Computational Techniques Coursework 6

Solution 1

We perform six iterations of the power method using l_1 norm

$$x_1 = \frac{Ax_0}{\|Ax_0\|} = \frac{1}{14}\begin{bmatrix} -10 \\ -4 \end{bmatrix} = \begin{bmatrix} -\frac{5}{7} \\ -\frac{2}{7} \end{bmatrix}$$
$$x_2 = \frac{Ax_1}{\|Ax_1\|} = \frac{7}{19}\begin{bmatrix} 2 \\ \frac{5}{7} \end{bmatrix} = \begin{bmatrix} \frac{14}{19} \\ \frac{5}{19} \end{bmatrix}$$
$$x_3 = \frac{Ax_2}{\|Ax_2\|} = \frac{19}{43}\begin{bmatrix} -\frac{32}{19} \\ -\frac{11}{19} \end{bmatrix} = \begin{bmatrix} -\frac{32}{43} \\ -\frac{11}{43} \end{bmatrix}$$
$$x_4 = \frac{Ax_3}{\|Ax_3\|} = \frac{43}{91}\begin{bmatrix} \frac{68}{43} \\ \frac{23}{43} \end{bmatrix} = \begin{bmatrix} \frac{68}{91} \\ \frac{23}{91} \end{bmatrix}$$
$$x_5 = \frac{Ax_4}{\|Ax_4\|} = \frac{91}{187}\begin{bmatrix} -\frac{140}{91} \\ -\frac{47}{91} \end{bmatrix} = \begin{bmatrix} -\frac{140}{187} \\ -\frac{47}{187} \end{bmatrix}$$
$$x_6 = \frac{Ax_5}{\|Ax_5\|} = \frac{187}{379}\begin{bmatrix} \frac{284}{187} \\ \frac{95}{187} \end{bmatrix} = \begin{bmatrix} \frac{284}{379} \\ \frac{95}{379} \end{bmatrix}$$

Hence the estimated dominant eigenvector of A is $\begin{bmatrix} \frac{284}{379} \\ \frac{95}{379} \end{bmatrix} \approx \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$

In the last iterationm we use the value of x_6 to approximate the dominant eigenvalue

$$\sigma = \frac{x_6^T A x_6}{x_6^T x_6} = -\frac{180593}{89681} \approx -2.014$$
$$|\lambda_{\text{dominant}}| \approx 2.014$$

Solution 2

We perform seven iterations of the power method using l_∞ norm instead

$$x_1 = \frac{Ax_0}{\|Ax_0\|} = \frac{1}{5}\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

$$x_2 = \frac{Ax_1}{\|Ax_1\|} = \frac{5}{11}\begin{bmatrix} 1 \\ 1 \\ \frac{11}{5} \end{bmatrix} = \begin{bmatrix} \frac{5}{11} \\ \frac{5}{11} \\ 1 \end{bmatrix}$$

$$x_3 = \frac{Ax_2}{\|Ax_2\|} = \frac{11}{31}\begin{bmatrix} \frac{15}{11} \\ \frac{17}{11} \\ \frac{31}{11} \end{bmatrix} = \begin{bmatrix} \frac{15}{31} \\ \frac{17}{31} \\ 1 \end{bmatrix}$$

$$x_4 = \frac{Ax_3}{\|Ax_3\|} = \frac{31}{97}\begin{bmatrix} \frac{49}{31} \\ \frac{18}{31} \\ \frac{97}{31} \end{bmatrix} = \begin{bmatrix} \frac{49}{97} \\ \frac{18}{97} \\ 1 \end{bmatrix}$$

$$x_5 = \frac{Ax_4}{\|Ax_4\|} = \frac{97}{200}\begin{bmatrix} \frac{85}{97} \\ \frac{114}{97} \\ \frac{200}{97} \end{bmatrix} = \begin{bmatrix} \frac{85}{200} \\ \frac{114}{200} \\ 1 \end{bmatrix}$$

$$x_6 = \frac{Ax_5}{\|Ax_5\|} = \frac{200}{627}\begin{bmatrix} \frac{313}{200} \\ \frac{344}{200} \\ \frac{627}{200} \end{bmatrix} = \begin{bmatrix} \frac{313}{627} \\ \frac{344}{627} \\ 1 \end{bmatrix}$$

$$x_7 = \frac{Ax_6}{\|Ax_6\|} = \frac{627}{1972}\begin{bmatrix} \frac{1001}{627} \\ \frac{772}{627} \\ \frac{1972}{627} \end{bmatrix} = \begin{bmatrix} \frac{1001}{1972} \\ \frac{772}{1972} \\ 1 \end{bmatrix}$$

The approximate dominant eigenvector is $\begin{bmatrix} 0.5 \\ 0.5 \\ 1 \end{bmatrix}$

