OPTIMAL TEST FUNCTIONS FOR THE HIGHER LEVEL DENSITIES

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ABSTRACT. Do this later.

CONTENTS

1. INTRODUCTION

Things to do later: Uncomment email addresses and addresses. Were very annoying while working... Also edit keywords and thanks... And update subject classifications at http://www.ams.org/msc/

Also do this later.

1.a. **History.** This too, will be done later.

1.b. **Applications to the Order of Vanishing at the Central Point.**

Let $\mathcal{F}$ be a family of cuspidal newforms, and to each $f \in \mathcal{F}$ write $f = \sum_{n=1}^{\infty} a_{n,f} q^n$ for the corresponding q-expansion, where $q(z) = e^{2\pi i z}$. To each $f \in \mathcal{F}$, we associate the L-function

$$L(s, f) \;=\; \sum_{1}^{\infty} \frac{a_{n,f}}{n^s}. \tag{1.1}$$

We assume GRH and write the nontrivial zeros of $L(s, f)$ as $\rho_f^{(j)} = \frac{1}{2} + \mathrm{i}\gamma_f^{(j)}$ where $\gamma_f^{(j)} \in \mathbb{R}$ and γ_f^i are increasingly ordered and centered about 0. Letting c_f denote the analytic conductor of f, we write

$$\tilde{\gamma}_f^{(j)} \;=\; \frac{\log(c_f)}{2\pi} \gamma_f^{(j)}. \tag{1.2}$$

The $\tilde{\gamma}_f^{(j)}$ scaled to have mean spacing 1 near the central point.

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For each $f \in \mathcal{F}$ and non-negative even function $\phi : \mathbb{R}^n \to \mathbb{R}$ of sufficient decay rate with $\phi(0) = 1$, we define

$$D_n(f;\phi) \ \coloneqq \ \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \phi(\tilde{\gamma}_f^{(j_1)}, \dots, \tilde{\gamma}_f^{(j_n)}). \tag{1.3}$$

Letting $\mathcal{F}(Q) \coloneqq \{f \in \mathcal{F} : c_f \leq Q\}$ and fixing ϕ, we consider the averages of $D_n(f;\phi)$ over $\mathcal{F}(Q)$:

$$\mathbb{E}(D_n(f;\phi), Q) \ \coloneqq \ \frac{1}{|\mathcal{F}(Q)|} \sum_{f \in \mathcal{F}(Q)} D_n(f;\phi). \tag{1.4}$$

Under suitable assumptions on $\mathcal{F}$ Maybe be more precise here? Need $\mathcal{F}$ to be large with many different conductors..., we may assume that

$$\lim_{Q\to\infty} \mathbb{E}(D_n(f;\phi), Q)) \ = \ \int_{\mathbb{R}^2} \phi(x) W(\mathcal{F}, n)(x)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \tag{1.5}$$

where $W(\mathcal{F}, n)$ is a distribution determined by the family $\mathcal{F}$.

Here, we are interested in the order of vanishing at the central point. Write r_f for the order of the zero of $L(s, f)$ at $s = \frac{1}{2}$, and $\Pr(N) = \Pr(f \in \mathcal{F} : r_f = N)$. Taking only those terms in (1.4) with $r_f = N$ and $\tilde{\gamma}_f^{(j_k)} = 0$ for all $1 \leq k \leq n$, (1.5) gives the bound

$$\begin{aligned} &\sum_{m=0}^{\infty} 2m(2m-2)\cdots(2m-2(n-1)) \Pr(2m) \\ &\qquad + \sum_{m=0}^{\infty} (2m+2-n) \cdot 2m \cdot (2m-2) \cdot \cdots \cdot (2m-2(n-2)) \Pr(2m+1) \\ &\qquad \leq \int_{\mathbb{R}^n} \phi(x) W(\mathcal{F})(x)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n. \end{aligned} \tag{1.6}$$

It is therefore of interest to choose ϕ optimally to obtain the best bound on the left-hand side of (1.6).

	Truncate after term	Bound for (1.6) with $n = 2$
$W_{2,\mathrm{O}}$	100	0.222483
$W_{2,\mathrm{SO(even)}}$	100	0.252298
$W_{2,\mathrm{SO(odd)}}$	100	0.130293
$W_{2,\mathrm{U}}$	100	0.493856
$W_{2,\mathrm{USp}}$	100	0.130293

TABLE 1. Bounds for (1.6) with $n = 2$. Update this table after code finishes.

The results in Table 1 give the following bounds for families $\mathcal{F}$ with $W(\mathcal{F}) = W_{2,\mathcal{G}}$ for $\mathcal{G} = \mathrm{O}, \mathrm{U}, \mathrm{USp}$:

$$\sum_{m=0}^{\infty} m(m-1)\Pr(2m) + \sum_{m=0}^{\infty} m^2 \Pr(2m+1) \;\le\; \frac{1}{4}\cdot 0.222483 \quad W(\mathcal{F},2) \;=\; W_{2,\mathrm{O}}, \tag{1.7}$$

$$\sum_{m=0}^{\infty} m(m-1)\Pr(2m) + \sum_{m=0}^{\infty} m^2 \Pr(2m+1) \;\le\; \frac{1}{4}\cdot 0.493856 \quad W(\mathcal{F},2) \;=\; W_{2,\mathrm{U}}, \tag{1.8}$$

$$\sum_{m=0}^{\infty} m(m-1)\Pr(2m) + \sum_{m=0}^{\infty} m^2 \Pr(2m+1) \;\le\; \frac{1}{4}\cdot 0.130293 \quad W(\mathcal{F},2) \;=\; W_{2,\mathrm{USp}}. \tag{1.9}$$

Subtracting these bounds from $\sum_{N=0}^{\infty}\Pr(N) = 1$, we obtain

$$\Pr(0)+\Pr(1)+\Pr(2) \;\ge\; 0.944379 \qquad W(\mathcal{F},2) \;=\; W_{2,\mathrm{O}} \tag{1.10}$$

$$\Pr(0)+\Pr(1)+\Pr(2) \;\ge\; 0.876536 \qquad W(\mathcal{F},2) \;=\; W_{2,\mathrm{U}} \tag{1.11}$$

$$\Pr(0)+\Pr(1)+\Pr(2) \;\ge\; 0.967427 \qquad W(\mathcal{F},2) \;=\; W_{2,\mathrm{USp}}. \tag{1.12}$$

Splitting O by sign of functional equation gives stronger results. If $\mathcal{G} = \mathrm{SO(even)}$, the zeros at $s = \frac{1}{2}$ are all of even order. Then for $n = 2$, (1.6) becomes

$$\sum_{m=0}^{\infty} \frac{m(m-1)}{2}\Pr(2m) \;\le\; \frac{1}{8}\int_{\mathbb{R}^2} \phi(x)W(\mathcal{F})(x)\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{1.13}$$

Subtracting (1.13) from $\sum_{m=0}^{\infty}\Pr(2m) = 1$ then gives

$$\Pr(0)+\Pr(2) \;\ge\; 0.968463 \qquad W(\mathcal{F},2) \;=\; W_{2,\mathrm{SO(even)}}. \tag{1.14}$$

Instead taking $\mathcal{G} = \mathrm{SO(odd)}$, then the zeros at $s = \frac{1}{2}$ all have odd order, hence (1.6) in the case $n = 2$ becomes

$$\sum_{m=0}^{\infty} m^2 \Pr(2m+1) \;\le\; \frac{1}{4}\int_{\mathbb{R}^2} \phi(x)W(\mathcal{F})(x)\,\mathrm{d}x_1\,\mathrm{d}x_2 \tag{1.15}$$

Subtracting (1.15) from $\sum_{m=0}^{\infty}\Pr(2m+1) = 1$ then gives

$$\Pr(1) \;\ge\; 0.967427 \qquad W(\mathcal{F},2) \;=\; W_{2,\mathrm{SO(odd)}}. \tag{1.16}$$

2. Bounding the Right-hand Side of (1.6)

Suppose we have fixed admissible ϕ_2, and we wish to choose admissible ϕ_1 to make

$$\frac{1}{\phi_1(0)\phi_2(0)}\int_{\mathbb{R}^2} \phi_1(x)\phi_2(y)W_{2,\mathcal{G}}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{\phi_1(0)}\int_{x=-1}^{1} \widehat{\phi_1}(x)\left[\frac{1}{\phi_2(0)}\int_{y=-1}^{1} \widehat{\phi_2}(y)\widehat{W_{2,\mathcal{G}}}(x,y)\,\mathrm{d}y\right]\mathrm{d}x \tag{2.1}$$

as small as possible. Choosing ϕ with Fourier transform of large support is ideal for the best results, but for computational convenience we will work with $\operatorname{supp}(\widehat{\phi_1}), \operatorname{supp}(\widehat{\phi_2}) \subset [-1, 1]$. Writing

$$\tilde{V}_{\mathcal{G}}(x) \;=\; \tilde{V}_{\phi_2,\mathcal{G}}(x) \;:=\; \frac{1}{\phi_2(0)} \int_{y=-1}^{1} \widehat{\phi_2}(y)\widehat{W_{2,\mathcal{G}}}(x,y)\,\mathrm{d}y, \tag{2.2}$$

(2.1) becomes

$$\frac{1}{\phi_1(0)} \int_{x=-1}^{1} \widehat{\phi_1}(x)\tilde{V}_{\mathcal{G}}(x)\,\mathrm{d}x \tag{2.3}$$

Expressions for $\tilde{V}_{\mathcal{G}}$ for each of the classical compact groups $\mathcal{G}$, obtained by computing (2.2) from the 2-level Fourier expansions in [**?**], are given below:

$$\begin{aligned}
\tilde{V}_{\mathrm{SO(even)}}(x) \;&=\; \left(\frac{\widehat{\phi_2}(0)}{\phi_2(0)} + \frac{1}{2}\right)\delta(x) + \frac{1}{2}\left(\frac{\widehat{\phi_2}(0)}{\phi_2(0)} + \frac{1}{2}\right)\chi_{[-1,1]}(x) \\
&\qquad + \frac{\widehat{\phi_2}(x)+\widehat{\phi_2}(-x)}{\phi_2(0)}(|x|-1)\chi_{[-1,1]}(x) - \int_{|x|-1}^{1-|x|} \widehat{\phi_2}(y)\,\mathrm{d}y
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\tilde{V}_{\mathrm{SO(odd)}}(x) \;&=\; \left(\frac{\widehat{\phi_2}(0)}{\phi_2(0)} - \frac{1}{2}\right)\delta(x) + \frac{1}{2}\left(-\frac{\widehat{\phi_2}(0)}{\phi_2(0)} + \frac{1}{2}\right)\chi_{[-1,1]}(x) \\
&\qquad + \frac{\widehat{\phi_2}(x)+\widehat{\phi_2}(-x)}{\phi_2(0)}(|x|-1)\chi_{[-1,1]}(x) + \int_{|x|-1}^{1-|x|} \widehat{\phi_2}(y)\,\mathrm{d}y
\end{aligned} \tag{2.5}$$

$$\tilde{V}_{\mathrm{O}}(x) \;=\; \frac{\widehat{\phi_2}(0)}{\phi_2(0)}\delta(x) + \frac{1}{4}\chi_{[-1,1]}(x) + \frac{\widehat{\phi_2}(x)+\widehat{\phi_2}(-x)}{\phi_2(0)}(|x|-1)\chi_{[-1,1]}(x) \tag{2.6}$$

$$\tilde{V}_{\mathrm{U}}(x) \;=\; \frac{\widehat{\phi_2}(0)}{\phi_2(0)}\delta(x) + \frac{\widehat{\phi_2}(-x)}{\phi_2(0)}(|x|-1)\chi_{[-1,1]}(x) \tag{2.7}$$

$$\begin{aligned}
\tilde{V}_{\mathrm{USp}}(x) \;&=\; \left(\frac{\widehat{\phi_2}(0)}{\phi_2(0)} - \frac{1}{2}\right)\delta(x) + \frac{1}{2}\left(-\frac{\widehat{\phi_2}(0)}{\phi_2(0)} + \frac{1}{2}\right)\chi_{[-1,1]}(x) \\
&\qquad + \frac{\widehat{\phi_2}(x)+\widehat{\phi_2}(-x)}{\phi_2(0)}(|x|-1)\chi_{[-1,1]}(x) + \int_{|x|-1}^{1-|x|} \widehat{\phi_2}(y)\,\mathrm{d}y
\end{aligned} \tag{2.8}$$

where δ is the Dirac delta. Observe that each $\tilde{V}_{\mathcal{G}}$ has the form $c_{\mathcal{G}}\delta(x) - \tilde{m}_{\mathcal{G}}(x)$ where $\tilde{m}_{\mathcal{G}} \in L^2[-1,1]$ and $c_{\mathcal{G}} \in \mathbb{R}$. In the cases of $\mathcal{G} = \mathrm{SO(even)}, \mathrm{SO(odd)}, \mathrm{O}$, or USp, every choice of ϕ_2 has $m_{\mathcal{G}}$ even. In the case of U, if ϕ_2 itself is even, then $\widehat{\phi_2}$ is even, hence m_{U} is even. Furthermore, under suitable assumptions on ϕ_2, we will have $c_{\mathcal{G}} > 0$ in each case ($\phi_2 \geq 0$ is sufficient for $\mathrm{SO(even)}, \mathrm{O}, \mathrm{U}$, and not sure if there is a general criteria for SO(odd) and USp...). In this case, we will define

$$V_{\mathcal{G}}(x) \;:=\; \frac{1}{c_{\mathcal{G}}}\tilde{V}_{\mathcal{G}}(x) \tag{2.9}$$

$$m_{\mathcal{G}}(x) \;:=\; \frac{1}{c_{\mathcal{G}}}\tilde{m}(x) \tag{2.10}$$

and the minimization of (2.1) will be equivalent to finding ϕ_1 which minimizes

$$\frac{1}{\phi_1(0)} \int_{x=-1}^{1} \widehat{\phi_1}(x) V_{\mathcal{G}}(x)\,\mathrm{d}x. \tag{2.11}$$

