Computational Techniques Coursework 1

1) Below is the proof for Q1:

Let $\mathbf{x}$ be any vector in $\mathbb{R}^n$, then

$$\begin{aligned} \|\mathbf{Ax}\|_1 &= |a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n| + |a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n| + \ldots + |a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n| \\ &= \sum_{i=1}^{m} |\sum_{j=1}^{n}(a_{ij} \cdot x_j)| \leq \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij} \cdot x_j| \qquad \text{(due to triangular inequality: } |a+b| \leq |a| + |b| \text{ for any } a, b \in \mathbb{R}) \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij} \cdot x_j| &= \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}||x_j| \qquad \text{(due to the fact that } |ab| = |a||b| \text{ for any } a, b \in \mathbb{R}) \\ &= |a_{11}||x_1| + |a_{12}||x_2| + \ldots + |a_{21}||x_1| + |a_{22}||x_2| + \ldots + |a_{mn}||x_n| \\ &= |x_1|(|a_{11}| + |a_{21}| + |a_{31}| + \ldots + |a_{m1}|) + |x_2|(|a_{12}| + |a_{22}| + a_{32} + \ldots + |a_{m2}|) + \ldots + |x_n|(|a_{1n}| + \ldots + |a_{mn}|) \\ &= \sum_{j=1}^{n} |x_j|(\sum_{i=1}^{m} |a_{ij}|) = \sum_{j=1}^{n}(\sum_{i=1}^{m} |a_{ij}|)|x_j| \end{aligned}$$

which has proved that $\|\mathbf{Ax}\|_1 \leq \sum_{j=1}^{n}(\sum_{i=1}^{m} |a_{ij}|)|x_j|$for all $\mathbf{x} \in \mathbb{R}^n$

2) Let $\mathbf{x}$ be any nonzero vector in $\mathbb{R}^n$, then

$$\begin{aligned} \|\mathbf{A}\|_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}| \\ \|\mathbf{x}\|_1 &= \sum_{i=1}^{n} |x_i| \\ \|\mathbf{Ax}\|_1 &= \sum_{i=1}^{m} |\sum_{j=1}^{n}(a_{ij} \cdot x_j)| \end{aligned}$$

Let kth column be the one with maximum absolute column sum in the matrix $\mathbf{A}$, then

$$\begin{aligned} \|\mathbf{A}\|_1 &= \sum_{i=1}^{m} |a_{ik}| \\ \|\mathbf{Ax}\|_1 &\leq \sum_{j=1}^{n}(\sum_{i=1}^{m} |a_{ij}|)|x_j| \qquad \text{(From Q1)} \\ &= |x_1|(|a_{11}| + |a_{21}| + |a_{31}| + \ldots + |a_{m1}|) + |x_2|(|a_{12}| + |a_{22}| + a_{32} + \ldots + |a_{m2}|) + \ldots + |x_n|(|a_{1n}| + \ldots + |a_{mn}|) \\ &\leq |x_1|(|a_{1k}| + |a_{2k}| + |a_{3k}| + \ldots + |a_{mk}|) + |x_2|(|a_{1k}| + |a_{2k}| + a_{3k} + \ldots + |a_{m2}|) + \ldots + |x_n|(|a_{1k}| + \ldots + |a_{mk}|) \\ &\qquad \text{(due to the fact that the sum of } k\text{th column is bigger than or equal to any other column)} \\ &= (\sum_{i=1}^{n} |x_i|)(\sum_{i=1}^{m} |a_{ik}|) = \|\mathbf{x}\|_1\|\mathbf{A}\|_1 = \|\mathbf{A}\|_1\|\mathbf{x}\|_1 \end{aligned}$$

Therefore, $\|\mathbf{Ax}\|_1 \leq \|\mathbf{A}\|_1\|\mathbf{x}\|_1$ and that $\frac{\|\mathbf{Ax}\|_1}{\|\mathbf{x}\|_1} \leq \|\mathbf{A}\|_1$ for all nonzero vectors $\mathbf{x} \in \mathbb{R}^n$.

3) Let kth column of matrix $\mathbf{A}$ be the one with the maximum absolute column sum, and then we construct a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$x_k = c, \text{where } c \in \mathbb{R}$$
$$x_i = 0, \text{where } i \neq k, 1 \leq i \leq n$$

Then we know that

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i| = c$$

Since only $x_k = c$ and other entries are all zero, we know that $\mathbf{Ax}$ will be the column vector of kth column of $\mathbf{A}$ with each entry multiplied by c. Hence,
$\|\mathbf{Ax}\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij} \cdot x_j| = |x_k| \sum_{i=1}^{m} |a_{ik}| = c \sum_{i=1}^{m} |a_{ik}| = c\|\mathbf{A}\|_1$. Therefore, we have:

$$\frac{\|\mathbf{Ax}\|_1}{\|\mathbf{x}\|_1} = \frac{c\|\mathbf{A}\|_1}{c} = \|\mathbf{A}\|_1$$

From 2) we know that for all nonzero $\mathbf{x} \in \mathbb{R}^n$, $\frac{\|\mathbf{Ax}\|_1}{\|\mathbf{x}\|_1} \leq \|\mathbf{A}\|_1$. Therefore, the equality above holds only for certain vectors $\mathbf{x} \in \mathbb{R}^n$. Hence, $\|\mathbf{A}\|_1 = \max_{\mathbf{x} \neq 0} \frac{\|\mathbf{Ax}\|_1}{\|\mathbf{x}\|_1}$.

4) The second equality can be proved by letting $c = 1$ in the proof shown in 3).

Same as in question 3), we construct a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ with $x_k = 1$ and all other entries being 0, where the kth column of matrix $\mathbf{A}$ is the column with the maximum absolute column sum.

Then we know that

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i| = 1$$

Same as in question 3), we have

$$\frac{\|\mathbf{Ax}\|_1}{\|\mathbf{x}\|_1} = \|\mathbf{Ax}\|_1 = \|\mathbf{A}\|_1 = \max_{\mathbf{x} \neq 0} \frac{\|\mathbf{Ax}\|_1}{\|\mathbf{x}\|_1}$$

In this case, $\|\mathbf{Ax}\|_1 (\text{with c} = 1) = \max_{\|\mathbf{x}\|_1 = 1} \|\mathbf{Ax}\|_1$, since if we set another entry of $\mathbf{x}$ to 1 and all others to 0, then it will not produce the column with maximum column sum of $\mathbf{A}$, which will make the value smaller than $\|\mathbf{A}\|_1$.

We have proved the second equality.