DETERMINING OPTIMAL TEST FUNCTIONS FOR 2-LEVEL DENSITIES

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Abstract. Katz and Sarnak conjectured a correspondence between the n-level density statistics of zeros from families of L-functions with eigenvalues from random matrix ensembles, and in many cases the sums of smooth test functions, whose Fourier transforms are finitely supported over scaled zeros in a family, converge to an integral of the test function against a density $W_{n,G}$ depending on the symmetry G of the family (unitary, symplectic or orthogonal). This integral bounds the average order of vanishing at the central point of the corresponding family of L-functions.

We can obtain better estimates on this vanishing in two ways. The first is to do more number theory, and prove results for larger n and greater support; the second is to do functional analysis and obtain better test functions to minimize the resulting integrals. We pursue the latter here when $n = 2$, minimizing

$$\frac{1}{\Phi(0,0)} \int_{\mathbb{R}^2} W_{2,G}(x,y)\Phi(x,y)dxdy$$

over test functions $\Phi : \mathbb{R}^2 \to [0,\infty)$ with compactly supported Fourier transform. We study a restricted version of this optimization problem, imposing that our test functions take the form $\phi(x)\psi(y)$ for some fixed admissible $\psi(y)$ and $\operatorname{supp}\widehat{\phi} \subseteq [-1,1]$. Extending results from the 1-level case, namely the functional analytic arguments of Iwaniec, Luo and Sarnak and the differential equations method introduced by Freeman and Miller, we explicitly solve for the optimal ϕ for appropriately chosen fixed test function ψ.

Contents

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1. Introduction

1.1. n-level density. Let $\mathcal{F}$ be a family of cuspidal newforms, and to each $f \in \mathcal{F}$ we associate the L-function

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_{n,f}}{n^s}.$$

We assume that the Riemann hypothesis holds for each $L(s, f)$ and for all Dirichlet L-functions, that is, we can enumerate the non-trivial zeros of $L(s, f)$ by

$$\rho_f^{(j)} = \frac{1}{2} + i\gamma_f^{(j)}$$

for $\gamma_f^{(j)} \in \mathbb{R}$ increasingly ordered and centered about zero. By arguments due to Riemann, the number of zeros with $|\gamma_f^{(j)}|$ bounded by an absolute large constant is of order $\log c_f$ for some constant $c_f > 1$ known as the *analytic conductor*. It is of interest to study the statistics of these "low-lying" zeros of $L(s, f)$, and to this end [ILS00] introduced the *n-level density*,

$$D_n(f; \Phi) := \sum_{\substack{j_1,\ldots,j_n \\ j_i \neq \pm j_k}} \Phi\left(\frac{\log c_f}{2\pi}\gamma_f^{(j_1)}, \ldots, \frac{\log c_f}{2\pi}\gamma_f^{(j_n)}\right) \tag{1.1}$$

for *test functions* $\Phi : \mathbb{R}^n \to \mathbb{R}$, which we take to be non-negative even Schwartz class functions with compactly supported Fourier transform and $\Phi(0) > 0$. In practice the sum (1.1) is impossible to evaluate asymptotically, since by choice of Φ it essentially captures only a bounded number of zeros. Instead we study averages over finite subfamilies $\mathcal{F}(Q) := \{f \in \mathcal{F} : c_f \leq Q\}$, namely

$$\mathbb{E}(D_n(f; \Phi), Q) := \frac{1}{\#\mathcal{F}(Q)} \sum_{f \in \mathcal{F}(Q)} D(f; \Phi). \tag{1.2}$$

If $\mathcal{F}$ is a complete family of cuspidal newforms in a spectral sense, there exists a distribution $W_{n,\mathcal{F}}$ such that

$$\lim_{Q\to\infty} \mathbb{E}(D_n(f; \Phi), Q) = \frac{1}{\Phi(0, \ldots, 0)} \int_{\mathbb{R}^n} \Phi(x_1, \ldots, x_n) W_{n,\mathcal{F}}(x_1, \ldots, x_n) dx_1 \cdots dx_n. \tag{1.3}$$

Katz and Sarnak conjectured that $W_{n,\mathcal{F}}$ depends on a corresponding symmetry group $G(\mathcal{F})$, the scaling limit of one of the classical compact groups, so for the remainder we shall write $W_{n,G}$ in place of $W_{n,\mathcal{F}}$.