Noting that $V_{\mathcal{G}}(x) = \delta(x) - m_{\mathcal{G}}(x)$ has the same form as the Fourier transforms of the 1-level weights, we may minimize (2.11) using the same technique as that given in Appendix A of [**?**], which we do as follows. Define the compact self-adjoint operator $K_{\mathcal{G}} : L^2[-\frac{1}{2},\frac{1}{2}] \to L^2[-\frac{1}{2},\frac{1}{2}]$ by

$$(K_{\mathcal{G}}g)(x) \;:=\; \int_{y=-\frac{1}{2}}^{\frac{1}{2}} m_{\mathcal{G}}(x-y)g(y)\,\mathrm{d}y. \tag{2.12}$$

If 1 is an eigenvalue of $K_{\mathcal{G}}$ and g is a 1-eigenvector such that $\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}, g\rangle_{L^2} \neq 0$, then $\widehat{\phi_1}(x) = (g * \breve{g})(x)$ optimizes (2.11) with a minimum value of 0, where we have written $\breve{g}(x) := \overline{g(-x)}$. On the other hand, if 1 is not an eigenvalue of $K_{\mathcal{G}}$, then there is a unique $g \in L^2[-\frac{1}{2},\frac{1}{2}]$ such that

$$g(x) - (K_{\mathcal{G}}g)(x) \;=\; \chi_{[-1,1]}(x). \tag{2.13}$$

Then $\widehat{\phi_1}(x) = (g * \breve{g})(x)$ is an optimizer of (2.11), and the minimum value of (2.11) is given by $\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}, g\rangle_{L^2}^{-1}$. Rescaling by $c_{\mathcal{G}}$, the minimum value of (2.1) is $c_{\mathcal{G}}\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}, g\rangle_{L^2}^{-1}$.

The equation (2.13) is a Fredholm equation of the second kind, and may be solved by iteration. For $n \in \mathbb{Z}_{\geq 1}$, define

$$k_{n,\mathcal{G}}(x) \;:=\; \int_{t_1,\dots,t_n=-\frac{1}{2}}^{\frac{1}{2}} m_{\mathcal{G}}(x-t_1)m_{\mathcal{G}}(t_1-t_2)\cdots m_{\mathcal{G}}(t_{n-1}-t_n)\,\mathrm{d}t_n \cdots \mathrm{d}t_1. \tag{2.14}$$

Then the solution to (2.13) is given by

$$g(x) = \chi_{[-1,1]}(x) + \sum_{n=1}^{\infty} k_{n,\mathcal{G}}(x) \tag{2.15}$$

assuming the series converges, and if we integrate term by terms, the corresponding minimum value of (2.1) is given by

$$c_{\mathcal{G}}\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}, g\rangle_{L^2}^{-1} = c_{\mathcal{G}}\left(1 + \sum_{n=1}^{\infty} \int_{x=-\frac{1}{2}}^{\frac{1}{2}} k_{n,\mathcal{G}}(x)\,\mathrm{d}x\right)^{-1}. \tag{2.16}$$

As $m_{\mathcal{G}}$ and $c_{\mathcal{G}}$ are determined by the choice of ϕ_2, the approach is to choose ϕ_2 which is computationally convenient, then estimate the value of (2.16) by computing as many terms as possible.

3. Preliminaries for the Computation of Table 1

Before computing (2.15), we must first choose ϕ_2, argue that we are in the non-degenerate case when 1 is not an eigenvalue of $K_{\mathcal{G}}$, then establish the convergence of (2.15). Experimentation with different choices of ϕ_2 having the form $\widehat{\phi_2} = g * \breve{g}$ for $g \in L^2[-1/2, 1/2]$ shows that the test function

$$\widehat{\phi_2}(x) \ \coloneqq\ (\chi_{[-\frac{1}{2},\frac{1}{2}]} * \breve{\chi}_{[-\frac{1}{2},\frac{1}{2}]})(x) \ =\ (1-|x|)\chi_{[-1,1]}(x) \tag{3.1}$$

yields relatively good results while being computationally simple. In this case, $\phi_2(0) = 1 = \widehat{\phi_2}(0)$, and the $m_{\mathcal{G}}$ are listed below

$$\begin{aligned} m_{\mathrm{SO(even)}}(x) \ &=\ -\frac{1}{2} + \frac{4}{3}(1-|x|)^2 + \frac{2}{3}(1-x^2) &(3.2)\\ m_{\mathrm{SO(odd)}}(x) \ =\ m_{\mathrm{USp}}(x) \ &=\ \frac{1}{2} + 4(1-|x|)^2 - 2(1-x^2) &(3.3)\\ m_{\mathrm{O}}(x) \ &=\ \left(2(1-|x|)^2 - \frac{1}{4}\right)\chi_{[-1,1]}(x) &(3.4)\\ m_{\mathrm{U}}(x) \ &=\ (1-|x|)^2\chi_{[-1,1]}(x). &(3.5)\\ & &(3.6) \end{aligned}$$

As the 2-level densities for $\mathrm{SO(odd)}$ and USp are identical for our range of Fourier transform support, we will use $\mathrm{SO(odd)}$ for notation referencing both groups. The next result establishes that 1 is not an eigenvalue of $K_{\mathrm{SO(even)}}$, K_{O}, or K_{U}. Appendix A establishes several useful integrals which are referenced throughout.

Proposition 3.1. *Letting $||\cdot||_{\mathcal{L}}$ denote the operator norm for the space of linear operators $\mathcal{L}(L^2)$ on $L^2[-1/2, 1/2]$,*

$$\begin{aligned} ||K_{\mathrm{SO(even)}}||_{\mathcal{L}} \ &\leq\ \sqrt{\frac{71}{135}} &(3.7)\\ ||K_{\mathrm{O}}||_{\mathcal{L}} \ &\leq\ \sqrt{\frac{43}{48}} &(3.8)\\ ||K_{\mathrm{U}}||_{\mathcal{L}} \ &\leq\ \sqrt{\frac{1}{3}}. &(3.9) \end{aligned}$$

In particular, 1 is not an eigenvalue of K_{O}, $K_{\mathrm{SO(even)}}$, or K_{U}.

Proof. Fix $g \in L^2[-\frac{1}{2}, \frac{1}{2}]$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} ||K_{\mathrm{SO(even)}}g||_{L^2}^2 \ &\leq\ \int_{x=-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{y=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(even)}}(x-y)| \cdot |g(y)| \, \mathrm{d}y \right)^2 \mathrm{d}x &(3.10)\\ &\leq\ \int_{x,y=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(even)}}(x-y)|^2 \, \mathrm{d}y \, \mathrm{d}x ||g||_{L^2}^2 &(3.11) \end{aligned}$$

$$= \left[\frac{16}{9}\int_{x,y=-\frac{1}{2}}^{\frac{1}{2}}(1-|x-y|)^4\,\mathrm{d}y\,\mathrm{d}x - \frac{4}{3}\int_{x,y=-\frac{1}{2}}^{\frac{1}{2}}(1-|x-y|)^2\,\mathrm{d}y\,\mathrm{d}x \right. \tag{3.12}$$

$$- \frac{2}{3}\int_{x,y=-\frac{1}{2}}^{\frac{1}{2}}(1-(x-y)^2)\,\mathrm{d}y\,\mathrm{d}x + \frac{4}{9}\int_{x,y=-\frac{1}{2}}^{\frac{1}{2}}(1-(x-y)^2)^2\,\mathrm{d}y\,\mathrm{d}x \tag{3.13}$$

$$\left. + \frac{16}{9}\int_{x,y=-\frac{1}{2}}^{\frac{1}{2}}(1-(x-y)^2)(1-|x-y|)^2\,\mathrm{d}y\,\mathrm{d}x + \frac{1}{4}\right]||g||_{L^2}^2. \tag{3.14}$$

We may apply (A.1) to compute the first two terms, (A.14) to compute the third, (A.18) for the fourth, and (A.23) for the fifth, yielding

$$= \left(\frac{16}{9}\frac{1}{3} - \frac{4}{3}\frac{1}{2} - \frac{2}{3}\frac{5}{6} + \frac{4}{9}\frac{11}{15} + \frac{16}{9}\frac{7}{15}\right)||g||_{L^2}^2 \tag{3.15}$$

$$= \frac{71}{135}||g||_{L^2}^2. \tag{3.16}$$

So $||K_{\mathrm{SO(even)}}||_{\mathcal{L}} \leq \sqrt{71/135}$, and the claim that 1 is not an eigenvalue of $K_{\mathrm{SO(even)}}$ follows from the fact that $\sqrt{71/135} < 1$.

Similar arguments work for O and U. Applying the Cauchy-Schwarz inequality,

$$\int_{x=-\frac{1}{2}}^{\frac{1}{2}}|K_{\mathrm{O}}g(x)|^2\,\mathrm{d}x \leq \int_{x=-\frac{1}{2}}^{\frac{1}{2}}\left(\int_{y=-\frac{1}{2}}^{\frac{1}{2}}|m_{\mathrm{O}}(x-y)|\cdot|g(y)|\,\mathrm{d}y\right)^2\mathrm{d}x \tag{3.17}$$

$$\leq \int_{x=-\frac{1}{2}}^{\frac{1}{2}}\int_{y=-\frac{1}{2}}^{\frac{1}{2}}|m_{\mathrm{O}}(x-y)|^2\,\mathrm{d}y\,\mathrm{d}x||g||_{L^2}^2 \tag{3.18}$$

$$= \int_{x=-\frac{1}{2}}^{\frac{1}{2}}\int_{y=-\frac{1}{2}}^{\frac{1}{2}}\left[4(1-|x-y|)^4 - (1-|x-y|)^2 + \frac{1}{16}\right]\mathrm{d}y\,\mathrm{d}x||g||_{L^2}^2 \tag{3.19}$$

for any $g \in L^2[-\frac{1}{2},\frac{1}{2}]$. Applying (A.1), the final line becomes

$$= \left(4\frac{2}{6} - \frac{2}{4} + \frac{1}{16}\right)||g||_{L^2}^2 \tag{3.20}$$

$$= \frac{43}{48}||g||_{L^2}^2. \tag{3.21}$$

Thus, $||K||_{\mathcal{L}} \leq \sqrt{43/48}$.

For U, again applying the Cauchy-Schwarz inequality, we have

$$||K_{\mathrm{U}} g||_{L^2}^2 \ \leq \ \int\limits_{x=-\frac{1}{2}}^{\frac{1}{2}} \int\limits_{y=-\frac{1}{2}}^{\frac{1}{2}} (1-|x-y|)^4 \, \mathrm{d}y \, \mathrm{d}x ||g||_{L^2}^2. \tag{3.22}$$

By (A.1),

$$\int\limits_{x=-\frac{1}{2}}^{\frac{1}{2}} \int\limits_{y=-\frac{1}{2}}^{\frac{1}{2}} (1-|x-y|)^4 \, \mathrm{d}y \, \mathrm{d}x ||g||_{L^2}^2 \ = \ \frac{1}{3} ||g||_{L^2}^2 \tag{3.23}$$

for arbitrary $g \in L^2[-\frac{1}{2}, \frac{1}{2}]$. It follows that $||K_{\mathrm{U}}||_{\mathcal{L}} \leq \sqrt{\frac{1}{3}}$.

□

Estimates similar to those used in Proposition 3.1 are too weak for $\mathrm{SO(odd)}$; more care is needed. The following result will establish that 1 is also not an eigenvalue of $K_{\mathrm{SO(odd)}}$. The argument applies to the other groups as well, but the estimates (3.8) will be needed later.

Proposition 3.2. 1 *is not an eigenvalue of* $K_{\mathrm{SO(odd)}}$.

