2021 Computational Technique Examination Solution

Disclaimer: this solution only serves as a reference. Some answer may be incorrect or insufficient. If you think something is wrong just send me a message or leave a comment :)

Another thing to mention: I'm really surprised by how much we have to write just in this time frame and I only finished like half of the paper :)))). This is not cool :))

1. a. i) We first find the matrix A^TA which has a smaller dimension than AA^T

$$A^TA = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

We then find the spectral decomposition of A^TA using its eigenvalues and eigenvectors. First we find the eigenvalues and eigenvectors of A^TA by letting $\det(A^TA - \lambda I) = 0$

$$\begin{vmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{vmatrix} = 0 \Rightarrow \begin{aligned} (17-\lambda)^2 &= 64 \\ 17-\lambda &= \pm 8 \\ \lambda &= 25, 9 \end{aligned}$$

Hence, $\lambda_1 = 25$, $\lambda_2 = 9$. We then find the corresponding eigenvectors:

$$E_{25} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E_9 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore we have the orthogonal matrix V:

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

after normalization.

We use the fact that $US = AV$ to find U

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{25}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{9}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$u_3 = u_1 \times u_2 = \begin{vmatrix} i & j & k \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} \end{vmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

Hence we have the SVD of A

$$A = USV^T = \frac{1}{3\sqrt{2}} \begin{bmatrix} \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -2 \\ 0 & 2\sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

ii) In principal component analysis, we have

$$w = \arg\max_{\|w\|=1} \{w^T A^T A w\} = v_1$$

where $v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which we have foudn in 1a i)

iii) Using SVD we have

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T = \frac{5}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 4 & -4 \end{bmatrix}$$

Both matrix have rank one as both of them have two rows that are the linear combination of the other row. (or columnwise: both columns are multiple of each other)

b. Matrix B does not have a cholesky decomposition. To find the cholesky decomposition of A (there are different solutions):

$$A = LL^T = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

From the corresponding values in matrix A we know that

$$\begin{aligned} l_{11}^2 &= 25 \Rightarrow l_{11} = 5 \\ l_{11} l_{21} &= 15 \Rightarrow l_{21} = \frac{15}{5} = 3 \\ l_{11} l_{31} &= -5 \Rightarrow l_{31} = \frac{-5}{5} = -1 \\ l_{21}^2 + l_{22}^2 &= 18 \Rightarrow l_{22} = \sqrt{18 - 9} = 3 \\ l_{21} l_{31} + l_{22} l_{32} &= 0 \Rightarrow l_{32} = 1 \\ l_{31}^2 + l_{32}^2 + l_{33}^2 &= 11 \Rightarrow l_{11} = 3 \end{aligned}$$

Hence we have

$$L = \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

c. We first try to find the eigenvectors of A by letting its characteristic polynomial equal to 0

$$\begin{aligned} (3-\lambda)(4-\lambda)(2-\lambda) + 3 &= 0 \\ -(\lambda - 3)^3 &= 0 \\ \lambda &= 3 \end{aligned}$$

Hence the algebraic multiplicity of λ is 3. However, the geometric multiplicity of λ is 2:

$$(A-3I)x = 0 \Rightarrow x = \text{span}\left(\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

Therefore, A is not diagnolisable since it has an eigenvalue whose algebraic multiplicity is not equal to its geometric multiplicity.

The JNF of A is

$$A = SJS^{-1} = \begin{bmatrix}0 & 0 & 1\\1 & 1 & 0\\1 & 0 & 0\end{bmatrix}\begin{bmatrix}3 & 1 & 0\\0 & 3 & 0\\1 & 0 & 3\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\0 & 1 & -1\\1 & 0 & 0\end{bmatrix}$$

where S is the basis we need to find out. Notice that reordering the columns can produce different results.

d. To show that u is an eigenvector of $A = uv^T$, we need to show that $Au = \lambda u$

$$Au = uv^T u = u(v^T u) = (v^T u)u$$

Since u,$v \in \mathbb{R}^3$, we know that $v^T u$ is a scalar and hence we can let $\lambda = (v^T u)$ to prove that u is an eigenvector of A

To find other eigenvalues (maximum is 3 since it's 3 dimensional), we can use the fact that

$$\det(A) = v^T u \cdot \lambda_2 \cdot \lambda_3$$
$$\text{trace}(A) = v^T u + \lambda_2 + \lambda_3$$

Since $A = uv^T$, the sum of the diagnol elements equal to $v^T u$ and hence

$$\text{trace}(A) = v^T u = v^T u + \lambda_2 + \lambda_3$$
$$0 = \lambda_2 + \lambda_3$$
$$\lambda_2 = -\lambda_3$$

Hence the other two eigenvalues are $\lambda_2 = \sqrt{-\frac{\det(A)}{v^T u}} = \sqrt{\frac{\det(A)}{v^T u}}i$ and $\lambda_3 = -\sqrt{\frac{\det(A)}{v^T u}}i$

