Determining optimal test functions for 2-level densities

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Abstract

It is of great interest to study the order of vanishing at the central point of L-functions, such as the famous Riemann zeta function and its generalizations. For example, the Birch and Swinnerton-Dyer conjecture relates the order of vanishing for an elliptic curve L-function to the rank of its group of rational solutions. Katz and Sarnak conjectured a correspondence between the n-level density statistics of zeros from families of L-functions (which essentially involves n-tuples of zeros) with eigenvalues from random matrix ensembles, and in many cases that sums of smooth test functions, whose Fourier transforms are finitely supported over scaled zeros in a family, converge to an integral of the test function against a density $W_{n,G}$ depending on the symmetry G of the family (unitary, symplectic or orthogonal). This integral bounds the average order of vanishing at the central point of the corresponding family of L-functions.

We can obtain better estimates on this vanishing in two ways. The first is to do more number theory, and prove results for larger n and greater support; the second is to do functional analysis and obtain better test functions to minimize the resulting integrals. We pursue the latter here when $n = 2$, minimizing

$$\frac{1}{\Phi(0,0)} \int_{\mathbb{R}^2} W_{2,G}(x,y)\Phi(x,y)dxdy$$

over test functions $\Phi : \mathbb{R}^2 \to [0,\infty)$ with compactly supported Fourier transform. We study a restricted version of this optimization problem, imposing that our test functions take the form $\phi(x)\psi(y)$ for some fixed admissible $\psi(y)$ and $\operatorname{supp}\widehat{\phi} \subseteq [-1,1]$, thereby reducing the problem to one analogous to the 1-level density case in optimizing over one-variable test functions $\phi(x)$. Continuing with the analogy, Devlin and Miller extended the functional analytic arguments of Iwaniec, Luo and Sarnak, converting the restricted optimization problem to finding the unique $g \in L^2[-1/2,1/2]$ satisfying a Fredholm integral equation of the second kind. From here we take two approaches. First, showing g satisfies a homogeneous linear system of differential equations, a method introduced by Freeman and Miller, from which we can derive a closed form expression for g. Second, iterating to obtain a series representation of g and truncating to compute explicit estimates on the minimum value. We conclude by discussing improvements to previous bounds.

Contents

1 Prepatory material

Denote by G any one of the classical compact matrix groups, namely the orthogonal group O, the even special orthogonal group $\mathsf{SO(even)}$, the odd special orthogonal group $\mathsf{SO(odd)}$, and the symplectic group Sp. The 1-level densities of each of these groups are known to be

$$\begin{aligned} W_{\mathsf{O}}(x) &= 1 + \frac{1}{2}\delta_0(x), \\ W_{\mathsf{SO(even)}}(x) &= 1 + \frac{\sin 2\pi x}{2\pi x}, \\ W_{\mathsf{SO(odd)}}(x) &= 1 - \frac{\sin 2\pi x}{2\pi x} + \delta_0(x), \\ W_{\mathsf{Sp}}(x) &= 1 - \frac{\sin 2\pi x}{2\pi x}. \end{aligned}$$

Here $\delta_0(x)$ is the Dirac delta centered at zero. The corresponding Fourier transforms take the form

$$\widehat{W}_G(\xi) = \delta_0(\xi) + m_G(\xi)$$

where

$$\begin{aligned} m_{\mathsf{O}}(\xi) &= \frac{1}{2}, \\ m_{\mathsf{SO(even)}}(\xi) &= \frac{1}{2}\mathbb{1}_{[-1,1]}(\xi), \\ m_{\mathsf{SO(odd)}}(\xi) &= 1 - \frac{1}{2}\mathbb{1}_{[-1,1]}(\xi), \\ m_{\mathsf{Sp}}(\xi) &= -\frac{1}{2}\mathbb{1}_{[-1,1]}(\xi). \end{aligned}$$

