = 1, \infty$ to use real interpolation. In particular, the weak-type Schur's test cannot furnish strong-type bounds on the diagonal.

Corollary 5 (Weak-type Young's convolution inequality)**.** *Let $1 < p, q, r < \infty$ satisfy*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

Then

$$||f * g||_{L^r(\mathbb{R}^d)} \lesssim_{p,q} ||f||_{L^p(\mathbb{R}^d)}||g||_{L^{q,\infty}(\mathbb{R}^d)}$$

uniformly in $f \in L^p(\mathbb{R}^d)$ and $g \in L^{q,\infty}(\mathbb{R}^d)$.

Proof. The result follows from the weak-type Schur's test for kernel $K(x,y) := g(x-y)$ where $g \in L^{q,\infty}(\mathbb{R}^d)$. Since the $L^{q,\infty}$-norm is translation invariant, we have

$$||K(x,y)||_{L^{q,\infty}_x} = ||K(x,y)||_{L^{q,\infty}_y} = ||g||_{L^{q,\infty}}.$$

Working through the numerology, we conclude the weak-type Young's convolution inequality. ☐

A useful application of the weak-type Schur's test to partial differential equations is in proving the Hardy-Littlewood-Sobolev inequality. Let $X = Y = \mathbb{R}^d$ and define

$$g(x) := \frac{1}{|x|^\alpha}$$

for $0 < \alpha < d$. Observe that $g \notin L^p(\mathbb{R}^d)$ for any $1 \le p \le \infty$, so we cannot apply the strong-type Schur's test. On the other hand, $g \in L^{d/\alpha,\infty}(\mathbb{R}^d)$, so it follows from the weak-type Young's convolution inequality that

Corollary 6 (Hardy-Littlewood-Sobolev)**.** *Let $1 < p < r < \infty$ and $0 < \alpha < d$ satisfy*

$$\frac{1}{p} + \frac{\alpha}{d} = 1 + \frac{1}{r}.$$

Then

$$||f * |x|^{-\alpha}||_{L^r} \lesssim ||f||_{L^p}$$

uniformly in $f \in L^p(\mathbb{R}^d)$.

2. Calderon-Zygmund theory

We turn our attention to kernels $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ which are *singular* along the diagonal $x = y$. More precisely, we are interested in kernels which "barely" fail to be integrable, the prototypical example of which is the kernel

$$K(x,y) := \frac{1}{\pi}\frac{1}{y-x}.$$

Integrating in either x or y, we see that K admits a logarithmic singularity in the regions near the diagonal $|x-y| \ll 1$ and away from the diagonal $|x-y| \gg 1$. It is therefore not clear whether the integral operator corresponding to K, known as the Hilbert transform,

$$Hf(y) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x)}{y-x}\,dx$$

is well-defined. Nonetheless, we can view H as an integral operator "away from the diagonal", observing that the integral converges absolutely when $f \in L^2(\mathbb{R})$ is compactly supported and x lies outside of the support of f.

2.1. **Calderon-Zygmund operators.** A CALDERON-ZYGMUND KERNEL is a function $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ satisfying the HORMANDER CONDITION:

$$\int_{|x-y|>2|y-z|} |K(x,y) - K(x,z)|\, dx \lesssim 1 \qquad \text{uniformly for a.e. } y \neq z,$$
$$\int_{|x-y|>2|x-w|} |K(x,y) - K(w,y)|\, dy \lesssim 1 \qquad \text{uniformly for a.e. } x \neq w.$$

We say that a bounded linear operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a CALDERON-ZYGMUND OPERATOR if there exists a Calderon-Zygmund kernel K for which

$$Tf(y) = \int_{\mathbb{R}^d} K(x,y) f(x)\, dx \tag{*}$$

whenever $f \in L^2(\mathbb{R}^d)$ is compactly supported and y lies outside the support of f.

Remark. The Hormander condition is sometimes known as a *smoothness* condition, as it is often stated in the form of the strictly stronger Holder-type regularity assumptions

$$|K(x,y) - K(x,z)| \lesssim \frac{|y-z|^\delta}{|x-y|^{d+\delta}}, \qquad \text{whenever } |x-y| > 2|y-z|,$$
$$|K(x,y) - K(w,y)| \lesssim \frac{|x-w|^\delta}{|x-y|^{d+\delta}}, \qquad \text{whenever } |x-y| > 2|x-w|,$$

for some Holder exponent $0 < \delta \leq 1$. When $\delta = 1$, these form Lipschitz-type regularity estimates, which in turn are implied via the fundamental theorem of calculus by the gradient estimates

$$|\nabla_x K(x,y)| \lesssim \frac{1}{|x-y|^{d+1}},$$
$$|\nabla_y K(x,y)| \lesssim \frac{1}{|x-y|^{d+1}}.$$

The integral representation (*) does not fully characterise a Calderon-Zygmund operator. For example, the derivative operator $Tf(y) := f'(y)$ has kernel zero, however it is not a bounded operator on $L^2(\mathbb{R})$. Furthermore, for any $b \in L^\infty(\mathbb{R}^d)$ the multiplication operator $Tf(y) := b(y)f(y)$ is a Calderon-Zygmund operator with kernel zero. Fortunately, this is the only source of ambiguity:

Proposition 7. *If $T_1, T_2 : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ are Calderon-Zygmund operators associated to the same kernel, then they differ by a pointwise multiplication operator $(T_1 - T_2)f = bf$ for some $b \in L^\infty(\mathbb{R}^d)$.*

Proof. By linearity it suffices to show that a Calderon-Zygmund operator $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^d)$ corresponding to the zero kernel takes the form $Tf(y) = b(y)f(y)$ for some $b \in L^\infty(\mathbb{R}^d)$. Observe that the measure

$$E \mapsto \int_E T\mathbb{1}_E(y)\, dy$$

is an absolutely continuous measure, so by Radon-Nikodym there exists $b \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that

$$\int_E T\mathbb{1}_E(y)\, dy = \int_E b(y)\, dy.$$

Fix a Lebesgue point $x \in \mathbb{R}^d$ of b, then by Cauchy-Schwartz and the strong-type $(2,2)$ inequality,

$$\begin{aligned} |b(x)| &\leq \limsup_{r\to 0} \frac{1}{|B_r(x)|} \left| \int_{B_r(x)} b(y)\, dy \right| \\ &\leq \limsup_{r\to 0} \frac{1}{|B_r(x)|} \int_{\mathbb{R}^d} \mathbb{1}_{B_r(x)}(y) |T\mathbb{1}_{B_r(x)}(y)|\, dy \leq \limsup_{r\to\infty} \frac{||\mathbb{1}_{B_r(x)}||_{L^2} ||T\mathbb{1}_{B_r(x)}||_{L^2}}{|B_r(x)|} \lesssim 1. \end{aligned}$$

