Homework 4

Lingfeng Zhao

LZ1973

2.

```
import numpy as np
from sklearn.linear_model import LinearRegression
from sklearn.model_selection import train_test_split
from sklearn.metrics import r2_score

utr, uts, ytr, yts = train_test_split(u, y, test_size=0.5)
dtest = range(1, 11)

rss = []
for i in dtest:
    x_tr = []
    x_ts = []
    for j in range(i + 1):
       x_tr.append(np.exp(-j/i * utr))
       x_ts.append(np.exp(-j/i * uts))

    X = np.concatenate(x_tr, axis=1)
    Xts = np.concatenate(x_ts, axis=1)

    model = LinearRegression()
    model.fit(X, ytr) # Fits a linear model for a data matrix X
    yhat = model.predict(Xts) # Predicts values

    rsq = r2_score(yts, yhat, multioutput='uniform_average')
    print(f"R^2 is {rsq}")
    rss.append(rsq)

order = np.argmin(rss) + 1
print(f"Model with lowest r^2 is {order}")
```

3.

(a)

The training data has no noise: $y_i = f(x_i, \beta_0)$.

$$
\begin{aligned}
Bias(x) &= \mathbb{E}(f(x,\hat{\beta})) - f(x,\beta_0) \\
&= \mathbb{E}(\hat{\beta}x^2) - f(x,\beta_0) \\
&= \mathbb{E}(\frac{\Sigma y_i}{\Sigma x_i^2}x^2) - f(x,\beta_0) \\
&= \mathbb{E}(\frac{\Sigma y_i}{\Sigma x_i^2})x^2 - f(x,\beta_0) \\
&= \beta_0 x^2 - \beta_0 x^2 \\
&= 0
\end{aligned}
$$

(b)

The training data is $y_i = f(x_i, \beta_0) + \epsilon_i$ where the noise is i.i.d. $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$.

$$
\begin{aligned}
Bias(x) &= \mathbb{E}(f(x,\hat{\beta})) - f(x,\beta_0) \\
&= \mathbb{E}(\hat{\beta}x^2) - f(x,\beta_0) \\
&= \mathbb{E}(\frac{\Sigma y_i}{\Sigma x_i^2}x^2) - f(x,\beta_0) \\
&= \mathbb{E}(\frac{\Sigma \beta_0 x_i^2 + \epsilon_i}{\Sigma x_i^2})x^2 - f(x,\beta_0) \\
&= \beta_0 x^2 - \beta_0 x^2 \\
&= 0
\end{aligned}
$$

(c)

The training data is $y_i = f(x_i + \epsilon_i, \beta_0)$ where the noise is i.i.d. $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$.

$$
\begin{aligned}
Bias(x) &= \mathbb{E}(f(x,\hat{\beta})) - f(x,\beta_0) \\
&= \mathbb{E}(\hat{\beta}x^2) - f(x,\beta_0) \\
&= \mathbb{E}(\frac{\Sigma y_i}{\Sigma x_i^2}x^2) - f(x,\beta_0) \\
&= \mathbb{E}(\frac{\Sigma \beta_0 (x_i + \epsilon_i)^2}{\Sigma x_i^2})x^2 - f(x,\beta_0) \\
&= \beta_0 \mathbb{E}(\frac{\Sigma x_i^2 + \epsilon_i^2 + 2x_i\epsilon}{\Sigma x_i^2})x^2 - f(x,\beta_0) \\
&= \beta_0 (1 + \frac{\sigma^2}{\Sigma x_i^2} + 0)x^2 - \beta_0 x^2 \\
&= \frac{\sigma^2 \beta_0}{\Sigma x_i^2}
\end{aligned}
$$

4.

(a)

This is a normal regression question. We can get the result:

$$\hat{\beta} = \arg\min_{\beta} \Sigma_{i=1}^{N} (y_i - (\beta_0 + \beta_1 x_i))^2$$

(b)

The same as (a)

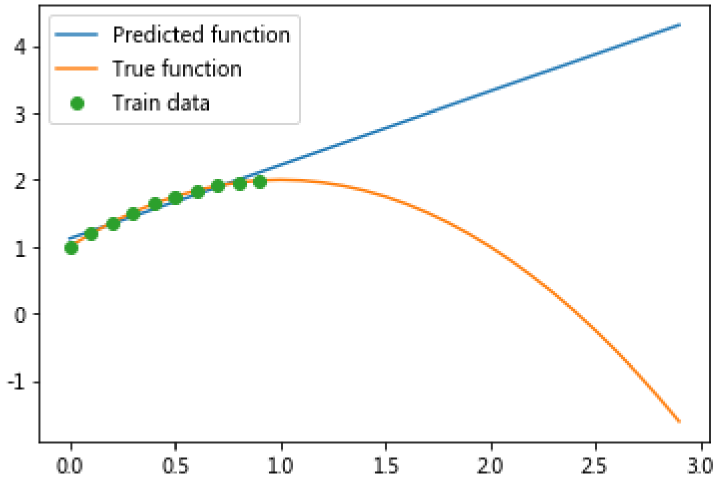
$$\hat{\beta} = \arg\min_{\beta} \Sigma_{i=1}^{N} (\beta_{00} + \beta_{01} x_i + \beta_{02} x_i^2 - (\beta_0 + \beta_1 x_i))^2$$

(c)

```
import numpy as np
from matplotlib import pyplot as plt
beta_0 = [-1, 2, 1]

x = np.arange(0, 1, 0.1)
y = np.polyval(beta_0, x)

beta_hat = np.polyfit(x, y, 1)
x_plot = np.arange(0, 3, 0.1)
y_pred = np.polyval(beta_hat, x_plot)
y_true = np.polyval(beta_0, x_plot)
plt.plot(x_plot, y_pred)
plt.plot(x_plot, y_true)
plt.plot(x, y, 'o')
plt.legend(['Predicted function', 'True function', 'Train data'])
plt.show()
```



(d)

When x = 3.

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(a)

Let us define $x_1 := cancer\ volume$, $x_2 := age$, $x_3 := if\ a\ cancer\ is\ Type1$

Model 1:

$$\hat{y} = \beta_0 + \beta_1 x_1$$

Model 2:

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

Model 3:

$$\hat{y} = \beta_0 + \beta_1 x_1 x_3 + \beta_2 x_1 (1 - x_3) + \beta_3 x_2$$

(b)

From Model 1 to Model 3, there are respectively 2, 3 and 4 parameters. Model 3 is the most complicated one.

(c)

Model 1:

$$\mathbf{A} = \begin{bmatrix} 1 & 0.7 \\ 1 & 1.3 \\ 1 & 1.6 \\ \vdots & \vdots \end{bmatrix}$$

Model 2:

$$\mathbf{A} = \begin{bmatrix} 1 & 0.7 & 55 \\ 1 & 1.3 & 65 \\ 1 & 1.6 & 70 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

Model 3:

$$\mathbf{A} = \begin{bmatrix} 1 & 0.7 & 0 & 55 \\ 1 & 0 & 1.3 & 65 \\ 1 & 0 & 1.6 & 70 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

(d)

$SE = stdRSS/\sqrt{K-1}$. For three model, Model 3 has the least RSS 0.7, so $SE = 0.05/\sqrt{10-1} \approx 0.0167$. We get the RSS target of $0.7 + 0.0167 = 0.7167$. Only Model 3 is less than this target. So we choose model 3.