Computational Techniques Assessed Coursework 4

Solution 1

a) The basis function are $f_1(t) = 1$ and $f_2(t) = t$

b) The matrix A is

$$A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \\ \dots & \dots \\ 1 & t_i \end{bmatrix}$$

c) In order to have a unique solution, $\mathbf{A}^T\mathbf{A}$ has to be positive definite and thus $\mathbf{A}$ has to be a full-rank matrix.

We know that the normal equation on $\mathbf{A}$ is $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$. Using Cholesky factorization we can find a lower triangular matrix $\mathbf{L}$ such that $\mathbf{A}^T\mathbf{A} = \mathbf{L}\mathbf{L}^T$. If $\mathbf{A}^T\mathbf{A}$ is positive definite, then the Cholesky factorization is unique and therefore we can find a unique solution to $\mathbf{L}\mathbf{y} = \mathbf{A}^T\mathbf{b}$ by using forward substitution to get $\mathbf{y}$. Then we find a unique solution to $\mathbf{L}^T\mathbf{x} = \mathbf{y}$ by using backward substitution. Since both forward and backward substitution generate unique solutions, we know that the solution $\mathbf{x}$ is also unique.

Since $\mathbf{A}^T\mathbf{A}$ is positive definite, we know that for any $\mathbf{x} \in \mathbb{R}^2$

$$\begin{gathered} \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} > 0 \\ (\mathbf{A}\mathbf{x})^T\mathbf{A}\mathbf{x} > 0 \end{gathered}$$

Therefore, it is not possible to find a non-zero solution to the equation $\mathbf{A}\mathbf{x} = \mathbf{0}$ and thus the columns of $\mathbf{A}$ are linearly independent, which indicates that $\mathbf{A}$ has full-rank.

We can then expand this ineuqality and get the

$$\begin{aligned} \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} &> 0 \\ \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} m & \sum_{i=1}^m t_i \\ \sum_{i=1}^m t_i & \sum_{i=1}^m t_i^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &> 0 \\ mx_1^2 + 2\sum_{i=1}^m t_i \cdot x_1x_2 + \sum_{i=1}^m t_i^2 \cdot x_2^2 &> 0 \end{aligned}$$

This indicates that $(2\sum_{i=1}^m t_i)^2 - 4m\sum_{i=1}^m t_i^2 < 0$. We then expand this inequality

$$
\begin{aligned}
(2\sum_{i=1}^{m} t_i)^2 - 4m\sum_{i=1}^{m} t_i^2 &< 0 \\
4(\sum_{i=1}^{m} t_i)^2 &< 4m\sum_{i=1}^{m} t_i^2 \\
(\sum_{i=1}^{m} t_i)^2 &< m\sum_{i=1}^{m} t_i^2
\end{aligned}
$$

where m denotes the number of rows of $\mathbf{A}$, or the number of data points in $\mathbf{A}$.

From the fact that $\mathbf{A}$'s columns are linearly independent, there are at least 2 dinstinct values of t_i that are different.

d) The matrix $\mathbf{A}^T\mathbf{A}$ is as follows

$$
\mathbf{A}^T\mathbf{A} = \begin{bmatrix} m & \sum_{i=1}^{m} t_i \\ \sum_{i=1}^{m} t_i & \sum_{i=1}^{m} t_i^2 \end{bmatrix} = \begin{bmatrix} m & S_t \\ S_t & S_{t^2} \end{bmatrix}
$$

e) The determinant is as follows

$$
\det(\mathbf{A}^T\mathbf{A}) = \begin{vmatrix} m & \sum_{i=1}^{m} t_i \\ \sum_{i=1}^{m} t_i & \sum_{i=1}^{m} t_i^2 \end{vmatrix} = m\sum_{i=1}^{m} t_i^2 - (\sum_{i=1}^{m} t_i)^2 = mS_{t^2} - S_t^2
$$