Define

$$\mathfrak{I}_G(\sigma) = \inf_{\phi} \frac{1}{\phi(0)} \int_{\mathbb{R}} \phi(x) W_G(x) dx$$

where the infimum is taken over non-negative Schwarz functions $\phi : \mathbb{R} \to [0, \infty)$ with compactly supported Fourier transform satisfying $\mathrm{supp}(\widehat{\phi}) \subseteq [-2\sigma, 2\sigma]$. We refer to such ϕ as TEST FUNCTIONS. We are interested in studying the following:

a. Finding test functions witnessing the infimum $\mathfrak{I}_G(\sigma)$ for fixed σ and group G, i.e. finding ϕ satisfying $\mathrm{supp}(\widehat{\phi}) \subseteq [-2\sigma, 2\sigma]$ and

$$\mathfrak{I}_G(\sigma) = \frac{1}{\phi(0)} \int_{\mathbb{R}} \phi(x) W_G(x) dx$$

b. Computing the value of $\mathfrak{I}_G(\sigma)$.

c. Analyzing the behavior of $\mathfrak{I}_G$ as a real function on $(0, \infty)$. In particular, Freeman conjectured that $\mathfrak{I}_G$ is continuous and real analytic except at integers and half-integers.

Fix $\sigma > 0$ and one of our aforementioned compact groups G; for brevity we suppress the subscripts $W := W_G$ and $m := m_G$. Moreover, whenever we define a new object in terms of W and m, there is an implicit subscript G to denote the dependence on the group.

In the literature, no one tries to directly find the optimal test function ϕ, instead appealing to functional analytic arguments. For example, it follows from a theorem of Ahiezer and the Paley-Wiener theorem that the optimal test function admits the form $\phi(z) = |h(z)|^2$, where $h : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type 1 and $h \in L^2(\mathbb{R})$. That is, its Fourier transform admits the form

$$\widehat{\phi}(\xi) = (g * g^{\smile})(\xi)$$

where $\operatorname{supp} g \subseteq [-\sigma, \sigma]$ and $g \in L^2[-\sigma, \sigma]$ and $g^{\smile}(\xi) = \overline{g(-\xi)}$. Notice that the Fourier transforms of the distributions W are significantly easier to work with as they are step functions, so naturally we appeal to the Plancharel theorem to write

$$\frac{1}{\phi(0)} \int_{\mathbb{R}} \phi(x) W(x) dx = \frac{1}{\int_{\mathbb{R}} \widehat{\phi}(\xi) d\xi} \int_{\mathbb{R}} \widehat{\phi}(\xi) \widehat{W}(\xi) d\xi.$$

Rewriting $\widehat{\phi}$ in terms of $g \in L^2[-\sigma, \sigma]$ converts the problem to an equivalent optimization over the Hilbert space L^2 (for details of derivation, cf. Freeman Proposition 3.2), namely

$$\mathfrak{I}_G(\sigma) = \inf_{g \in L^2[-\sigma,\sigma]} R(g),$$

where we define a quadratic form $R : L^2[-\sigma, \sigma] \to \mathbb{R}$ and a self-adjoint operator $K : L^2[-\sigma, \sigma] \to L^2[-\sigma, \sigma]$ by

$$R(g) = \frac{\langle (I + K)g, g \rangle}{|\langle g, 1 \rangle|^2}, \qquad (Kg)(x) = \int_{-\sigma}^{\sigma} m(x - y) g(y) dy,$$

where I is the identity operator on $L^2[-\sigma, \sigma]$. By functional analysis hocus pocus, the constant function one is in the image of $I + K$. Moreover, there exists $f \in (\ker(I + K))^{\perp}$ satisfying the equation

$$(I + K)f \equiv 1.$$

This implies $A := \langle (I + K)f, f \rangle = \langle 1, f \rangle$ is a positive constant. ILS proved that in fact

$$\mathfrak{I}_G(\sigma) = \inf_{g \in L^2[-\sigma,\sigma]} R(g) = R(f) = \frac{1}{A}.$$