This shows $b \in L^\infty(\mathbb{R}^d)$. It remains to show $Tf = bf$. Since T corresponds to the zero kernel, $T\mathbb{1}_E$ is supported in E. It follows that if $E, F \subseteq \mathbb{R}^d$ have measure zero boundary, then $\mathbb{1}_F T\mathbb{1}_E = \mathbb{1}_F \left[T\mathbb{1}_{E\cap F} + T\mathbb{1}_{E\setminus F} \right] = T\mathbb{1}_{E\cap F}$. In particular, this result holds when E and F are dyadic cubes. We can write

$$\langle b\mathbb{1}_E, \mathbb{1}_F \rangle = \int_{E\cap F} b(y)\, dy = \int_{E\cap F} T\mathbb{1}_{E\cap F}(y)\, dy = \langle T\mathbb{1}_E, \mathbb{1}_F \rangle,$$

so by linearity, density of simple functions, and boundedness of T, we conclude $Tf = bf$. ☐

It is of interest to show that a Calderon-Zygmund operator is strong-type (p,p) for $1<p<\infty$. To this end, note that the adjoint is also a Calderon-Zygmund operator and recall the operator is strong-type $(2,2)$ by definition. We can therefore reduce the problem to showing a weak-type $(1,1)$ inequality, as Marcinkiewicz interpolation would furnish $1<p<2$, which by duality would furnish $2<p<\infty$.

To motivate the proof of the weak-type $(1,1)$ inequality, suppose that $f\in L^1(\mathbb{R}^d)$ is supported on the ball $|x-x_0|<r$ and has mean zero, i.e. $\int f=0$, then we can write

$$Tf(y)=\int_{|x-x_0|<r}K(x,y)f(x)\,dx=\int_{|x-x_0|<r}(K(x,y)-K(x_0,y))f(x)\,dx$$

whenever $|y-x_0|\geq 2r$. By Fubini's theorem and the Hormander condition,

$$||Tf||_{L^1_y(|y-x_0|\geq 2r)}\leq\int_{|y-x_0|\geq 2r}\int_{|x-x_0|<r}|K(x,y)-K(x_0,y)|\,|f(x)|\,dxdy\lesssim||f||_{L^1_x(|x-x_0|<r)}.$$

To prove the weak-type $(1,1)$ inequality, we decompose a generic function $f\in L^1(\mathbb{R}^d)$ into a bounded "good" part controlled by Chebyshev's inequality and the strong-type $(2,2)$ inequality, and localised "bad" parts with mean zero controlled by the argument above.

Lemma 8 (Calderon-Zygmund decomposition)**.** *Let $f\in L^1(\mathbb{R}^d)$ and $\lambda>0$, there exists a decomposition $f=g+b$, where g is the "good" part and b is the "bad" part, such that*

(a) $|g|\leq 2^d\lambda$ *a.e.,*
(b) $b=f\mathbb{1}_{\bigcup_k Q_k}$, *where $\{Q_k\}_k$ is a collection of cubes with pair-wise disjoint interiors satisfying*

$$\frac{1}{|Q_k|}\int_{Q_k}|b(y)|dy\leq 2^{d+1}\lambda,\qquad\int_{Q_k}b(y)\,dy=0.$$

Proof. Since $f\in L^1(\mathbb{R}^d)$, we can sub-divide $\mathbb{R}^d$ into dyadic cubes $Q\subseteq\mathbb{R}^d$ satisfying

$$\frac{1}{|Q|}\int_Q|f(y)|dy\leq\lambda.$$

We run the following algorithm: fixing one such cube Q, we sub-divide it into 2^d congruent dyadic cubes. Consider one of these smaller cubes $Q'\subseteq Q$, if it satisfies

$$\frac{1}{|Q'|}\int_{Q'}|f(y)|dy>\lambda \tag{*}$$

then we stop the algorithm and add Q' to the collection of cubes in the support of b. Such a cube satisfies

$$\lambda<\frac{1}{|Q'|}\int_{Q'}|f(y)|dy\leq\frac{2^d}{|Q|}\int_Q|f(y)|dy\leq 2^d\lambda.$$

If Q' does not satisfy (*), we continue the algorithm, further sub-dividing Q' into 2^d congruent dyadic cubes and examining each one. Define the "good" part by

$$g(x)=\begin{cases}f(x), & \text{if } x\notin\bigcup_k Q_k,\\ \frac{1}{|Q_k|}\int_{Q_k}f(y)\,dy & \text{if } x\in Q_k.\end{cases}$$

The properties of $b:=f-g$ are easily verified, so it remains to check $|g|\leq 2^d\lambda$ a.e. The inequality follows by construction for $x\in Q_k$, so suppose $x\notin\bigcup_k Q_k$. Again, by construction the average of $|f|$ is bounded by λ for any dyadic cube containing x. Moreover, there exists a family of such dyadic cubes with diameter tending to zero, so we conclude from the dyadic Lebesgue differentiation theorem

$$|f(x)|\leq\lim_{x\in Q,\,\mathrm{diam}\,Q\to 0}\frac{1}{|Q|}\int_Q|f(y)|\,dy<\lambda$$

for a.e. $x\notin\bigcup_k Q_k$. □

Theorem 9. *If $T:L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ is a Calderon-Zygmund operator, then it satisfies the weak-type $(1,1)$ and strong-type (p,p) inequalities for $1<p<\infty$.*

Proof. Assuming the weak-type $(1,1)$ inequality, Marcinkiewicz interpolation furnishes the strong-type (p,p) inequalities for $1 < p < 2$. We obtain the inequality for $2 < p < \infty$ via duality, using Holder's inequality and observing the adjoint T^* is a Calderon-Zygmund operator which we have just shown is strong-type (p', p'),

$$\begin{aligned}||Tf||_{L^p} &= \sup_{||g||_{L^{p'}}\leq 1} \langle Tf, g\rangle = \sup_{||g||_{L^{p'}}\leq 1} \langle f, T^* g\rangle \\ &\leq \sup_{||g||_{L^{p'}}\leq 1} ||f||_{L^p} ||T^* f||_{L^{p'}} \lesssim \sup_{||g||_{L^{p'}}\leq 1} ||f||_{L^p} ||g||_{L^{p'}} \leq ||f||_{L^p}.\end{aligned}$$

It remains then to show the weak-type $(1,1)$ inequality. For $f \in L^1(\mathbb{R}^d)$, we perform a Calderon-Zygmund decomposition $f = g + b$ at the level $\lambda > 0$. By the triangle inequality,

$$|\{y \in \mathbb{R}^d : Tf(y) > \lambda\}| \leq |\{y \in \mathbb{R}^d : Tg(y) > \lambda/2\}| + |\{y \in \mathbb{R}^d : Tb(y) > \lambda/2\}|.$$

