Computational Techniques Assessed Coursework 7

Solution 1

1. By the definition of Laplace transformation, we know that

$$L_{k+1} = \int_0^\infty e^{-st} f(t; k+1, \theta) dt$$

   We also observed the following property of f

$$\begin{aligned} f'(t; k+1, \theta) &= \frac{d}{dt}\left(\frac{t^k e^{-\frac{t}{\theta}}}{\theta^{k+1}\Gamma(k+1)}\right) \\ &= \frac{d}{dt}\left(\frac{t^k e^{-\frac{t}{\theta}}}{\theta^{k+1}k\Gamma(k)}\right) \\ &= \frac{kt^{k-1} \cdot e^{-\frac{t}{\theta}} - \frac{1}{\theta} \cdot t^k \cdot e^{-\frac{t}{\theta}}}{\theta^{k+1} \cdot k \cdot \Gamma(k)} \\ &= \frac{1}{\theta} \cdot \frac{t^{k-1}e^{-\frac{t}{\theta}}}{\theta^k \Gamma(k)} - \frac{1}{\theta} \cdot \frac{t^k \cdot e^{-\frac{t}{\theta}}}{\theta^{k+1} \cdot k \cdot \Gamma(k)} \\ &= \frac{1}{\theta}(f(t; k, \theta) - f(t; k+1, \theta)) \end{aligned}$$

   By the nature of Laplace transformation, we know that

$$(\mathcal{L}f'(t; k+1, \theta))(s) = sL_{k+1}(s) - f(0; k+1, \theta) = sL_{k+1}(s)$$

   Hence we have

$$\begin{aligned} sL_{k+1}(s) &= \int_0^\infty \frac{e^{-st}}{\theta}(f(t; k, \theta) - f(t; k+1, \theta))dt \\ sL_{k+1}(s) &= \frac{1}{\theta}(L_k(s) - L_{k+1}(s)) \\ \theta s L_{k+1}(s) + L_{k+1}(s) &= L_k(s) \\ L_{k+1}(s) &= \frac{L_k(s)}{1+\theta s} \end{aligned}$$

2. When $k = 1$, $L_1(s) = \int_0^\infty e^{-st} \frac{e^{-\frac{t}{\theta}}}{\theta} dt = \frac{1}{\theta} \int_0^\infty e^{-t(s+\frac{1}{\theta})} dt = \frac{1}{\theta} \cdot \frac{\theta}{1+s\theta} = \frac{1}{1+\theta s}$

   From 1 we have $L_{k+1}(s) = \frac{L_k(s)}{1+\theta s}$. Hence we have

$$L_k(s) = \frac{L_{k-1}(s)}{1+\theta s} = \frac{1}{1+\theta s} \cdot \frac{L_{k-2}(s)}{1+\theta s} = \ldots = \frac{L_1(s)}{(1+\theta s)^{k-1}} = \left(\frac{1}{1+\theta s}\right)^k$$

Solution 2

The Laplace transform of e^{iwt} is as follows

$$\mathcal{L}(s) = \int_0^\infty e^{-st} e^{iwt}\, dt = \int_0^\infty e^{-t(s-iw)}\, dt = (-\frac{e^{-t(s-iw)}}{s-iw})|_0^\infty = \frac{1}{s-iw}$$

By Euler's formula, we know that $e^{iwt} = \cos wt + i \sin wt$

Hence the Laplace transformation of $\cos wt$ and $\sin wt$ are

$$\begin{aligned}(\mathcal{L}\cos wt)(s) + (\mathcal{L} i \sin wt)(s) &= \frac{1}{s-iw} \\ (\mathcal{L}\cos wt)(s) + (\mathcal{L} i \sin wt)(s) &= \frac{s+iw}{(s-iw)(s+iw)} \\ (\mathcal{L}\cos wt)(s) + (\mathcal{L} i \sin wt)(s) &= \frac{s+iw}{s^2+w^2} \\ (\mathcal{L}\cos wt)(s) + (\mathcal{L} i \sin wt)(s) &= \frac{s}{s^2+w^2} + \frac{iw}{s^2+w^2}\end{aligned}$$

By taking the real parts we have $(\mathcal{L}\cos wt)(s) = \frac{s}{s^2+w^2}$ and by taking the imaginary parts we have $(\mathcal{L}\sin wt)(s) = \frac{w}{s^2+w^2}$