ILS showed that if $\sigma = 1$, then the optimal functions $f \in L^2[-1, 1]$ are even, and are trigonometric polynomials when restricted to $[0, 1]$. Freeman extended this result to arbitrary $\sigma > 0$. The brute force approach to this problem is to to find a trigonometric polynomial $f : [0, \sigma] \to \mathbb{R}$, i.e. taking the form

$$f(x) = \sum_{n \geq 0} a_n \cos(nx) + b_n \sin(nx)$$

where $a_n, b_n = 0$ for all $n \geq N$, such that $(I + K)f$ is a constant. Normalizing by this constant gives the desired equation $(I + K)f \equiv 1$. Since f is even, we can extend it to a function on $[-\sigma, \sigma]$ to recover the desired optimal function.

2 One level

The key observation in deriving the optimal $f \in L^2[-\sigma, \sigma]$ is, as remarked earlier, it suffices to find f such that $(I + K)f$ is constant. Differentiating gives us a delay diffferential equation which can be solved by hand. Here we work out the example of $\mathsf{SO}(\mathsf{even})$ for $\sigma = 1$. We want to find $f \in L^2[-1, 1]$ such that

$$f(x) + \frac{1}{2} \int_{-1}^{1} f(y) \mathbb{1}_{[-1.1]}(x - y) dy = 1 \tag{1}$$

whenever $x \in [0,1]$. We know f is even, so symmetrizing allows us to reconstruct f on $[-1,0]$. First we rewrite the equation above in a more workable form; split the integral

$$\int_{-1}^{1} f(y)\mathbb{1}_{[-1.1]}(x-y)dy = \int_{-1}^{0} f(y)\mathbb{1}_{[-1.1]}(x-y)dy + \int_{0}^{1} f(y)\mathbb{1}_{[-1.1]}(x-y)dy.$$

For the second integral on the right, $x, y \in [0,1]$ so $x - y \in [-1,1]$, so the characteristic function term is redundant. In the first integral on the left, $y \in [-1,0]$ and $x \in [0,1]$ implies $x - y \in [-1,1]$ if and only if $y \in [x-1,0]$. By evenness, this is equivalent to integrating $f(y)$ for $y \in [0,1-x]$. Thus (1) is equivalent to

$$f(x) + \frac{1}{2}\int_0^1 f(y)dy + \frac{1}{2}\int_0^{1-x} f(y)dy = 1 \tag{2}$$

for $x \in [0,1]$. We want to find f such that the left hand side is constant with respect to x. Afterwards, setting $x = 1$ allows us to compute the desired normalizing constant. Differentiating (2) with respect to x gives the delay differential equation

$$f'(x) - \frac{1}{2}f(1-x) = 0 \tag{3}$$

for all $x \in [0,1]$. Divine inspiration and a routine check using the identity $\sin(x) = \cos(x - \pi/2)$ shows that

$$f(x) = \cos\left(\frac{x}{2} - \frac{\pi+1}{4}\right)$$

is a solution. Of course, for those who are not divinely inspired, we can solve by differentiating (3) to obtain

$$f''(x) = -\frac{1}{2}f'(1-x) = -\frac{1}{4}f(x). \tag{4}$$

the second equality follows directly from (3). This is a standard linear differential equation with a two-parameter family of solutions given by $f(x) = a\cos(x/2) + b\sin(x/2)$. Substituting these solutions into (3) yields the linear system

$$\begin{pmatrix} 2\cos(1/2) & 2\sin(1/2)-1 \\ 2\sin(1/2)+1 & -2\cos(1/2) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The matrix has determinant zero with non-trivial rank so the kernel of the matrix is a one-dimensional subspace, i.e. all solutions to (3) are scalar multiples of our divinely inspired answer.

3 Quadratic kernel

There are three key observations. First, the kernels take the form of quadratic polynomials in $|x|$ on the interval $[-1,1]$, i.e. $m_{G,\psi}(x) = -a - b|x| - c|x|^2$ whenever $x \in [-1,1]$. Second, the solutions found in [FM] and [ILS] to the 1-level density case for varying support took the form of piecewise trigonometric polynomials, and in particular continuously differentiable everywhere except finitely many points. Third, the right-hand side of (9) is constant with respect to $x \in [-1/2,1/2]$. Therefore, assuming g is sufficiently smooth, we can differentiate the Fredholm integral equation (9) to obtain a corresponding system of linear homogeneous differential equations.