Proof. Suppose 1 is an eigenvector for $K_{\mathrm{SO(odd)}}$, and let $g \in L^2[-\frac{1}{2}, \frac{1}{2}]$ be a 1-eigenvector. This means

$$0 \ = \ g(x) - \int\limits_{y=-\frac{1}{2}}^{\frac{1}{2}} m_{\mathrm{SO(odd)}}(x-y) g(y) \, \mathrm{d}y. \tag{3.24}$$

This is a Fredholm equation of the second kind, and the unique continuous solution is given by the corresponding Liouville-Neumann series which, in this case, is just 0. On the other hand, notice that $m_{\mathrm{SO(odd)}}$ is continuous and therefore uniformly continuous on $[-1, 1]$. So for fixed $\epsilon > 0$ there is $\delta > 0$ such that if $|h| < \delta$, then $|m_{\mathrm{SO(odd)}}(\alpha+h) - m_{\mathrm{SO(odd)}}(\alpha)| < \epsilon$ whenever $\alpha, \alpha+h \in [-1, 1]$. In particular, if $|h| < \delta$ and $x, x+h \in [-\frac{1}{2}, \frac{1}{2}]$, then by the Cauchy-Schwarz inequality,

$$\begin{aligned} |g(x+h) - g(x)| \ &\leq \ \int\limits_{y=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x+h-y) - m_{\mathrm{SO(odd)}}(x-y)| \cdot |g(y)| \, \mathrm{d}y & (3.25) \\ &\leq \ \epsilon \int\limits_{y=-\frac{1}{2}}^{\frac{1}{2}} |g(y)| \, \mathrm{d}y & (3.26) \\ &\leq \ \epsilon ||g||_{L^2} & (3.27) \end{aligned}$$

It follows that g is continuous, so by uniqueness, $g = 0$, a contradiction.

□

With the establishment that 1 is not an eigenvalue of $K_{\mathcal{G}}$ in any case, we move to argue the convergence of (2.15). For $\mathrm{SO(even)}, \mathrm{O}$, and U, this will follow easily from the estimates (3.8). For $\mathrm{SO(odd)}$, we need more careful estimates which are obtained in Appendix A.

Proposition 3.3. *For* $\mathcal{G} = \mathrm{SO(even)}, \mathrm{SO(odd)}, \mathrm{O}$*, and* U*, the series* (2.15) *converges absolutely and uniformly on* $[-1,1]$.

Proof. The argument is the same for $\mathrm{SO(even)}, \mathrm{O}$, and U, relying only on the fact that all three bounds in (3.8) are less than 1 and that each $m_{\mathcal{G}}$ is bounded on $[-1,1]$. We do the case of $\mathrm{SO(even)}$. Since $|m_{\mathrm{SO(even)}}| \leq 3/2$ on $[-1,1]$, for any $x \in [-1,1]$ we have

$$|k_{n,\mathrm{SO(even)}}(x)| \leq \int\limits_{t_1,\ldots,t_n=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(even)}}(x-t_1)| \cdot |m_{\mathrm{SO(even)}}(t_1-t_2)| \cdots |m_{\mathrm{SO(even)}}(t_{n-1}-t_n)| \, \mathrm{d}t_n \cdots \mathrm{d}t_1 \tag{3.28}$$

$$\leq \frac{3}{2} \int\limits_{t_1=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(even)}}(t_1-t_2)| \cdots \int\limits_{t_n=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(even)}}(t_{n-1}-t_n)| \, \mathrm{d}t_n \, \mathrm{d}t_{n-1} \cdots \mathrm{d}t_1 \tag{3.29}$$

$$= \frac{3}{2} ||\mathcal{K}_{\mathrm{SO(even)}}^{n-1} \chi_{[-1,1]}||_{L^1} \tag{3.30}$$

where $\mathcal{K}_{\mathrm{SO(even)}} \in \mathcal{L}(L^2)$ is defined by

$$(\mathcal{K}_{\mathrm{SO(even)}} g)(x) := \int\limits_{y=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(even)}}(x-y)| g(y) \, \mathrm{d}y. \tag{3.31}$$

The same arguments as Proposition 3.1 show that $||\mathcal{K}_{\mathrm{SO(even)}}||_{\mathcal{L}} = ||K_{\mathrm{SO(even)}}||_{\mathcal{L}}$, so by the Cauchy-Schwarz inequality,

$$\frac{3}{2} ||\mathcal{K}_{\mathrm{SO(even)}}^{n-1} \chi_{[-1,1]}||_{L^1} \leq \frac{3}{2} ||\mathcal{K}_{\mathrm{SO(even)}}^{n-1} \chi_{[-1,1]}||_{L^2} \tag{3.32}$$

$$\leq \frac{3}{2} \left(\frac{71}{135} \right)^{\frac{n}{2}}. \tag{3.33}$$

By the Weierstrass M-test, (2.15) converges absolutely and uniformly on $[-1,1]$ for $\mathrm{SO(even)}$.

Don't need (A.63) for this. Sufficient to use L^2 norm of (A.33) instead and argue similarly to other groups. For $\mathrm{SO(odd)}$, noting that (A.63) is bounded by $3511/3645$ on $[-1,1]$ is sufficient:

$$|k_{2n,\mathrm{SO(odd)}}(x)| \leq \int\limits_{t_1,t_2=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x-t_1) m_{\mathrm{SO(odd)}}(t_1-t_2)| \int\limits_{t_3,t_4=-\frac{1}{2}}^{\frac{1}{2}} \cdots \tag{3.34}$$

$$\cdots \int\limits_{t_{2n},t_{2n+1}=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(t_{2n-2}-t_{2n-1}) m_{\mathrm{SO(odd)}}(t_{2n-1}-t_{2n})| \, \mathrm{d}t_{2n} \, \mathrm{d}t_{2n-1} \cdots \mathrm{d}t_1 \tag{3.35}$$

$$\leq \left(\frac{3511}{3645} \right)^n \tag{3.36}$$

and similarly for $k_{2n+1,\mathrm{SO(odd)}}$ while accounting for the extra term using the bound $|m_{\mathrm{SO(odd)}}(x)| \leq 5/2$.

□

The uniform convergence also ensures that integrating term by term as done in (2.16) is justified. The next result would also establish this.

Proposition 3.4. *For $\mathcal{G}$ any of the five groups and any $n \in \mathbb{Z}_{\geq 1}$, $k_{n,\mathcal{G}}$ is even, increasing on $[-1/2, 0]$, and decreasing on $[0, 1/2]$. Furthermore, for $x \in [-1/2, 1/2]$ arbitrary,*

$$k_{n,\mathcal{G}}(x) \ \geq \ 0. \tag{3.37}$$

Proposition 3.4 implies that each term in (2.16) is non-negative, so truncating after finitely many terms yields an upper bound for (2.16). For U, the proposition is obvious since m_{U} is itself non-negative. For the other groups, this is not the case. The general idea of the proof is the same for each group, but with different technical details. We include all three here.

Proof. For $\mathcal{G} = \mathrm{SO(even)}$. We induct. By (4.9) and (**??**), we have

$$\begin{aligned}
k_{1,\mathrm{SO(even)}}(x) \ &= \ \int\limits_{y=-\frac{1}{2}}^{\frac{1}{2}} \left[-\frac{1}{2} + \frac{4}{3}(1-|x-y|)^2 + \frac{2}{3}(1-(x-y)^2)\right] \mathrm{d}y && (3.38)\\
&= \ -\frac{1}{2} + \frac{4}{3}\left(\frac{7}{12} - x^2\right) + \frac{2}{3}\left(\frac{11}{12} - x^2\right) && (3.39)\\
&= \ -2x^2 + \frac{8}{9} && (3.40)
\end{aligned}$$

which clearly satisfies the proposition. Assume that $k_{n-1,\mathrm{SO(even)}}$ satisfies the proposition. Then since $m_{\mathrm{SO(even)}}$ and $k_{n-1,\mathrm{SO(even)}}$ are even, taking $u = -y$ gives

$$\begin{aligned}
k_{n,\mathrm{SO(even)}}(-x) \ &= \ \int\limits_{y=-\frac{1}{2}}^{\frac{1}{2}} m_{\mathrm{SO(even)}}(-x-y) k_{n-1,\mathrm{SO(even)}}(y)\, \mathrm{d}y && (3.41)\\
&= \ \int\limits_{y=-\frac{1}{2}}^{\frac{1}{2}} m_{\mathrm{SO(even)}}(x+y) k_{n-1,\mathrm{SO(even)}}(-y)\, \mathrm{d}y && (3.42)\\
&= \ \int\limits_{u=-\frac{1}{2}}^{\frac{1}{2}} m_{\mathrm{SO(even)}}(x-u) k_{n-1,\mathrm{SO(even)}}(u)\, \mathrm{d}u && (3.43)\\
&= \ k_{n,\mathrm{SO(even)}}(x) && (3.44)
\end{aligned}$$

for $x \in [-1/2, 1/2]$, proving that $k_{n,\mathrm{SO(even)}}$ is even on $[-1/2, 1/2]$.

To see that $k_{n,\mathrm{SO(even)}}$ is increasing on $[-1/2, 0]$, choose a sequence $(f_\epsilon)_{\epsilon>0}$ of smooth, even functions on $[-1, 1]$ which are increasing on $[-1, 0]$, decreasing on $[0, 1]$, and which increases to

$m_{\mathrm{SO(even)}}$ pointwise as $\epsilon \to 0$ and is pointwise bounded in modulus by $2|m_{\mathrm{SO(even)}}(x)|$. Define

$$k_{n,\mathrm{SO(even)},\epsilon}(x) \;:=\; \int_{y=-\frac{1}{2}}^{\frac{1}{2}} f_\epsilon(x-y)k_{n-1,\mathrm{SO(even)}}(y)\,\mathrm{d}y. \tag{3.45}$$

Then by the Leibniz Integral Rule, we have

$$\begin{aligned} \frac{\mathrm{d}k_{n,\mathrm{SO(even)},\epsilon}}{\mathrm{d}x}(x) \;&=\; \int_{y=-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-y)k_{n-1,\mathrm{SO(even)}}(y)\,\mathrm{d}y \\ &=\; \int_{y=-\frac{1}{2}}^{x} \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-y)k_{n-1,\mathrm{SO(even)}}(y)\,\mathrm{d}y \\ &\qquad + \int_{y=x}^{2x+\frac{1}{2}} \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-y)k_{n-1,\mathrm{SO(even)}}(y)\,\mathrm{d}y \\ &\qquad + \int_{y=2x+\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-y)k_{n-1,\mathrm{SO(even)}}(y)\,\mathrm{d}y \end{aligned} \tag{3.46}$$

We assume $x \in (-1/2, 0)$. Note that since f_ϵ is even, the derivative factor in the integrand, as a function of y, is odd around x. It is non-negative on $(x, 1/2)$, non-positive on $(-1/2, x)$, and in particular,

$$\frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-(x+t)) \;=\; -\frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-(x-t)) \tag{3.47}$$

for all $t \in [0, x+1/2)$. As $x \le 0$, and $k_{n-1,\mathrm{SO(even)}}(y)$ is increasing on $[-1/2, 0]$ and symmetric about 0, we have $k_{n-1,\mathrm{SO(even)}}(x+t) \ge k_{n-1,\mathrm{SO(even)}}(x-t)$ for $t \in [0, x+1/2)$, hence

$$\frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-(x+t))k_{n-1,\mathrm{SO(even)}}(x+t) \;\ge\; -\frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-(x-t))k_{n-1,\mathrm{SO(even)}}(x-t). \tag{3.48}$$

It follows that the middle integral in (3.46) is at least as large in modulus as the first, and has the opposite sign. So the sum of the first two terms of (3.46) is non-negative. The third term is also non-negative, so $k_{n,\mathrm{SO(even)},\epsilon}$ is increasing on $[-1/2, 0]$. Passing to the limit $\lim_{\epsilon\to 0} k_{n,\mathrm{SO(even)},\epsilon}(x)$ using Dominated Convergence proves $k_{n,\mathrm{SO(even)}}$ is increasing on $[-1/2, 0]$, and is decreasing on $[0, 1/2]$ by symmetry about $x = 0$.

Finally, by the arguments above, to prove that $k_{n,\mathrm{SO(even)}}$ is non-negative, we need only prove that $k_{n,\mathrm{SO(even)}}(-1/2) \ge 0$. Observe that $m_{\mathrm{SO(even)}}$ is convex on $(-1, 0)$. Indeed, on $(-1, 0)$, we have

$$\frac{\mathrm{d}^2 m_{\mathrm{SO(even)}}}{\mathrm{d}x^2} \;=\; \frac{4}{3}. \tag{3.49}$$

So the derivative of $m_{\mathrm{SO(even)}}$ is increasing on $(-1, 0)$ from $4/3$ to $8/3$. As $m_{\mathrm{SO(even)}}$ is non-positive on $[-1, \sqrt{7}/2-2]$ and non-negative on $[\sqrt{7}/2-2, 0]$, this implies that $|m_{\mathrm{SO(even)}}(\sqrt{7}/2-2-t)| \le$

$m_{\mathrm{SO(even)}}(\sqrt{7}/2-2+t)$ for all $t\in[0,\sqrt{7}/2-1]$. In particular, we have

$$\left|m_{\mathrm{SO(even)}}\left(-\frac{1}{2}-\left(\frac{3}{2}-\frac{\sqrt{7}}{2}+t\right)\right)\right| \;\le\; m_{\mathrm{SO(even)}}\left(-\frac{1}{2}-\left(\frac{3}{2}-\frac{\sqrt{7}}{2}-t\right)\right) \tag{3.50}$$

for all $t\in[0,\sqrt{7}/2-1]$. Furthermore, since $k_{n-1,\mathrm{SO(even)}}$ is decreasing on $[0,1/2]$, symmetric about 0, and $3/2-\sqrt{7}/2\in[0,1/2]$, we have

$$0\;\le\; k_{n-1,\mathrm{SO(even)}}\left(\frac{3}{2}-\frac{\sqrt{7}}{2}+t\right)\;\le\; k_{n-1,\mathrm{SO(even)}}\left(\frac{3}{2}-\frac{\sqrt{7}}{2}-t\right) \tag{3.51}$$

for all $t\in[0,\sqrt{7}/2-1]$. Combining, we have

$$\left|m_{\mathrm{SO(even)}}\left(-\frac{1}{2}-\left(\frac{3}{2}-\frac{\sqrt{7}}{2}+t\right)\right)k_{n-1,\mathrm{SO(even)}}\left(\frac{3}{2}-\frac{\sqrt{7}}{2}+t\right)\right| \tag{3.52}$$

$$\le\; m_{\mathrm{SO(even)}}\left(-\frac{1}{2}-\left(\frac{3}{2}-\frac{\sqrt{7}}{2}-t\right)\right)k_{n-1,\mathrm{SO(even)}}\left(\frac{3}{2}-\frac{\sqrt{7}}{2}-t\right) \tag{3.53}$$

for all $t\in[0,\sqrt{7}/2-1]$.

