

$$\begin{aligned}
\mathbb{E}[\hat{\Sigma}] &= \mathbb{E} \left[\frac{1}{n} \sum_{k=1}^n (x_k - \hat{\mu})(x_k - \hat{\mu})^T \right] \\
&= \frac{1}{n} \mathbb{E} \left[\sum_{k=1}^n (x_k x_k^T - x_k \hat{\mu}^T - \hat{\mu} x_k^T + \hat{\mu} \hat{\mu}^T) \right] \\
&= \frac{1}{n} \sum_{k=1}^n \mathbb{E} [x_k x_k^T - x_k \hat{\mu}^T - \hat{\mu} x_k^T + \hat{\mu} \hat{\mu}^T] \\
&= \frac{1}{n} \sum_{k=1}^n \mathbb{E} \left[x_k x_k^T - x_k \left(\frac{1}{n} \sum_{j=1}^n x_j^T \right) - \left(\frac{1}{n} \sum_{j=1}^n x_j \right) x_k^T + \left(\frac{1}{n} \sum_{j=1}^n x_j \right) \left(\frac{1}{n} \sum_{j=1}^n x_j^T \right) \right] \\
&= \frac{1}{n} \sum_{k=1}^n \left\{ \mathbb{E} [x_k x_k^T] - \frac{1}{n} \sum_{j=1}^n \mathbb{E} [x_k x_j^T] - \frac{1}{n} \sum_{j=1}^n \mathbb{E} [x_j x_k^T] + \frac{1}{n^2} \sum_{j=1}^n \sum_{i=1}^n x_j x_i^T \right\} \\
&= \frac{1}{n} \sum_{k=1}^n \left\{ \mathbb{E} [x_k x_k^T] - \frac{1}{n} \left(\mathbb{E} [x_k x_k^T] + \sum_{j=1, j \neq k}^n \mathbb{E} [x_k x_j^T] \right) - \frac{1}{n} \left(\mathbb{E} [x_k x_k^T] + \sum_{j=1, j \neq k}^n \mathbb{E} [x_j x_k^T] \right) \right. \\
&\quad \left. + \frac{1}{n^2} \left(\sum_{j=1}^n \mathbb{E} [x_j x_j^T] + \sum_{j=1}^n \sum_{i=1, i \neq j}^n \mathbb{E} [x_j x_i^T] \right) \right\} \\
&= \frac{1}{n} \sum_{k=1}^n \left\{ \frac{n-2}{n} \mathbb{E} [x_k x_k^T] - \frac{2}{n} \sum_{j=1, j \neq k}^n \mu \mu^T + \frac{1}{n} \sum_{k=1}^n \left\{ \frac{1}{n^2} \left(\sum_{j=1}^n \mathbb{E} [x_j x_j^T] + \sum_{j=1}^n \sum_{i=1, i \neq j}^n \mathbb{E} [x_j x_i^T] \right) \right\} \right\} \\
&= \frac{1}{n} \sum_{k=1}^n \left\{ \frac{n-1}{n} \mathbb{E} [x_k x_k^T] - \frac{2(n-1)}{n} \mu \mu^T + \frac{n-1}{n} \mu \mu^T \right\} \\
&= \frac{n-1}{n} \frac{1}{n} \sum_{k=1}^n \{ \mathbb{E} [x_k x_k^T] - 2 \mu \mu^T + \mu \mu^T \} \\
&= \frac{n-1}{n} \frac{1}{n} \sum_{k=1}^n \mathbb{E} \{ x_k x_k^T - x \mu^T - \mu x^T + \mu \mu^T \} = \frac{n-1}{n} \frac{1}{n} \sum_{k=1}^n \mathbb{E} [(x - \mu)(x - \mu)^T] = \frac{n-1}{n} \Sigma
\end{aligned}$$