Probability and Statistics Exercise 1 - Event

Solution 1

From Axiom 3 we know that

$$\begin{aligned} P(E) &= P(E \cap F) + P(E \cap \bar{F}) \\ P(F) &= P(F \cap E) + P(F \cap \bar{E}) \\ P(E \cup F) &= P(E \cap F) + P(E \cap \bar{F}) + P(F \cap \bar{E}) \end{aligned}$$

Therefore, we have

$$\begin{aligned} P(E \cup F) &= P(E \cap F) + P(E) - P(E \cap F) + P(F) - P(E \cap F) \\ &= P(E) + P(F) - P(E \cap F) \end{aligned}$$

Solution 2

E and F will be independent if they satisfy the following property

$$P(E \cap F) = P(E)P(F)$$

Since E and F are mutually exclusive, it must be that

$$P(E \cap F) = P(\emptyset) = 0$$

Therefore, either $P(E) = 0$ or $P(F) = 0$.

Solution 3

a) $P(\text{odd number}) = \frac{1}{2}$

b) $P(\text{odd number less than 4}) = \frac{1}{3}$

Solution 4

a) $P(\text{two sixes}) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.

b) $P(\text{total of 3}) = \frac{2}{36} = \frac{1}{18}$

Solution 5

$P(\text{problem solved}) = 1 - P(\text{A failed} \cap \text{B failed}) = 1 - P(\text{A failed})P(\text{B failed}) = 1 - (1 - \frac{2}{5})(1 - \frac{1}{3}) = \frac{3}{5}$

Solution 6

$P(\mathrm{AX.XB}<\frac{3}{16}) = P(x(1-x) < \frac{3}{16}) = \frac{1}{2}$

Solution 7

a) $P(\text{odd outcome}) = \frac{18}{37}$

b) i. $P(\text{first is red}) = \frac{x}{x+y}$

ii. $P(\text{second is red}) = \frac{x(x+y-1)}{(x+y)(x+y-1)} = \frac{x}{x+y}$

iii. $P(\text{first two are red}) = \frac{x}{x+y} \cdot \frac{x}{x+y-1} = \frac{x^2}{(x+y)(x+y-1)}$

iv. $P(\text{last but one is red}) = \frac{x}{x+y}$

Solution 8

a) $P(\text{coin head} \cap \text{odd number}) = \frac{1}{2} \cdot \frac{3}{6} = \frac{1}{4}$

b) Since there are 30% females in the class, there are 70% males in the class. The final ratio of the class that passed exam is $P(\text{passed}) = 0.9 \cdot 0.7 + 0.8 \cdot 0.3 = 0.87$, which is 87%.

Solution 9

a) The events of rain on two consecutive days are NOT independent since $P(\text{first day rain})P(\text{second day rain}) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} \neq \frac{1}{10}$.

b) Let $E = \text{chance of rain today}$ and $F = \text{chance of rain tomorrow}$, then

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{\frac{1}{10}}{\frac{1}{4}} = \frac{2}{5}$$

c) Let $E = \text{chance of rain today}$ and $F = \text{chance of rain tomorrow}$, then

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{10}}{\frac{1}{4}} = \frac{2}{5}$$

Solution 10

a) $P(\text{leave it in the fourth shop}) = (1 - \frac{1}{4})^3 \cdot \frac{1}{4} = \frac{27}{256}$

b) Let $E = \text{he left his umbrella}$ and $F = \text{he left it in the fourth shop}$, then

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{\frac{27}{256}}{\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4}} = \frac{\frac{27}{256}}{\frac{175}{256}} = \frac{27}{175}$$

c) Let $E = \text{he left his umbrella after the first shop}$ and $F = \text{he left it in the fourth shop}$, then

$$P(E) = \frac{175}{256} - \frac{1}{4} = \frac{111}{256}$$

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{\frac{27}{256}}{\frac{111}{256}} = \frac{9}{37}$$