Now write

$$k_{n,\mathrm{SO(even)}}\left(-\frac{1}{2}\right)\;=\;\int_{y=-\frac{1}{2}}^{\frac{5}{2}-\sqrt{7}} m_{\mathrm{SO(even)}}\left(-\frac{1}{2}-y\right)k_{n-1,\mathrm{SO(even)}}(y)\,\mathrm{d}y \tag{3.54}$$

$$+\int_{\frac{5}{2}-\sqrt{7}}^{\frac{3}{2}-\frac{\sqrt{7}}{2}} m_{\mathrm{SO(even)}}\left(-\frac{1}{2}-y\right)k_{n-1,\mathrm{SO(even)}}(y)\,\mathrm{d}y \tag{3.55}$$

$$+\int_{y=\frac{3}{2}-\frac{\sqrt{7}}{2}}^{\frac{1}{2}} m_{\mathrm{SO(even)}}\left(-\frac{1}{2}-y\right)k_{n-1,\mathrm{SO(even)}}(y)\,\mathrm{d}y. \tag{3.56}$$

On $[-1/2,1/2-2/\sqrt{8}]$, the function $m_{\mathrm{SO(even)}}(-1/2-y)$ is non-negative, so the first term of (3.56) is non-negative. The second term is also non-negative for the same reason, and from (3.53) it is at least as large in modulus as the third term. Therefore, (3.56) is non-negative, proving the claim.

□

Proof. For $\mathrm{SO(odd)}$. We again induct. By (4.9) and (4.15), we have

$$k_{1,\mathrm{SO(odd)}}(x)\;=\;\int_{y=-\frac{1}{2}}^{\frac{1}{2}}\left[\frac{1}{2}+4(1-|x-y|)^2-2(1-(x-y)^2)\right]\,\mathrm{d}y \tag{3.57}$$

$$=\;\frac{1}{2}+4\left(\frac{7}{12}-x^2\right)-2\left(\frac{11}{12}-x^2\right) \tag{3.58}$$

$$=\;-2x^2+1 \tag{3.59}$$

which clearly satisfies the proposition. Assume that $k_{n-1,\mathrm{SO(odd)}}$ satisfies the proposition. The same argument as for $\mathrm{SO(even)}$ proves that $k_{n,\mathrm{SO(odd)}}$ is even.

To prove that $k_{n,\mathrm{SO(odd)}}$ is decreasing on $[0,1/2]$ requires greater care than $\mathrm{SO(even)}$ since $m_{\mathrm{SO(odd)}}$ begins to increase on $[2/3,1]$. Choose sequence $(f_\epsilon)_{\epsilon>0}$ of smooth functions increasing to $m_{\mathrm{SO(odd)}}$ pointwise which are even, increasing on $[-2/3,0]\cup[2/3,1]$, decreasing on $[-1,-2/3]\cup[0,2/3]$, bounded pointwise by $2|m_{\mathrm{SO(odd)}}(x)|$, and equal to $m_{\mathrm{SO(odd)}}$ on the region $[-1,-1/3]\cup[1/3,1]$. Define

$$k_{n,\mathrm{SO(odd)},\epsilon}(x) \ := \int\limits_{y=-\frac{1}{2}}^{\frac{1}{2}} f_\epsilon(x-y)k_{n-1,\mathrm{SO(odd)}}(y)\,\mathrm{d}y. \tag{3.60}$$

By the Leibniz Integral Rule, we have

$$\frac{\mathrm{d}k_{n,\mathrm{SO(odd)},\epsilon}}{\mathrm{d}x}(x) \ = \int\limits_{y=-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-y)k_{n-1,\mathrm{SO(odd)}}(y)\,\mathrm{d}y. \tag{3.61}$$

We first consider x lying in the region $(0,1/6)$. In this case, write

$$\begin{aligned}\frac{\mathrm{d}k_{n,\mathrm{SO(odd)},\epsilon}}{\mathrm{d}x}(x) \ &= \int\limits_{y=-\frac{1}{2}}^{2x-\frac{1}{2}} \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-y)k_{n-1,\mathrm{SO(odd)}}(y)\,\mathrm{d}y \\ &\qquad + \int\limits_{y=2x-\frac{1}{2}}^{x} \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-y)k_{n-1,\mathrm{SO(odd)}}(y)\,\mathrm{d}y \\ &\qquad + \int\limits_{y=x}^{\frac{1}{2}} \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-y)k_{n-1,\mathrm{SO(odd)}}(y)\,\mathrm{d}y.\end{aligned} \tag{3.62}$$

By hypothesis, $k_{n-1,\mathrm{SO(odd)}}\ge 0$ on $[-1/2,1/2]$. As y increases through $[-1/2,2x-1/2]$, the value $x-y$ decreases through $[1/2-x,1/2+x]\subset[0,2/3]$. f_ϵ is decreasing on $[1/2-x,1/2+x]$, so the left-most integral is non-positive. As y increases through $[2x-1/2,x]$, the value $x-y$ decreases through $[0,1/2-x]\subset[0,2/3]$, so similarly to before, the middle integral is also non-positive. As y increases through $[x,1/2]$, $x-y$ decreases through $[x-1/2,0]\subset[-2/3,0]$. Since f_ϵ is increasing on $[x-1/2,0]$, the right-most integral is non-negative. Note however that the derivative terms, as functions of y, are odd about x, so

$$-\frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-(x-t)) \ = \ \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-(x+t)) \tag{3.63}$$

for $t\in[0,1/2-x)$. In particular, since $k_{n-1,\mathrm{SO(odd)}}$ is even and decreasing on $[0,1/2]$, this implies that

$$-\frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-(x-t))k_{n-1,\mathrm{SO(odd)}}(x-t) \ = \ \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-(x+t))k_{n-1,\mathrm{SO(odd)}}(x+t) \tag{3.64}$$

for all $t\in[0,1/2-x)$. Therefore, the middle integral in (3.62) is larger in modulus than the third integral, hence, $k_{n,\mathrm{SO(odd)},\epsilon}$ is decreasing on $(0,1/6)$.

Next consider the region $(1/6, 1/2)$. Write

$$\frac{\mathrm{d}k_{n,\mathrm{SO(odd)},\epsilon}}{\mathrm{d}x}(x) = \int_{y=-\frac{1}{2}}^{x-\frac{2}{3}} \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-y)k_{n-1,\mathrm{SO(odd)}}(y)\,\mathrm{d}y + \int_{y=x-\frac{2}{3}}^{2x-\frac{5}{6}} \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-y)k_{n-1,\mathrm{SO(odd)}}(y)\,\mathrm{d}y \tag{3.65}$$

$$+ \int_{y=2x-\frac{5}{6}}^{2x-\frac{1}{2}} \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-y)k_{n-1,\mathrm{SO(odd)}}(y)\,\mathrm{d}y \tag{3.66}$$

$$+ \int_{y=2x-\frac{1}{2}}^{x} \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-y)k_{n-1,\mathrm{SO(odd)}}(y)\,\mathrm{d}y + \int_{y=x}^{\frac{1}{2}} \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-y)k_{n-1,\mathrm{SO(odd)}}(y)\,\mathrm{d}y \tag{3.67}$$

Using similar arguments to above, the first and fifth integrals are non-negative, and the second, third, and fourth integrals are non-positive. Since $k_{n-1,\mathrm{SO(odd)}}$ is non-negative, decreasing on $[0, 1/2]$, and symmetric about $x = 0$, we have

$$-\frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-(x-t))k_{n-1,\mathrm{SO(odd)}}(x-t) \geq \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x-(x+t))k_{n-1,\mathrm{SO(odd)}}(x+t) \tag{3.68}$$

for all $t \in [0, 1/2 - x)$. Therefore, the sum of the fourth and fifth integrals is non-positive.

Note that since $f_\epsilon = m_{\mathrm{SO(odd)}}$ on $[1/3, 1]$, we have

$$\frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}(x) = 12x - 8 \qquad x \in \left(\frac{1}{3}, 1\right) \tag{3.69}$$

In particular,

$$-\frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}\left(x - \left(x - \frac{2}{3} + t\right)\right) = \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}\left(x - \left(x - \frac{2}{3} - t\right)\right) \tag{3.70}$$

for all $t \in [0, 1/3)$. So since $k_{n-1,\mathrm{SO(odd)}}$ is non-negative, decreasing on $[0, 1/2]$, and symmetric about $x = 0$, we have

$$\begin{aligned} -\frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}\left(x - \left(x - \frac{2}{3} + t\right)\right) k_{n-1,\mathrm{SO(odd)}}\left(x - \frac{2}{3} + t\right) \\ \geq \frac{\mathrm{d}f_\epsilon}{\mathrm{d}x}\left(x - \left(x - \frac{2}{3} - t\right)\right) k_{n-1,\mathrm{SO(odd)}}\left(x - \frac{2}{3} - t\right) \end{aligned} \tag{3.71}$$

for all $t \in [0, x - 1/6)$. Thus, the sum of the first and second integrals is non-positive as well. So since the third is also non-positive, we may conclude that $k_{n,\mathrm{SO(odd)},\epsilon}$ is also decreasing on $(1/6, 1/2)$, and by continuity, it is decreasing on $[0, 1/2]$. Passing to the limit $\lim_{\epsilon\to 0} k_{n,\mathrm{SO(odd)},\epsilon}$ using Dominated Convergence, $k_{n,\mathrm{SO(odd)}}$ is also decreasing on $[0, 1/2]$, hence increasing on $[-1/2, 0]$ by symmetry.

Finally, using the previous arguments, to show that $k_{n,\mathrm{SO(odd)}}$ is non-negative, we need only show that $k_{n,\mathrm{SO(odd)}}(1/2) \geq 0$. Write

$$k_{n,\mathrm{SO(odd)}}\left(\frac{1}{2}\right) = \int_{y=-\frac{1}{2}}^{-\frac{1}{3}} m_{\mathrm{SO(odd)}}\left(\frac{1}{2}-y\right) k_{n-1,\mathrm{SO(odd)}}(y)\,\mathrm{d}y + \int_{y=-\frac{1}{3}}^{0} m_{\mathrm{SO(odd)}}\left(\frac{1}{2}-y\right) k_{n-1,\mathrm{SO(odd)}}(y)\,\mathrm{d}y \tag{3.72}$$

$$+ \int_{y=0}^{\frac{1}{3}} m_{\mathrm{SO(odd)}}\left(\frac{1}{2}-y\right) k_{n,\mathrm{SO(odd)}}(y)\,\mathrm{d}y + \int_{y=\frac{1}{3}}^{\frac{1}{2}} m_{\mathrm{SO(odd)}}\left(\frac{1}{2}-y\right) k_{n,\mathrm{SO(odd)}}(y)\,\mathrm{d}y. \tag{3.73}$$

By hypothesis, $k_{n-1,\mathrm{SO(odd)}}(y) \geq 0$ on $[-1/2, 1/2]$. Since $m_{\mathrm{SO(odd)}}(y) \geq 0$ on $[0, 1/2] \cup [5/6, 1]$ and $m_{\mathrm{SO(odd)}}(y) \leq 0$ on $[1/2, 5/6]$, the second integral is non-positive while the first, third, and fourth are non-negative. $k_{n-1,\mathrm{SO(odd)}}$ is symmetric about $y = 0$, so $k_{n-1,\mathrm{SO(odd)}}(-y) = k_{n-1,\mathrm{SO(odd)}}(y)$ for all $y \in [0, 1/3]$. On the other hand, $m_{\mathrm{SO(odd)}}$ is convex on $(0, 1)$:

$$\frac{\mathrm{d}^2 m_{\mathrm{SO(odd)}}}{\mathrm{d}x^2}(x) = 12. \tag{3.74}$$

Therefore,

$$m_{\mathrm{SO(odd)}}\left(\frac{1}{2}-y\right) \geq -m_{\mathrm{SO(odd)}}\left(\frac{1}{2}+y\right) \tag{3.75}$$

for all $y \in [0, 1/3]$. It follows that the sum of the second and third integrals is non-negative, so since the first and fourth are also non-negative, we have $k_{n,\mathrm{SO(odd)}}(1/2) \geq 0$ as claimed.

