Computational Techniques Assessed CW2

Solution 1

i) Since $\mathbf{v}$ is an eigenvector of $\mathbf{A}^T\mathbf{A}$, we have

$$(\mathbf{A}^T\mathbf{A})\mathbf{v} = \lambda\mathbf{v}, \lambda \neq 0$$

We then multiply $\mathbf{A}$ to both sides

$$\begin{aligned} \mathbf{A}(\mathbf{A}^T\mathbf{A})\mathbf{v} &= \lambda\mathbf{A}\mathbf{v} \\ (\mathbf{A}\mathbf{A}^T)(\mathbf{A}\mathbf{v}) &= \lambda\mathbf{A}\mathbf{v} \end{aligned}$$

Hence $\mathbf{A}\mathbf{v}$ is an eigenvector of $\mathbf{A}\mathbf{A}^T$ with the same eigenvalue $\lambda \neq 0$.

ii) We need to prove that $\mathbf{A}\mathbf{v}$ is not an eigenvector of $\mathbf{A}\mathbf{A}^T$ if $\mathbf{v}$ is an eigenvector of $\mathbf{A}^T\mathbf{A}$ when $\lambda = 0$. We will prove this by showing a counterexample.

If $\lambda = 0$, we have

$$(\mathbf{A}^T\mathbf{A})\mathbf{v} = \mathbf{0}$$

We then multiply $\mathbf{v}^T$ to both sides of the equation

$$\begin{aligned} \mathbf{v}^T(\mathbf{A}^T\mathbf{A})\mathbf{v} &= \mathbf{0} \\ \mathbf{v}^T\mathbf{A}^T\mathbf{A}\mathbf{v} &= \mathbf{0} \\ (\mathbf{A}\mathbf{v})^T(\mathbf{A}\mathbf{v}) &= \mathbf{0} \end{aligned}$$

The equation above thus implies that $\mathbf{A}\mathbf{v}$ is the zero vector. Thus the statement in i) is false when $\lambda = 0$.

iii) Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal eigenvectors, we have

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= 0 \\ \mathbf{A}^T\mathbf{A} \cdot \mathbf{v}_1 &= \lambda_1\mathbf{v}_1 \\ \mathbf{A}^T\mathbf{A} \cdot \mathbf{v}_2 &= \lambda_2\mathbf{v}_2 \end{aligned}$$

From i) we know that $\mathbf{A}\mathbf{v}_1$ is an eigenvector of $\mathbf{A}\mathbf{A}^T$ with the same eigenvalue λ_1 and $\mathbf{A}\mathbf{v}_2$ is an eigenvector of $\mathbf{A}\mathbf{A}^T$ with the same eigenvalue λ_2. Thus we have

$$\begin{aligned} (\mathbf{A}\mathbf{A}^T)(\mathbf{A}\mathbf{v}_1) &= \lambda_1\mathbf{v}_1 \qquad (1) \\ (\mathbf{A}\mathbf{A}^T)(\mathbf{A}\mathbf{v}_2) &= \lambda_2\mathbf{v}_2 \qquad (2) \end{aligned}$$

We then multiply both sides of (1) by both sides of (2)

$$
\begin{aligned}
(\mathbf{A}\mathbf{A}^T)(\mathbf{A}\mathbf{v}_1) \cdot (\mathbf{A}\mathbf{A}^T)(\mathbf{A}\mathbf{v}_2) &= \lambda_1 \mathbf{v}_1 \cdot \lambda_2 \mathbf{v}_2 \\
\mathbf{A}(\mathbf{A}^T\mathbf{A}\mathbf{v}_1) \cdot \mathbf{A}(\mathbf{A}^T\mathbf{A}\mathbf{v}_2) &= \lambda_1 \lambda_2 (\mathbf{v}_1 \cdot \mathbf{v}_2) \\
\lambda_1 \mathbf{A}\mathbf{v}_1 \cdot \lambda_2 \mathbf{A}\mathbf{v}_2 &= \lambda_1 \lambda_2 (\mathbf{v}_1 \cdot \mathbf{v}_2) \\
\mathbf{A}\mathbf{v}_1 \cdot \mathbf{A}\mathbf{v}_2 &= 0
\end{aligned}
$$

Hence, $\mathbf{A}\mathbf{v}_1$ and $\mathbf{A}\mathbf{v}_2$ are orthogonal.

iv) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal eigenvectors of $\mathbf{A}\mathbf{A}^T$, then $\mathbf{A}^T\mathbf{v}_1$ and $\mathbf{A}^T\mathbf{v}_2$ are orthogonal.

We need to prove this statement.