For control over the "good" part, it follows from Chebyshev's inequality, the strong-type $(2,2)$ inequality, and the "good" inequality $|g| \lesssim \lambda$ that

$$|\{y \in \mathbb{R}^d : Tg(y) > \lambda/2\}| \leq \frac{||Tg||_{L^2}^2}{(\lambda/2)^2} \lesssim \frac{||g||_{L^2}^2}{\lambda^2} \lesssim \frac{||f||_{L^1}}{\lambda}.$$

For control over the "bad" part, we claim that

$$\int_{\mathbb{R}^d \setminus 2Q_k} |T(b\mathbb{1}_{Q_k})(y)|\, dy \lesssim \int_{Q_k} |b(y)|\, dy.$$

This would complete the proof, as, writing $b = \sum_k b\mathbb{1}_{Q_k}$, it follows from sub-additivity of the Lebesgue measure, Chebyshev's inequality and construction of Q_k that

$$\begin{aligned}|\{y \in \mathbb{R}^d : Tg(y) > \lambda/2\}| &\leq \sum_k |2Q_k| + |\{y \notin \bigcup_k 2Q_k : Tb(y) > \lambda/2\}| \\ &\leq \frac{2^{d+1}}{\lambda} \sum_k \int_{Q_k} |b(y)|\, dy + \frac{2}{\lambda} \sum_k \int_{\mathbb{R}^d \setminus 2Q_k} |T(b\mathbb{1}_{Q_k})(y)|\, dy \lesssim \frac{||f||_{L^1}}{\lambda}.\end{aligned}$$

Denote $w_k \in Q_k$ the center of the cube, then since the "bad" part has zero integral on Q_k we can write

$$T(b\mathbb{1}_{Q_k})(y) = \int_{Q_k} K(x,y) b(x)\, dx = \int_{Q_k} (K(x,y) - K(w_k, y))\, b(x)\, dx$$

for all $y \notin 2Q_k$. It follows from Fubini's theorem and the Hormander condition that

$$\int_{\mathbb{R}^d \setminus 2Q_k} |T(b\mathbb{1}_{Q_k})(y)|\, dy \leq \int_{Q_k} |b(y)| \left(\int_{\mathbb{R}^d \setminus 2Q_k} |K(x,y) - K(w_k, y)| dy \right) dx \lesssim \int_{Q_k} |b(x)|\, dx,$$

proving the claim and thereby concluding the proof. □

Remark. The strong-type inequality fails at the endpoints $p = 1, \infty$. For example, the Hilbert transform of the characteristic function of $[a,b]$ takes the form

$$H\mathbb{1}_{[a,b]}(y) = \frac{1}{\pi} \int_a^b \frac{1}{y-x}\, dx = \frac{1}{\pi} \log \left| \frac{x-a}{x-b} \right|$$

whenever $y \notin [a,b]$, so it is neither integrable nor bounded.

2.2. **Convolution operators.** A Calderon-Zygmund convolution kernel is a tempered distribution $K \in \mathcal{S}'$ which coincides with a locally integrable function on $\mathbb{R}^d \setminus 0$, satisfies the boundedness condition $\widehat{K} \in L^\infty(\mathbb{R}^d)$ and the Hormander condition

$$\int_{|x| > 2|y|} |K(x-y) - K(x)|\, dx \lesssim 1 \qquad \text{uniformly for a.e. } y \neq 0.$$

Define the convolution operator

$$Tf(y) := (K * f)(y) = \overline{\langle f_y, K\rangle},$$

where $f_y(x) = \overline{f(y-x)}$ and $f \in C_c^\infty(\mathbb{R}^d)$. Observe that T commutes with translation, satisfies a strong-type $(2,2)$ inequality by Plancharel's theorem and the boundedness condition, and takes the form

$$Tf(y) = \overline{\langle f_y, K\rangle} = \int_{\mathbb{R}^d} K(x) f(y-x)\, dx = \int_{\mathbb{R}^d} K(y-x) f(x)\, dx$$

whenever y lies outside the support of f. The kernel $(x, y) \mapsto K(y - x)$ satisfies the general Hormander condition, and so it follows that T extends to a Calderon-Zygmund operator. Conversely, a Calderon-Zygmund operator which commutes with translation necessarily takes the form of a convolution with a Calderon-Zygmund kernel; more generally,

Proposition 10. *Let $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a bounded linear operator commuting with translation. Then there exists a unique kernel $K \in \mathcal{S}'(\mathbb{R}^d)$ such that $Tf = K * f$ for all $f \in \mathcal{S}(\mathbb{R}^d)$ and $\widehat{K} \in L^\infty(\mathbb{R}^d)$.*

We want to find conditions under which a kernel $K \in L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus 0)$ can be identified with a Calderon-Zygmund kernel. Obviously K must satisfy the Hormander condition, so it remains to establish when $\widehat{K} \in L^\infty(\mathbb{R}^d)$ holds. To make sense of this problem, we must first identify K with a tempered distribution on $\mathbb{R}^d$, as *a priori* it only defines a distribution on $\mathbb{R}^d \setminus 0$. Define the PRINCIPAL VALUE DISTRIBUTION of K by

$$\langle \phi, \mathrm{pv}\, K \rangle := \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} K(x)\phi(x)\, dx$$

provided the limit exists and is continuous with respect to $\phi \in \mathcal{S}(\mathbb{R}^d)$.

Lemma 11. *Let $K \in L^1_{loc}(\mathbb{R}^d \setminus 0)$ be a kernel satisfying the size condition*

$$\int_{R < |x| < 2R} |K(x)|\, dx \lesssim 1 \qquad \text{uniformly in } 0 < R < \infty.$$

Then the principal value distribution $\mathrm{pv}\, K$ *exists if and only if*

$$\lim_{\varepsilon \to 0} \int_{\varepsilon < |x| < 1} K(x)\, dx$$

exists.

Proof. Suppose $\mathrm{pv}\, K$ exists and choose $\phi \in \mathcal{S}(\mathbb{R}^d)$ such that $\phi \equiv 1$ on the unit ball $|x| < 1$. Formally, we can write

$$\langle \phi, \mathrm{pv}\, K \rangle = \lim_{\varepsilon \to 0} \int_{\varepsilon < |x| < 1} K(x)\, dx + \int_{|x| > 1} K(x)\phi(x)\, dx,$$

where the limit exists provided that the second integral on the right exists. Indeed, decomposing the region $|x| > 1$ dyadically and applying the size condition, we obtain

$$\begin{aligned} \left| \int_{|x|>1} K(x)\phi(x)\, dx \right| &\lesssim \sum_{N \in 2^{\mathbb{N}}} \int_{N \le |x| \le 2N} \frac{|x|}{N} |\phi(x)|\, |K(x)|\, dx \\ &\lesssim || |x|\, \phi ||_{L^\infty} \sum_{N \in 2^{\mathbb{N}}} \frac{1}{N} \int_{N \le |x| \le 2N} |K(x)|\, dx \lesssim || |x|\, \phi ||_{L^\infty}. \end{aligned}$$