2. a. i) Using the l_1 norm we have

$$\text{cond}(A) = \|A\|_1 \|A^{-1}\|_1 = 18 \cdot \frac{164}{152} \approx 19.42$$

Using l_∞ norm we have

$$\text{cond}(A) = \|A\|_\infty \|A^{-1}\|_\infty = 18 \cdot \frac{164}{152} \approx 19.42$$

Using l_2 norm we have

$$\text{cond}(A) = \|A\|_2 \|A^{-1}\|_2 = 15.5994 \cdot \frac{1}{1.15373} = 13.52$$

using the fact that the eigenvalues of A^{-1} is the inverse of eigenvalues of A

b. i) Below is the proof

$$\|\sum_{i=m+1}^{n} \frac{M^i t^i}{i!}\| = \sum_{i=m+1}^{n} \frac{\|M^i\| t^i}{i!} \le \sum_{i=m+1}^{n} \frac{\|M\|^i t^i}{i!}$$

assuming that the norm is subordinate and thus $\|AB\| \le \|A\|\|B\|$

ii) Let $m < n < N$ for some large number $N \in \mathbb{R}$. Since we know that the exponential series defined on real numbers is convergent, we thus know that for any $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

We can then set $x = \|M\|t$ and thus $\lim_{i\to\infty} \frac{(\|M\|t)^i}{i!} = 0$. For sufficiently large number $m < n < N$, we have

$$\lim_{m,n\to\infty} \sum_{i=m+1}^{n} \frac{\|M\|^i t^i}{i!} = \lim_{m,n\to\infty} (\sum_{i=0}^{n} \frac{\|M\|^i t^i}{i!} - \sum_{i=0}^{m} \frac{\|M\|^i t^i}{i!}) = e^{\|M\|t} - e^{\|M\|t} = 0$$

iii) The proof is shown below

$$\begin{aligned} \|e^{Mt}\| = \|\sum_{i=0}^{\infty} \frac{M^i t^i}{i!}\| \le \sum_{i=0}^{\infty} \frac{\|M\|^i t^i}{i!} && \text{(From i)} \\ = e^{\|M\|t} && \text{(From the definition of exponential series)} \end{aligned}$$

iv) The proof of $\frac{d}{dt} e^{Mt} = M e^{Mt}$ is as follows

$$\begin{aligned} \frac{d}{dt} e^{Mt} = \frac{d}{dt} (\sum_{i=0}^{\infty} \frac{M^i t^i}{i!}) &= \frac{d}{dt} (I + Mt + \frac{M^2 t^2}{2!} + \frac{M^3 t^3}{3!} + \dots) \\ &= \mathbf{0} + M + tM^2 + \frac{t^2 M^3}{2!} + \dots \\ &= M(I + tM + \frac{t^2 M^2}{2!} + \dots) \\ &= M \sum_{i=0}^{\infty} \frac{M^i t^i}{i!} = M e^{Mt} \end{aligned}$$

We can plug in $\vec{f}(t) = e^{Mt} \vec{f}(0)$ in to the differential equation to verify

$$\begin{aligned} \frac{d}{dt}(e^{Mt} \vec{f}(0)) &= M e^{Mt} \vec{f}(0) + e^{Mt} \cdot 0 \\ &= M e^{Mt} \vec{f}(0) \\ &= M \vec{f}(0) \end{aligned}$$

v) This question is supposed to relate to iv) but who knows :)

The answer is as follows

$$\begin{aligned} y_1(t) &= \frac{1}{2} e^{3t} + \frac{1}{2} e^{-3t} \\ y_2(t) &= \frac{1}{6} e^{3t} - \frac{1}{6} e^{-3t} \end{aligned}$$

c. The coefficient can be found as follows

$$
\begin{aligned}
a_k &= \frac{1}{2}(\int_0^1 xe^{-i\pi kx}dx + \int_1^2 (2-x)e^{-i\pi kx}dx) \\
&= \frac{1}{2}(-\frac{1}{\pi ki} + 2\int_1^2 e^{-i\pi kx}dx - \int_1^2 xe^{-i\pi kx}dx) \\
&= \frac{1}{2}(-\frac{1}{\pi ki} + 2\frac{i}{\pi k}(e^{-2i\pi k} - e^{-i\pi k}) - (-3e^{-2i\pi k} + 2e^{-i\pi k})) \\
&= -\frac{1}{2\pi ki} + \frac{i}{\pi k}((-1)^{2k} - (-1)^k) - \frac{1}{2}(-3\cdot(-1)^{2k} + 2\cdot(-1)^k) \\
&= -\frac{1}{2\pi ki} - (-1)^k\frac{i}{\pi k} + \frac{3}{2} - (-1)^k
\end{aligned}
$$