Lemma 1. *If $g \in L^2[-1/2,1/2]$ is smooth and solves*

$$1 = g(x) + \int_{-1/2}^{1/2} (a + b|x-y| + c|x-y|^2)g(y)dy, \tag{5}$$

then it satisfies for $x \in [-1/2, 1/2]$ the following system of equations,

$$1 = g(0) + a \int_{-1/2}^{1/2} |y| g(y) dy, \tag{6}$$

$$0 = g''(x) + 2bg(x) + 2c \int_{-1/2}^{1/2} g(y) dy, \tag{7}$$

$$0 = g'''(x) + 2bg'(x). \tag{8}$$

Proof. Notice that

$$\int_{-1/2}^{1/2} |x-y| g(y) dy = \int_{-1/2}^{x} (x-y) g(y) dy + \int_{x}^{1/2} (y-x) g(y) dy.$$

Differentating under the integral sign gives

$$\frac{d}{dx} \int_{-1/2}^{1/2} |x-y| g(y) dy = \int_{-1/2}^{x} g(y) dy - \int_{x}^{1/2} g(y) dy,$$
$$\frac{d^2}{dx^2} \int_{-1/2}^{1/2} |x-y| g(y) dy = 2g(x)$$

and

$$\frac{d}{dx} \int_{-1/2}^{1/2} (x-y)^2 g(y) dy = \int_{-1/2}^{1/2} (2x-2y) g(y) dy,$$
$$\frac{d^2}{dx^2} \int_{-1/2}^{1/2} (x-y)^2 g(y) dy = 2 \int_{-1/2}^{1/2} g(y) dy.$$

And on the First Day of Genesis, He the God of Math deemed the rest to be trivial. $\square$

Theorem 2. *The solutions take the form*

$$g(x) = \frac{6b^{3/2}(b+c)\cos(\sqrt{2b}x) - 6\sqrt{2}bc \sin\sqrt{b/2}}{6\sqrt{b}(b+c)^2 \cos\sqrt{b/2} + \sqrt{2}(6ab^2 + 3b^3 + 3b^2c + bc(c-12) - 6c^2)\sin(\sqrt{b/2})}.$$

Proof. Assuming g is even, has solutions of (8) take the form $A\cos(\sqrt{2b}x) + C$. This has two degrees of freedom. Substituting into (7) reduces to one degree of freedom,

$$\begin{aligned} 0 &= g''(x) + 2bg(x) + 2c \int_{-1/2}^{1/2} g(y) dy \\ &= 2C(b+c) + \frac{4Ac}{\sqrt{2b}} \sin\left(\frac{\sqrt{2b}}{2}\right). \end{aligned}$$

This shows that g is a scalar multiple of

$$\cos(\sqrt{2b}x) - \frac{2c}{b+c} \frac{\sin\left(\sqrt{b/2}\right)}{\sqrt{2b}}.$$

Obviously this implies that $b \geq 0$ if we want to avoid nasty complexifications. Pumping into Wolfram Alpha, we obtain

$$g(x) = \frac{6b^{3/2}(b+c)\cos(\sqrt{2b}x) - 6\sqrt{2}bc \sin\sqrt{b/2}}{6\sqrt{b}(b+c)^2 \cos\sqrt{b/2} + \sqrt{2}(6ab^2 + 3b^3 + 3b^2c + bc(c-12) - 6c^2)\sin(\sqrt{b/2})}.$$