□

Proof. For O. We proceed similarly to above. By (4.9), we have

$$k_{1,\mathrm{O}}(x) = \int_{y=-\frac{1}{2}}^{\frac{1}{2}} \left[2\left(1-|x-y|\right)^2 - \frac{1}{4}\right] \mathrm{d}y \tag{3.76}$$

$$= 2\left(-x^2 + \frac{7}{12}\right) - \frac{1}{4} \tag{3.77}$$

$$= \frac{11}{12} - 2x^2 \tag{3.78}$$

This clearly satisfies the proposition. Assume $k_{n-1,\mathrm{O}}$ satisfies the proposition. There is no special casework for O, so the same arguments as SO(even) proves the even and increasing/decreasing claims for O. Then to prove that $k_{n,\mathrm{O}}$ is non-negative, we need only prove that $k_{n,\mathrm{O}}(-1/2) \geq 0$. Observe that m_{O} is convex on $(-1, 0)$. Indeed, on $(-1, 0)$, we have

$$\frac{\mathrm{d}^2 m_{\mathrm{O}}}{\mathrm{d}x^2}(x) = 4. \tag{3.79}$$

So the derivative of m_{O} is increasing on $(-1, 0)$ from 0 to 4. As m_{O} is non-positive on $[-1, -1 + 8^{-1/2}]$ and non-negative on $[-1 + 8^{-1/2}, 0]$, this implies that $|m_{\mathrm{O}}(-1 + 8^{-1/2} - t)| \leq m_{\mathrm{O}}(-1 +$

$8^{-1/2}+t)$ for all $t \in [0, 8^{-1/2}]$. In particular, we have

$$\left| m_{\mathrm{O}}\left(-\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{\sqrt{8}}+t\right)\right)\right| \;\leq\; m_{\mathrm{O}}\left(-\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{\sqrt{8}}-t\right)\right) \tag{3.80}$$

for all $t \in [0, 8^{-1/2}]$. Furthermore, since $k_{n-1,\mathrm{O}}$ is decreasing on $[0, 1/2]$, symmetric about 0, and $1/2 - 8^{-1/2} \in [0, 1/2]$, we have

$$0 \;\leq\; k_{n-1,\mathrm{O}}\left(\frac{1}{2}-\frac{1}{\sqrt{8}}+t\right) \;\leq\; k_{n-1,\mathrm{O}}\left(\frac{1}{2}-\frac{1}{\sqrt{8}}-t\right) \tag{3.81}$$

for all $t \in [0, 8^{-1/2}]$. Combining,

$$\begin{aligned} &\left| m_{\mathrm{O}}\left(-\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{\sqrt{8}}+t\right)\right)\right| k_{n-1,\mathrm{O}}\left(\frac{1}{2}-\frac{1}{\sqrt{8}}+t\right) \\ &\qquad\qquad \leq\; m_{\mathrm{O}}\left(-\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{\sqrt{8}}-t\right)\right) k_{n-1,\mathrm{O}}\left(\frac{1}{2}-\frac{1}{\sqrt{8}}-t\right) \end{aligned} \tag{3.82}$$

for all $t \in [0, 8^{-1/2}]$.

Now write

$$\begin{aligned} k_{n,\mathrm{O}}\left(-\frac{1}{2}\right) \;=\; &\int_{y=-\frac{1}{2}}^{\frac{1}{2}-\frac{2}{\sqrt{8}}} m_{\mathrm{O}}\left(-\frac{1}{2}-y\right) k_{n-1,\mathrm{O}}(y)\,\mathrm{d}y \\ &+ \int_{\frac{1}{2}-\frac{2}{\sqrt{8}}}^{\frac{1}{2}-\frac{1}{\sqrt{8}}} m_{\mathrm{O}}\left(-\frac{1}{2}-y\right) k_{n-1,\mathrm{O}}(y)\,\mathrm{d}y \\ &+ \int_{y=\frac{1}{2}-\frac{1}{\sqrt{8}}}^{\frac{1}{2}} m_{\mathrm{O}}\left(-\frac{1}{2}-y\right) k_{n-1,\mathrm{O}}(y)\,\mathrm{d}y. \end{aligned} \tag{3.83}$$

On $[-1/2, 1/2 - 2/\sqrt{8}]$, the function $m_{\mathrm{O}}(-1/2 - y)$ is non-negative, so the first term is non-negative. The second term is also non-negative for the same reason, and from (3.82) it is at least as large in modulus as the third. Therefore, $k_{n,\mathrm{O}}(-1/2) \geq 0$, proving the claim.

□

4. Algorithm for Computing Table 1

The previous section establishes that we may compute finitely many terms of (2.16) in order to upper bound the true value. This section addresses how to do this. We begin with a few lemmas.

Lemma 4.1. *If $x \in [-1/2, 1/2]$ and $n \in \mathbb{Z}_{\geq 0}$, then*

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}} y^n |x-y|\,\mathrm{d}y = \left(\frac{2}{n+1}-\frac{2}{n+2}\right)x^{n+2}+[(-1)^n-1]\,\frac{1}{n+1}\frac{1}{2^{n+1}}x+[(-1)^n+1]\,\frac{1}{n+2}\frac{1}{2^{n+2}}. \tag{4.1}$$

Proof. We divide the interval as $[-1/2, 1/2] = [-1/2, x] \cup [x, 1/2]$ and integrate over each region:

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}} y^n |x-y| \,\mathrm{d}y = \int_{y=-\frac{1}{2}}^{x} (y^n x - y^{n+1}) \,\mathrm{d}y + \int_{y=x}^{\frac{1}{2}} (y^{n+1} - y^n x) \,\mathrm{d}y \tag{4.2}$$

$$= \frac{x}{n+1}\left(x^{n+1} + (-1)^n \frac{1}{2^{n+1}}\right) - \frac{1}{n+2}\left(x^{n+2} + (-1)^{n+1}\frac{1}{2^{n+2}}\right) \tag{4.3}$$

$$+ \frac{1}{n+2}\left(\frac{1}{2^{n+2}} - x^{n+2}\right) - \frac{x}{n+1}\left(\frac{1}{2^{n+1}} - x^{n+1}\right) \tag{4.4}$$

$$= \left(\frac{2}{n+1} - \frac{2}{n+2}\right) x^{n+2} + [(-1)^n - 1]\frac{1}{n+1}\frac{1}{2^{n+1}} x + [(-1)^n + 1]\frac{1}{n+2}\frac{1}{2^{n+2}}. \tag{4.5}$$

□

Lemma 4.2. *Define*

$$a_j := \frac{4}{2j+2} - \frac{4}{2j+1} \tag{4.6}$$

$$b_j := \frac{1}{2j+1}\frac{1}{4^j} \tag{4.7}$$

$$c_j := \frac{1}{2j+1}\frac{1}{4^j} - \frac{1}{2j+2}\frac{1}{4^j} + \frac{1}{2j+3}\frac{1}{4^{j+1}}. \tag{4.8}$$

Then for $x \in [-1/2, 1/2]$ and $n \in \mathbb{Z}_{\geq 0}$, we have

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}} (1 - |x-y|)^2 y^{2n} \,\mathrm{d}y = a_n x^{2n+2} + b_n x^2 + c_n \tag{4.9}$$

Proof. Apply (4.1):

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}} (1-|x-y|)^2 y^{2n} \,\mathrm{d}y = \int_{y=-\frac{1}{2}}^{\frac{1}{2}} y^{2n} \,\mathrm{d}y - 2\int_{y=-\frac{1}{2}}^{\frac{1}{2}} y^{2n}|x-y| \,\mathrm{d}y$$
$$+ x^2 \int_{y=-\frac{1}{2}}^{\frac{1}{2}} y^{2n} \,\mathrm{d}y - 2x \int_{y=-\frac{1}{2}}^{\frac{1}{2}} y^{2n+1} \,\mathrm{d}y + \int_{y=-\frac{1}{2}}^{\frac{1}{2}} y^{2n+2} \,\mathrm{d}y \tag{4.10}$$

$$= \frac{1}{2n+1}\frac{1}{4^n} - \frac{4}{2n+1}x^{2n+2} + \frac{2}{n+1}x^{2n+2}$$
$$- \frac{2}{n+1}\frac{1}{4^{n+1}} + \frac{1}{2n+1}\frac{1}{4^n}x^2 + \frac{1}{2n+3}\frac{1}{4^{n+1}} \tag{4.11}$$

$$= a_n x^{2n+2} + b_n x^2 + c_n \tag{4.12}$$

□

j	a_j	b_j	c_j
0	-2	1	$\frac{7}{12}$
1	$-\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{30}$
2	$-\frac{2}{15}$	$\frac{1}{80}$	$\frac{29}{6720}$
3	$-\frac{1}{14}$	$\frac{1}{448}$	$\frac{23}{32256}$
4	$-\frac{2}{45}$	$\frac{1}{2304}$	$\frac{67}{506880}$
5	$-\frac{1}{33}$	$\frac{1}{11264}$	$\frac{23}{878592}$

TABLE 2. The first few values of a_j, b_j, and c_j. Need to double check these with the code at some point.

Lemma 4.3. *For $n \in \mathbb{Z}_{\geq 0}$, define*

$$\alpha_n \ := \ -\frac{1}{2n+1}\frac{1}{4^n} \tag{4.13}$$

$$\beta_n \ := \ -\frac{1}{2n+3}\frac{1}{4^{n+1}} + \frac{1}{2n+1}\frac{1}{4^n}. \tag{4.14}$$

Then

$$\int\limits_{y=-\frac{1}{2}}^{\frac{1}{2}} (1-(x-y)^2)y^{2n}\,\mathrm{d}y \ = \ \alpha_n x^2 + \beta_n \tag{4.15}$$

Proof.

$$\int\limits_{y=-\frac{1}{2}}^{\frac{1}{2}} (1-(x-y)^2)y^{n}\,\mathrm{d}y \ = \ \frac{1}{2n+1}\frac{1}{4^n} - \int\limits_{y=-\frac{1}{2}}^{\frac{1}{2}} (x^2-2xy+y^2)y^{2n}\,\mathrm{d}y \tag{4.16}$$

$$= \ \frac{1}{2n+1}\frac{1}{4^n} - x^2\frac{1}{2n+1}\frac{1}{4^n} - \frac{1}{2n+3}\frac{1}{4^{n+1}} \tag{4.17}$$

$$= x^2\alpha_n + \beta_n \tag{4.18}$$

□

Using the lemmas, we can recursively compute $k_{n,\mathcal{G}}$ for each $\mathcal{G}$. The simplest example is $\mathcal{G} = \mathrm{U}$.

$$k_{1,\mathrm{U}}(x) \ = \ \int\limits_{t_1=-\frac{1}{2}}^{\frac{1}{2}} (1-|x-t_1|)^2\,\mathrm{d}t_1 = a_0x^2 + b_0x^2 + c_0 = -x^2 + \frac{7}{12} \tag{4.19}$$

$$k_{2,\mathrm{U}}(x) \ = \ \int\limits_{t_1=-\frac{1}{2}}^{\frac{1}{2}} (1-|x-t_1|)^2 \int\limits_{t_2=-\frac{1}{2}}^{\frac{1}{2}} (1-|t_1-t_2|)^2\,\mathrm{d}t_2\,\mathrm{d}t_1 \tag{4.20}$$

$$= \int_{t_1=-\frac{1}{2}}^{\frac{1}{2}} (1-|x-t_1|)^2 \left(-t_1^2 + \frac{7}{12}\right) \mathrm{d}t_1 \tag{4.21}$$

$$= (-1)(a_1x^4 + b_1x^2 + c_1) + \frac{7}{12}(a_0x^2 + b_0x^2 + c_0) \tag{4.22}$$

$$= \frac{1}{3}x^4 - \frac{2}{3}x^2 + \frac{221}{720}. \tag{4.23}$$

The process is similar for the other groups. The $k_{n,\mathcal{G}}$ are polynomials of degree $2n$ whose coefficients are polynomials in the coefficients of $k_{n-1,\mathcal{G}}$.

Appendix A. Some Useful Lemmas

Clean this out after I figure out which results I actually need to keep. Here, we present some useful lemmas which will simplify later calculations.

