西湖大學
WESTLAKE UNIVERSITY

Adacanced Electromagnetic

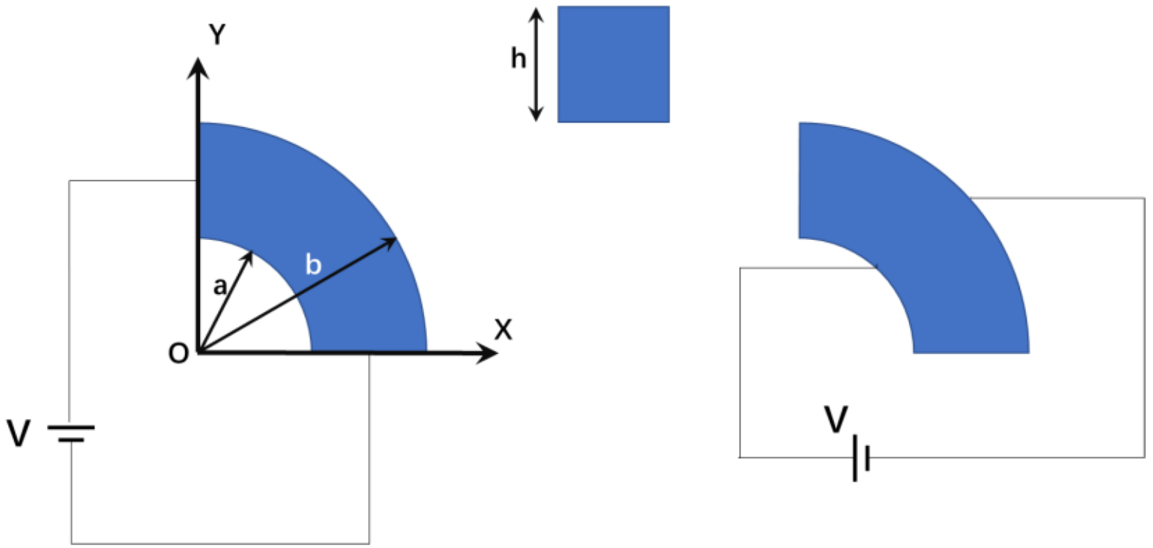
School of Engineering

September 27, 2024

Problem 1

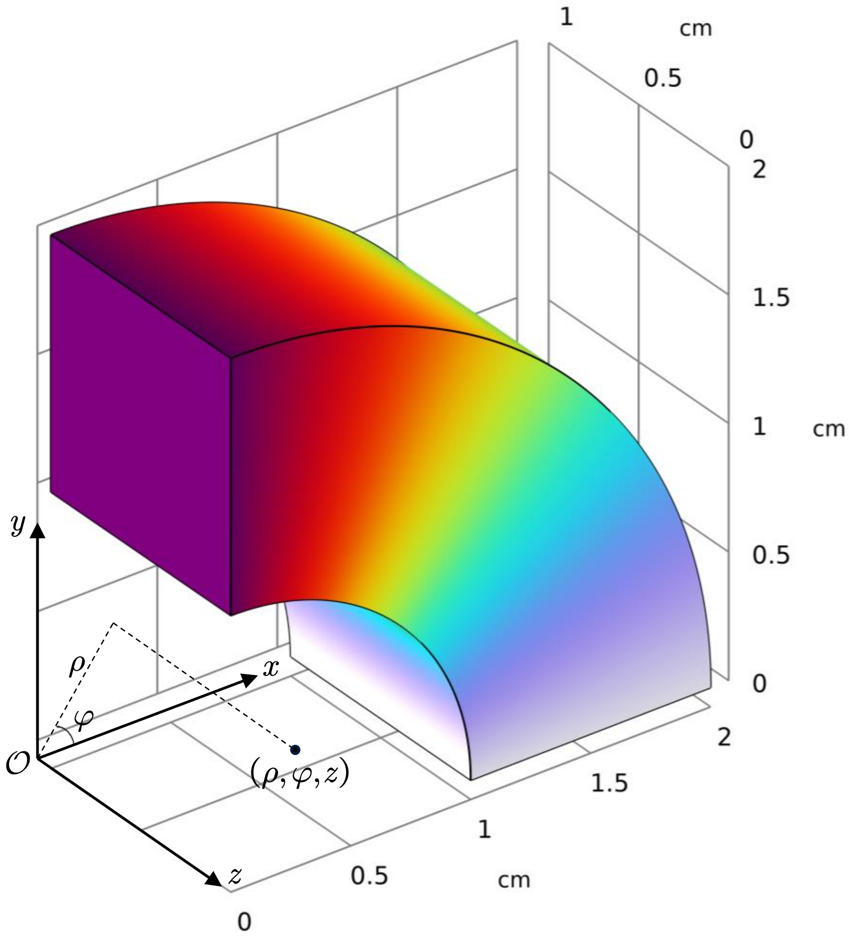
Problem 1

- A conducting material of thickness h and uniform conductivity σ has the shape of a quarter of a square tube as shown in the figure below, with inner radius a and outer radius b. Calculate the steady electric currents and the resistances in the two cases.



