Probability and Stats Exercise 3 - Continuous Random Variables

Solution 1

Since the function satisfies $f(-x) = f(x)$, we know that

$$\int_{-\infty}^{0} f(t)dt = \int_{0}^{\infty} f(t)dt$$

and we also know

$$\int_{-x}^{0} f(t)dt = \int_{0}^{x} f(t)dt$$

Thus we have the following property

$$\begin{gathered}
\int_{-\infty}^{0} f(t)dt - \int_{-x}^{0} f(t)dt = \int_{0}^{\infty} f(t)dt - \int_{0}^{x} f(t)dt \\
\int_{-\infty}^{x} f(t)dt = \int_{-x}^{\infty} f(t)dt \\
F(x) - F(-\infty) = F(\infty) - F(-x) \\
F(x) - 0 = 1 - F(-x) \\
F(-x) = 1 - F(x)
\end{gathered}$$

Solution 2

a) From geometry, we know that $P(X < r) = \frac{\text{the are of circle with radius r}}{\text{the area of the entire circle}} = \frac{\pi r^2}{\pi \cdot 1^2} = r^2$

Hence the cumulative distribution function

$$F_X(x) = r^2$$

b) Below is the answer

$$P(r < X < s) = F_X(s) - F_X(r) = s^2 - r^2$$

c) The pdf of X is therefore

$$f_X(x) = \frac{d}{dx} F_X(x) = 2r$$

d) The mean can be calculated by the following integration between interval $[0, 1]$

$$E_X(X) = \int_{0}^{1} x f_X(x) dx = \int_{0}^{1} 2r^2 dr = \frac{2}{3}$$

Solution 3

The mean of an $\exp(\lambda)$ can be calculated as

$$\begin{aligned}
E_X(X) &= \int_0^\infty x \cdot \lambda e^{-\lambda x} dx \\
&= F(\infty) - F(0) \qquad \text{where } F(x) = -xe^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda} \\
&= 0 - (-\frac{1}{\lambda}) \\
&= \frac{1}{\lambda}
\end{aligned}$$

Then the variance can be calculated as

$$\begin{aligned}
Var_X(X) &= E(X^2) - (E(X))^2 \\
&= \int_0^\infty x^2 \cdot \lambda e^{-\lambda x} dx - (\frac{1}{\lambda})^2 \\
&= F(\infty) - F(0) - \frac{1}{\lambda^2}
\end{aligned}$$

where $F(x) = -x^2 e^{-\lambda x} - \frac{2xe^{-\lambda x}}{\lambda} - \frac{2e^{-\lambda x}}{\lambda^2}$

To find the value of $F(\infty)$, we have to use limit with L'Hopital rule

$$\lim_{x\to\infty} -x^2 e^{-\lambda x} = \lim_{x\to\infty} \frac{-x^2}{e^{\lambda x}} = \lim_{x\to\infty} \frac{-2x}{\lambda e^{\lambda x}} = \lim_{x\to\infty} \frac{-2}{\lambda^2 e^{\lambda x}} = 0$$

Hence

$$\begin{aligned}
Var_X(X) &= -F(0) - \frac{1}{\lambda^2} \\
&= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2}
\end{aligned}$$

Solution 4

$$m(t) = \int_0^\infty e^{tx} f(x) dx$$

a) the mgf of function $xf(x)$ is

$$m_{xf(x)}(t) = \int_0^\infty e^{tx} x f(x) dx$$

Therefore, assuming differentiation inside the integral is valid

$$m'(t) = \frac{d}{dt}(\int_0^\infty e^{tx} f(x) dx) = \int_0^\infty (\frac{d}{dt} e^{tx} f(x)) dx = \int_0^\infty x e^{tx} f(x) dx = m_{xf(x)}(t)$$

b) The nth derivative can be obtained as

$$m^{(n)}(t) = \int_0^\infty x^n e^{tx} f(x)dx = \frac{k \int_0^\infty x^n e^{tx} f(x)dx}{k} = k \int_0^\infty \frac{x^n e^{tx} f(x)}{k} dx$$

where k could be the $m^{(n)}(0)$

c) X being exponential indicates that

$$f(x) = \lambda e^{-\lambda x}$$

The mgt and its nth derivative of X is

$$\begin{aligned}
m(t) &= \int_0^\infty e^{tx}(\lambda e^{-\lambda x})dx \\
m^{(n)}(t) &= \int_0^\infty x^n e^{tx}(\lambda e^{-\lambda x})dx \\
m^{(n)}(0) &= \int_0^\infty x^n(\lambda e^{-\lambda x})dx \\
&= (-e^{-\lambda x}(\sum_{i=0}^{n} \frac{i! \cdot x^{n-i}}{\lambda^i}))\Big|_0^\infty
\end{aligned}$$

