Computational Techniques Coursework 5

Solution 1

Using the l_1 norm, we can find the condition number using

$$\text{cond}(A) = \|A^{-1}\|_1 \|A\|_1$$

To find A^{-1}, we need to use the inverse matrix of a 2×2 matrix

$$\begin{aligned} A^{-1} &= \frac{1}{\frac{1}{15} - \frac{1}{16}} \begin{bmatrix} \frac{1}{5} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{3} \end{bmatrix} \\ &= 240 \cdot \begin{bmatrix} \frac{1}{5} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{3} \end{bmatrix} \\ &= \begin{bmatrix} 48 & -60 \\ -60 & 80 \end{bmatrix} \end{aligned}$$

Then we find $\|A^{-1}\|_1$ and $\|A\|_1$

$$\begin{aligned} \|A^{-1}\| &= 140 \\ \|A\| &= \frac{7}{12} \end{aligned}$$

Hence the condition number of A using the l_1 norm is $\frac{245}{3}$

i) The condition of A under l_1 norm is shown above. The condition number of A is a relatively small figure($\frac{245}{3} \approx= 81.67$), so this matrix is well-conditioned. The output size will change according to the input with a factor of $\frac{245}{3}$.

ii) The l_∞ norm of A is $\frac{7}{12}$ and the l_2 norm of A is $\frac{\sqrt{497}}{30\sqrt{2}}$. The corresponding condition numbers of A are as follows

$$\text{cond}_{l_2}(A) = \frac{\sqrt{497}}{30\sqrt{2}} \cdot \sqrt{15904} = \frac{1988}{30} = \frac{994}{15}$$

$$\text{cond}_{l_\infty}(A) = \frac{7}{12} \cdot 140 = \frac{245}{3}$$

Solution 2

Using l_∞ norm of A, we can find the condition number of A as follows

$$\text{cond(A)} = \|A\|_\infty \|A^{-1}\|_\infty = 4 \cdot 3 = 12$$

Solution 3

Using the l_1 norm of A, we can find its condition number as follows

$$A^{-1} = -\frac{1}{13}\begin{bmatrix} 6 & 1 & 6 \\ 1 & -2 & 1 \\ 6 & 1 & 19 \end{bmatrix}$$

$$\text{cond(A)} = \|A\|_1 \|A^{-1}\|_1 = 5 \cdot 0 = 0$$

Solution 4

To show that x_n converges to 1, we need to prove that $\forall \epsilon > 0 \exists N \in \mathbb{R}\ s.t. \forall n > N, |x_n - 1| < \epsilon$.

We know that

$$\begin{aligned} |1 + \frac{1}{n} - 1| &< \epsilon \\ \frac{1}{n} &< \epsilon \\ \frac{1}{\epsilon} &< n \end{aligned}$$

Let $N = \lceil \frac{1}{\epsilon} \rceil$, then this value of N can always satisfy the statement above.

Hence, the sequence x_n converges to 1

Solution 5

Using the Cauchy-test, we need to show that $\forall \epsilon > 0 \exists N \in R$ such that $\forall n > m > N, |a_n - a_m| < \epsilon$

$$\begin{aligned} |a_n - a_m| &< \epsilon \\ |\sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{m} \frac{1}{i}| &< \epsilon \\ \sum_{i=m+1}^{n} \frac{1}{i} &< \epsilon \end{aligned}$$

No matter how big N is, we can always know that

$$\sum_{i=m+1}^{n} \frac{1}{i} > \sum_{i=m+1}^{n} \frac{1}{n}$$

Hence we can always find a smaller value ϵ than $|a_n - a_m|$, which proves that there is no N satisfying the given condition and the series diverges.

