# Computational Techniques Coursework 8

## Solution 1

i) To compute the local extreme point, we will make the second term in Taylor's series equal to 0

$$\frac{\partial}{\partial x} f(x, y) = -y = 0 \Rightarrow y = 0$$
$$\frac{\partial}{\partial y} f(x, y) = -x = 0 \Rightarrow x = 0$$

The Hessian matrix of $f(x, y)$ is

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Since the eigenvalue of $H$ is $\pm 1$, we know that $H$ is indefinite, and hence $f(x, y)$ has no extreme point(max or min) exists at point $(0, 0)$.

ii) Similarly as computed in i):

$$\frac{\partial}{\partial x} f(x, y) = 2x = 0 \Rightarrow x = 0$$
$$\frac{\partial}{\partial y} f(x, y) = -3y^2 = 0 \Rightarrow y = 0$$

the Hessian matrix of $f(x, y)$ is

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -6y \end{bmatrix}$$

When $y = 0$, the determinant of $H$ is 0 and hence $H$ is indefinite and hence there is no extreme point exists at point $(0, 0)$.

iii) Similarly as computed above

$$\frac{\partial}{\partial x} f(x, y) = 2x = 0 \Rightarrow x = 0$$
$$\frac{\partial}{\partial y} f(x, y) = 2y = 0 \Rightarrow y = 0$$

the Hessian matrix of $f(x, y)$ is

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Since the only eigenvalues of $H$ is $2$ and its determinant is positive, we know that $H$ is positive definite and has a local minimum at $(0, 0)$ with value $f(0, 0) = 0$.

## Solution 2

a) We first compute the Hessian matrix of $f_1(x, y)$

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Since the only eigenvalues of $H$ is $2$ and its determinant is positive, we know that $H$ is positive definite and has a local minimum.

We then set $\frac{\partial}{\partial x} f_1(x, y) = 0$ and $\frac{\partial}{\partial y} f_1(x, y) = 0$

$$\frac{\partial}{\partial x} f_1(x, y) = 2x = 0 \Rightarrow x = 0$$
$$\frac{\partial}{\partial y} f_1(x, y) = 2y = 0 \Rightarrow y = 0$$

which indicates that $f_1(x, y)$ has a minimum at point $(0, 0)$.

b) We first set $\frac{\partial}{\partial x} f_2(x, y) = 0$ and $\frac{\partial}{\partial y} f_2(x, y) = 0$

$$\frac{\partial}{\partial x} f_2(x, y) = -2x = 0 \Rightarrow x = 0$$
$$\frac{\partial}{\partial y} f_2(x, y) = 2y = 0 \Rightarrow y = 0$$

then we compute the Hessian matrix of $f_2$

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

Since the eigenvalue of $H$ is $\pm 2$, we know that $f_2$ does not have maximum or minimum at $(0, 0)$. Since the determinant is $-4 < 0$, we know that $f_2$ has a saddle point at $(0, 0)$.

c) We first set $\frac{\partial}{\partial x} f_3(x, y) = 0$ and $\frac{\partial}{\partial y} f_3(x, y) = 0$

$$\frac{\partial}{\partial x} f_3(x, y) = 3x^2 + 6y = 0$$
$$\frac{\partial}{\partial y} f_3(x, y) = -3y^2 + 6x = 0$$

which leads to the solution

$$x_1 = 0, y_1 = 0$$
$$x_2 = 2, y_2 = -2$$

We now check the nature of those two critical points.

We then compute the Hessian matrix of $f_3$

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} 6x & 6 \\ 6 & -6y^2 \end{bmatrix}$$

when $x = 0, y = 0$, the eigenvalues of $H$ are $\pm 6$ and the determinant of $H$ is $-36 < 0$. Hence the point $(0, 0)$ is a saddle point.

when $x = 2, y = -2$, the determinant of $H$ is $-324 < 0$ and its eigenvalue is around $12.97$. Hence $H$ is indefinite and $(2, -2)$ forms a saddle point.