Probability and Stats Assessed Coursework Jointly Distributed Random Variables

Solution 1

a) Since the integration over the joint pdf must equal to 0, we have

$$\begin{aligned}\int_{x=0}^{2}\int_{y=0}^{2} k(x+y)dydx &= 1\\ \int_{x=0}^{2}(2kx+2k)dx &= 1\\ 4k+4k &= 1\\ k &= \frac{1}{8}\end{aligned}$$

b) The marginal pdfs of X and Y can be found as follows

$$\begin{aligned}p_X(x) &= \int_{y=-\infty}^{\infty} f(x,y)dy = \int_0^2 \frac{1}{8}(x+y)dy = \frac{1}{4}x+\frac{1}{4}, 0<x<2\\ p_Y(y) &= \int_{x=-\infty}^{\infty} f(x,y)dx = \int_0^2 \frac{1}{8}(x+y)dx = \frac{1}{4}y+\frac{1}{4}, 0<y<2\end{aligned}$$

c) From b) we know that

$$p_X(x)p_Y(y) = (\frac{1}{4}x+\frac{1}{4})(\frac{1}{4}y+\frac{1}{4}) = \frac{1}{16}(xy+x+y+1) \neq f(x,y)$$

which means that X and Y are NOT independent.

Solution 2

We can determine $P(X>Y)$ as follows

$$\begin{aligned}P(X>Y) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{x} f(x,y)dydx &= \int_{x=0}^{\infty}\int_{y=0}^{x} abe^{-(ax+by)}dydx\\ &= \int_{x=0}^{\infty} a(e^{-ax}-e^{-(a+b)x})dx\\ &= 1-\frac{a}{a+b} = \frac{b}{a+b} = \frac{6}{11}\end{aligned}$$

Solution 3

a) The total integration of the pmf must be 1, hence we have

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=1}^{3} cx_i y_j &= 1\\ \sum_{i=1}^{3} 6cx_i &= 1\\ 36c &= 1\\ c &= \frac{1}{36}\end{aligned}$$

b) $P(X=2, Y=3) = \frac{1}{36}\cdot 2\cdot 3 = \frac{1}{6}$

c) $P(X\leq 2, Y\leq 2) = \frac{1}{36}(1\cdot 1 + 1\cdot 2 + 2\cdot 1 + 2\cdot 2) = \frac{1}{4}$

d) $P(X\geq 2) = \frac{1}{36}(2+3)(1+2+3) = \frac{5}{6}$

e) $P(Y<2) = \frac{1}{36}(1+2+3) = \frac{1}{6}$

f) $P(X=1) = \frac{1}{36}(1+2+3) = \frac{1}{6}$

g) $P(Y=3) = \frac{1}{36}\cdot 3\cdot (1+2+3) = \frac{1}{2}$

Solution 4

a) Again, the total area/integration of pdf has to be 1

$$\begin{aligned}\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f(x,y)dydx &= 1\\ \int_{x=0}^{1}\int_{y=0}^{1} c(x^2+y^2)dydx &= 1\\ \int_{x=0}^{1} c(x^2+\frac{1}{3})dx &= 1\\ \frac{2}{3}c &= 1\\ c &= \frac{3}{2}\end{aligned}$$

b) $P(X<\frac{1}{2}, Y>\frac{1}{2}) = \int_{x=0}^{\frac{1}{2}}\int_{y=\frac{1}{2}}^{1}\frac{3}{2}(x^2+y^2)dydx = \frac{3}{2}\int_{x=0}^{\frac{1}{2}}\frac{1}{2}x^2+\frac{7}{24}dx = \frac{1}{4}$

c) $P(\frac{1}{4}<X<\frac{3}{4}) = \int_{x=\frac{1}{4}}^{\frac{3}{4}}\int_{y=0}^{1}\frac{3}{2}(x^2+y^2)dydx = \frac{3}{2}\int_{x=\frac{1}{4}}^{\frac{3}{4}} x^2+\frac{1}{3}dx = \frac{29}{64}$

d) $P(Y<\frac{1}{2}) = \frac{3}{2}\int_{x=0}^{1}\int_{y=0}^{\frac{1}{2}}(x^2+y^2)dydx = \frac{3}{2}\int_{x=0}^{1}\frac{1}{2}x^2+\frac{1}{24}dx = \frac{5}{16}$

e) We know that

$$
\begin{aligned}
p_X(x) &= \int_{y=0}^{1} \frac{3}{2}(x^2+y^2)dy = \frac{3}{2}x^2 + \frac{1}{2} \\
p_Y(y) &= \int_{x=0}^{1} \frac{3}{2}(x^2+y^2)dx = \frac{3}{2}y^2 + \frac{1}{2} \\
p_X(x)p_Y(y) &= (\frac{3}{2}x^2 + \frac{1}{2})(\frac{3}{2}y^2 + \frac{1}{2}) = \frac{1}{4}(9x^2y^2 + 3x^2 + 3y^2 + 1) \neq f(x,y)
\end{aligned}
$$

Hence X and Y are not independent.