Solution 3

1) Using Laplace transformation we have

$$\begin{aligned}s(\mathcal{L}x)(s) - x(0) &= 2(\mathcal{L}x)(s) + (\mathcal{L}y)(s) + \int_0^\infty e^{-st} e^{-t} dt \\ s(\mathcal{L}y)(s) - y(0) &= 4(\mathcal{L}x)(s) - (\mathcal{L}y)(s)\end{aligned}$$

In the form of linear system using matrix

$$\begin{aligned}\begin{bmatrix} s-2 & -1 \\ -4 & s+1 \end{bmatrix} \begin{bmatrix} (\mathcal{L}x)(s) \\ (\mathcal{L}y)(s) \end{bmatrix} &= \begin{bmatrix} \frac{1}{s+1} \\ 0 \end{bmatrix} \\ \begin{bmatrix} (\mathcal{L}x)(s) \\ (\mathcal{L}y)(s) \end{bmatrix} &= \frac{1}{(s-2)(s+1)-4} \begin{bmatrix} s+1 & 1 \\ 4 & s-2 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} \\ 0 \end{bmatrix} \\ \begin{bmatrix} (\mathcal{L}x)(s) \\ (\mathcal{L}y)(s) \end{bmatrix} &= \frac{1}{(s-3)(s+2)} \begin{bmatrix} 1 \\ \frac{4}{s+1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{(s-3)(s+2)} \\ \frac{4}{(s-3)(s+2)(s+1)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{5(s-3)} - \frac{1}{5(s+2)} \\ \frac{1}{5(s-3)} + \frac{4}{5(s+2)} - \frac{1}{s+1} \end{bmatrix}\end{aligned}$$

We can then obtain that $x(t) = \frac{e^{3t}}{5} - \frac{e^{-2t}}{5}$ and $y(t) = \frac{e^{3t}}{5} + \frac{4e^{-2}}{5} - e^{-t}$

2a) Taking the Laplace transform directly, we have

$$\begin{aligned} s(\mathcal{L}x')(s) - x'(0) &= -w^2(\mathcal{L}x)(s) \\ s(s(\mathcal{L}x)(s) - x(0)) - y_0 &= -w^2(\mathcal{L}x)(s) \\ s^2(\mathcal{L}x)(s) - x_0 s - y_0 &= -w^2(\mathcal{L}x)(s) \\ (\mathcal{L}x)(s) &= \frac{x_0 s + y_0}{s^2 + w^2} \end{aligned}$$

From Problem 2 we know that $x(t) = x_0 \cos wt + \frac{y_0}{w} \sin wt$

2b) Take $y(t) = x'(t)$, we have

$$\begin{aligned} y' &= -w^2 x \\ y &= x' \end{aligned}$$

Then we take the Laplace transform

$$\begin{aligned} s(\mathcal{L}y)(s) - y(0) &= -w^2(\mathcal{L}x)(s) \\ (\mathcal{L}y)(s) &= s(\mathcal{L}x)(s) - x(0) \end{aligned}$$

which, in matrix form, is

$$\begin{aligned} \begin{bmatrix} w^2 & s \\ -s & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} &= \begin{bmatrix} y_0 \\ x_0 \end{bmatrix} \\ \begin{bmatrix} X \\ Y \end{bmatrix} &= \frac{1}{s^2 + w^2} \begin{bmatrix} 1 & -s \\ s & w^2 \end{bmatrix} \begin{bmatrix} y_0 \\ x_0 \end{bmatrix} \\ \begin{bmatrix} X \\ Y \end{bmatrix} &= \begin{bmatrix} \frac{y_0 - x_0 s}{s^2 + w^2} \\ \frac{y_0 s + x_0 w^2}{s^2 + w^2} \end{bmatrix} \end{aligned}$$

Solution 4

1) We will partition the fraction as follows

$$\frac{s-1}{(s+1)(s-2)} = \frac{a}{s+1} + \frac{b}{s-2}$$

From the left hand side we know that

$$\begin{aligned} as - 2a + bs + b &= s - 1 \\ (a+b)s - 2a + b &= s - 1 \end{aligned}$$

$$\Rightarrow \begin{cases} a + b = 1 \\ -2a + b = -1 \end{cases} \Rightarrow \begin{cases} a = \frac{2}{3} \\ b = \frac{1}{3} \end{cases}$$