Solution 3

We first prove the Sherman-Morrison formula directly by multiplying $(\mathbf{A} + \mathbf{u}\mathbf{v}^T)$ on the right hand side

$$\begin{aligned}
(\mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1+\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}})(\mathbf{A} + \mathbf{u}\mathbf{v}^T) &= \mathbf{A}^{-1}(\mathbf{A} + \mathbf{u}\mathbf{v}^T) - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1+\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}}(\mathbf{A} + \mathbf{u}\mathbf{v}^T) \\
&= \mathbf{A}^{-1}\mathbf{A} + \mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}\mathbf{A}}{1+\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T}{1+\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} \\
&= \mathbf{I} + \mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T}{1+\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} - \frac{(\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T)(\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T)}{1+\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} \\
&= \mathbf{I} + \frac{(\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T)(1+\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u})}{1+\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} - \frac{(\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T)(1+\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u})}{1+\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} \\
&= \mathbf{I}
\end{aligned}$$

which is exactly the same result we get when we multiply $(\mathbf{A}{+}\mathbf{u}\mathbf{v}^T)$ to the left hand side

$$(\mathbf{A}{+}\mathbf{u}\mathbf{v}^T)^{-1}(\mathbf{A}{+}\mathbf{u}\mathbf{v}^T) = \mathbf{I}$$

The same result follows if we multiply the right hanside like this

$$(\mathbf{A}+\mathbf{u}\mathbf{v}^T)(\mathbf{A}^{-1}-\frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1+\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}})=\mathbf{I}$$

a) In the Sherman-Morrison formula, we let $\mathbf{A}=\mathbf{I}$ and we can easily get the result

$$\begin{aligned}(\mathbf{I}+\mathbf{u}\mathbf{v}^T)^{-1} &= \mathbf{I}^{-1}-\frac{\mathbf{I}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{I}^{-1}}{1+\mathbf{v}^T\mathbf{I}^{-1}\mathbf{u}} \\ &= \mathbf{I}-\frac{\mathbf{u}\mathbf{v}^T}{1+\mathbf{v}^T\mathbf{u}}\end{aligned}$$

b) In the equation we have just proved in a), we further choose $\mathbf{v}=-\frac{2}{\mathbf{u}^T\mathbf{u}}\mathbf{u}$. Them the left hand side becomes

$$\begin{aligned}(\mathbf{I}+\mathbf{u}\mathbf{v}^T)^{-1} &= (\mathbf{I}+\mathbf{u}(\frac{-2}{\mathbf{u}^T\mathbf{u}}\mathbf{u})^T)^{-1} \\ &= (\mathbf{I}+(\frac{-2}{\mathbf{u}^T\mathbf{u}})\mathbf{u}\mathbf{u}^T)^{-1} \\ &= (\mathbf{I}-2\frac{\mathbf{u}\mathbf{u}^T}{\mathbf{u}^T\mathbf{u}})^{-1}\end{aligned}$$

and the right hand side becomes

$$\begin{aligned}\mathbf{I}-\frac{\mathbf{u}\mathbf{v}^T}{1+\mathbf{v}^T\mathbf{u}} &= \mathbf{I}-\frac{\mathbf{u}(\frac{-2}{\mathbf{u}^T\mathbf{u}}\mathbf{u})^T}{1+(\frac{-2}{\mathbf{u}^T\mathbf{u}}\mathbf{u})^T\mathbf{u}} \\ &= \mathbf{I}-\frac{(\frac{-2}{\mathbf{u}^T\mathbf{u}})\mathbf{u}\mathbf{u}^T}{1+\frac{-2}{\mathbf{u}^T\mathbf{u}}\mathbf{u}^T\mathbf{u}} \\ &= \mathbf{I}-\frac{(\frac{-2}{\mathbf{u}^T\mathbf{u}})\mathbf{u}\mathbf{u}^T}{1-2} \\ &= \mathbf{I}-2\frac{\mathbf{u}\mathbf{u}^T}{\mathbf{u}^T\mathbf{u}}\end{aligned}$$

c) Let $\mathbf{w}=\mathbf{A}\mathbf{u}$ and hence $\mathbf{A}+\mathbf{w}\mathbf{v}^T=\mathbf{A}(\mathbf{I}+\mathbf{u}\mathbf{v}^T)$

We plug in those equations into a)

$$\begin{aligned}(\mathbf{A}+\mathbf{w}\mathbf{v}^T)^{-1} &= (\mathbf{A}(\mathbf{I}+\mathbf{u}\mathbf{v}^T))^{-1} \\ &= (\mathbf{I}+\mathbf{u}\mathbf{v}^T)^{-1}\mathbf{A}^{-1} && \text{(By the fact that } (\mathbf{A}\mathbf{B})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}\text{)} \\ &= (\mathbf{I}-\frac{\mathbf{u}\mathbf{v}^T}{1+\mathbf{v}^T\mathbf{u}})\mathbf{A}^{-1} && \text{(From a)} \\ &= \mathbf{A}^{-1}-\frac{\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1+\mathbf{v}^T\mathbf{u}} \\ &= \mathbf{A}^{-1}-\frac{\mathbf{A}^{-1}\mathbf{w}\mathbf{v}^T\mathbf{A}^{-1}}{1+\mathbf{v}^T\mathbf{A}^{-1}\mathbf{w}} && \text{(By substituting } \mathbf{u} \text{ with } \mathbf{A}^{-1}\mathbf{w}\text{)}\end{aligned}$$

which has proved the Sherman-Morrison formula