$\square$

4 Worked example: two-level $\mathsf{SO(even)}$ for $\sigma = 1/2$

For the two level density, we are interested in finding an optimal test function $\Phi : \mathbb{R}^2 \to \mathbb{R}$ with Fourier transform supported in $[-2\sigma, -2\sigma] \times [-2\sigma, -2\sigma]$ minimizing

$$\frac{1}{\Phi(0,0)} \int_{\mathbb{R}^2} \Phi(x,y) W_G(x,y) dx dy.$$

Putting restrictions on Φ, such as assuming it is of the form $\Phi(x,y) = \phi(x)\psi(y)$, makes the problem much easier while hopefully giving us a good upper bound on the true value of the minimum. Charlie was interested in $\sigma = 1/2$ with

$$\psi(x) = \left(\frac{\sin(\pi x)}{\pi x}\right)^2, \qquad \widehat{\psi}(x) = (1-|x|)\mathbb{1}_{[-1,1]}(x).$$

Analogous to the set-up in Appendix A of ILS and Jesse Freeman's thesis, Charlie found that finding the optimal $\phi : \mathbb{R} \to \mathbb{R}$ was equivalent to finding the unique solution $g \in L^2[-1/2, 1/2]$ satisfying equation

$$g(x) - \int_{-1/2}^{1/2} m_G(x-y) g(y) dy = 1, \tag{9}$$

for all $x \in [-1/2, 1/2]$. Charlie proved that g is even, increasing on $[-1/2, 0]$ and decreasing on $[0, 1/2]$. This seems to suggest that solutions are of the form $g(x) = a\cos(px) + c$ for some period $p > 0$ and constants $a, c \in \mathbb{R}$. We show that this is indeed the case for $\mathsf{SO(even)}$, which has corresponding kernel

$$m_{\mathsf{SO(even)}}(x) = -\frac{1}{2} + \frac{4}{3}(1-|x|)^2 + \frac{2}{3}(1-x^2). \tag{10}$$

Substituting (10) into (9) gives

$$-\frac{3}{2}\int_{-1/2}^{1/2} g(y)dy + \frac{8}{3}\int_{-1/2}^{1/2} |x-y| g(y) dy - \frac{2}{3}\int_{-1/2}^{1/2} (x-y)^2 g(y) dy = 1$$

for all $x \in [-1/2, 1/2]$. Notice that

$$\int_{-1/2}^{1/2} |x-y| g(y) dy = \int_{-1/2}^{x} (x-y) g(y) dy + \int_{x}^{1/2} (y-x) g(y) dy.$$

Differentating under the integral sign gives

$$\begin{aligned} \frac{d}{dx}\int_{-1/2}^{1/2} |x-y| g(y) dy &= \int_{-1/2}^{x} g(y) dy - \int_{x}^{1/2} g(y) dy, \\ \frac{d^2}{dx^2}\int_{-1/2}^{1/2} |x-y| g(y) dy &= 2g(x) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dx}\int_{-1/2}^{1/2} (x-y)^2 g(y) dy &= \int_{-1/2}^{1/2} (2x-2y) g(y) dy, \\ \frac{d^2}{dx^2}\int_{-1/2}^{1/2} (x-y)^2 g(y) dy &= 2\int_{-1/2}^{1/2} g(y) dy. \end{aligned}$$

Thus, (9) becomes, after differentiating three times with respect to x,

$$g'''(x) + \frac{16}{3} g'(x) = 0$$

for all $x \in [-1/2, 1/2]$. The solution set consists of the trigonometric polynomials $a\cos(4x/\sqrt{3}) + b\sin(4x/\sqrt{3}) + c$ for some constants $a, b, c \in \mathbb{R}$. We know g is even, so this forces $b = 0$. To obtain the remaining coefficients, we differentiate (9) twice to obtain

$$\begin{aligned} 0 &= g''(x) + \frac{16}{3}g(x) - \frac{4}{3}\int_{-1/2}^{1/2} g(y)dy \\ &= 4c - \frac{4}{3}a\int_{-1/2}^{1/2} \cos(4y/\sqrt{3})dy. \end{aligned}$$

Rearranging,

$$c = \frac{1}{3}a\int_{-1/2}^{1/2} \cos(4y/\sqrt{3})dy = \frac{2}{3}a\int_{0}^{1/2} \cos(4y/\sqrt{3})dy = \frac{\sin(2/\sqrt{3})}{2\sqrt{3}}a.$$

This shows that g is a scalar multiple of $\cos(4x/\sqrt{3}) + \frac{\sin(2/\sqrt{3})}{2\sqrt{3}}$. Plugging into (**??**) for $x = 0$ should give the desired multiplicative constant.