The inverse is as follows

$$
\begin{aligned}
(\mathbf{A}^T\mathbf{A})^{-1} &= \frac{1}{\det(\mathbf{A}^T\mathbf{A})}\begin{bmatrix} S_{t^2} & -S_t \\ -S_t & m \end{bmatrix} \\
&= \begin{bmatrix} \frac{S_{t^2}}{mS_{t^2}-S_t^2} & -\frac{S_t}{mS_{t^2}-S_t^2} \\ -\frac{S_t}{mS_{t^2}-S_t^2} & \frac{m}{mS_{t^2}-S_t^2} \end{bmatrix}
\end{aligned}
$$

f) The matrix $\mathbf{A}^T\mathbf{b}$ is as follows

$$
\mathbf{A}^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ t_1 & t_2 & t_3 & \dots & t_m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \dots \\ y_m \end{bmatrix} = \begin{bmatrix} S_y \\ S_{ty} \end{bmatrix}
$$

g) We know that the normal equations give solution to the least square problem. Now we find the solution for $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$

$$
\begin{aligned}
\mathbf{A}^T\mathbf{A}\mathbf{x} &= \mathbf{A}^T\mathbf{b} \\
\mathbf{x} &= (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} \\
\mathbf{x} &= \begin{bmatrix} \frac{S_{t^2}}{mS_{t^2}-S_t^2} & -\frac{S_t}{mS_{t^2}-S_t^2} \\ -\frac{S_t}{mS_{t^2}-S_t^2} & \frac{m}{mS_{t^2}-S_t^2} \end{bmatrix} \begin{bmatrix} S_y \\ S_{ty} \end{bmatrix} \\
\mathbf{x} &= \begin{bmatrix} \frac{S_{t^2}S_y - S_tS_{ty}}{mS_{t^2}-S_t^2} \\ \frac{mS_{ty} - S_tS_y}{mS_{t^2}-S_t^2} \end{bmatrix}
\end{aligned}
$$

h) We can obtain two suitably chosen data pairs t_1, y_1 and t_2, y_2 satisfying (1), which must be distinct pairs

$$\begin{aligned} y_1 &= p_1 + p_2 t_1 \\ y_2 &= p_1 + p_2 t_2 \end{aligned}$$

so that

$$\begin{aligned}
\mathbf{x} = \begin{bmatrix} \frac{S_{t^2} S_y - S_t S_{ty}}{m S_{t^2} - S_t^2} \\ \frac{m S_{ty} - S_t S_y}{m S_{t^2} - S_t^2} \end{bmatrix} &= \begin{bmatrix} \frac{(t_1^2+t_2^2)(2p_1+p_2(t_1+t_2)) - (t_1+t_2)(p_1(t_1+t_2)+p_2(t_1^2+t_2^2))}{2(t_1^2+t_2^2)-(t_1+t_2)^2} \\ \frac{2(p_1(t_1+t_2)+p_2(t_1^2+t_2^2)) - (t_1+t_2)(2p_1+p_2(t_1+t_2))}{2(t_1^2+t_2^2)-(t_1+t_2)^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2p_1(t_1^2+t_2^2)+p_2(t_1+t_2)(t_1^2+t_2^2)-p_1(t_1+t_2)^2-p_2(t_1+t_2)(t_1^2+t_2^2)}{2t_1^2+2t_2^2-t_1^2-t_2^2-2t_1t_2} \\ \frac{2p_1(t_1+t_2)+2p_2(t_1^2+t_2^2)-2p_1(t_1+t_2)-p_2(t_1+t_2)^2}{2t_1^2+2t_2^2-t_1^2-t_2^2-2t_1t_2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{p_1(2t_1^2+2t_2^2-(t_1+t_2)^2)}{(t_1-t_2)^2} \\ \frac{p_2(2t_1^2+2t_2^2-(t_1+t_2)^2)}{(t_1-t_2)^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{p_1(t_1-t_2)^2}{(t_1-t_2)^2} \\ \frac{p_2(t_1-t_2)^2}{(t_1-t_2)^2} \end{bmatrix} \\
&= \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}
\end{aligned}$$