Define

$$K(y) := \frac{\sin(\pi y)}{\pi y}, \qquad K_\varepsilon(x, y) := K(x - y) + \varepsilon K(x + y)$$

for $\varepsilon = 0, \pm 1$. The corresponding n-level densities, as referenced in [HM] and derived in [KS99], have the following distinct closed form determinant expansions,

$$W_{n,\mathsf{SO(even)}}(x) = \det\left(K_1(x_i, x_j)\right)_{i,j \le n}, \tag{1.4}$$

$$W_{n,\mathsf{SO(odd)}}(x) = \det\left(K_{-1}(x_i, x_j)\right)_{i,j,\le n} + \sum_{k=1}^{n} \delta(x_k) \det\left(K_{-1}(x_i, x_j)\right)_{i,j,\neq k}, \tag{1.5}$$

$$W_{n,\mathsf{O}}(x) = \frac{1}{2} W_{n,\mathsf{SO(even)}}(x) + \frac{1}{2} W_{n,\mathsf{SO(odd)}}(x), \tag{1.6}$$

$$W_{n,\mathsf{U}}(x) = \det\left(K_0(x_i, x_j)\right)_{i,j,\le n}, \tag{1.7}$$

$$W_{n,\mathsf{Sp}}(x) = \det\left(K_{-1}(x_i, x_j)\right)_{i,j,\le n}. \tag{1.8}$$

1.2. **Main result.** It is discussed in [FM15] and [ILS00] that the 1-level density gives estimates on the average order of vanishing of L-functions at the central point in a family. Here we deal with the 2-level densities, which has the advantage of giving better estimates on higher vanishing at the central point. Writing r_f for the order of the zero of $L(s, f)$ at $s = 1/2$ and $\Pr(m) := \Pr(f \in \mathcal{F} : r_f = m)$, we have

$$\sum_{m=0}^{\infty} m(m-1) \Pr(m) \ \le \ \frac{1}{\Phi(0,0)} \int_{\mathbb{R}^2} \Phi(x, y) W_{2,\mathcal{F}}(x, y) dx dy. \tag{1.9}$$

It is therefore of interest to choose Φ optimally to obtain the best bound on the left-hand side of (1.9). Rather than minimizing over test functions of two variables, we instead fix a single variable test function ψ and, imposing the restriction $\Phi(x, y) = \phi(x)\psi(y)$, minimize over single variable test functions ϕ with $\operatorname{supp} \widehat{\phi} \subseteq [-1, 1]$. For our fixed ψ, we consider

$$\psi(y) = \left(\frac{\sin(\pi y)}{\pi y}\right)^2. \tag{1.10}$$

Iwaniec, Luo and Sarnak [ILS00] showed that the optimal test functions with Fourier transforms supported in $[-1, 1]$ for the 1-level densities are exactly scalar multiples of ψ, making it the natural choice of fixed test function. Our main result is to solve this restricted optimization problem for ψ as defined above.

Theorem 1. *Let ψ be as in (1.10). For each of the classical compact groups $G = \mathsf{SO(even)}, \mathsf{SO(odd)}, \mathsf{U}, \mathsf{O}$, and Sp, there exists an optimal square integrable function $g_{G,\psi} \in L^2[-1/2, 1/2]$ and constant $c_{G,\psi}$ such that*

$$\frac{c_{G,\psi}}{\int_{-1/2}^{1/2} g(x) dx} = \inf_{\phi} \frac{1}{\phi(0)\psi(0)} \int_{\mathbb{R}^2} \phi(x)\psi(y) W_{2,G}(x, y) dx dy, \tag{1.11}$$

where the infimum is taken over test functions ϕ with Fourier transform satisfying $\operatorname{supp} \widehat{\phi} \subseteq [-1, 1]$. The constants and optimal square integrable functions are given by

$$c_{G,\psi} = \begin{cases} \frac{1}{2}, & \text{if } G = \mathsf{Sp}, \\ 1, & \text{if } G = \mathsf{U}, \\ \frac{3}{2}, & \text{if } G = \mathsf{SO(even)}, \mathsf{SO(odd)}, \mathsf{O}, \end{cases} \tag{1.12}$$