Without using the Cauchy-test, we can say that

$$
\begin{aligned}
\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots &= 1+(\frac{1}{2})+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})+\ldots \\
&> 1+(\frac{1}{2})+(\frac{1}{4}+\frac{1}{4})+(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8})+\ldots \\
&= 1+\frac{1}{2}+\frac{1}{2}+\ldots = \infty
\end{aligned}
$$

Solution 6

i) The matrix M and vector $\vec{c}$ are as follows

$$
\begin{aligned}
\vec{v}_{n+1} &= \mathbf{M}\vec{v}_n + \vec{c} \\
\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} &= \mathbf{M}\begin{bmatrix} a_n \\ b_n \end{bmatrix} + \vec{c} \\
\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} &= \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} a_n \\ b_n \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\end{aligned}
$$

Therefore

$$
M = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \vec{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}
$$

ii) We can prove that $\vec{v}_n$ will converge as $n \to \infty$ by showing that $f(\vec{v}) = \mathbf{M}\vec{v} + \vec{c}$ is a contraction.

$$
\begin{aligned}
\|f(\vec{v}_1) - f(\vec{v}_2)\| &= \|\mathbf{M}\vec{v}_1 + \vec{c} - \mathbf{M}\vec{v}_2 - \vec{c}\| \\
&= \|\mathbf{M}(\vec{v}_1 - \vec{v}_2)\| \\
&\le \|\mathbf{M}\|\|\vec{v}_1 - \vec{v}_2\|
\end{aligned}
$$

Since $\alpha, \beta < 1$, we have $\|M\|_1 < 1$ and $\|M\|_\infty < 1$. Using either of the two (consistent) norm we can confirm that $d(f(\vec{v}_1), f(\vec{v}_2)) \le \sigma(\vec{v}_1, \vec{v}_2)$, where $\sigma < 1$. This proves that $f(\vec{v})$ is a contraction. By the Fixed Point Theorem, there exists a $\vec{v}_n$ such that $\vec{v}_n = \mathbf{M}\vec{v}_n + \vec{c}$.

$$
\begin{aligned}
\begin{bmatrix} a_n \\ b_m \end{bmatrix} &= \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} a_n \\ b_n \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
\begin{bmatrix} (1-\alpha)a_n \\ (1-\beta)b_n \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
\begin{bmatrix} a_n \\ b_n \end{bmatrix} &= \begin{bmatrix} \frac{1}{1-\alpha} \\ \frac{2}{1-\beta} \end{bmatrix}
\end{aligned}
$$

Hence $\vec{v}_n$ converges to $\begin{bmatrix} \frac{1}{1-\alpha} \\ \frac{2}{1-\beta} \end{bmatrix}$ when $n \to \infty$.

The reason that a_n and b_n converge is as follows

$$
\begin{aligned}
a_n = \alpha a_{n-1} + 1 &= \alpha(\alpha a_{n-2} + 1) + 1 \\
&= \alpha^2 a_{n-2} + \alpha + 1 \\
&= \alpha^3 a_{n-3} + \alpha^2 + \alpha + 1 \\
&= \alpha^{n-1} a_1 + \alpha^{n-2} + \alpha^{n-3} + \ldots + 1
\end{aligned}
$$

When $n \to \infty$, $\alpha^{n-1} \to 0$ and $\alpha^{n-2} + \alpha^{n-3} + \ldots + 1 = \frac{1}{1-\alpha}$. Hence $a_n = \frac{1}{1-\alpha}$ when $n \to \infty$. Same reasoning for the convergence of b_n.

iii) From the information given in the question, we have

$$
\begin{aligned}
a_{n+1} &= \alpha a_n + 1 \\
b_{n+1} &= \beta b_n + 2 \\
c_{n+1} &= a_{n+1} + b_{n+1} = \alpha a_n + \beta b_n + 3 = (\alpha - \beta) a_n + \beta(a_n + b_n) + 3 = (\alpha - \beta) a_n + \beta c_n + 3
\end{aligned}
$$

Hence we can construct the linear system

$$
\begin{aligned}
\begin{bmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{bmatrix} &= \begin{bmatrix} \alpha a_n & 0 & 0 \\ 0 & \beta b_n & 0 \\ (\alpha - \beta) a_n & 0 & \beta c_n \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ \alpha - \beta & 0 & \beta \end{bmatrix} \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
\end{aligned}
$$

which proves the statement.