Moreover, this shows the left-hand side defines a tempered distribution. For the converse, suppose the limit exists, then formally we can write

$$\langle \phi, \mathrm{pv}\, K \rangle = \phi(0) \lim_{\varepsilon \to 0} \int_{\varepsilon < |x| < 1} K(x)\, dx + \lim_{\varepsilon \to 0} \int_{\varepsilon < |x| < 1} K(x)(\phi(x) - \phi(0))\, dx + \int_{|x| > 1} K(x)\phi(x)\, dx.$$

The first term on the right is a constant multiple of the Dirac distribution, the third term defines a tempered distribution as remarked earlier, so it remains to verify the second term is a tempered distribution. It follows from the mean value theorem that $|\phi(x) - \phi(0)| \le ||\nabla\phi||_{L^\infty} |x|$. Decomposing $|x| < 1$ dyadically and applying the size condition, we obtain

$$\begin{aligned} \left| \int_{\varepsilon < |x| < 1} K(x)(\phi(x) - \phi(0))\, dx \right| &\le ||\nabla\phi||_{L^\infty} \int_{|x| < 1} |x|\, |K(x)|\, dx \\ &\lesssim ||\nabla\phi||_{L^\infty} \sum_{N \in 2^{\mathbb{N}}} \frac{1}{N} \int_{1/2N < |x| < 1/N} |K(x)|\, dx \lesssim ||\nabla\phi||_{L^\infty}. \end{aligned}$$

This completes the proof. □

Remark. The size condition is named as such since it is often stated in the form of the strictly stronger estimate

$$|K(x)| \lesssim \frac{1}{|x|^d}, \qquad \text{uniformly in } x \ne 0.$$

For example, the kernel of the Hilbert transform $K(x) = 1/\pi x$ satisfies the size condition. Furthermore, the only homogeneous Calderon-Zygmund kernels on $\mathbb{R}^d$ are those of degree $-d$.

Now that we have made sense of our kernel as a tempered distribution, we need to verify the boundedness condition $\widehat{\mathrm{pv}\,K} \in L^\infty(\mathbb{R}^d)$. Collecting our results, we could then conclude the convolution operator $Tf := \mathrm{pv}\,K * f$ is a Calderon-Zygmund operator. To this end, it is convenient to analyse the truncated kernels

$$K_\varepsilon := K\mathbb{1}_{\varepsilon<|x|<1/\varepsilon}.$$

Observe that $K_\varepsilon \in L^1(\mathbb{R}^d)$ and therefore define tempered distributions. By dominated convergence theorem, $K_\varepsilon \to \mathrm{pv}\,K$ in the sense of distributions and thus $K_\varepsilon * f \to \mathrm{pv}\,K * f$ pointwise for $f \in \mathcal{S}(\mathbb{R}^d)$. Moreover by Plancharel's theorem and Holder's inequality we have

$$\langle f, \widehat{\mathrm{pv}\,K}\rangle = \lim_{\varepsilon\to 0}\langle f, \widehat{K_\varepsilon}\rangle \leq \lim_{\varepsilon\to 0} ||f||_{L^1}||\widehat{K_\varepsilon}||_{L^\infty}.$$

Thus if the truncated kernels satisfy the boundedness condition uniformly, duality furnishes $\widehat{\mathrm{pv}\,K} \in L^\infty(\mathbb{R}^d)$.

Lemma 12. *Let $K \in L^1_{loc}(\mathbb{R}^d \setminus 0)$ satisfy the cancellation, size, and smoothness conditions given respectively by*

$$\left|\int_{R_1<|x|<R_2} K(x)\,dx\right| \lesssim 1 \qquad \text{uniformly in } 0 < R_1 < R_2 < \infty,$$
$$\int_{R<|x|<2R} |K(x)|\,dx \lesssim 1 \qquad \text{uniformly in } 0 < R < \infty,$$
$$\int_{|x|>2|y|} |K(x-y) - K(x)|\,dx \lesssim 1 \qquad \text{uniformly in } y \neq 0.$$

Then the truncated kernels K_ε are Calderon-Zygmund convolution kernels satisfying the boundedness condition and Hormander condition uniformly in $\varepsilon > 0$.

Proof. We first show that K_ε satisfies the Hormander condition uniformly in ε by dividing the region $|x| > 2|y|$ into three regions,

$$\begin{aligned}\int_{|x|>2|y|} |K_\varepsilon(x-y) - K_\varepsilon(x)|\,dx &= \int_{\substack{|x|>2|y| \\ \varepsilon<|x|<1/\varepsilon \\ \varepsilon<|x-y|<1/\varepsilon}} |K(x-y)-K(x)|\,dx \\ &\quad + \int_{\substack{|x|>2|y| \\ \varepsilon<|x|<1/\varepsilon \\ |x-y|<\varepsilon \text{ or } |x-y|>1/\varepsilon}} |K(x)|\,dx \\ &\quad + \int_{\substack{|x|>2|y| \\ \varepsilon<|x-y|<1/\varepsilon \\ |x|<\varepsilon \text{ or } |x|>1/\varepsilon}} |K(x-y)|\,dx =: A + B + C.\end{aligned}$$

It is clear that A is controlled uniformly in $y \neq 0$ and ε by the smoothness condition. To control B, suppose that $|x| > 2|y|$ and $\varepsilon < |x| < 1/\varepsilon$. If $|x-y| < \varepsilon$ or $|x-y| > 1/\varepsilon$, then by the triangle inequality we have $|x| \leq |x-y| + |y| < \varepsilon + |x|/2 < 2\varepsilon$ or $|x| > |x-y| - |y| > 1/\varepsilon - |x|/2 > 1/2\varepsilon$ respectively. Thus

$$B \lesssim \int_{\varepsilon<|x|<2\varepsilon} |K(x)|\,dx + \int_{1/2\varepsilon<|x|<1/\varepsilon} |K(x)|\,dx \lesssim 1$$

by the size condition. Arguing analogously gives the result for C.