and

$$g_{\mathsf{SO(even)},\psi}(x) = \frac{216\cos(4x/\sqrt{3}) + 36\sqrt{3}\sin(2/\sqrt{3})}{162\cos(2/\sqrt{3}) - 5\sqrt{3}\sin(2/\sqrt{3})}, \tag{1.13}$$

$$g_{\mathsf{SO(odd)},\psi}(x) = \frac{8\cos(4x/\sqrt{3}) + 12\sqrt{3}\sin(2/\sqrt{3})}{11\sqrt{3}\sin(2/\sqrt{3}) + 2\cos(2/\sqrt{3})}, \tag{1.14}$$

$$g_{\mathsf{U},\psi}(x) = \frac{6\cos(2x) + 6\sin(1)}{3\cos(1) + 4\sin(1)}, \tag{1.15}$$

$$g_{\mathsf{O},\psi}(x) = \frac{36\cos(4x/\sqrt{3}) + 18\sqrt{3}\sin(2/\sqrt{3})}{18\cos(2/\sqrt{3}) + 13\sqrt{3}\sin(2/\sqrt{3})}, \tag{1.16}$$

$$g_{\mathsf{Sp},\psi}(x) = \frac{8\cos(4x) + 12\sin(2)}{2\cos(2) + 3\sin(2)}. \tag{1.17}$$

Additionally, the optimal test function $\phi_{G,\psi}$ realizing the infimum in (1.11) satisfies $\widehat{\phi_{G,\psi}} = g_{G,\psi} * g_{G,\psi}$.

The test function ψ is used in Section 1 of [ILS00] to obtain naive bounds on the average order of vanishing. Similarly, we can compute naive bounds for the 2-level densities by taking $\Phi(x,y) = \psi(x)\psi(y)$. Table 1 shows that the bounds derived from Theorem 1 significantly improve the naive bounds.

Family	**Naive bounds**	**Closed form of (1.11)**
$\mathsf{SO(even)}$	$\frac{5}{12} \approx 0.416666$	$\frac{1}{96}\left(54\sqrt{3}\cot\left(\frac{2}{\sqrt{3}}\right) - 5\right) \approx 0.378448$
$\mathsf{SO(odd)}$	$\frac{13}{12} \approx 1.083333$	$\frac{1}{32}\left(33 + 2\sqrt{3}\cot\left(\frac{2}{\sqrt{3}}\right)\right) \approx 1.07909$
O	$\frac{3}{4} \approx 0.75$	$\frac{1}{24}\left(13 + 6\sqrt{3}\cot\left(\frac{2}{\sqrt{3}}\right)\right) \approx 0.733014$
U	$\frac{1}{2} \approx 0.5$	$\frac{1}{12}(4 + 3\cot(1)) \approx 0.493856$
Sp	$\frac{1}{12} \approx 0.083333$	$\frac{1}{32}(3 + 2\cot(2)) \approx 0.0651464$

TABLE 1. Comparing naive bounds taking $\phi = \psi$ with the optimal value from (1.11) for each of the classical compact groups.

2. PROOF OF THEOREM 1

2.1. **Functional analytic setup.** Prior literature on the optimization problem, such as [FM15] and [Fre17], dealt with the 1-level densities following the functional analytic approach outlined in Appendix A of [ILS00]. We want to impose restrictions so that such an approach is amenable to the 2-level density optimization problem. To that end, we consider the optimization over test functions of the form $\Phi(x,y) = \phi(x)\psi(y)$ for fixed admissible $\psi(y)$ with $\operatorname{supp}\widehat{\psi} \subseteq [-1,1]$. This reduces the problem to one analogous to the 1-level density, where we are interested in finding an optimal one-variable test function. Explicitly, we want to compute

$$\inf_{\phi} \frac{1}{\phi(0)\psi(0)} \int_{\mathbb{R}^2} \phi(x)\psi(y)W_{2,G}(x,y)dxdy \tag{2.1}$$

where the infimum is taken over test functions $\phi : \mathbb{R} \to \mathbb{R}$ with $\operatorname{supp}\widehat{\phi} \subseteq [-1,1]$. Attacking the optimization problem via the Fourier transform is more promising than a direct approach. On the transform side, assumptions on the support reduce an integration over the entire plane

