2020 Computational Technique Examination Solution

1. a. i) To find the singular value decomposition of A, we need to first find $A^T A$. Notice that in this case we can also choose AA^T since they have the same dimension.

$$A^T A = \begin{bmatrix} 8 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then we find the spectral decomposition of $A^T A$. First we need to compute the eigenvalues and eigenvectors of $A^T A$ in order to find its spectral decomposition.

The characteristic polynomial is

$$\begin{aligned} -\lambda(8-\lambda)(5-\lambda)+4\lambda &= 0 \\ \lambda(4-40+13\lambda-\lambda^2) &= 0 \\ -\lambda(\lambda-4)(\lambda-9) &= 0 \end{aligned}$$

Hence the eigenvalues aree $\lambda_1 = 0, \lambda_2 = 4, \lambda_3 = 9$

The corresponding normalized eigenvectors are

$$e_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, e_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, e_3 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

We then put them together by the order of eigenvalue from highest to lowest to form V.

$$V = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$$

Then we use the fact that $US = AV$ to calculate U

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{9}} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{4}{3} \\ \frac{5}{3} \\ -\frac{2}{3} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{4}} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

We can calculate u_3 by using the cross product which will reesult in a vector perpendicular to u_1 and u_2

$$u_3 = u_1 \times u_2 = \begin{vmatrix} i & j & k \\ \frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{vmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$$

Thus we have found the singular value decomposition of A

$$\begin{aligned} A = USV^T &= \begin{bmatrix} \frac{4}{3\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{2}{3} \\ \frac{5}{3\sqrt{5}} & 0 & -\frac{2}{3} \\ -\frac{2}{3\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{4}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \sqrt{5} & 0 & 0 \\ -\frac{2}{\sqrt{5}} & \frac{4}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \end{aligned}$$

ii) $\|A\|_2 = 3$, which is the largest singular value of A

b. i) Notice that A is a symmetric matrix. We will find the Cholesky factorisation as follows

$$A = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Hence we know that $l_{11}^2 = 4 \Rightarrow l_{11} = 2$, $l_{11} l_{21} = 12 \Rightarrow l_{21} = 6$, $l_{11} l_{31} = -16 \Rightarrow l_{31} = -8$, $l_{21}^2 + l_{22}^2 = 37 \Rightarrow l_{22} = 1$, $l_{21} l_{31} + l_{22} l_{32} = -43 \Rightarrow l_{32} = 5$, $l_{31}^2 + l_{32}^2 + l_{33}^2 = 98 \Rightarrow 3$

which is shown as below

$$L=\begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{bmatrix}$$

ii) Since all elements on the diagnal of L are all positive, A is positive definite.

c. We perform a QR decomposition as follows

$$u_1=a_1=\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, e_1=\frac{u_1}{\|u_1\|}=\frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$e_1\cdot a_1=2, e_1\cdot a_2=4$$

$$u_2=a_2-(e_1\cdot a_2)e_1=\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, e_2=\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, e_2\cdot a_2=2$$

Hence we have

$$Q=[e_1,e_2]=\frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, R=\begin{bmatrix} e_1\cdot a_1 & e_1\cdot a_2 \\ 0 & e_2\cdot a_2 \end{bmatrix}=\begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}$$

where

$$QR=\frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}=\begin{bmatrix} -1 & -1 \\ 1 & 3 \\ -1 & -1 \\ 1 & 3 \end{bmatrix}=A$$

d. We show this property as follows. w_{max} is the biggest element of w

$$
\begin{aligned}
|v \cdot w| &= |v_1 w_1 + v_2 w_2 + \ldots + v_m w_m| \\
&\le |v_1 w_{max} + v_2 w_{max} + \ldots + v_m w_{max}| \\
&= |w_{max}||v_1 + v_2 + \ldots + v_m| \\
&= \|w\|_\infty \|v\|_1
\end{aligned}
$$

e. The columns of A must be perpendicular to each other and are unit vectors. If the columns of A satisfy these two conditions, then we have

$$
\begin{aligned}
A^T A &= \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{a}_1 & \mathbf{a}_1 \cdot \mathbf{a}_2 & \ldots \\ \mathbf{a}_2 \cdot \mathbf{a}_1 & \mathbf{a}_2 \cdot \mathbf{a}_2 & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix} \\
&= I
\end{aligned}
$$

where $\mathbf{a}_j$ denotes the jth column of A. Since columns of A are perpendicular to each other, we have

$$
\mathbf{a}_i \cdot \mathbf{a}_j = 0, i \neq j
$$

Since the columns of A are all unit vectors, we have

$$
\mathbf{a}_j \cdot \mathbf{a}_j = 1
$$

Therefore we know that $A^T = A^{-1}$ and $A^T A = AA^T = I$.

Since $AA^T = \begin{bmatrix} \mathbf{a}^1 \cdot \mathbf{a}^1 & \mathbf{a}^1 \cdot \mathbf{a}^2 & \ldots \\ \mathbf{a}^2 \cdot \mathbf{a}^1 & \mathbf{a}^2 \cdot \mathbf{a}^2 & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix} = I$ wheree $\mathbf{a}^i$ denotes the ith row of A, we know that $\mathbf{a}^i \cdot \mathbf{a}^i = 1$ and $\mathbf{a}^i \cdot \mathbf{a}^j = 0$ for $i \neq j$. Hence the row of A (which is also the columns of A^T) are perpedicular to ecah other and they are all unit vectors.