Now we need to show that K_ε satisfies the boundedness condition uniformly in ε. Fix $\xi \in \mathbb{R}^d$, we divide $\mathbb{R}^d_x$ into the regions of low oscillation $|x| < 1/|\xi|$ and low oscillation $|x| > 1/|\xi|$,

$$\widehat{K}_\varepsilon(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot x} K_\varepsilon(x)dx = \left(\int_{|x|<1/|\xi|} + \int_{|x|>1/|\xi|}\right) e^{-2\pi i\xi\cdot x}K_\varepsilon(x)dx.$$

For control over the region of low oscillation,

$$\begin{aligned}\left|\int_{|x|<1/|\xi|} e^{-2\pi i\xi\cdot x}K_\varepsilon(x)dx\right| &\leq \left|\int_{|x|<1/|\xi|} K_\varepsilon(x)\,dx\right| + \left|\int_{|x|<1/|\xi|} (e^{-2\pi i\xi\cdot x} - 1)K_\varepsilon(x)\,dx\right| \\ &\lesssim \left|\int_{\varepsilon<|x|<1/\varepsilon} K(x)\,dx\right| + \int_{|x|<1/|\xi|} |x|\,|\xi|\,|K(x)|\,dx \\ &\lesssim 1 + |\xi| \sum_{N\in 2^{\mathbb{N}}} \int_{1/N|\xi|<|x|<2/N|\xi|} |x|\,|K(x)|\,dx \lesssim 1,\end{aligned}$$

where the first inequality follows from the triangle inequality, the second from the basic estimate $|e^{i\theta}-1| \lesssim 1$, the third from the cancellation condition, and the last from applying the size condition to dyadic annuli arising from decomposing the region $|x| < 1/|\xi|$. For control over the region of high oscillation, we write

$$\begin{aligned}
\int_{|x|>1/|\xi|} e^{-2\pi i\xi\cdot x} K_\varepsilon(x)dx &= \int_{|x|>1/|\xi|} \frac{1}{2}(e^{-2\pi i\xi\cdot x} - e^{-2\pi i\xi\cdot(x-\xi/2|\xi|^2)})K_\varepsilon(x)\,dx \\
&= \frac{1}{2}\int_{|x|>1/|\xi|} e^{-2\pi i\xi x} K_\varepsilon(x)\,dx - \frac{1}{2}\int_{|x-\xi/2|\xi|^2|>1/|\xi|} e^{-2\pi i\xi\cdot x} K_\varepsilon(x-\xi/2|\xi|^2)\,dx \\
&= \frac{1}{2}\int_{|x|>1/|\xi|} e^{-2\pi i\xi x}(K_\varepsilon(x) - K_\varepsilon(x-\xi/2|\xi|^2))\,dx \\
&\quad + \frac{1}{2}\int_{|x|\le 1/|\xi|\le|x-\xi/2|\xi|^2|} e^{-2\pi i\xi\cdot x} K_\varepsilon(x-\xi/2|\xi|^2)\,dx \\
&\quad - \frac{1}{2}\int_{|x-\xi/2|\xi|^2|\le 1/|\xi|\le|x|} e^{-2\pi i\xi\cdot x} K_\varepsilon(x-\xi/2|\xi|^2)\,dx =: \mathrm{I}+\mathrm{II}+\mathrm{III},
\end{aligned}$$

where the first equality follows from identity $e^{-2\pi i\xi\cdot\xi/2|\xi|^2} = e^{-\pi i} = -1$, the second from a change of variables $x \mapsto x - \xi/2|\xi|^2$, and the third from decomposing the region $|x-\xi/2|\xi|^2| > 1/|\xi|$.

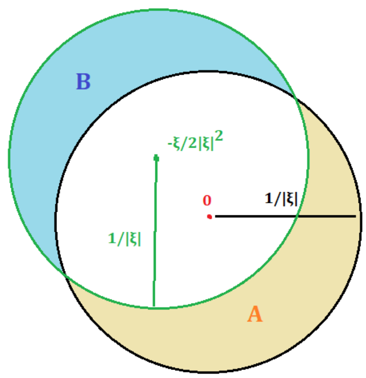

FIGURE 1. A is the region of integration $|x-\xi/2|\xi|^2| \le 1/|\xi| \le |x|$ and B is the region of integration $|x| \le 1/|\xi| \le |x-\xi/2|\xi|^2|$.

The first integral I is controlled by the uniform Hormander condition for K_ε,

$$|\mathrm{I}| \lesssim \int_{|x|>1/|\xi|} |K_\varepsilon(x) - K_\varepsilon(x-\xi/2|\xi|^2)|\,dx \lesssim 1.$$

By the triangle inequality, $|x| \le 1/|\xi| \le |x-\xi/2|\xi|^2|$ implies $1/|\xi| \le |x-\xi/2|\xi|^2| \le 3/2|\xi|$. Thus the second integral II is controlled by the size condition,

$$|\mathrm{II}| \lesssim \int_{1/2|\xi|\le|x-\xi/2|\xi|^2|\le 3/2|\xi|} |K_\varepsilon(x-\xi/2|\xi|^2)|\,dx \lesssim 1.$$

Arguing analogously gives the result for III. We conclude $||\widehat{K_\varepsilon}||_{L^\infty} \lesssim 1$ uniformly in ε, as desired. $\square$

Theorem 13. *Let $K \in L^1_{loc}(\mathbb{R}^d \setminus 0)$ satisfy the cancellation, size, and smoothness conditions,*

$$\begin{aligned}
\left|\int_{R_1<|x|<R_2} K(x)\,dx\right| &\lesssim 1 \qquad \text{uniformly in } 0<R_1<R_2<\infty, \\
\int_{R<|x|<2R} |K(x)|\,dx &\lesssim 1 \qquad \text{uniformly in } 0<R<\infty, \\
\int_{|x|>2|y|} |K(x-y)-K(x)|\,dx &\lesssim 1 \qquad \text{uniformly in } y \ne 0,
\end{aligned}$$

and suppose further that the principal value distribution pv K *exists, i.e.*

$$\lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<1} K(x)\,dx.$$

Then the convolution operator $Tf := \mathrm{pv}\,K * f$ *is a Calderon-Zygmund operator such that* $K_\varepsilon * f \to Tf$ *pointwise a.e. for* $f \in \mathcal{S}(\mathbb{R}^d)$.

Remark. The cancellation condition and existence of the principal value distribution are implied by the stronger cancellation condition

$$\int_{R_1<|x|<R_2} K(x)\,dx = 0 \qquad \text{for all } 0 < R_1 < R_2 < \infty.$$

As an example, the kernel of the Hilbert transform satisfies this strong cancellation condition. Another useful example with applications to partial differential equations are the Riesz transforms, which correspond to the kernels $K_j(x) := x_j/|x|^{d+1}$.

2.3. **Singular integrals.** A SINGULAR KERNEL is a function $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ satisfying the Holder-type regularity estimates

$$|K(x,y) - K(x,z)| \lesssim \frac{|y-z|^\delta}{|x-y|^{d+\delta}}, \qquad \text{whenever } |x-y| > 2|y-z|,$$
$$|K(x,y) - K(w,y)| \lesssim \frac{|x-w|^\delta}{|x-y|^{d+\delta}}, \qquad \text{whenever } |x-y| > 2|x-w|,$$

for some Holder exponent $0 < \delta \le 1$ and the decay estimate

$$|K(x,y)| \lesssim \frac{1}{|x-y|^d}.$$

Similar to the case of the Hilbert transform, singular kernels admit a logarithmic singularity along the diagonal $x = y$, so we cannot apply Schur's test to prove boundedness of the corresponding operator. As one might expect, showing a singular kernel gives rise to a Calderon-Zygmund operator is far more subtle than the translation-invariant case, and so we will reserve this story for another time and assume there exists a Calderon-Zygmund operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with K as its kernel.