$\mathbb{R}^2$ to an integration over the square $[-1,1]\times[-1,1]$, and the 2-level densities themselves are unwieldy to work with, while their Fourier transforms are sums of linear polynomials in $|x|$ and Dirac delta functions. Moreover, Gallagher [Gal85] noted that a correspondence exists between admissible test functions ϕ and square-integrable functions. Namely, it follows by the Ahiezer and Paley-Wiener theorems that ϕ is a test function with $\operatorname{supp}\widehat{\phi}\subseteq[-1,1]$ if and only if there exists $f\in L^2[-1/2,1/2]$ such that

$$\widehat{\phi}(x)=(f*\breve{f})(x), \tag{2.2}$$

where

$$\breve{f}(x)=\overline{f(-x)}. \tag{2.3}$$

Thus rather than minimizing a functional over test functions, we can instead view the problem as minimizing a functional $\tilde{R}_{G,\psi}$ on a subset of $L^2[-1/2,1/2]$, defined by

$$\tilde{R}_{G,\psi}(f):=\frac{1}{\phi(0)\psi(0)}\int_{\mathbb{R}^2}\phi(x)\psi(y)W_{2,G}(x,y)dxdy. \tag{2.4}$$

This perspective gives access to more functional analytic tools, namely Fredholm theory. Motivated by our earlier remarks on the Fourier transform, we apply the Plancharel theorem to write

$$\tilde{R}_{G,\psi}(f)=\frac{1}{\phi(0)}\int_{-1}^{1}\widehat{\phi}(x)\tilde{V}_{G,\psi}(x)dx, \tag{2.5}$$

where we have a weight function $\tilde{V}_{G,\psi}$ given by

$$\tilde{V}_{G,\psi}(x)=\frac{1}{\psi(0)}\int_{-1}^{1}\widehat{\psi}(y)\widehat{W_{2,G}}(y)dy. \tag{2.6}$$

In the 1-level case, the role of the weight function is played by the Fourier transforms of the 1-level distributions (1.4) - (1.8), which take the form $\delta+m_G$. Analogously, following calculations due to Hughes and Miller [HM], for each of the classical compact groups the weight function (2.6) takes the form

$$\tilde{V}_{G,\psi}(x)=c_{G,\psi}\delta(x)+\tilde{m}_{G,\psi}(x)\mathbb{1}_{[-1,1]}(x) \tag{2.7}$$

for constants $c_{G,\psi}\in\mathbb{R}$ and kernel $\tilde{m}_{G,\psi}\in L^2[-1,1]$ depending on our choice of initial test function ψ and the classical compact group G, namely

$$c_{G,\psi}=\frac{\widehat{\psi}(0)}{\psi(0)}+\begin{cases}-\frac{1}{2}, & \text{if } G=\mathsf{Sp},\\ 0, & \text{if } G=\mathsf{U},\\ \frac{1}{2}, & \text{if } G=\mathsf{SO(even)},\mathsf{SO(odd)},\mathsf{O},\end{cases} \tag{2.8}$$

and

$$\tilde{m}_{\mathsf{SO(even)},\psi}(x) = \frac{1}{2}\left(\frac{\widehat{\psi}(0)}{\psi(0)} + \frac{1}{2}\right) + 2\frac{\widehat{\psi}(x)}{\psi(0)}(|x|-1) - \frac{\int_{|x|-1}^{1-|x|}\widehat{\psi}(y)dy}{\psi(0)}, \tag{2.9}$$

$$\tilde{m}_{\mathsf{SO(odd)},\psi}(x) = \frac{1}{2}\left(\frac{\widehat{\psi}(0)}{\psi(0)} - \frac{3}{2}\right) + 2\frac{\widehat{\psi}(x)}{\psi(0)}(|x|-1) + \frac{\int_{|x|-1}^{1-|x|}\widehat{\psi}(y)dy}{\psi(0)}, \tag{2.10}$$

$$\tilde{m}_{\mathsf{O},\Psi}(x) = \frac{1}{2}\left(\frac{\widehat{\psi}(0)}{\psi(0)} - \frac{1}{2}\right) + 2\frac{\widehat{\psi}(x)}{\psi(0)}(|x|-1), \tag{2.11}$$

$$\tilde{m}_{\mathsf{U},\psi}(x) = \frac{\widehat{\psi}(x)}{\psi(0)}(|x|-1), \tag{2.12}$$

$$\tilde{m}_{\mathsf{Sp},\psi}(x) = -\frac{1}{2}\left(\frac{\widehat{\psi}(0)}{\psi(0)} - \frac{1}{2}\right) + 2\frac{\widehat{\psi}(x)}{\psi(0)}(|x|-1) + \frac{\int_{|x|-1}^{1-|x|}\widehat{\psi}(y)dy}{\psi(0)}. \tag{2.13}$$