To determine the limitation when x approaches to ∞, we again use L'Hopital's rule to generalize the limit to all the sumation terms in the equation above

$$\lim_{x \to \infty} -e^{-\lambda x}(\frac{i! \cdot x^{n-i}}{\lambda^i}) = \lim_{x \to \infty} -\frac{i! \cdot x^{n-i}}{\lambda^i e^{\lambda x}} = \lim_{x \to \infty} -\frac{n!}{\lambda^n e^{\lambda x}} = 0$$

Hence $m(t)$ becomes

$$m(t) = 0 - (-1 \cdot \frac{n!}{\lambda^n}) = \frac{n!}{\lambda^n}$$

Solution 5

a) We can define T_n directly by the definition of exponential random variable

$$T_n = T_1 + T_2 + T_3 + \ldots = \sum_{i=1}^{n} \lambda e^{-\lambda x}$$

Hence the moment generating function of pdf of T_n is

$$\begin{aligned}
M_{T_n}(t) &= (\int_0^\infty e^{tx}(\lambda e^{-\lambda x})dx)^n \\
&= (\int_0^\infty \lambda e^{(t-\lambda)x} dx)^n \\
&= ((\frac{\lambda}{t-\lambda} e^{(t-\lambda)x})\Big|_{x=0}^\infty)^n \\
&= (0 - \frac{\lambda}{t-\lambda})^n \\
&= (\frac{\lambda}{\lambda - t})^n
\end{aligned}$$

b) Using moment generating function, we can determine the probability density function as follow

$$
\begin{aligned}
(\frac{\lambda}{\lambda - t})^n &= \int_0^{\infty} e^{tx} f(x) dx \\
\frac{\lambda^{n-1}}{(n-1)!} \frac{d^{n-1}}{dt^{n-1}} (\frac{\lambda}{\lambda - t}) &= \frac{d^{n-1}}{dt^{n-1}} (\int_0^{\infty} e^{tx} f(x) dx) \\
\frac{d^{n-1}}{dt^{n-1}} \frac{(n-1)!}{\lambda^{n-1}} (\frac{\lambda}{\lambda - t}) &= \int_0^{\infty} (\frac{d^{n-1}}{dt^{n-1}} e^{tx} f(x)) dx) \\
\frac{d^{n-1}}{dt^{n-1}} \frac{(n-1)!}{\lambda^{n-1}} (\frac{\lambda}{\lambda - t}) &= \int_0^{\infty} x^{n-1} e^{tx} f(x) dx
\end{aligned}
$$

Solution 6

a) It is easy to show that

$$
\begin{aligned}
\int_0^x g_{n+1}(u) du &= \int_0^x \frac{\lambda^{n+1} u^n}{n!} e^{-\lambda u} du \\
&= -\frac{\lambda^n u^n e^{-\lambda u}}{n!} + \int_0^x \frac{e^{-\lambda u}}{\lambda} \cdot \frac{\lambda^{n+1} u^{n-1}}{(n-1)!} du \\
&= -\frac{(\lambda u)^n}{n!} e^{-\lambda u} + \int_0^x \frac{e^{-\lambda u} \lambda^n u^{n-1}}{(n-1)!} dx \\
&= \int_0^x g_n(u) du - \frac{(\lambda u)^n}{n!} e^{-\lambda u}
\end{aligned}
$$

b) From 5b and 6a we know that

$$
\begin{aligned}
T_n \leq x \Rightarrow g_n(x) &\leq x \\
\Rightarrow \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x} &\leq x \\
\Rightarrow \frac{\lambda^n x^n}{n!} e^{-\lambda x} &\leq \frac{x^2}{n} \\
\Rightarrow \frac{\lambda^n x^n}{n!} e^{-\lambda x} &\leq \frac{x}{\lambda} \\
\Rightarrow \frac{\lambda^n x^n}{n!} e^{-\lambda x} &\geq \frac{\lambda}{x} = n
\end{aligned}
$$

$$
\begin{aligned}
P(N_x = n) &= \frac{e^{-\lambda x} (\lambda x)^n}{n!} \\
P(T_n \leq x) - P(T_{n+1} \leq x) &= \int_0^x g_n(x) dx - \int_0^x g_n(x) dx + \frac{(\lambda x)^n}{n!} e^{-\lambda x} = \frac{(\lambda x)^n}{n!} e^{-\lambda x}
\end{aligned}
$$

c) From b) we know that $P(N_x = n)$ follows Poisson distribution by substituting λx into the Poisson distribution equation