2. a. i) The condition number of matrix A is

$$
\text{cond}(A) = \|A\|_\infty \|A^{-1}\|_\infty = 30 \cdot \frac{913}{268} = 102.201
$$

ii) From the definition of vector norms and vector operations, we have

$$\begin{aligned}
&\|x-y\|>0, x\neq y \\
&\|x-x\|=0 \\
&\|y-x\|=\|-(x-y)\|=|-1|\|x-y\|=\|x-y\| \\
&\|x-y\|=\|(x-z)+(z-y)\|\leq\|x-z\|+\|z-y\|
\end{aligned}$$

which fits properly the definition of metric space:

$$\begin{aligned}
&d(x,x)=0 \\
&d(x,y)>0, x\neq y \\
&d(x,y)=d(y,x) \\
&d(x,y)\leq d(x,z)+d(z,y)
\end{aligned}$$

The matrix norm definition is contructed in similar way so that the proof of it defining a metric is also similar.

iii) Define the function f as $f(\vec{v})=M\vec{v}$ and we will prove that f is a contraction. Since the given norm is subordinate, we know that $\|M\vec{v}\|\leq\|M\|\|\vec{v}\|$ for any vector $\vec{v}\in\mathbb{R}^n$.

$$\begin{aligned}
\|M\overrightarrow{v_1}-M\overrightarrow{v_2}\| &= \|M(\overrightarrow{v_1}-\overrightarrow{v_2})\| \\
&\leq \|M\|\|\overrightarrow{v_1}-\overrightarrow{v_2}\|
\end{aligned}$$

Since $\|M\|=1$, by definition f is a contraction. We construct a sequence $p_n=M^n\vec{v}$. By fixed point theorem, we know that as $n\to\infty$ there is an n such that $f(p_n)=p_n$. Then we have:

$$\begin{aligned}
Mp_n &= p_n \\
(M-I)p_n &= 0
\end{aligned}$$

which indicates that either $M-I=0$ or $p_n=0$. M might not possible be I since under some subordinate norm $\|I\|=1$ (e.g. l_1 norm of matrix). Then we have $p_n=0$ as $n\to\infty$.

b. i) By the definition of Laplace transformation, we have

$$\begin{aligned}
(\mathcal{L}f_1)(s) &= \int_0^\infty e^{-st}f(t)dt \\
&= \int_0^\infty e^{-st}dt \\
&= (-\frac{e^{-st}}{s})|_0^\infty \\
&= 0-(-\frac{1}{s})=\frac{1}{s}
\end{aligned}$$

By the fact that if $f(t)$ has LT $F(s)$ then $tf(t)$ has LT $-\frac{dF}{ds}$, we can easily obtain $(\mathcal{L}f_2)(s)=-\frac{d}{ds}(\mathcal{L}f_1)(s)=\frac{1}{s^2}$

ii) By taking Laplace transformation on both sides, we obtain

$$\begin{aligned}
sF_1(s)-y_1(0) &= 25F_2(s)+\tfrac{1}{s^2} \\
sF_2(s)-y_2(0) &= F_1(s)
\end{aligned}
\Rightarrow
\begin{aligned}
sF_1(s) &= 25F_2(s)+\tfrac{1}{s^2} \\
sF_2(s) &= F_1(s)
\end{aligned}$$

Using substitution we can obtain

$$
\begin{aligned}
s^2F_2(s) &= 25F_2(s) + \frac{1}{s^2} \\
(s^2-25)F_2(s) &= \frac{1}{s^2} \\
F_2(s) &= \frac{1}{s^2(s+5)(s-5)} = -\frac{1}{25s^2} - \frac{1}{250(s+5)} + \frac{1}{250(s-5)}
\end{aligned}
$$

and hence

$$
y_2(t) = -\frac{t}{25} - \frac{e^{-5t}}{250} + \frac{e^{5t}}{250}
$$

c. i) The proof is as follows

$$
\begin{aligned}
\frac{\partial f(\vec{x})}{\partial \vec{x}} &= \frac{1}{2}(A\vec{x} + \vec{x}^T A) - \vec{b} + 0 \\
&= \frac{1}{2}(A\vec{x} + A\vec{x}) - \vec{b} \\
&= A\vec{x} - \vec{b}
\end{aligned}
$$

ii) The steepest descent vector is

$$
-\nabla f(x_0, y_0) = -(2x_0 + 2\gamma y_0) = -2(x_0 + \gamma y_0) \Rightarrow -2(x_0, \gamma y_0)
$$

iii) We need to minimize the following equation

$$
\begin{aligned}
f(x_1, x_0) &= (1-c)^2 x_0^2 + \gamma(1-c\gamma)^2 y_0^2 \\
\frac{df}{dc} &= -2(1-c)x_0^2 - 2\gamma^2(1-c\gamma)y_0^2 \\
&= -2(x_0^2 - cx_0^2 + \gamma^2 y_0^2 - c\gamma^3 y_0^2) = 0 \\
\Rightarrow x_0^2 + \gamma^2 y_0^2 &= c(x_0^2 + \gamma^3 y_0^2) \\
c &= \frac{x_0^2 + \gamma^2 y_0^2}{x_0^2 + \gamma^3 y_0^2}
\end{aligned}
$$

iv) When $\gamma = 1$, the deepest descent is $\begin{bmatrix} -2x_0 \\ -2y_0 \end{bmatrix}$. From the definition of $f(x, y)$ we know that $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Thereby, the next iteration will produce

$$
x_1 = x_0 + \frac{4x_0^2 + 4y_0^2}{8x_0^2 + 8y_0^2} \begin{bmatrix} -2x_0 \\ -2y_0 \end{bmatrix} = 0, y_1 = 0
$$

From iii) we know that when $c = \frac{x_0^2+\gamma^2 y_0^2}{x_0^2+\gamma^3 y_0^2}$ the stationary point will occur. When $\gamma = 1$, $c = 1$ and thus $x_1 = (1-1)x_0 = 0$ and $y_0 = (1-1)y_0 = 0$.

Hence it will only take one iteration.