To complete the analogy with the 1-level case, we consider the normalized functional, weight function and kernel:

$$R_{G,\psi}(f) := \frac{\tilde{R}_{G,\psi}(f)}{c_{G,\psi}}, \tag{2.14}$$

$$V_{G,\psi}(x) := \frac{\tilde{V}_{G,\psi}(x)}{c_{G,\psi}}, \tag{2.15}$$

$$m_{G,\psi}(x) := \frac{\tilde{m}_{G,\psi}(x)}{c_{G,\psi}}. \tag{2.16}$$

Consider the compact self-adjoint operator $K_{G,\psi} : L^2[-1/2,1/2] \to L^2[-1/2,1/2]$ (see Section 6.6 of [RS96]) defined by

$$(K_{G,\psi}f)(x) := \int_{-1/2}^{1/2} m_{G,\psi}(x-y)f(y)dy. \tag{2.17}$$

Then the optimization problem (2.1) is equivalent to the minimization of the quadratic form

$$R_{G,\psi}(f) = \frac{\langle (I+K_{G,\psi})f, f\rangle}{|\langle f,1\rangle|^2} \tag{2.18}$$

subject to the linear constraint $\langle f,1\rangle \neq 0$. We know $R_{G,\psi} \geq 0$, which implies $I + K_{G,\psi}$ is positive definite. If $I + K_{G,\psi}$ is non-singular, that is, -1 is not an eigenvalue of $K_{G,\psi}$, then it follows from the Fredholm alternative that there exists a unique $g_{G,\psi} \in L^2[-1/2,1/2]$ satisfying the integral equation

$$(I+K_{G,\psi})(g_{G,\psi})(x) = g_{G,\psi}(x) + \int_{-1/2}^{1/2} m_{G,\psi}(x-y)g_{G,\psi}(y)dy = 1 \tag{2.19}$$

for $x \in [-1/2,1/2]$, and by Proposition A.1 of [ILS00],

$$\inf_f R_{G,\psi}(f) = \frac{1}{\langle 1, g_{G,\psi}\rangle}. \tag{2.20}$$

Equation (1.11) in Theorem 1 follows directly from (2.4), (2.14) and (2.20). It remains to check the following lemma:

Lemma 2. *$I + K_{G,\psi}$ is non-singular, that is, -1 is not an eigenvalue of $K_{G,\psi}$.*

Proof. Let $f \in L^2[-1/2, 1/2]$ such that $K_{G,\psi} f = -f$, i.e.,

$$0 = f(x) + \int_{-1/2}^{1/2} m_{G,\psi}(x-y) f(y) dy.$$

This is a Fredholm equation of the second kind, and the unique continuous solution is given by the corresponding Liouville-Neumann series, which, in this case, is the constant zero function. On the other hand, $m_{G,\psi}$ is uniformly continuous on $[-1,1]$, so for fixed $\varepsilon > 0$, there exists $\delta > 0$ witnessing the uniform continuity. Then

$$|f(x+h) - f(x)| \leq \int_{-1/2}^{1/2} |m_{G,\psi}(x+h-y) - m_{G,\psi}(x-y)||f(y)|dy \leq \varepsilon ||f||_{L^2}$$

whenever $|h| < \delta$. It follows that f is continuous, so by uniqueness, $f = 0$. □

Remark. A similar argument shows that the solution $g_{G,\psi}$ to Equation (2.19) is continuous.