We can show that c_n converges by showing that $\vec{w}_n$ converges

Again, we will show that $f(\vec{w}_n) = \mathbf{M}' \vec{w}_n + \vec{c}'$ is a contraction

$$
\begin{aligned}
\|f(\vec{w}_1) - f(\vec{w}_2)\| &= \|\mathbf{M}' \vec{w}_1 + \vec{c}' - \mathbf{M}' \vec{w}_2 - \vec{c}'\| \\
&= \|\mathbf{M}'(\vec{w}_1 - \vec{w}_2)\| \\
&\leq \|\mathbf{M}'\| \|\vec{w}_1 - \vec{w}_2\|
\end{aligned}
$$

By choose a consistent norm such as l_∞ norm, we can show that $\|M'\| < 1$.

$$
l_\infty = \max_i \sum_j w_{ij} = \alpha < 1
$$

which means tht $f(\vec{w})$ is a contraction and that there exists a $\vec{w}_n$ such that $\vec{w}_n = \mathbf{M}' \vec{w}_n + \vec{c}'$

$$
\begin{aligned}
(\mathbf{I}_3 - \mathbf{M}')\vec{w}_n &= \vec{c}' \\
\begin{bmatrix} 1 - \alpha & 0 & 0 \\ 0 & 1 - \beta & 0 \\ \beta - \alpha & 0 & 1 - \beta \end{bmatrix} \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
\begin{bmatrix} (1 - \alpha) a_n \\ (1 - \beta) b_n \\ (\beta - \alpha) a_n + (1 - \beta) c_n \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} &= \begin{bmatrix} \frac{1}{1-\alpha} \\ \frac{2}{1-\beta} \\ \frac{3 - \frac{\beta - \alpha}{1 - \alpha}}{1 - \beta} \end{bmatrix} = \begin{bmatrix} \frac{1}{1-\alpha} \\ \frac{2}{1-\beta} \\ \frac{3 - 2\alpha - \beta}{(1-\alpha)(1-\beta)} \end{bmatrix}
\end{aligned}
$$

Hence c_n will converges to $\frac{3-2\alpha-\beta}{(1-\alpha)(1-\beta)}$ when $n \to \infty$

Solution 7

Metric space is a non-empty set S of points together with a mapping $d : S \times S \to \mathbb{R}$ satisfying the following properties

$$\begin{aligned} &d(x,x) = 0 \\ &d(x,y) > 0 \text{ if } x \neq y \\ &d(x,y) = d(y,x) \\ &d(x,y) \leq d(x,z) + d(z,y) \end{aligned}$$

The norm of the difference between two vectors satisfies those properties. Let A, B and C be three different matrices, then

$$\begin{aligned} &\|A - A\| = \|\mathbf{0}\| = 0 \\ &\|A - B\| > 0 && \text{(by the definition of norm e.g.} \sum_{ij}(x_{ij}^p)^{\frac{1}{p}}) \\ &\|A - B\| = \|B - A\| && \text{(since the norms are taken with absolute values)} \\ &\|A - C\| \leq \|A - B\| + \|B - C\| && \text{(by triangular inequality)} \end{aligned}$$