The decay estimate implies that the kernel is locally integrable in each variable on $\mathbb{R}^d \setminus 0$. We can therefore define the truncated operator

$$T_\varepsilon f(y) := \int_{|x-y|>\varepsilon} K(x,y) f(x)\,dx$$

for $f \in C_c^\infty(\mathbb{R}^d)$. We say that T is a SINGULAR INTEGRAL OPERATOR if

$$\lim_{\varepsilon\to 0} T_\varepsilon f(y) = Tf(y)$$

for a.e. $y \in \mathbb{R}^d$ and $f \in L^p(\mathbb{R}^d)$ for $1 < p < \infty$. To make sense of the problem, we need to establish conditions under which the limit on the left exists. Following an argument analogous to the proof of Lemma 11, we obtain

Lemma 14. *Let* $K : \mathbb{R}^d \to \mathbb{R}^d \to \mathbb{C}$ *be a singular kernel and* $f \in C_c^\infty(\mathbb{R}^d)$. *Then the limit*

$$\lim_{\varepsilon\to 0} T_\varepsilon f(y)$$

exists if and only if the limit

$$\lim_{\varepsilon\to 0}\int_{\varepsilon<|x-y|<1} K(x,y)\,dx$$

exists.

It is typical to have $T_\varepsilon f \to Tf$ pointwise a.e. for $f \in C_c^\infty(\mathbb{R}^d)$, such as in the case of convolution kernels in Theorem 13. This would reduce the problem to showing the set of functions for which $T_\varepsilon f \to Tf$ pointwise a.e. forms a closed subspace of $L^p(\mathbb{R}^d)$, which in turn is implied by weak-type bounds for the corresponding MAXIMAL OPERATOR

$$T^* f(y) := \sup_{\varepsilon>0} |T_\varepsilon f(y)|.$$

Lemma 15 (Cotlar's inequality)**.** *Let $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be a Calderon-Zygmund operator with singular kernel and let $0 < \nu \leq 1$. Then*

$$T^* f(y) \lesssim_\nu M(|Tf|^\nu)(y)^{1/\nu} + Mf(y)$$

uniformly for $f \in C_c^\infty(\mathbb{R}^d)$ and a.e. $y \in \mathbb{R}^d$.

Proof. Fix $\varepsilon > 0$, we aim to show

$$T_\varepsilon f(y) \lesssim_\nu M(|Tf|^\nu)(y)^{1/\nu} + Mf(y).$$

Suppose that $|y - z| < \varepsilon/2$, then applying the Holder-type estimates on the kernel, the second follows from a dyadic decomposition of the region $|x - y| > \varepsilon$ we obtain

$$\begin{aligned} |T(f\mathbb{1}_{|x-y|>\varepsilon})(z) - T(f\mathbb{1}_{\mathbb{1}_{|x-y|>\varepsilon}})(y)| &\leq \int_{|x-y|>\varepsilon} |K(x,z) - K(x,y)|\, |f(x)|\, dx \\ &\lesssim |y-z|^\delta \int_{|x-y|>\varepsilon} \frac{|f(x)|}{|x-y|^{d+\delta}}\, dx \\ &\lesssim \varepsilon^\delta \sum_{N \in 2^{\mathbb{N}}} \int_{N\varepsilon<|x-y|<2N\varepsilon} \frac{|f(x)|}{|x-y|^{d+\delta}}\, dx \\ &\lesssim \sum_{N \in 2^{\mathbb{N}}} N^{-\delta} \frac{1}{(N\varepsilon)^d} \int_{|x-y|<2N\varepsilon} |f(x)|\, dx \lesssim Mf(y). \end{aligned}$$

Observe that $T(f\mathbb{1}_{|x-y|>\varepsilon})(y) = T_\varepsilon f(y)$. Hence by the triangle inequality and the inequality above we have

$$|T_\varepsilon f(y)| \leq Mf(y) + |Tf(z)| + |T(f\mathbb{1}_{|x-y|<\varepsilon})(z)|.$$

It remains to choose z such that the last two terms are controlled by $M(|Tf|^\nu)(y)^{1/\nu}$ and $Mf(y)$ respectively. To control $|Tf(z)|$, we apply Chebyshev's inequality,

$$|\{z : |z-y| < \varepsilon \text{ and } |Tf(z)| > \lambda\}| \leq \frac{1}{\lambda^\nu} \int_{|z-y|<\varepsilon} |Tf(z)|^\nu\, dz \leq \frac{|B_\varepsilon(y)|}{\lambda^\nu} M(|Tf|^\nu))(y).$$

Choosing $\lambda > 4^{1/\nu} M(|Tf|^\nu)(y)^{1/\nu}$, we obtain

$$|\{z : |z-y| < \varepsilon \text{ and } |Tf(z)| > \lambda\}| \leq \frac{1}{4} |B_\varepsilon(y)|.$$

To control $|T(f\mathbb{1}_{|x-y|<\varepsilon})(z)|$, it follows from the weak-type $(1,1)$ inequality for T that

$$|\{z : |z-y| < \varepsilon \text{ and } |T(f\mathbb{1}_{|x-y|<\varepsilon})(z)| > \lambda\}| \leq \frac{C}{\lambda} \int_{|x-y|<\varepsilon} |f(x)|\, dx \leq \frac{C\, |B_\varepsilon(y)|}{\lambda} Mf(y).$$

Choosing $\lambda > 4CMf(y)$, we obtain

$$|\{z : |z-y| < \varepsilon \text{ and } |T(f\mathbb{1}_{|x-y|<\varepsilon})(z)| > \lambda\}| \leq \frac{1}{4} |B_\varepsilon(y)|.$$

Thus there exists $z \in B_\varepsilon(y)$, i.e. $|z - y| < \varepsilon$, such that $|Tf(z)| < \lambda$ and $|T(f\mathbb{1}_{|x-y|<\varepsilon})| < \lambda$. Choosing λ optimally completes the proof. ☐

Theorem 16. *Let $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be a Calderon-Zygmund operator with singular kernel, then its maximal operator T^* is weak-type $(1,1)$ and strong-type (p,p) for $1 < p < \infty$.*

Proof. The strong-type (p,p) inequality follows immediately from Cotlar's inequality for $\nu = 1$ since the Hardy-Littlewood maximal function and T are strong-type (p,p). For the weak-type $(1,1)$ inequality, fix $\nu < 1$, then it suffices by Cotlar's inequality and the Hardy-Littlewood weak-type $(1,1)$ inequality to show

$$||M(|Tf|^\nu)^{1/\nu}||_{L^{1,\infty}} \lesssim ||f||_{L^1}.$$

By Hunt's interpolation theorem, the Hardy-Littlewood maximal operator is bounded on $L^{1/\nu,\infty}(\mathbb{R}^d)$. Combined with the weak-type $(1,1)$ inequality for T, we obtain

$$||M(|Tf|^\nu)^{1/\nu}||_{L^{1,\infty}} \sim ||M(|Tf|^\nu)||_{L^{1/\nu,\infty}}^{1/\nu} \lesssim |||Tf|^\nu||_{L^{1/\nu,\infty}}^{1/\nu} \sim ||Tf||_{L^{1,\infty}} \lesssim ||f||_{L^1}$$

as desired. ☐

Theorem 17. *Let $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be a Calderon-Zygmund operator with singular kernel such that*

$$\lim_{\varepsilon\to 0} T_\varepsilon f = Tf$$

pointwise a.e. for all $f \in C_c^\infty(\mathbb{R}^d)$. Then the convergence above continues to hold for $f \in L^p(\mathbb{R}^d)$ both pointwise and in norm for $1 < p < \infty$.