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal eigenvectors, we have

$$
\begin{aligned}
\mathbf{v}_1 \cdot \mathbf{v}_2 &= 0 \\
\mathbf{A}\mathbf{A}^T \cdot \mathbf{v}_1 &= \lambda_1 \mathbf{v}_1 \\
\mathbf{A}\mathbf{A}^T \cdot \mathbf{v}_2 &= \lambda_2 \mathbf{v}_2
\end{aligned}
$$

From i) we know that $\mathbf{A}\mathbf{v}_1$ and $\mathbf{A}\mathbf{v}_2$ are eigenvectors of $\mathbf{A}^T\mathbf{A}$ with the same eigenvalues of λ_1 and λ_2, respectively. Thus we have

$$
\begin{aligned}
(\mathbf{A}^T\mathbf{A})(\mathbf{A}^T\mathbf{v}_1) &= \lambda_1 \mathbf{v}_1 \qquad (1) \\
(\mathbf{A}^T\mathbf{A})(\mathbf{A}^T\mathbf{v}_2) &= \lambda_2 \mathbf{v}_2 \qquad (2)
\end{aligned}
$$

We then multiply both sides of (1) by both sides of (2)

$$
\begin{aligned}
(\mathbf{A}^T\mathbf{A})(\mathbf{A}^T\mathbf{v}_1) \cdot (\mathbf{A}^T\mathbf{A})(\mathbf{A}^T\mathbf{v}_2) &= \lambda_1 \mathbf{v}_1 \cdot \lambda_2 \mathbf{v}_2 \\
\mathbf{A}^T(\mathbf{A}\mathbf{A}^T\mathbf{v}_1) \cdot \mathbf{A}^T(\mathbf{A}\mathbf{A}^T\mathbf{v}_2) &= \lambda_1 \mathbf{v}_1 \cdot \lambda_2 \mathbf{v}_2 \\
\lambda_1 \mathbf{A}^T\mathbf{v}_1 \cdot \lambda_2 \mathbf{A}^T\mathbf{v}_2 &= \lambda_1 \mathbf{v}_1 \cdot \lambda_2 \mathbf{v}_2 \\
\mathbf{A}^T\mathbf{v}_1 \cdot \mathbf{A}^T\mathbf{v}_2 &= 0
\end{aligned}
$$

Hence we have proved the statement above.

v) From i) we know that for every non-zero eigenvalue λ of $\mathbf{A}^T\mathbf{A}$ and corresponding eigenvector $\mathbf{v}$, we can find the eigenvector $\mathbf{A}\mathbf{v}$ of $\mathbf{A}\mathbf{A}^T$ with the same eigenvalue λ. Similarly, for every non-zero eigenvalue λ of $\mathbf{A}\mathbf{A}^T$ (this can be regarded as $(\mathbf{A}^T)^T\mathbf{A}^T$) and corresponding eigenvector $\mathbf{v}$, we can find the eigenvector $\mathbf{A}\mathbf{v}$ of $\mathbf{A}^T\mathbf{A}$ with the same eigenvalue λ. This implies that $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ have the same set of non-zero eigenvalues.

We can further prove this by a simple contradiction: assume that $\mathbf{A}^T\mathbf{A}$ has an eigenvalue λ with corresponding eigenvector v that is not belong to the set of eigenvalues of $\mathbf{A}\mathbf{A}^T$. From i), we can get an eigenvector $\mathbf{A}\mathbf{v}$ of $\mathbf{A}\mathbf{A}^T$ whose eigenvalue is the same as λ, which contradicts with our assumption.

Solution 2

i) We first calculate $\mathbf{B}$

$$
\mathbf{B} = \mathbf{A}^T\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}
$$

We first obtain the eigenvalues of $\mathbf{B}$. In order to do this, we need to find all non-zero solutions of the following equation

$$(\mathbf{B} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

which is equivalent to find the value of λ in $\det(\mathbf{B} - \lambda\mathbf{I}) = 0$. The process is as follows

$$\begin{aligned}
\begin{bmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{bmatrix} &= 0 \\
(11-\lambda)^2 - 1 &= 0 \\
(11-\lambda)^2 &= 1 \\
\lambda &= 10, 12
\end{aligned}$$

Then we need to find the eigenvectors by using the eigenvalues above and substitute into the equation $(\mathbf{B} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$.

$$\begin{aligned}
(\mathbf{B} - 10\mathbf{I})\mathbf{x} &= \mathbf{0} \\
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{x} &= \mathbf{0} \\
E_{10} = \mathbf{x} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{B} - 12\mathbf{I})\mathbf{x} &= \mathbf{0} \\
\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\mathbf{x} &= \mathbf{0} \\
E_{12} = \mathbf{x} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}$$

ii) From the properties we have obtained in Problem 1, we can now directly get the eigenvalues of $\mathbf{C}$, which is the same as $\mathbf{B}$: $\lambda = 10, 12$.

The corresponding eigenvectors can be obtained as described in Problem 1 i):

$$E_{10} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$$

$$E_{12} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

iii) Since $\mathbf{B}$ is a symmetric matrix, we can obtain a orthonormal set of eigenvectors as shown below. We can directly obtain $\mathbf{Q}$ by using the eigenvectors computed above

$$\mathbf{Q} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

By the fact that for any symmetric matrix $\mathbf{M}$, $\mathbf{M}^{-1} = \mathbf{M}^T$, we can easily obtain $\mathbf{Q}^{-1}$:

$$\mathbf{Q}^{-1} = \mathbf{Q}^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Now, we can compute $\Lambda = \mathbf{Q}^T\mathbf{B}\mathbf{Q}$:

$$\Lambda = \mathbf{Q}^T\mathbf{B}\mathbf{Q} = (\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}) \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} (\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}) = \frac{1}{2}\begin{bmatrix} 10 & -10 \\ 12 & 12 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 12 \end{bmatrix}$$