Lemma A.1. *If* $x \in [-\frac{1}{2}, \frac{1}{2}]$, *then*

$$\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=-\frac{1}{2}}^{\frac{1}{2}} (1-|x-y|)^n \,\mathrm{d}y\,\mathrm{d}x = \frac{2}{n+2}. \tag{A.1}$$

Proof. We split the y integral into a sum of integrals over the regions $[-\frac{1}{2}, x]$ and $[x, \frac{1}{2}]$:

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}} (1-|x-y|)^n \,\mathrm{d}y = \int_{y=x}^{\frac{1}{2}} (1-y+x)^n \,\mathrm{d}y + \int_{y=-\frac{1}{2}}^{x} (1-x+y)^n \,\mathrm{d}y \tag{A.2}$$

$$= -\frac{1}{n+1}(1-y+x)^{n+1}\Big|_{y=x}^{\frac{1}{2}} + \frac{1}{n+1}(1-x+y)^{n+1}\Big|_{y=-\frac{1}{2}}^{x} \tag{A.3}$$

$$= -\frac{1}{n+1}\left(x+\frac{1}{2}\right)^{n+1} + \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+1}\left(-x+\frac{1}{2}\right)^{n+1}. \tag{A.4}$$

Therefore,

$$\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=-\frac{1}{2}}^{\frac{1}{2}} (1-|x-y|)^n \,\mathrm{d}y\,\mathrm{d}x = -\frac{1}{n+1}\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \left(x+\frac{1}{2}\right)^{n+1} \mathrm{d}x + \frac{2}{n+1} - \frac{1}{n+1}\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \left(-x+\frac{1}{2}\right)^{n+1} \mathrm{d}x \tag{A.5}$$

$$= -\frac{1}{(n+1)(n+2)}\left(x+\frac{1}{2}\right)^{n+2}\Big|_{x=-\frac{1}{2}}^{\frac{1}{2}} + \frac{2}{n+1} + \frac{1}{(n+1)(n+2)}\left(-x+\frac{1}{2}\right)^{n+2}\Big|_{x=}^{\frac{1}{2}} \tag{A.6}$$

$$= \frac{2}{n+1} - \frac{2}{(n+1)(n+2)} \tag{A.7}$$

$$= \frac{2}{n+2}. \tag{A.8}$$

□

Lemma A.2. *For $x \in [-1, 1]$, we have*

$$\int_{|x|-1}^{1-|x|} (1 - |y|)\,\mathrm{d}y = (1 - x^2)\chi_{[-1,1]}(x). \tag{A.9}$$

Proof. Observe that

$$\int_{|x|-1}^{1-|x|} (1 - |y|)\,\mathrm{d}y = (1 - |x|) - (|x| - 1) - \int_{y=0}^{1-|x|} y\,\mathrm{d}y + \int_{y=|x|-1}^{0} y\,\mathrm{d}y \tag{A.10}$$

$$= 2 - 2|x| - \frac{1}{2}(1 - |x|)^2 - \frac{1}{2}(1 - |x|)^2 \tag{A.11}$$

$$= 2 - 2|x| - (1 - 2|x| + x^2) \tag{A.12}$$

$$= 1 - x^2. \tag{A.13}$$

□

Lemma A.3.

$$\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=-\frac{1}{2}}^{\frac{1}{2}} (1 - (x - y)^2)\,\mathrm{d}y\,\mathrm{d}x = \frac{5}{6}. \tag{A.14}$$

Proof.

$$\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=-\frac{1}{2}}^{\frac{1}{2}} (1 - (x - y)^2)\,\mathrm{d}y = 1 + \frac{1}{3}\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \left(x - \frac{1}{2}\right)^3 - \frac{1}{3}\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \left(x + \frac{1}{2}\right)^3 \mathrm{d}x \tag{A.15}$$

$$= 1 - \frac{1}{3}\frac{1}{4} - \frac{1}{3}\frac{1}{4} \tag{A.16}$$

$$= \frac{5}{6}. \tag{A.17}$$

□

Lemma A.4.

$$\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=-\frac{1}{2}}^{\frac{1}{2}} (1 - (x - y)^2)^2\,\mathrm{d}y\,\mathrm{d}x = \frac{11}{15}. \tag{A.18}$$

Proof. From (A.14), we deduce that $\int_{x,y=-1/2}^{1/2} (x - y)^2\,\mathrm{d}y\,\mathrm{d}x = 1/6$. Therefore,

$$\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=-\frac{1}{2}}^{\frac{1}{2}} (1 - (x - y)^2)^2\,\mathrm{d}y\,\mathrm{d}x = 1 - 2\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=-\frac{1}{2}}^{\frac{1}{2}} (x - y)^2\,\mathrm{d}y\,\mathrm{d}x + \int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=-\frac{1}{2}}^{\frac{1}{2}} (x - y)^4\,\mathrm{d}y\,\mathrm{d}x \tag{A.19}$$

$$= 1 - \frac{2}{6} - \frac{1}{5}\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \left(x - \frac{1}{2}\right)^5 + \frac{1}{5}\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \left(x + \frac{1}{2}\right)^5 \mathrm{d}x \tag{A.20}$$

$$= \frac{2}{3} + \frac{1}{5}\frac{1}{6} + \frac{1}{5}\frac{1}{6} \tag{A.21}$$

$$= \frac{11}{15}. \tag{A.22}$$

□

Lemma A.5.

$$\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{y=-\frac{1}{2}}^{\frac{1}{2}} (1 - |x-y|)^2(1-(x-y)^2)\,\mathrm{d}y\,\mathrm{d}x = \frac{7}{15}. \tag{A.23}$$

Proof. We first expand out the integrand:

$$\int_{x,y=-\frac{1}{2}}^{\frac{1}{2}} (1-|x-y|)^2(1-(x-y)^2)\,\mathrm{d}y\,\mathrm{d}x = 1 - 2\int_{x=-\frac{1}{2}}^{\frac{1}{2}}\int_{y=-\frac{1}{2}}^{\frac{1}{2}} |x-y|\,\mathrm{d}y\,\mathrm{d}x + 2\int_{x=-\frac{1}{2}}^{\frac{1}{2}} x^2 \int_{y=-\frac{1}{2}}^{\frac{1}{2}} |x-y|\,\mathrm{d}y\,\mathrm{d}x \tag{A.24}$$

$$-4\int_{x=-\frac{1}{2}}^{\frac{1}{2}} x \int_{y=-\frac{1}{2}}^{\frac{1}{2}} y|x-y|\,\mathrm{d}y\,\mathrm{d}x \tag{A.25}$$

$$+2\int_{x=-\frac{1}{2}}^{\frac{1}{2}}\int_{y=-\frac{1}{2}}^{\frac{1}{2}} y^2|x-y|\,\mathrm{d}y\,\mathrm{d}x \tag{A.26}$$

$$-\int_{x,y=-\frac{1}{2}}^{\frac{1}{2}} (x-y)^4\,\mathrm{d}y\,\mathrm{d}x. \tag{A.27}$$

In the calculation of (A.18), we made the the computation $\int_{x,y=-1/2}^{1/2}(x-y)^4\,\mathrm{d}y\,\mathrm{d}x = 1/15$. Apply (4.1) to the first, second, third, and fourth terms to obtain

$$= 1 - 2\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \left(x^2 + \frac{1}{4}\right)\mathrm{d}x + 2\int_{x=-\frac{1}{2}}^{\frac{1}{2}} x^2\left(x^2 + \frac{1}{4}\right)\mathrm{d}x \tag{A.28}$$

$$-4\int_{x=-\frac{1}{2}}^{\frac{1}{2}} x\left(\frac{1}{3}x^3 - \frac{1}{4}x\right)\mathrm{d}x + 2\int_{x=-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{6}x^4 + \frac{1}{32}\right)\mathrm{d}x - \frac{1}{15} \tag{A.29}$$

$$= 1 + \int_{x=-\frac{1}{2}}^{\frac{1}{2}} \left(x^4 - \frac{1}{2}x^2 - \frac{7}{16} \right) \mathrm{d}x - \frac{1}{15} \tag{A.30}$$

$$= 1 + \frac{1}{5}\frac{1}{16} - \frac{1}{6}\frac{1}{4} - \frac{7}{16} - \frac{1}{15} \tag{A.31}$$

$$= \frac{7}{15}. \tag{A.32}$$

□

Lemma A.6.

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x-y)| \,\mathrm{d}y = \begin{cases} -4|x|^3 + 1 & |x| \leq \frac{1}{3} \\ -2x^2 + \frac{29}{27} & \frac{1}{3} \leq |x| \leq \frac{1}{2}. \end{cases} \tag{A.33}$$

Proof. Note that $m_{\mathrm{SO(odd)}} \leq 0$ on $[-5/6, -1/2] \cup [1/2, 5/6]$ and $m_{\mathrm{SO(odd)}} \geq 0$ elsewhere. First, assume $x \in [0, 1/3]$. Then write

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x-y)| \,\mathrm{d}y = -\int_{y=-\frac{1}{2}}^{x-\frac{1}{2}} m_{\mathrm{SO(odd)}}(x-y) \,\mathrm{d}y + \int_{y=x-\frac{1}{2}}^{\frac{1}{2}} m_{\mathrm{SO(odd)}}(x-y) \,\mathrm{d}y. \tag{A.34}$$

We compute each integral individually.

$$\int_{y=-\frac{1}{2}}^{x-\frac{1}{2}} m_{\mathrm{SO(odd)}}(x-y) \,\mathrm{d}y = \int_{y=-\frac{1}{2}}^{x-\frac{1}{2}} \left[\frac{1}{2} + 4(1-x+y)^2 - 2(1-(x-y)^2) \right] \mathrm{d}y \tag{A.35}$$

$$= -\frac{3}{2}\left(x - \frac{1}{2} + \frac{1}{2} \right) + \frac{4}{3}(1-x+y)^3 \Big|_{y=-\frac{1}{2}}^{x-\frac{1}{2}} - \frac{2}{3}(x-y)^3 \Big|_{y=-\frac{1}{2}}^{x-\frac{1}{2}} \tag{A.36}$$

$$= -\frac{3}{2}x + \frac{4}{3}\frac{1}{8} - \frac{4}{3}\left(\frac{1}{2} - x \right)^3 - \frac{2}{3}\frac{1}{8} + \frac{2}{3}\left(x + \frac{1}{2} \right)^3 \tag{A.37}$$

$$= 2x^3 - x^2 \tag{A.38}$$

$$\int_{y=x-\frac{1}{2}}^{\frac{1}{2}} m_{\mathrm{SO(odd)}}(x-y) \,\mathrm{d}y = \int_{y=x-\frac{1}{2}}^{x} \left[\frac{1}{2} + 4(1-x+y)^2 - 2(1-(x-y)^2) \right] \mathrm{d}y \tag{A.39}$$

$$+ \int_{y=x}^{\frac{1}{2}} \left[\frac{1}{2} + 4(1-y+x)^2 - 2(1-(x-y)^2) \right] \mathrm{d}y \tag{A.40}$$

$$= -\frac{3}{2}\left(x - x + \frac{1}{2} \right) + \frac{4}{3}(1-x+y)^3 \Big|_{y=x-\frac{1}{2}}^{x} - \frac{2}{3}(x-y)^3 \Big|_{y=x-\frac{1}{2}}^{x} \tag{A.41}$$

$$-\frac{3}{2}\left(\frac{1}{2}-x\right)-\frac{4}{3}(1-y+x)^3\Big|_{y=x}^{\frac{1}{2}}-\frac{2}{3}(x-y)^3\Big|_{y=x}^{\frac{1}{2}} \tag{A.42}$$

$$=-\frac{3}{2}\frac{1}{2}+\frac{4}{3}-\frac{4}{3}\frac{1}{8}+\frac{2}{3}\frac{1}{8}-\frac{3}{2}\left(\frac{1}{2}-x\right)-\frac{4}{3}\left(\frac{1}{2}+x\right)^3+\frac{4}{3}-\frac{2}{3}\left(x-\frac{1}{2}\right)^3 \tag{A.43}$$

$$=-2x^3-x^2+1. \tag{A.44}$$

Combining, we obtain

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}}|m_{\mathrm{SO(odd)}}(x-y)|\,\mathrm{d}y=-4x^3+1. \tag{A.45}$$

Now assume $x\in[1/3,1/2]$. Then write

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}}|m_{\mathrm{SO(odd)}}(x-y)|\,\mathrm{d}y=\int_{y=-\frac{1}{2}}^{x-\frac{5}{6}}m_{\mathrm{SO(odd)}}(x-y)\,\mathrm{d}y-\int_{y=x-\frac{5}{6}}^{x-\frac{1}{2}}m_{\mathrm{SO(odd)}}(x-y)\,\mathrm{d}y+\int_{y=x-\frac{1}{2}}^{\frac{1}{2}}m_{\mathrm{SO(odd)}}(x-y)\,\mathrm{d}y. \tag{A.46}$$

We again compute each integral individually.