Proof. Pointwise convergence follows from boundedness of the maximal operator. Convergence in norm follows from dominated convergence theorem and $T^* f \in L^p(\mathbb{R}^d)$. □

Remark. We cannot use the same argument to establish convergence in norm for $p = 1$. Nevertheless, this result still holds provided that $Tf \in L^1(\mathbb{R}^d)$; this is due to Calderon and Capri.

2.4. **Fourier multipliers.** If $m : \mathbb{R}^d \to \mathbb{C}$ is a tempered distribution, we can define the FOURIER MULTIPLIER $m(D) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)^*$ implicitly in frequency space by

$$\widehat{m(D)f}(\xi) := m(\xi)\widehat{f}(\xi),$$

or explicitly in physical space by

$$m(D)f(x) := (\check{m} * f)(x).$$

The function m is known as the SYMBOL of the operator $m(D)$. We formally have the multiplier calculus

$$m(D)^* = \overline{m}(D),$$
$$m_1(D) + m_2(D) = (m_1 + m_2)(D),$$
$$m_1(D)m_2(D) = (m_1 m_2)(D).$$

In particular, Fourier multipliers commute with each other. Just as in Section 2.2 we determined conditions on the convolution kernel under which the corresponding operator formed a Calderon-Zygmund operator, we want to find conditions on m under which the operator $m(D)$ forms a Calderon-Zygmund operator.

Theorem 18 (Hormander-Mikhlin multiplier theorem)**.** *Let $m \in C^{d+2}_{loc}(\mathbb{R}^d \setminus 0)$ obey the homogeneous symbol estimate of order zero*

$$|D^\alpha_\xi m(\xi)| \lesssim |\xi|^{-|\alpha|}$$

uniformly in $\xi \neq 0$ for all $0 \le |\alpha| \le d+2$. Then $m(D)$ is a Calderon-Zygmund operator.

Proof. The boundedness condition $m \in L^\infty(\mathbb{R}^d)$ is clearly satisfied. *A priori*, we only know that the kernel $\check{m}$ is a tempered distribution. We claim that it is in fact a singular kernel satisfying the gradient estimate, which would complete the proof. To this end, we localise in frequency space, choosing a non-negative bump function $\phi \in C_c^\infty(\mathbb{R}^d)$ supported on the unit ball, and defining

$$\psi_N(\xi) := \phi(\xi/2N) - \phi(\xi/N)$$

for $N \in 2^{\mathbb{Z}}$. By construction, ψ_N are localised at dyadic frequencies $|\xi| \sim N$ and form a partition of unity $\sum_N \psi_N \equiv 1$. We can write

$$m = \sum_{N\in 2^{\mathbb{Z}}} m\psi_N =: \sum_{N\in 2^{\mathbb{Z}}} m_N$$

with convergence pointwise and in the sense of tempered distributions. It follows from the Paley-Wiener theorem that the kernels $\widetilde{m_N}$ are smooth. Furthermore, they satisfy by the Fourier transform strong-type $(1,\infty)$ inequality

$$||x^\alpha \widetilde{m_N}||_{L^\infty_x} \lesssim ||\partial^\alpha_\xi m_N||_{L^1_\xi},$$
$$||x^\alpha \nabla\widetilde{m_N}||_{L^\infty_x} \lesssim ||\partial^\alpha_\xi (\xi m_N)||_{L^1_\xi}.$$

Using the product rule and the control on the derivatives of m, the right-hand sides are controlled pointwise by

$$\left|\partial^\alpha_\xi m_N\right| \lesssim \sum_{\beta+\gamma=\alpha} |\partial^\beta_\xi m|\,|\partial^\gamma_\xi \psi_N| \lesssim_\alpha \sum_{\beta+\gamma=\alpha} |\xi|^{-|\beta|} N^{-|\gamma|} |\partial^\gamma_\xi \psi(\xi/N)|,$$
$$\left|\partial^\alpha_\xi (\xi m_N)\right| \lesssim \sum_{\beta+\gamma=\alpha} |\partial^\beta_\xi (\xi m)|\,|\partial^\gamma_\xi \psi_N| \lesssim_\alpha \sum_{\beta+\gamma=\alpha} |\xi|^{1-|\beta|} N^{-|\gamma|} |\partial^\gamma_\xi \psi(\xi/N)|.$$

Therefore

$$||\partial_\xi^\alpha m_N||_{L^1_\xi} \lesssim \sum_{\beta+\gamma=\alpha} \int_{|\xi|\sim N} |\xi|^{-|\beta|} N^{-|\gamma|} d\xi \lesssim \sum_{\beta+\gamma=\alpha} N^{d-|\beta|-|\gamma|} \sim N^{d-|\alpha|},$$

$$||\partial_\xi^\alpha (\xi m_N)||_{L^1_\xi} \lesssim \sum_{\beta+\gamma=\alpha} \int_{|\xi|\sim N} |\xi|^{1-|\beta|} N^{-|\gamma|} d\xi \lesssim \sum_{\beta+\gamma=\alpha} N^{1+d-|\beta|-|\gamma|} \sim N^{1+d-|\alpha|}.$$

Collecting the inequalities above and taking $|\alpha| = 0$ and $|\alpha| = d+2$, we obtain

$$|\widetilde{m_N}(x)| \lesssim \min\{N^d, |x|^{-d-2}\}$$
$$|\nabla \widetilde{m_N}(x)| \lesssim \min\{N^{d+1}, N^{-2}|x|^{-d-2}\}.$$

These inequalities imply that the convergence $\check{m} = \sum_N \widetilde{m_N}$ holds in $C^1_{\text{loc}}(\mathbb{R}^d \setminus 0)$ and furthermore

$$|\check{m}(x)| \lesssim \sum_{N\in 2^{\mathbb{Z}}} |\widetilde{m_N}(x)| \lesssim \sum_{N\leq |x|^{-1}} N^d + \sum_{N>|x|^{-1}} N^{-2}|x|^{-d-2} \lesssim |x|^{-d},$$

$$|\nabla\check{m}(x)| \lesssim \sum_{N\in 2^{\mathbb{Z}}} |\nabla\widetilde{m_N}(x)| \lesssim \sum_{N\leq |x|^{-1}} N^{d+1} + \sum_{N>|x|^{-1}} N^{-1}|x|^{-d-2} \lesssim |x|^{-d-1}.$$

This proves the claim and thereby the theorem. $\square$

Remark. As an application to partial differential equations, define the RIESZ TRANSFORMS $R_j := iD_j/|D|$ as multipliers with symbols $m_j := i\xi_j/|\xi|$. It is easy to verify that m_j obey the homogeneous symbol estimates, so, writing $\partial_j\partial_j = -R_jR_k\Delta$, we obtain the *elliptic regularity estimate*

$$||\partial_j\partial_k f||_{L^p} \lesssim_{d,p} ||\Delta f||_{L^p}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$ and $1 < p < \infty$.