This test works because any two points can determine a line. In other words, the solution to the following linear system is unique since that t_1, t_2, y_1, y_2 are known and $t_1 \neq t_2$

$$\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Solution 2

From matrix $\mathbf{A}$, we know that

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

We can directly set $\mathbf{u}_1 = \mathbf{a}_1$ and then compute $\mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}$

$$\mathbf{e}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \mathbf{e}_1 \cdot \mathbf{a}_1 = 2, \mathbf{e}_1 \cdot \mathbf{a}_2 = 1, \mathbf{e}_1 \cdot \mathbf{a}_3 = 1, \mathbf{e}_1 \cdot \mathbf{e}_4 = \frac{3}{2}$$

Then we can obtain $\mathbf{u}_2 = \mathbf{a}_2 - (\mathbf{e}_1 \cdot \mathbf{a}_2)\mathbf{e}_1$

$$\mathbf{u}_2 = \mathbf{a}_2 - (\mathbf{e}_1 \cdot \mathbf{a}_2)\mathbf{e}_1 = \begin{bmatrix} 1 - \frac{1}{2} \\ 0 - \frac{1}{2} \\ 1 - \frac{1}{2} \\ 0 - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Again, we can compute $\mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}$

$$\mathbf{e}_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \mathbf{e}_2 \cdot \mathbf{a}_2 = 1, \mathbf{e}_2 \cdot \mathbf{a}_3 = 0, \mathbf{e}_2 \cdot \mathbf{a}_4 = \frac{1}{2}$$

Then we can obtain $\mathbf{u}_3 = \mathbf{a}_3 - (\mathbf{e}_1 \cdot \mathbf{a}_3)\mathbf{e}_1 - (\mathbf{e}_2 \cdot \mathbf{a}_3)\mathbf{e}_2$

$$\mathbf{u}_3 = \begin{bmatrix} 0 - \frac{1}{2} \\ 0 - \frac{1}{2} \\ 1 - \frac{1}{2} \\ 1 - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Again, we can compute $\mathbf{e}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|}$

$$\mathbf{e}_3 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \mathbf{e}_3 \cdot \mathbf{a}_3 = 1, \mathbf{e}_3 \cdot \mathbf{a}_4 = -\frac{1}{2}$$

Then we can obtain $\mathbf{u}_4 = \mathbf{a}_4 - (\mathbf{e}_1 \cdot \mathbf{a}_4)\mathbf{e}_1 - (\mathbf{e}_2 \cdot \mathbf{a}_4)\mathbf{e}_2 - (\mathbf{e}_3 \cdot \mathbf{a}_4)\mathbf{e}_3$

$$\mathbf{u}_4 = \begin{bmatrix} 1 - \frac{3}{4} - \frac{1}{4} - \frac{1}{4} \\ 1 - \frac{3}{4} + \frac{1}{4} - \frac{1}{4} \\ 1 - \frac{3}{4} - \frac{1}{4} + \frac{1}{4} \\ 0 - \frac{3}{4} + \frac{1}{4} + \frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}$$

Again, we can compute $\mathbf{e}_4 = \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|}$

$$\mathbf{e}_4 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \mathbf{e}_4 \cdot \mathbf{a}_4 = \frac{1}{2}$$

Then we can put $\mathbf{Q} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \end{bmatrix}$

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

and $\mathbf{R}$ as follows

$$\mathbf{R} = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{a}_1 & \mathbf{e}_1 \cdot \mathbf{a}_2 & \mathbf{e}_1 \cdot \mathbf{a}_3 & \mathbf{e}_1 \cdot \mathbf{a}_4 \\ 0 & \mathbf{e}_2 \cdot \mathbf{a}_2 & \mathbf{e}_2 \cdot \mathbf{a}_3 & \mathbf{e}_2 \cdot \mathbf{a}_4 \\ 0 & 0 & \mathbf{e}_3 \cdot \mathbf{a}_3 & \mathbf{e}_3 \cdot \mathbf{a}_4 \\ 0 & 0 & 0 & \mathbf{e}_4 \cdot \mathbf{a}_4 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

we can also verify that

$$\begin{aligned} \mathbf{QR} &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \mathbf{A} \end{aligned}$$