Solution 7

The cdf of X is

$$
F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}
$$

The pdf of the transfofrmed variable $Y = e^X$ is

$$p_Y(x) = \begin{cases} 0 & x < 0 \\ \dfrac{e^x}{e-1} & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

Solution 8

$X - \mu$ will result in a left shift of the random variable, with itself still being a normal distribution. Divided by σ will result in the graph being shrunk but still it will be a normal distribution. According to the linear transformation of mean and variance, the new mean will be $\mu - \mu = 0$ and the new variance will be $\frac{\sigma^2}{\sigma^2} = 1$

Hence, $Y \sim N(0, 1)$

Solution 9

a) Using the definition of pdf, we can easily find $F_Y(y)$

$$F_Y(y) = F_X(\frac{y-b}{a})$$

b) Using chain rule

$$\begin{aligned} f_Y(y) = F'_Y(y) &= F'_X(\frac{y-b}{a}) \cdot \frac{1}{a} \\ &= \frac{1}{|a|} f_X(\frac{y-b}{a}) \end{aligned}$$

Solution 10

a) $z = 1.16$

b) $z = 1.09$

c) $z = -1.35$

Solution 11

a) $area = 0.38493$

b) $area = 0.25175$

c) $area = 0.66369$

d) $area = 0.18278$

e) $area = 0.89973$

Solution 12

a) We first find the pmf of the Bernoulli random variable $\tilde{B} = p^x(1-p)^{1-x}$

$$p_{\tilde{B}}(x) = \begin{cases} p & x = \sqrt{\dfrac{1-p}{p}} \\ 1-p & x = -\sqrt{\dfrac{p}{1-p}} \\ 0 & \end{cases}$$

The mean can be calculated as follows

$$\begin{aligned} E(\tilde{B}) &= p\sqrt{\frac{1-p}{p}} - (1-p)\sqrt{\frac{p}{1-p}} \\ &= \sqrt{p(1-p)} - \sqrt{(1-p)p} = 0 \end{aligned}$$

The variance can be calculated as follows

$$\begin{aligned} Var(\tilde{B}) &= E(\tilde{B}^2) - (E(\tilde{B}))^2 \\ &= \frac{1-p}{p} \cdot p + \frac{p}{1-p} \cdot (1-p) - 0 \\ &= 1 - p + p = 1 \end{aligned}$$

b) The characteristic function of $\tilde{B}$ is

$$\phi_X(t) = E(e^{itx}) = \sum_{k=1}^{n} e^{itk} p_{\tilde{B}}(k) = e^{it\sqrt{\frac{1-p}{np}}} p + e^{-it\sqrt{\frac{p}{n(1-p)}}}(1-p)$$

and thus the characteristic function of $S = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tilde{B}$ is

$$\phi_S(t) = \prod_{j=1}^{n} \phi_{X_i}(t) = (E(e^{itx}))^n = (e^{it\sqrt{\frac{1-p}{np}}} p + e^{-it\sqrt{\frac{p}{n(1-p)}}}(1-p))^n$$

c) As $n \to \infty$, the number of samples become large enough to allow sample mean to converge to normal distribution according to Central Limit Theorem. $e^{-\frac{t^2}{2}}$ is the characteristic function of normal distribution, which conforms the previous statement.

Solution 13

a) The probability generating function $G_{\text{Bin}}(z)$ is

$$\begin{aligned} G_{\text{Bin}}(z) = E_X(z^X) &= \sum_x p_X(x) z^x \\ &= \sum_x \binom{n}{x} (pz)^x (1-p)^{n-x} \\ &= (px + 1 - p)^n \end{aligned}$$

As $n \to \infty$, we have

$$
\begin{aligned}
\lim_{n\to\infty} G_{\text{Bin}}(z) &= \lim_{n\to\infty}(px+1-p)^n \\
&= \lim_{n\to\infty}(1+\frac{\lambda z-\lambda}{n})^n \\
&= e^{\lambda x-\lambda} \\
&= e^{-\lambda(1-z)}
\end{aligned}
$$

b) Similar as above, the pgf of a sequence of Binomial pmfs is the same as the pgf of Poisson distribution $e^{\lambda(z-1)}$, which indicates that as $n\to\infty$, the pmfs will tend to a Poisson pmf.