Theorem 3. *The solutions to the integral equation*

$$g_G(x) - \int_{-1/2}^{1/2} m_G(x-y)g_G(y)dy = 1$$

for $x \in [-1/2, 1/2]$ are

$$\begin{aligned} g_{\mathsf{SO(even)}}(x) &= \frac{216\cos(4x/\sqrt{3}) + 36\sqrt{3}\sin(2/\sqrt{3})}{-162\cos(2/\sqrt{3}) + 5\sqrt{3}\sin(2/\sqrt{3}}), \\ g_{\mathsf{SO(odd)}}(x) &= \frac{8\cos(4x) + 12\sin(2)}{2\cos(2) + 3\sin(2)}, \\ g_{\mathsf{U}}(x) &= \frac{6\cos(2x) + 6\sin(1)}{3\cos(1) + 4\sin(1)}, \\ g_{\mathsf{O}}(x) &= \frac{6\cos(2x\sqrt{2}) + 3\sqrt{2}\sin(\sqrt{2})}{3\cos(\sqrt{2}) + \sqrt{2}\sin(\sqrt{2})} \end{aligned}$$

5 Iteration

We have solved the minimization problem for fixed ψ; our goal now is to generalize by optimizing over a larger class of ψ. Supposing that we can write $\widehat{\psi} = g * g$ for some even real valued function $g \in L^2[-1/2, 1/2]$, we consider g which are solutions to the earlier minimization problem, namely functions of the form

$$g(x) = a\cos(bx) + c.$$

Since the problem is scaling invariant, we can normalize the constant term $c = 1$. For brevity, denote $h : \mathbb{R} \to \mathbb{R}$ the Fourier inverse of the map $x \mapsto \cos(bx)\mathbb{1}_{[-1/2,1/2]}(x)$. WolframAlpha gives us explicitly

$$h(x) = \int_{-1/2}^{1/2} e^{2\pi ixy}\cos(by)dy = \frac{2b\sin(b/2)\cos(\pi x) - 4\pi x\cos(b/2)\sin(\pi x)}{b^2 - 4\pi^2x^2}. \tag{11}$$

For notational convenience, denote $\mathcal{F}$ the Fourier transform operator, i.e. $\mathcal{F}\phi = \widehat{\phi}$. Viewing g as a function on $\mathbb{R}$ supported on $[-1/2, 1/2]$, its Fourier inverse takes the form

$$\mathcal{F}^{-1}(g)(x) = a\int_{-1/2}^{1/2} e^{2\pi ixy}\cos(by)dy + \int_{-1/2}^{1/2} e^{2\pi ixy}dy = ah(x) + \frac{\sin(\pi x)}{\pi x}.$$

As the Fourier transform and its inverse take convolutions to products and vice versa, we have

$$\psi(x) = \mathcal{F}^{-1}g * g(x) = \left[\mathcal{F}^{-1}(g)(x)\right]^2 = [ah(x)]^2 + 2ah(x)\frac{\sin(\pi x)}{\pi x} + \left(\frac{\sin(\pi x)}{\pi x}\right)^2. \tag{12}$$

Notice the last term is exactly our original ψ that was used earlier. This is unsurprising since we are pushing around convolutions and Fourier transforms of trigonometric polynomials, so by linearity the constant term should pop out.

$$\widehat{\psi}(x) = a\mathcal{F}\left([h(x)]^2\right) + a\mathcal{F}\left(h(x)\frac{\sin(\pi x)}{\pi x}\right) + (1-|x|)\mathbb{1}_{[-1,1]}(x).$$

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