2.2. **Solving a Fredholm integral equation with quadratic kernel.** Let ψ be as in (1.10), which has Fourier transform

$$\widehat{\psi}(x) = (\mathbb{1}_{[-1/2,1/2]} * \mathbb{1}_{[-1/2,1/2]})(x) = (1 - |x|)\mathbb{1}_{[-1,1]}(x). \tag{2.21}$$

Not only is this the natural choice of fixed test function ψ, it also lends to an elegant derivation of the corresponding optimal $g_{G,\psi}$, as the kernels $m_{G,\psi}$ take the form of quadratic polynomials in $|x|$ on the interval $[-1,1]$. Namely,

$$m_{\mathsf{SO(even)},\psi}(x) = -\frac{3}{2} + \frac{8}{3}|x| - \frac{2}{3}x^2, \tag{2.22}$$

$$m_{\mathsf{SO(odd)},\psi}(x) = -\frac{5}{6} + \frac{8}{3}|x| - 2x^2, \tag{2.23}$$

$$m_{\mathsf{O},\psi}(x) = -\frac{7}{6} + \frac{8}{3}|x| - \frac{4}{3}2x^2, \tag{2.24}$$

$$m_{\mathsf{U},\psi}(x) = -1 + 2|x| - x^2, \tag{2.25}$$

$$m_{\mathsf{Sp},\psi}(x) = -\frac{5}{2} + 8|x| - 6x^2, \tag{2.26}$$

Prior experience with the analogous 1-level problem in [ILS00], [Fre17] and [FM15] suggests that $g_{G,\psi}$ takes the form of an even trigonometric polynomial. Indeed, not only does this hold in Theorem 1, this holds in generality for Fredholm integral equation where the kernel is an even quadratic polynomial in $|x|$, as Lemma 2 relies only on uniform continuity of the kernel.

Theorem 3. *Let $a, b, c \in \mathbb{R}$ with $b \geq 0$. The following Fredholm integral equation with quadratic kernel,*

$$1 = g(x) + \int_{-1/2}^{1/2} (a + b|x-y| + c|x-y|^2) g(y) dy, \tag{2.27}$$

admits the unique continuous solution

$$g(x) = \frac{6b^{3/2}(b+c)\cos(\sqrt{2b}x) - 6\sqrt{2}bc\sin(\sqrt{b/2})}{6\sqrt{b}(b+c)^2\cos(\sqrt{b/2}) + \sqrt{2}(6ab^2 + 3b^3 + 3b^2c + bc(c-12) - 6c^2)\sin(\sqrt{b/2})}. \tag{2.28}$$

Theorem 1 follows as an immediate corollary. We prove Theorem 3 following a differential equations argument due to Freeman and Miller. Observe that the left-hand side of Equation (2.27) is constant, so derivatives of the expression on the right vanish. Assuming g is sufficiently smooth, we differentiate under the integral sign to obtain

$$\begin{aligned}\frac{d}{dx}\int_{-1/2}^{1/2}|x-y|g(y)dy &= \frac{d}{dx}\left(\int_{-1/2}^{x}(x-y)g(y)dy+\int_{x}^{1/2}(y-x)g(y)dy\right)\\ &= \int_{-1/2}^{x}g(y)dy-\int_{x}^{1/2}g(y)dy,\\ \frac{d^2}{dx^2}\int_{-1/2}^{1/2}|x-y|g(y)dy &= 2g(x),\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dx}\int_{-1/2}^{1/2}(x-y)^2g(y)dy &= \int_{-1/2}^{1/2}(2x-2y)g(y)dy,\\ \frac{d^2}{dx^2}\int_{-1/2}^{1/2}(x-y)^2g(y)dy &= 2\int_{-1/2}^{1/2}g(y)dy.\end{aligned}$$

We thereby obtain the corresponding system of linear homogeneous differential equations,

$$1 = g(0)+a\int_{-1/2}^{1/2}|y|g(y)dy, \tag{2.29}$$

$$0 = g''(x)+2bg(x)+2c\int_{-1/2}^{1/2}g(y)dy, \tag{2.30}$$

$$0 = g'''(x)+2bg'(x). \tag{2.31}$$

Equation (2.29) is exactly (2.27) taking $x = 0$. We obtain equations (2.30) and (2.31) by differentiating (2.27) under the integral sign twice and thrice respectively. Assuming g is even, solutions to equation (2.31) take the form $g(x) = A\cos(\sqrt{2b}x)+C$ for some constants $A, C \in \mathbb{R}$. Substituting into equation (2.30) reduces these two degrees of freedom to one,

$$0 = g''(x)+2bg(x)+2c\int_{-1/2}^{1/2}g(y)dy = 2C(b+c)+\frac{4Ac}{\sqrt{2b}}\sin\left(\frac{\sqrt{2b}}{2}\right). \tag{2.32}$$

This shows that

$$g(x) = A\cos(\sqrt{2b}x)-A\frac{2c}{b+c}\frac{\sin\left(\sqrt{b/2}\right)}{\sqrt{2b}}. \tag{2.33}$$

Substituting the above into (2.29) allows us to solve for A explicitly, which completes the derivation of (2.28). □

Remark. The same differential equations method can be applied for kernels which take the form of of higher order polynomials in $|x|$ on $[-1,1]$, that is, $m(x) = p(x)$ for some degree n polynomial p. Future approaches to the optimization problem may want to consider optimizing over fixed test functions ψ which produce kernels of such form.