$$\int_{y=-\frac{1}{2}}^{x-\frac{5}{6}}m_{\mathrm{SO(odd)}}(x-y)\,\mathrm{d}y=\int_{y=-\frac{1}{2}}^{x-\frac{5}{6}}\left[\frac{1}{2}+4(1-x+y)^2-2(1-(x-y)^2)\right]\,\mathrm{d}y \tag{A.47}$$

$$=-\frac{3}{2}\left(x-\frac{5}{6}+\frac{1}{2}\right)+\frac{4}{3}(1-x+y)^3\Big|_{y=-\frac{1}{2}}^{x-\frac{5}{6}}-\frac{2}{3}(x-y)^3\Big|_{y=-\frac{1}{2}}^{x-\frac{5}{6}} \tag{A.48}$$

$$=-\frac{3}{2}\left(x-\frac{1}{3}\right)+\frac{4}{3}\frac{1}{6^3}-\frac{4}{3}\left(\frac{1}{2}-x\right)^3-\frac{2}{3}\left(\frac{5}{6}\right)^3+\frac{2}{3}\left(x+\frac{1}{2}\right)^3 \tag{A.49}$$

$$=2x^3-x^2+\frac{1}{27} \tag{A.50}$$

$$\int_{y=x-\frac{5}{6}}^{x-\frac{1}{2}}m_{\mathrm{SO(odd)}}(x-y)\,\mathrm{d}y=\int_{y=x-\frac{5}{6}}^{x-\frac{1}{2}}\left[\frac{1}{2}+4(1-x+y)^2-2(1-(x-y)^2)\right]\,\mathrm{d}y \tag{A.51}$$

$$=-\frac{3}{2}\left(x-\frac{1}{2}-x+\frac{5}{6}\right)+\frac{4}{3}(1-x+y)^3\Big|_{y=x-\frac{5}{6}}^{x-\frac{1}{2}}-\frac{2}{3}(x-y)^3\Big|_{y=x-\frac{5}{6}}^{x-\frac{1}{2}} \tag{A.52}$$

$$=-\frac{3}{2}\frac{1}{3}+\frac{4}{3}\frac{1}{8}-\frac{4}{3}\frac{1}{6^3}-\frac{2}{3}\frac{1}{8}+\frac{2}{3}\left(\frac{5}{6}\right)^3 \tag{A.53}$$

$$= -\frac{1}{27} \tag{A.54}$$

$$\int_{y=x-\frac{1}{2}}^{\frac{1}{2}} m_{\mathrm{SO(odd)}}(x-y)\,\mathrm{d}y = \int_{y=x-\frac{1}{2}}^{x} \left[\frac{1}{2} + 4(1-x+y)^2 - 2(1-(x-y)^2)\right]\,\mathrm{d}y \tag{A.55}$$

$$+ \int_{y=x}^{\frac{1}{2}} \left[\frac{1}{2} + 4(1-y+x)^2 - 2(1-(x-y)^2)\right]\,\mathrm{d}y \tag{A.56}$$

$$= -\frac{3}{2}\left(x - x + \frac{1}{2}\right) + \frac{4}{3}(1-x+y)^3\Big|_{y=x-\frac{1}{2}}^{x} - \frac{2}{3}(x-y)^3\Big|_{y=x-\frac{1}{2}}^{x} \tag{A.57}$$

$$-\frac{3}{2}\left(\frac{1}{2} - x\right) - \frac{4}{3}(1-y+x)^3\Big|_{y=x}^{\frac{1}{2}} - \frac{2}{3}(x-y)^3\Big|_{y=x}^{\frac{1}{2}} \tag{A.58}$$

$$= -\frac{3}{2}(1-x) + \frac{4}{3} - \frac{4}{3}\frac{1}{8} + \frac{2}{3}\frac{1}{8} - \frac{4}{3}\left(\frac{1}{2}+x\right)^3 + \frac{4}{3} - \frac{2}{3}\left(x - \frac{1}{2}\right)^3 \tag{A.59}$$

$$= -2x^3 - x^2 + 1. \tag{A.60}$$

Combining gives

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x-y)|\,\mathrm{d}y = -2x^2 + \frac{29}{27}. \tag{A.61}$$

To obtain (A.33), use the fact that $m_{\mathrm{SO(odd)}}$ is even, then make the substitution $u = -y$:

$$\int_{y=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(-x-y)|\,\mathrm{d}y = \int_{y=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x+y)|\,\mathrm{d}y = \int_{u=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x-u)|\,\mathrm{d}u. \tag{A.62}$$

□

Lemma A.7.

$$\int_{y,z=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x-y)m_{\mathrm{SO(odd)}}(y-z)|\,\mathrm{d}z\,\mathrm{d}y = \begin{cases} 4|x|^5 - \frac{8}{3}x^4 - \frac{26}{27}|x|^3 - \frac{14}{9}x^2 + \frac{3511}{3645} & |x| \leq \frac{1}{6} \\ -\frac{4}{5}x^6 + \frac{32}{5}|x|^5 - 5x^4 - \frac{7}{4}x^2 + \frac{31}{1620}|x| + \frac{77}{80} & \frac{1}{6} \leq |x| \leq \\ -\frac{4}{5}x^6 + \frac{28}{5}|x|^5 + \frac{1}{3}x^4 + \frac{26}{27}|x|^3 - \frac{419}{36}x^2 + \frac{111}{20}|x| + \frac{1393}{11664} & \frac{1}{3} \leq |x| \leq \end{cases} \tag{A.63}$$

Proof. Applying (A.33), we have

$$|m_{\mathrm{SO(odd)}}(x-y)| \int_{z=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(y-z)|\,\mathrm{d}z = \begin{cases} |m_{\mathrm{SO(odd)}}(x-y)|(-4|y|^3+1) & |y| \leq \frac{1}{3} \\ |m_{\mathrm{SO(odd)}}(x-y)|(-2y^2 + \frac{29}{27}) & \frac{1}{3} \leq |y| \leq \frac{1}{2}. \end{cases} \tag{A.64}$$

Assume $x \in [0, 1/6]$. We first compute

$$\int_{y=-\frac{1}{3}}^{\frac{1}{3}} |m_{\mathrm{SO(odd)}}(x-y)|(-4|y|^3+1)\,\mathrm{d}y = \int_{y=-\frac{1}{3}}^{0} m_{\mathrm{SO(odd)}}(x-y)(4y^3+1)\,\mathrm{d}y + \int_{y=0}^{\frac{1}{3}} m_{\mathrm{SO(odd)}}(x-y)\left(-4y^3+1\right)\,\mathrm{d}y. \tag{A.65}$$

We compute each term individually.

$$\int_{y=-\frac{1}{3}}^{0} m_{\mathrm{SO(odd)}}(x-y)(4y^3+1)\,\mathrm{d}y = \int_{y=-\frac{1}{3}}^{0} \left[\frac{1}{2} + 4(1-x+y)^2 - 2(1-(x-y)^2)\right](4y^3+1)\,\mathrm{d}y \tag{A.66}$$

$$= \int_{y=-\frac{1}{3}}^{0} \Big[24x^2y^3 + 6x^2 - 48xy^4 - 32xy^3 - 12xy \tag{A.67}$$

$$- 8x + 24y^5 + 32y^4 + 10y^3 + 6y^2 + 8y + \frac{5}{2}\Big]\,\mathrm{d}y \tag{A.68}$$

$$= -6\frac{1}{81}x^2 + 6\frac{1}{3}x^2 - \frac{48}{5}\frac{1}{243}x + 8\frac{1}{81}x + 6\frac{1}{9}x \tag{A.69}$$

$$- 8\frac{1}{3}x - 4\frac{1}{729} + \frac{32}{5}\frac{1}{243} - \frac{5}{2}\frac{1}{81} + 2\frac{1}{27} - 4\frac{1}{9} + \frac{5}{2}\frac{1}{3} \tag{A.70}$$

$$= \frac{52}{27}x^2 - \frac{262}{135}x + \frac{1651}{3645} \tag{A.71}$$

$$\int_{y=0}^{\frac{1}{3}} m_{\mathrm{SO(odd)}}(x-y)(-4y^3+1)\,\mathrm{d}y = \int_{y=0}^{x} \left[\frac{1}{2} + 4(1-x+y)^2 - 2(1-(x-y)^2)\right](-4y^3+1)\,\mathrm{d}y \tag{A.72}$$

$$+ \int_{y=x}^{\frac{1}{3}} \left[\frac{1}{2} + 4(1-y+x)^2 - 2(1-(x-y)^2)\right](-4y^3+1)\,\mathrm{d}y \tag{A.73}$$

$$= \int_{y=0}^{x} \Big[-24y^5 + (48x-32)y^4 + (-24x^2+32x-10)y^3 \tag{A.74}$$

$$+ 6y^2 + (12x+8)y + \left(6x^2 - 8x + \frac{5}{2}\right)\Big]\,\mathrm{d}y \tag{A.75}$$

$$+\int_{y=x}^{\frac{1}{3}} \Big[-24y^5 + (48x+32)y^4 + (-24x^2-32x-10)y^3 \tag{A.76}$$

$$+ 6y^2 + (12x-8)y + \left(6x^2+8x+\frac{5}{2}\right)\Big]\,\mathrm{d}y \tag{A.77}$$

$$= -4x^6 + \frac{1}{5}(48x-32)x^5 + \left(-6x^2+8x-\frac{5}{2}\right)x^4 + 2x^3 \tag{A.78}$$

$$+ (6x+4)x^2 + \left(6x^2-8x+\frac{5}{2}\right)x \tag{A.79}$$

$$-4\left(\frac{1}{729}-x^6\right) + \frac{1}{5}(48x+32)\left(\frac{1}{243}-x^5\right) + \left(-6x^2-8x-\frac{5}{2}\right)\left(\frac{1}{81}-\right. \tag{A.80}$$

$$+2\left(\frac{1}{27}-x^3\right) + (6x-4)\left(\frac{1}{9}-x^2\right) + \left(6x^2+8x+\frac{5}{2}\right)\left(\frac{1}{3}-x\right) \tag{A.81}$$

$$= \frac{16}{5}x^5 - \frac{164}{27}x^2 + \frac{262}{135}x + \frac{1651}{3645}. \tag{A.82}$$

Combining gives

$$\int_{y=-\frac{1}{3}}^{\frac{1}{3}} |m_{\mathrm{SO(odd)}}(x-y)|(-4|y|^3+1)\,\mathrm{d}y = \frac{16}{5}x^5 - \frac{112}{27}x^2 + \frac{3302}{3645}. \tag{A.83}$$

We next compute

$$\int_{y=\frac{1}{3}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x-y)| \left(-2y^2+\frac{29}{27}\right)\mathrm{d}y = \int_{y=\frac{1}{3}}^{\frac{1}{2}} \left[\frac{1}{2} + 4(1-y+x)^2 - 2(1-(x-y)^2)\right]\left(-2y^2+\frac{29}{27}\right)\mathrm{d}y \tag{A.84}$$

$$= \frac{13}{18}x^2 + \frac{239}{648}x + \frac{209}{7290}. \tag{A.85}$$

Finally, compute

$$\int_{y=-\frac{1}{2}}^{x-\frac{1}{2}} m_{\mathrm{SO(odd)}}(x-y)\left(-2y^2+\frac{29}{27}\right)\mathrm{d}y = \int_{y=-\frac{1}{2}}^{x-\frac{1}{2}} \left[\frac{1}{2} + 4(1-x+y)^2 - 2(1-(x-y)^2)\right]\left(-2y^2+\frac{29}{27}\right)\mathrm{d}y \tag{A.86}$$

$$= -\frac{2}{5}x^5 + \frac{4}{3}x^4 + \frac{13}{27}x^3 - \frac{31}{54}x^2 \tag{A.87}$$

$$\int_{y=x-\frac{1}{2}}^{-\frac{1}{3}} m_{\mathrm{SO(odd)}}(x-y)\left(-2y^2+\frac{29}{27}\right)\mathrm{d}y = \int_{y=x-\frac{1}{2}}^{-\frac{1}{3}} \left[\frac{1}{2}+4(1-x+y)^2-2(1-(x-y)^2)\right]\left(-2y^2+\frac{29}{27}\right)\mathrm{d}y \tag{A.88}$$

$$= \frac{2}{5}x^5-\frac{4}{3}x^4-\frac{13}{27}x^3+\frac{35}{27}x^2-\frac{239}{648}x+\frac{209}{7290} \tag{A.89}$$

$$\int_{y=-\frac{1}{2}}^{-\frac{1}{3}} |m_{\mathrm{SO(odd)}}(x-y)|\left(-2y^2+\frac{29}{27}\right)\mathrm{d}y = -\int_{y=-\frac{1}{2}}^{x-\frac{1}{2}} m_{\mathrm{SO(odd)}}(x-y)\left(-2y^2+\frac{29}{27}\right)\mathrm{d}y \tag{A.90}$$

$$+\int_{y=x-\frac{1}{2}}^{-\frac{1}{3}} m_{\mathrm{SO(odd)}}(x-y)\left(-2y^2+\frac{29}{27}\right)\mathrm{d}y \tag{A.91}$$

$$= \frac{4}{5}x^5-\frac{8}{3}x^4-\frac{26}{27}x^3+\frac{101}{54}x^2-\frac{239}{648}x+\frac{209}{7290}. \tag{A.92}$$

So for $x \in [0, 1/6]$, we have

$$\int_{y,z=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x-y)m_{\mathrm{SO(odd)}}(y-z)|\,\mathrm{d}z\,\mathrm{d}y = 4x^5-\frac{8}{3}x^4-\frac{26}{27}x^3-\frac{14}{9}x^2+\frac{3511}{3645}. \tag{A.93}$$