3. VECTOR-VALUED OPERATORS

We can generalise much of the preceding discussion concerning scalar-valued functions to functions taking values in Banach spaces. Let X and Y be Banach spaces and denote by $B(X,Y)$ the bounded linear maps from X to Y endowed with the usual operator norm. For $f : \mathbb{R}^d \to X$, a *vector-valued integral operator* takes the form

$$Tf(y) := \int_{\mathbb{R}^d} K(x,y) f(x)\, dx$$

where $K : \mathbb{R}^d \times \mathbb{R}^d \to B(X,Y)$ is the *kernel*.

3.1. **Vector-valued Calderon-Zygmund operators.**

A VECTOR-VALUED CALDERON-ZYGMUND KERNEL is a function $K : \mathbb{R}^d \times \mathbb{R}^d \to B(X,Y)$ satisfying the HORMANDER CONDITION:

$$\int_{|x-y|>2|y-z|} ||K(x,y) - K(x,z)||_{B(X,Y)}\, dx \lesssim 1 \qquad \text{uniformly for a.e. } y \neq z,$$

$$\int_{|x-y|>2|x-w|} ||K(x,y) - K(w,y)||_{B(X,Y)}\, dy \lesssim 1 \qquad \text{uniformly for a.e. } x \neq w.$$

We say that a bounded linear operator $T : L^2(\mathbb{R}^d; X) \to L^2(\mathbb{R}^d; Y)$ is a VECTOR-VALUED CALDERON-ZYGMUND OPERATOR if there exists a vector-valued Calderon-Zygmund kernel K for which

$$Tf(y) = \int_{\mathbb{R}^d} K(x,y) f(x)\, dx$$

whenever $f \in L^2(\mathbb{R}^d; X)$ is compactly supported and y lies outside the support of f. The Calderon-Zygmund decomposition and real interpolation continues to hold in the vector-valued setting, as the proofs depended only on the norm of the function. The only tool which does not freely carry over is duality $L^p(\mathbb{R}^d; Y)^* = L^{p'}(\mathbb{R}^d; Y^*)$, and so a little more work is needed to establish the strong-type (p,p) inequality for $2 < p < \infty$. We leave this as an exercise for the reader.

Theorem 19. *If $T : L^2(\mathbb{R}^d; X) \to L^2(\mathbb{R}^d; Y)$ is a vector-valued Calderon-Zygmund operator, then it satisfies the weak-type $(1,1)$ and strong-type (p,p) inequalities for $1 < p < \infty$.*

As an application, we can establish the *Littlewood-Paley inequality*. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ satisfy $0 \leq \phi \leq 1$ and

$$\phi(x) := \begin{cases} 1, & |x| \leq 1.4, \\ 0, & |x| > 1.42. \end{cases}$$

Define

$$\psi_N(\xi) := \phi(\xi/N) - \phi(2\xi/N)$$

for $N \in 2^{\mathbb{Z}}$. By construction, ψ_N are localised at dyadic frequencies $|\xi| \sim N$ and form a partition of unity $\sum_N \psi_N \equiv 1$. Given $f \in \mathcal{S}(\mathbb{R}^d)^*$, we define its Littlewood-Paley projection to frequency $|\xi| \sim N$ by

$$P_N f := \psi_N(D) f.$$

The name "projection" is a bit of a misnomer; the multipliers P_N fail to be true projections in the sense that by choosing smooth cutoffs in frequency space rather than sharp cutoffs, we have $P_N P_N \neq P_N$. Nevertheless, a slightly modified statement holds; define the fattened Littlewood-Paley projections to frequencies $|\xi| \sim N$ and their corresponding symbols by

$$\widetilde{P_N} := P_{\frac{N}{2}} + P_N + P_{2N}, \qquad \widetilde{\psi_N} := \psi_{\frac{N}{2}} + \psi_N + \psi_{2N}.$$

Since $\widetilde{\psi_N} \equiv 1$ on the support of ψ_N, it follows that $\widetilde{P_N} P_N = P_N$.

Theorem 20 (Littlewood-Paley inequality). *Let $1 < p < \infty$ and $f \in L^p(\mathbb{R}^d)$, define the* Littlewood-Paley square function *by*

$$Sf := \left(\sum_{N \in 2^{\mathbb{Z}}} |P_N f|^2 \right)^{1/2}.$$

Then

$$||Sf||_{L^p} \sim ||f||_{L^p}.$$

Proof. The inequality $||Sf||_{L^p} \lesssim ||f||_{L^p}$ is equivalent to establishing that the operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d; \ell^2)$ defined by

$$Tf := (P_N f)_{N \in 2^{\mathbb{Z}}}$$

is a Calderon-Zygmund operator. The corresponding kernel $K : \mathbb{R}^d \to B(\mathbb{C}, \ell^2)$ is

$$K(x) = (\widecheck{\psi_N}(x))_{N \in 2^{\mathbb{Z}}}.$$

Since $\sum_N \psi_N \equiv 1$, it follows from Plancharel's theorem that T is strong-type $(2, 2)$. The symbols ψ_N obey the estimates from the Hormander-Mikhlin multiplier theorem, so following the proof we conclude K satisfies the Hormander condition and therefore T is a Calderon-Zygmund operator.

For the reverse inequality, we argue by duality, remarking that $\widetilde{P}_N$ is self-adjoint and the argument above continues to hold replacing the square function Sf with the fattened square function $\widetilde{S}f$. The convergence $f = \sum_N P_N f = \sum_N \widetilde{P_N} P_N f$ holds in $L^p(\mathbb{R}^d)$, so by duality, Cauchy-Schwartz in N, and Holder's inequality in x,

$$||f||_{L^p} = \sup_{||g||_{L^{p'}} \leq 1} \sum_N \langle \widetilde{P}_N P_N f, g \rangle = \sup_{||g||_{L^{p'}} \leq 1} \sum_N \langle P_N f, \widetilde{P}_N g \rangle \leq \sup_{||g||_{L^{p'}} \leq 1} ||Sf||_{L^p} ||\widetilde{S} g||_{L^{p'}} \lesssim ||Sf||_{L^p}.$$

This completes the proof. $\square$