3. Iteration

One can improve the bounds found in Table 1 by choosing $\phi_{G,\psi}$ as our new fixed test function and optimizing accordingly. As $\widehat{\phi_{G,\psi}}$ takes the form of a piecewise trigonometric polynomial for each of the classical compact groups, the methods used in Section 2.2 are not applicable. We instead appeal to the standard approach to solving Fredholm integral equations by method of iteration. Suppose an even continuous kernel $m : [-1,1] \to \mathbb{R}$ satisfies

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}|m(x-y)|^2dxdy < 1. \tag{3.1}$$

Define a self-adjoint compact operator $K : L^2[-1/2, 1/2] \to L^2[-1/2, 1/2]$ by

$$(Kf)(x) := \int_{-1/2}^{1/2} m(x-y)f(y)dy. \tag{3.2}$$

It follows from the Cauchy-Schwarz inequality that the operator norm satisfies $||K||_{L^2 \to L^2} < 1$, that is, K is a contraction mapping, since

$$||Kg||_2^2 = \int_{-1/2}^{1/2} \left(\int_{-1/2}^{1/2} m(x-y)g(y)dy \right)^2 dx \leq ||g||_2^2 \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} |m(x-y)|^2 dxdy < 1. \tag{3.3}$$

Thus by the Weierstrass M-test, the series

$$g(x) := \sum_{n=0}^{\infty} (-1)^n K^n(1)(x) \tag{3.4}$$

converges absolutely and uniformly on the interval $[-1/2, 1/2]$. Moreover, it is the unique continuous solution to the Fredholm integral equation $(I+K)g = 1$. Since the series converges absolutely, we can integrate term by term to obtain the corresponding minimum value,

$$\frac{c_{G,\phi}}{\langle 1, g \rangle} = c_{G,\phi} \left(\sum_{n=0}^{\infty} (-1)^n \int_{-1/2}^{1/2} K^n(1)(x)dx \right)^{-1}. \tag{3.5}$$

Unfortunately, this method of deriving new bounds is computationally intensive as we need to compute n-dimensional integrals of unwieldy expressions. Additionally, depending on the rate of convergence we may need to compute a large number of terms to obtain meaningful degrees of accuracy. For the purposes of this paper we focus on the unitary group, where these challenges can be avoided.

For brevity, denote $\phi := \phi_{\mathrm{U},\psi}$. In this case we know the series converges, as

$$\int_{-1/2}^{1/2} \int_{-1/2}^{1/2} |m_{\mathrm{U},\phi}(x-y)|^2 dxdy = \frac{2\sin^2(1)(128 - 110\cos(2) - 37\sin(2))}{3(-8 + 6\cos(2) - \sin(2))^2} < 1. \tag{3.6}$$

Moreover, $\widehat{\phi}$ is non-negative, so it follows that $(-1)^n K^n_{\mathrm{U},\phi}(1)$ is non-negative. We can therefore truncate the series in (3.5) to obtain a legitimate upper bound, as the terms are non-negative. Summing five terms gives the bound

$$\inf_{\Phi} \frac{1}{\Phi(0,0)} \int_{\mathbb{R}^2} \Phi(x,y) W_{2,\mathrm{U}}(x,y) dxdy \leq c_{G,\psi} \left(\sum_{n=0}^{5} (-1)^n \int_{-1/2}^{1/2} K^n(1)(x)dx \right)^{-1} \approx 0.4888, \tag{3.7}$$

a small improvement on our previous bound in Table 1.

4. Conclusion

Conjecture. *Iterating, that is, fix ϕ_1 solve for ϕ_2, fix ϕ_2 solve for ϕ_3, etc., converges to the global/local minimum.*

Comparing with brute force might give some insight whether this conjecture is true or not. Strong version is global minimum, weak version is local minimum.

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