Solution 5

a) Using the cdf of X and Y we can find that

$$
\begin{aligned}
P(X < Y+1) &= \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{y+1} f_{XY}(x,y)dxdy \\
&= \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{y+1} f_X(x)f_Y(y)dxdy && (X \text{ and } Y \text{ are independent}) \\
&= \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{y+1} f_{X|Y}(x|y)f_Y(y)dxdy && (\text{By the partition rule of joint RV}) \\
&= \int_{y=-\infty}^{\infty} F_{X|Y}(y+1|y)f_Y(y)dy \\
&= \int_{y=-\infty}^{\infty} P(X < y+1|Y=y)f_Y(y)dy && (\text{By the definition of cdf})
\end{aligned}
$$

b) The probability of $X < Y + 1$ is

$$
\begin{aligned}
P(X < Y+1) &= \int_{-\infty}^{\infty} P(X < y+1|Y=y)f_Y(y)dy \\
&= \int_{-\infty}^{\infty}(\int_{-\infty}^{y+1} f_X(x)dx)f_Y(y)dy \\
&= \int_{-1}^{\infty}(\int_{0}^{y+1} \lambda e^{-\lambda x}dx)f_Y(y)dy \\
&= \int_{-1}^{\infty}(1-e^{-\lambda(y+1)})(\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}})dy \\
&= \int_{-1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy - \int_{-1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}-\lambda y-\lambda}dy \\
&= 1-\Phi(-1) - \int_{-1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}-\lambda y-\lambda}dy \\
&= \Phi(1) - \int_{-1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}-\lambda y-\lambda}dy
\end{aligned}
$$

When $\lambda = 1$,

$P(X < Y+1) = \Phi(1) - \int_{-1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}-y-1}dy = \Phi(1) - \frac{1}{\sqrt{2\pi}}\cdot\sqrt{\frac{\pi}{2e}} = \Phi(1) - \frac{1}{2\sqrt{e}}$

Solution 6

a) $T_n \leq x$ indicates that the nth occurance is within time length x, which means that there must be at least n occurances of the event within time period x. This is exactly the meaning of $N_x \geq n$.

b) With rate μ and time period of length x, the average number of events generated is thus μx

Since $T_n \leq x \Leftrightarrow N_x \geq n$, we have $P(T_n \leq x) = P(N_x \geq n)$.

$$P(N_x \geq n) = 1 - P(N_x < n) = 1 - \sum_{k=0}^{n-1} \frac{e^{-\mu x}(\mu x)^k}{k!} = 1 - e^{-\mu x} \sum_{k=0}^{n-1} \frac{(\mu x)^k}{k!}$$

c) Assume that the arrival of red bus and green bus are independent. Let X be the time needed for a red bus to come next, and Y be the time needed for a green bus to come next. We need to find $P(X < Y)$

$$\begin{aligned}
P(X < Y) &= \int_0^5 f_R(x)dx + \int_{y=5}^{\infty} \int_{x=5}^{y} f_R(x) f_R(y) dx dy \\
&= \int_0^5 \lambda e^{-\lambda x} dx + \int_{y=5}^{\infty} \int_{x=5}^{y} (\lambda e^{-\lambda x})(\mu e^{-\mu(y-5)}) dx dy \\
&= (1 - e^{-5\lambda}) + \int_{y=5}^{\infty} (-\mu e^{-\lambda y - \mu y + 5\mu} + \mu e^{-5\lambda - \mu y + 5\mu}) dy \\
&= 1 - e^{-5\lambda} - \frac{\lambda e^{-5\lambda}}{\lambda + \mu} \\
&= 1 - \frac{2\lambda + \mu}{\lambda + \mu} e^{-5\lambda}
\end{aligned}$$

Solution 7

a) Let Q be the the random varaible of the number of minutes that the professor being late. Then we need to find $P(Q > m)$

According to the partition rule of random variable

$$\begin{aligned}
P(Q > m) &= P(Q > m) \\
&= P(Z < -1) P(X > m) + P(Z \geq -1) P(Y > m) \\
&= \int_{-\infty}^{-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \int_m^{\infty} 0.1 e^{-0.1x} dx + \int_{-1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \int_m^{\infty} 0.05 e^{-0.05x} dx \\
&= 0.15865 \cdot e^{-0.1m} + (1 - 0.15865) \cdot e^{-0.05m} \\
&= 0.15865 \cdot e^{-0.1m} + 0.84134 \cdot e^{-0.05m}
\end{aligned}$$

b) The probability that the professor arrives at Victoria before 07:30 is $P(Z < -7)$

$$P(Z < -7) = \int_{-\infty}^{-7} p_Z(x) dx = 3.208 \cdot 10^{-12}$$

Solution 8

The definition of joint probability distribution is as follows

$$P_{XY}(B_X, B_Y) = P(X(s) \in B_X \wedge Y(s) \in B_Y | s \in S)$$