Hence we have

$$\frac{s-1}{(s+1)(s-2)} = \frac{2}{3(s+1)} + \frac{1}{3(s-2)}$$

2) We take the Laplace transform on both sides of the equation

$$\begin{aligned} &s(\mathcal{L}y_1)(s) - y_1(0) = 4(\mathcal{L}y_2)(s) + \int_0^\infty 2e^{-t}e^{-st}\,dt \\ &s(\mathcal{L}y_2)(s) - y_2(0) = (\mathcal{L}y_1)(s) \end{aligned}$$

which, in the matrix form, is (notice that X denotes the Laplace transform of y_1 and Y denotes the Laplace transform of y_2)

$$
\begin{aligned}
\begin{bmatrix} s & -4 \\ -1 & s \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} &= \begin{bmatrix} 1+\frac{2}{1+s} \\ -0.5 \end{bmatrix} \\
\begin{bmatrix} X \\ Y \end{bmatrix} &= \frac{1}{s^2-4} \begin{bmatrix} s & 4 \\ 1 & s \end{bmatrix} \begin{bmatrix} 1+\frac{2}{1+s} \\ -0.5 \end{bmatrix} \\
\begin{bmatrix} X \\ Y \end{bmatrix} &= \frac{1}{(s+2)(s-2)} \begin{bmatrix} \frac{s^2+s-2}{1+s} \\ -\frac{s^2-s-6}{2(1+s)} \end{bmatrix} \\
\begin{bmatrix} X \\ Y \end{bmatrix} &= \frac{1}{(s+2)(s-2)} \begin{bmatrix} \frac{(s+2)(s-1)}{1+s} \\ -\frac{(s-3)(s+2)}{2(1+s)} \end{bmatrix} \\
\begin{bmatrix} X \\ Y \end{bmatrix} &= \begin{bmatrix} \frac{s-1}{(s-2)(s+1)} \\ -\frac{s-3}{2(s+1)(s-2)} \end{bmatrix} = \begin{bmatrix} \frac{1}{3(s-2)} + \frac{2}{3(s+1)} \\ -\frac{1}{2}(\frac{4}{3(s+1)} - \frac{1}{3(s-2)}) \end{bmatrix}
\end{aligned}
$$

We can now calculate the solution as follows

$$
\begin{aligned}
y_1(t) &= \frac{2e^{-t}}{3} + \frac{e^{2t}}{3} \\
y_2(t) &= -\frac{2e^{-t}}{3} + \frac{e^{2t}}{6}
\end{aligned}
$$

Solution 7

a) The Fourier series expansion of the following square wave function is

$$
f^N(x) = \sum_{k=0}^{N} a_k e^{iw_k x} = \sum_{k=0}^{N} a_k e^{\pi k x} = \sum_{k=0}^{N} c_k \cos \pi k x + s_k \sin \pi k x
$$

where c_k and s_k can be calculated as follows

$$
\begin{aligned}
c_k &= \frac{1}{2} \int_0^2 f(x) \cos k\pi x dx \\
&= \frac{1}{2}(\int_0^1 \cos k\pi x dx - \int_1^2 \cos k\pi x dx) \\
&= \frac{\sin k\pi}{k\pi} \\
s_k &= \frac{1}{2}(\int_0^2 f(x) \sin k\pi x dx) \\
&= \frac{1}{2}(\int_0^1 \sin k\pi x dx - \int_1^2 \sin k\pi x dx) \\
&= \frac{2(1-\cos k\pi)}{k\pi}
\end{aligned}
$$

then we calculate the value of c_k

$$c_0 = \frac{1}{2} \int_0^2 f(x) dx = 0$$

Hence the Fourier series expansion of $f(x)$ is

$$\begin{aligned} f^N(x) &= \sum_{k=1}^{N} \frac{\sin k\pi}{k\pi} \cos k\pi x + \frac{2(1 - \cos k\pi)}{k\pi} \sin k\pi x \\ &= \sum_{k=1}^{N} \frac{2 \sin k\pi x - 2 \cos k\pi \sin k\pi x}{k\pi} \\ &= \frac{2}{\pi} \sum_{k=1}^{N} \frac{1}{k} \cdot \sin k\pi x (1 - \cos k\pi) \end{aligned}$$