Now assume $x \in [1/6, 1/3]$. Then

$$\int_{y=0}^{x} m_{\mathrm{SO(odd)}}(x-y)(-4y^3+1)\,\mathrm{d}y = \int_{y=0}^{x} \left[\frac{1}{2}+4(1-x+y)^2-2(1-(x-y)^2)\right](-4y^3+1)\,\mathrm{d}y \tag{A.94}$$

$$= -\frac{2}{5}x^6+\frac{8}{5}x^5-\frac{5}{2}x^4+2x^3-4x^2+\frac{5}{2}x \tag{A.95}$$

$$\int_{y=x}^{\frac{1}{3}} m_{\mathrm{SO(odd)}}(x-y)(-4y^3+1)\,\mathrm{d}y = \int_{y=x}^{\frac{1}{3}} \left[\frac{1}{2}+4(1-y+x)^2-2(1-(x-y)^2)\right](-4y^3+1)\,\mathrm{d}y \tag{A.96}$$

$$= \frac{2}{5}x^6+\frac{8}{5}x^5+\frac{5}{2}x^4-2x^3-\frac{56}{27}x^2-\frac{151}{270}x+\frac{1651}{3645} \tag{A.97}$$

$$\int_{y=-\frac{1}{3}}^{x-\frac{1}{2}} m_{\mathrm{SO(odd)}}(x-y)(4y^3+1)\,\mathrm{d}y = \int_{y=-\frac{1}{3}}^{x-\frac{1}{2}} \left[\frac{1}{2}+4(1-x+y)^2-2(1-(x-y)^2)\right](4y^3+1)\,\mathrm{d}y \tag{A.98}$$

$$= \frac{2}{5}x^6-\frac{8}{5}x^5+\frac{5}{2}x^4-\frac{259}{216}x^2+\frac{97}{270}x-\frac{3301}{116640} \tag{A.99}$$

$$\int_{y=x-\frac{1}{2}}^{0} m_{\mathrm{SO(odd)}}(x-y)(4y^3+1)\,\mathrm{d}y = \int_{y=x-\frac{1}{2}}^{0} \left[\frac{1}{2}+4(1-x+y)^2-2(1-(x-y)^2)\right](4y^3+1)\,\mathrm{d}y \tag{A.100}$$

$$= -\frac{2}{5}x^6+\frac{8}{5}x^5-\frac{5}{2}x^4+\frac{25}{8}x^2-\frac{23}{10}x+\frac{77}{160}. \tag{A.101}$$

Therefore,

$$\int_{y=-\frac{1}{3}}^{\frac{1}{3}} |m_{\mathrm{SO(odd)}}(x-y)|(-4|y|^3+1)\,\mathrm{d}y = \int_{y=0}^{x} m_{\mathrm{SO(odd)}}(x-y)(-4y^3+1)\,\mathrm{d}y + \int_{y=x}^{\frac{1}{3}} m_{\mathrm{SO(odd)}}(x-y)(-4y^3+1)\,\mathrm{d}y \tag{A.102}$$

$$-\int_{y=-\frac{1}{3}}^{x-\frac{1}{2}} m_{\mathrm{SO(odd)}}(x-y)(4y^3+1)\,\mathrm{d}y + \int_{y=x-\frac{1}{2}}^{0} m_{\mathrm{SO(odd)}}(x-y)(4y^3+1 \tag{A.103}$$

$$= -\frac{4}{5}x^6+\frac{32}{5}x^5-5x^4-\frac{7}{4}x^2-\frac{97}{135}x+\frac{77}{80}. \tag{A.104}$$

Next we compute

$$\int_{y=\frac{1}{3}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x-y)|\left(-2y^2+\frac{29}{27}\right)\,\mathrm{d}y = \int_{y=\frac{1}{3}}^{\frac{1}{2}} \left[\frac{1}{2}+4(1-y+x)^2-2(1-(x-y)^2)\right]\left(-2y^2+\frac{29}{27}\right)\,\mathrm{d}y \tag{A.105}$$

$$= \frac{13}{18}x^2+\frac{239}{648}x+\frac{209}{7290}. \tag{A.106}$$

Finally, we compute

$$\int_{y=-\frac{1}{2}}^{-\frac{1}{3}} |m_{\mathrm{SO(odd)}}(x-y)|\left(-2y^2+\frac{29}{27}\right)\,\mathrm{d}y = -\int_{y=-\frac{1}{2}}^{-\frac{1}{3}} \left[\frac{1}{2}+4(1-x+y)^2-2(1-(x-y)^2)\right]\left(-2y^2+\frac{29}{27}\right)\,\mathrm{d}y \tag{A.107}$$

$$= -\frac{13}{18}x^2+\frac{239}{648}x-\frac{209}{7290}. \tag{A.108}$$

Combining, for $x \in [1/6, 1/3]$, we have

$$\int_{y,z=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x-y)m_{\mathrm{SO(odd)}}(y-z)| \, \mathrm{d}z \, \mathrm{d}y = -\frac{4}{5}x^6+\frac{32}{5}x^5-5x^4-\frac{7}{4}x^2+\frac{31}{1620}x+\frac{77}{80} \quad (A.109)$$

Lastly, we assume $x \in [1/3, 1/2]$.

$$\int_{y=-\frac{1}{3}}^{x-\frac{1}{2}} m_{\mathrm{SO(odd)}}(x-y)(4y^3+1)\,\mathrm{d}y = \int_{y=-\frac{1}{3}}^{x-\frac{1}{2}} \left[\frac{1}{2}+4(1-x+y)^2-2(1-(x-y)^2)\right](4y^3+1)\,\mathrm{d}y \quad (A.110)$$

$$= \frac{2}{5}x^6-\frac{8}{5}x^5+\frac{5}{2}x^4-\frac{259}{216}x^2+\frac{97}{270}x-\frac{3301}{116640} \quad (A.111)$$

$$\int_{y=x-\frac{1}{2}}^{0} m_{\mathrm{SO(odd)}}(x-y)(4y^3+1)\,\mathrm{d}y = \int_{y=x-\frac{1}{2}}^{0} \left[\frac{1}{2}+4(1-x+y)^2-2(1-(x-y)^2)\right](4y^3+1)\,\mathrm{d}y \quad (A.112)$$

$$= -\frac{2}{5}x^6+\frac{8}{5}x^5-\frac{5}{2}x^4+\frac{25}{8}x^2-\frac{23}{10}x+\frac{77}{160} \quad (A.113)$$

$$\int_{y=0}^{x} m_{\mathrm{SO(odd)}}(x-y)(-4y^3+1)\,\mathrm{d}y = \int_{y=0}^{x} \left[\frac{1}{2}+4(1-x+y)^2-2(1-(x-y)^2)\right](-4y^3+1)\,\mathrm{d}y \quad (A.114)$$

$$= -\frac{2}{5}x^6+\frac{8}{5}x^5-\frac{5}{2}x^4+2x^3-4x^2+\frac{5}{2}x \quad (A.115)$$

$$\int_{y=x}^{\frac{1}{3}} m_{\mathrm{SO(odd)}}(x-y)(-4y^3+1)\,\mathrm{d}y = \int_{y=x}^{\frac{1}{3}} \left[\frac{1}{2}+4(1-y+x)^2-2(1-(x-y)^2)\right](-4y^3+1)\,\mathrm{d}y \quad (A.116)$$

$$= \frac{2}{5}x^6+\frac{8}{5}x^5+\frac{5}{2}x^4-2x^3-\frac{56}{27}x^2-\frac{151}{270}x+\frac{1651}{3645}. \quad (A.117)$$

Combining, we have

$$\int_{y=-\frac{1}{3}}^{\frac{1}{3}} |m_{\mathrm{SO(odd)}}(x-y)|(-4|y|^3+1) = -\int_{y=-\frac{1}{3}}^{x-\frac{1}{2}} m_{\mathrm{SO(odd)}}(x-y)(4y^3+1)\,\mathrm{d}y + \int_{y=x-\frac{1}{2}}^{0} m_{\mathrm{SO(odd)}}(x-y)(4y^3+1)\,\mathrm{d}y \quad (A.118)$$

$$+ \int_{y=0}^{x} m_{\mathrm{SO(odd)}}(x-y)(-4y^3+1)\,\mathrm{d}y + \int_{y=x}^{\frac{1}{3}} m_{\mathrm{SO(odd)}}(x-y)(-4y^3+1)\,\mathrm{d}y \quad (A.119)$$

$$= -\frac{4}{5}x^6 + \frac{32}{5}x^5 - 5x^4 - \frac{7}{4}x^2 - \frac{97}{135}x + \frac{77}{80}. \tag{A.120}$$

Next, observe that

$$\int\limits_{y=\frac{1}{3}}^{x} m_{\mathrm{SO(odd)}}(x-y)\left(-2y^2+\frac{29}{27}\right)\,\mathrm{d}y = \int\limits_{y=\frac{1}{3}}^{x} \left[\frac{1}{2} + 4(1-x+y)^2 - 2(1-(x-y)^2)\right]\left(-2y^2+\frac{29}{27}\right)\,\mathrm{d}y \tag{A.121}$$

$$= -\frac{2}{5}x^5 + \frac{4}{3}x^4 + \frac{13}{27}x^3 - \frac{170}{27}x^2 + \frac{971}{162}x - \frac{9703}{7290} \tag{A.122}$$

$$\int\limits_{y=x}^{\frac{1}{2}} m_{\mathrm{SO(odd)}}(x-y)\left(-2y^2+\frac{29}{27}\right)\,\mathrm{d}y = \int\limits_{y=x}^{\frac{1}{2}} \left[\frac{1}{2} + 4(1-y+x)^2 - 2(1-(x-y)^2)\right]\left(-2y^2+\frac{29}{27}\right)\,\mathrm{d}y \tag{A.123}$$

$$= \frac{2}{5}x^5 + \frac{4}{3}x^4 - \frac{13}{27}x^3 - \frac{85}{54}x^2 - \frac{7}{24}x + \frac{68}{135}. \tag{A.124}$$

Combining gives

$$\int\limits_{y=\frac{1}{3}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x-y)|\left(-2y^2+\frac{29}{27}\right)\,\mathrm{d}y = \int\limits_{y=\frac{1}{3}}^{x} m_{\mathrm{SO(odd)}}(x-y)\left(-2y^2+\frac{29}{27}\right)\,\mathrm{d}y + \int\limits_{y=x}^{\frac{1}{2}} m_{\mathrm{SO(odd)}}(x-y)\left(-2y^2+ \right. \tag{A.125}$$

$$= \frac{8}{3}x^4 - \frac{425}{54}x^2 + \frac{3695}{648}x - \frac{6031}{7290}. \tag{A.126}$$

Finally, we compute

$$\int\limits_{y=-\frac{1}{2}}^{x-\frac{5}{6}} m_{\mathrm{SO(odd)}}(x-y)\left(-2y^2+\frac{29}{27}\right)\,\mathrm{d}y = \int\limits_{y=-\frac{1}{2}}^{x-\frac{5}{6}} \left[\frac{1}{2} + 4(1-x+y)^2 - 2(1-(x-y)^2)\right]\left(-2y^2+\frac{29}{27}\right)\,\mathrm{d}y \tag{A.127}$$

$$= -\frac{2}{5}x^5 + \frac{4}{3}x^4 + \frac{13}{27}x^3 - \frac{35}{54}x^2 + \frac{8}{81}x + \frac{47}{7290} \tag{A.128}$$

$$\int\limits_{y=x-\frac{5}{6}}^{-\frac{1}{3}} m_{\mathrm{SO(odd)}}(x-y)\left(-2y^2+\frac{29}{27}\right)\,\mathrm{d}y = \int\limits_{y=x-\frac{5}{6}}^{-\frac{1}{3}} \left[\frac{1}{2} + 4(1-x+y)^2 - 2(1-(x-y)^2)\right]\left(-2y^2+\frac{29}{27}\right)\,\mathrm{d}y \tag{A.129}$$

$$= \frac{2}{5}x^5 - \frac{4}{3}x^4 - \frac{13}{27}x^3 + \frac{37}{27}x^2 - \frac{101}{216}x + \frac{1}{45}. \tag{A.130}$$

Thus,

$$\int_{y=-\frac{1}{2}}^{-\frac{1}{3}} |m_{\mathrm{SO(odd)}}(x-y)| \left(-2y^2+\frac{29}{27}\right) \mathrm{d}y = \int_{y=-\frac{1}{2}}^{x-\frac{5}{6}} m_{\mathrm{SO(odd)}}(x-y) \left(-2y^2+\frac{29}{27}\right) \mathrm{d}y - \int_{y=x-\frac{5}{6}}^{-\frac{1}{3}} m_{\mathrm{SO(odd)}}(x-y) \left(-2y \tag{A.131}$$

$$= -\frac{4}{5}x^5+\frac{8}{3}x^4+\frac{26}{27}x^3-\frac{109}{54}x^2+\frac{367}{648}x-\frac{23}{1458}. \tag{A.132}$$

Therefore, for $x \in [1/3, 1/2]$, we have

$$\int_{y,z=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x-y)m_{\mathrm{SO(odd)}}(y-z)|\,\mathrm{d}z\,\mathrm{d}y = -\frac{4}{5}x^6+\frac{28}{5}x^5+\frac{1}{3}x^4+\frac{26}{27}x^3-\frac{419}{36}x^2+\frac{111}{20}x+\frac{1393}{11664}. \tag{A.133}$$

We obtain (A.63) using the fact that $m_{\mathrm{SO(odd)}}$ is even and performing the substitution $u=-y$, $v=-z$:

$$\int_{y,z=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(-x-y)m_{\mathrm{SO(odd)}}(y-z)|\,\mathrm{d}z\,\mathrm{d}y = \int_{y,z=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x+y)m(y-z)|\,\mathrm{d}z\,\mathrm{d}y \tag{A.134}$$

$$= \int_{u,v=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x-u)m_{\mathrm{SO(odd)}}(-u+v)|\,\mathrm{d}v\,\mathrm{d}u \tag{A.135}$$

$$= \int_{u,v=-\frac{1}{2}}^{\frac{1}{2}} |m_{\mathrm{SO(odd)}}(x-u)m_{\mathrm{SO(odd)}}(u-v)|\,\mathrm{d}v\,\mathrm{d}u. \tag{A.136}$$

$\square$