Solution 11

a) $P(\text{component is satisfactory}) = 0.4 \cdot 0.8 + 0.4 \cdot 0.9 + 0.2 = 0.88$

b) $P(\text{exactly one out of two is satisfactory}) = 0.4 \cdot 0.8 \cdot 0.2 \cdot 2 + 0.4 \cdot 0.9 \cdot 0.1 \cdot 2 = 0.2$

c) Let $E = \text{pack with one out of two components tested as satisfactory and}$ $F = \text{selected pack contained medium quality components}$, then

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{0.4 \cdot 0.9 \cdot 0.1 \cdot 2}{0.2} = 0.36$$

d) Let $E = \text{pack with both components tested as satisfactory and}$ $F = \text{pack contained high quality components}$, then

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{0.2}{0.78} = \frac{10}{39}$$

Solution 12

From the Bayes Theorem we know that

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)} \Rightarrow \frac{P(A|B)}{P(B|A)} = \frac{P(A)}{P(B)}$$

Since $P(A) \geq P(B) > 0$, we know that $\frac{P(A)}{P(B)} \geq 1$ and hence $\frac{P(A|B)}{P(B|A)} \geq 1$, which means that $P(A|B) \geq P(B|A)$.

Solution 13

Since A, B, and C are independent, it is the case that

$$\begin{aligned} P(A \cap B \cap C) &= P(A)P(B)P(C) \\ P(A \cap B) &= P(A)P(B) \\ P(A \cap C) &= P(A)P(C) \\ P(B \cap C) &= P(B)P(C) \end{aligned}$$

Then we have

$$\begin{aligned} P(A \cap (B \cup C)) &= P((A \cap B) \cup (A \cap C)) && \text{set operation} \\ &= P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C)) && \text{From Q1} \\ &= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \\ &= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C) \\ &= P(A)(P(B) + P(C) - P(B)P(C)) \\ &= P(A)(P(B) + P(C) - P(B \cap C)) \\ &= P(A)P(B \cup C) && \text{From the reverse of Q1} \end{aligned}$$

Therefore, we have proved that A and $B \cup C$ are independent.

Solution 14

a) $P(A) = P(\{\mathrm{a}\}) + P(\{\mathrm{b}\}) = 0.2 + 0.3 = 0.5$

b) $P(B) = P(\{\mathrm{b}\}) + P(\{\mathrm{c}\}) + P(\{\mathrm{d}\}) = 0.3 + 0.4 + 0.1 = 0.8$

c) $P(\bar{A}) = 1 - P(A) = 0.5$

d) $P(A \cup B) = P(S) = 1$

e) $P(A \cap B) = P(\{\mathrm{b}\}) = 0.3$

Solution 15

Let $E = \text{random part to be defective}$ and $F = \text{random part came from factory 1}$, then

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{\frac{1}{30}}{\frac{1}{12}} = \frac{2}{5}$$

Solution 16

We first want to show the following properties

$$\begin{aligned} P(A \cap B) &= P(A)P(B) \\ P(A \cap C) &= P(A)P(C) \\ P(B \cap C) &= P(B)P(C) \end{aligned}$$

We know that $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{2}$ and that $P(A \cap B) = \frac{1}{4}$, thus proved the first property.

We know that $P(A) = \frac{1}{2}$ and $P(C) = \frac{1}{2}$ and that $P(A \cap C) = \frac{1}{4}$, thus proved the second property.

The rest follows from the facts.

We then show that $P(A \cap B \cap C) \neq P(A)P(B)P(C)$.

We know that $P(A \cap B \cap C) = 0$ since when both die are odd, the sum must be even. We also know that $P(A)P(B)P(C) = \frac{1}{8} \neq 0$.

Solution 17

a) $P(\text{random phone being defective}) = 0.5 \cdot 0.02 + 0.3 \cdot 0.05 + 0.2 \cdot 0.01 = 0.027$

b) $P(\text{defective from the second plant}) = \frac{0.3 \cdot 0.05}{0.027} = \frac{5}{9}$

Solution 18

$P(\text{win}) = \frac{8}{36} + \frac{24}{36} \cdot (P(\text{point before 7}))$