# Fourier analytic approach to deriving the optimal $g$

Jason Zhao

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Given a continuous kernel $m : [-1,1] \to \mathbb{R}$, we want to find $g \in L^2[-1/2, 1/2]$ satisfying

$$g(x) - \int_{-1/2}^{1/2} m(x-y)g(y)dy = 1$$

for $x \in [-1/2, 1/2]$. Writing $g$ as the sum of iterates of $m$ is one method, however, computing the resulting integral of $g$ up to $n$ terms involves computing along the order of $1 + 2 + \cdots + n$ many integrals. Very computationally inefficient. What if we instead computed the Fourier coefficients instead? Viewing $g$ and $m$ as 2-periodic functions, i.e. extending $g$ by zero on $[-1,1]$ and then $g$ and $m$ extended 2-periodically to $\mathbb{R}$, the integral equation becomes

$$g(x) - (m * g)(x) = \begin{cases} 1, & \text{if } |x| \leq 1/2, \\ -(m * g)(x), & \text{if } 1/2 \leq |x| \leq 1. \end{cases}.$$

Let $a(n), b(n), c(n), d(n)$ be the Fourier coefficients of $g, m, \mathbb{1}_{[-1/2,1/2]}, \mathbb{1}_{[-1,-1/2]\cup[1/2,1]}$, then since Fourier transforms take convolutions to products and vice versa, we obtain

$$a(n)(1 - b(n)) = c(n) - \sum_{k \in \mathbb{Z}} a(n-k)b(n-k)d(k).$$

This gives an infinite "linear" system of equations to solve for $a(n)$. If we can somehow hocus pocus that without too much computational difficult, then computing an approximation for the integral of $g$ up to $n$-terms in the Fourier expansion only involves $n$ or so integrals. Accounting for the convergence rate of the Fourier coefficients we can estimate how close this truncation is to the actual integral.