We now prove that $(\mathbf{I} - \mathbf{M})\mathbf{G}_n = \mathbf{I} - \mathbf{M}^{n+1}$

$$\begin{aligned} (\mathbf{I} - \mathbf{M})\mathbf{G}_n &= \mathbf{G}_n - \mathbf{M}\mathbf{G}_n \\ &= \mathbf{I} + \mathbf{M} + \mathbf{M}^2 + \mathbf{M}^3 + \ldots + \mathbf{M}^n - (\mathbf{M} + \mathbf{M}^2 + \mathbf{M}^3 + \ldots + \mathbf{M}^{n+1}) \\ &= \mathbf{I} + (\mathbf{M} - \mathbf{M}) + (\mathbf{M}^2 - \mathbf{M}^2) + \ldots + (\mathbf{M}^n - \mathbf{M}^n) - \mathbf{M}^{n+1} \\ &= \mathbf{I} - \mathbf{M}^{n+1} \end{aligned}$$

For any submultiplicative matrix norm, we have

$$\|Ax\| \leq \|A\|\|x\|$$

Hence for $\mathbf{M}^n$ we have

$$\|\mathbf{M}^n\| \leq \|\mathbf{M}^{n-1}\|\|\mathbf{M}\| \leq \|\mathbf{M}^{n-2}\|\|\mathbf{M}\|\|\mathbf{M}\| \leq \ldots \leq \|\mathbf{M}\|^n$$

Since $\|\mathbf{M}\| < 1$, when $n \to \infty$, $\|\mathbf{M}\|^n \to 0$ and hence $\|\mathbf{M}^n\| \to 0$, which means that $\mathbf{M}^n$ is the zero matrix.

From the previous equation of $(\mathbf{I} - \mathbf{M})\mathbf{G}_n = \mathbf{I} - \mathbf{M}^{n+1}$, when $n \to \infty$, it becomes

$$\begin{aligned} (\mathbf{I} - \mathbf{M})\mathbf{G}_\infty &= \mathbf{I} - \mathbf{0} \\ \mathbf{G}_\infty &= (\mathbf{I} - \mathbf{M})^{-1} \end{aligned}$$

Solution 8

In order to use the matrix $\mathbf{M}^2$, we need to construct $\vec{v}_{n+1} = \mathbf{M}^2\vec{v}_{n-1}$.

To prove the convergence of $\vec{v}_n$, we need to show that $f(\vec{v}) = \mathbf{M}^2\vec{v}$ is a contraction

$$\begin{aligned} \|f(\vec{v}_1) - f(\vec{v}_2)\| &= \|\mathbf{M}^2\vec{v}_1 - \mathbf{M}^2\vec{v}_2\| \\ &= \|\mathbf{M}^2(\vec{v}_1 - \vec{v}_2)\| \\ &\leq \|\mathbf{M}^2\|\|\vec{v}_1 - \vec{v}_2\| \end{aligned}$$

We now show that $\|\mathbf{M}^2\| < 1$ by choosing a suitable norm

$$\mathbf{M}^2 = \begin{bmatrix} 0.75 & 0.2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.75 & 0.2 \\ 1 & 0 \end{bmatrix} = \frac{1}{80} \begin{bmatrix} 61 & 12 \\ 60 & 16 \end{bmatrix}$$

We then choose the consistent l_∞ norm of the matrix and hence $\|\mathbf{M}^2\|_\infty = \frac{76}{80} < 1$

This proves that $f(\vec{v}_n)$ is a contraction and by the Fixed Point Theorem, ther exists a $\vec{v}_n$ such that $\vec{v}_n = \mathbf{M}^2 \vec{v}_n$

$$\begin{aligned} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} &= \frac{1}{80} \begin{bmatrix} 61 & 12 \\ 60 & 16 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} \\ 80 \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} &= \begin{bmatrix} 61 & 12 \\ 60 & 16 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} \\ &\Rightarrow \begin{cases} 80x_{n+1} = 61x_{n+1} + 12x_n \\ 80x_n = 60x_{n+1} + 16x_n \end{cases} \\ &\Rightarrow \begin{cases} x_{n+1} = \frac{76}{79} x_n \\ x_n = 0 \end{cases} \\ &\Rightarrow x_{n+1} = x_n = 0 \end{aligned}$$

Hence $\vec{v}_\infty = \vec{0}$ and $x_\infty = 0$