Problem 1

- Analytical solution for case 1



Laplace's equation (cylindrical coordinate system)

$$\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

$$\frac{\partial\phi}{\partial\rho} = 0 \;\&\; \frac{\partial\phi}{\partial z} = 0$$

$$\Rightarrow \quad \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} = 0$$

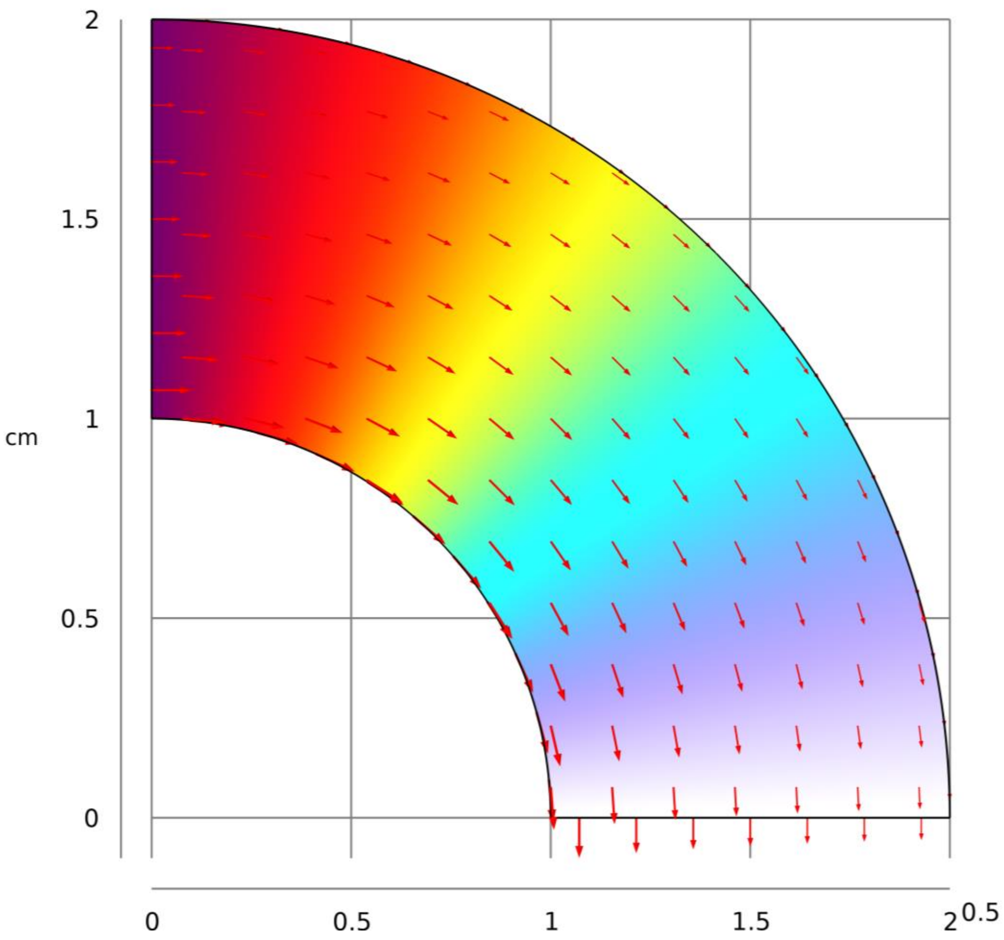
$$\Rightarrow \quad \phi(\varphi) = C_1\varphi + C_2$$

$$\phi(\varphi = 0) = 0 \;\&\; \phi(\varphi = \pi/2) = V_0$$

$$\Rightarrow \quad \phi(\varphi) = \frac{2V_0}{\pi}\varphi$$

Problem 1

- Analytical solution for case 1



The relationship between electric field and electric potential

$$\vec{E} = -\nabla\phi = -\frac{1}{\rho}\frac{2V_0}{\pi}\vec{e}_\varphi$$

$$\nabla = \vec{e}_\rho \frac{\partial}{\partial\rho} + \vec{e}_\varphi \frac{1}{\rho}\frac{\partial}{\partial\varphi} + \vec{e}_z \frac{\partial}{\partial z}$$

$$\Rightarrow \vec{J} = \sigma\vec{E} = \frac{2V_0\sigma}{\pi\rho}\vec{e}_\varphi$$

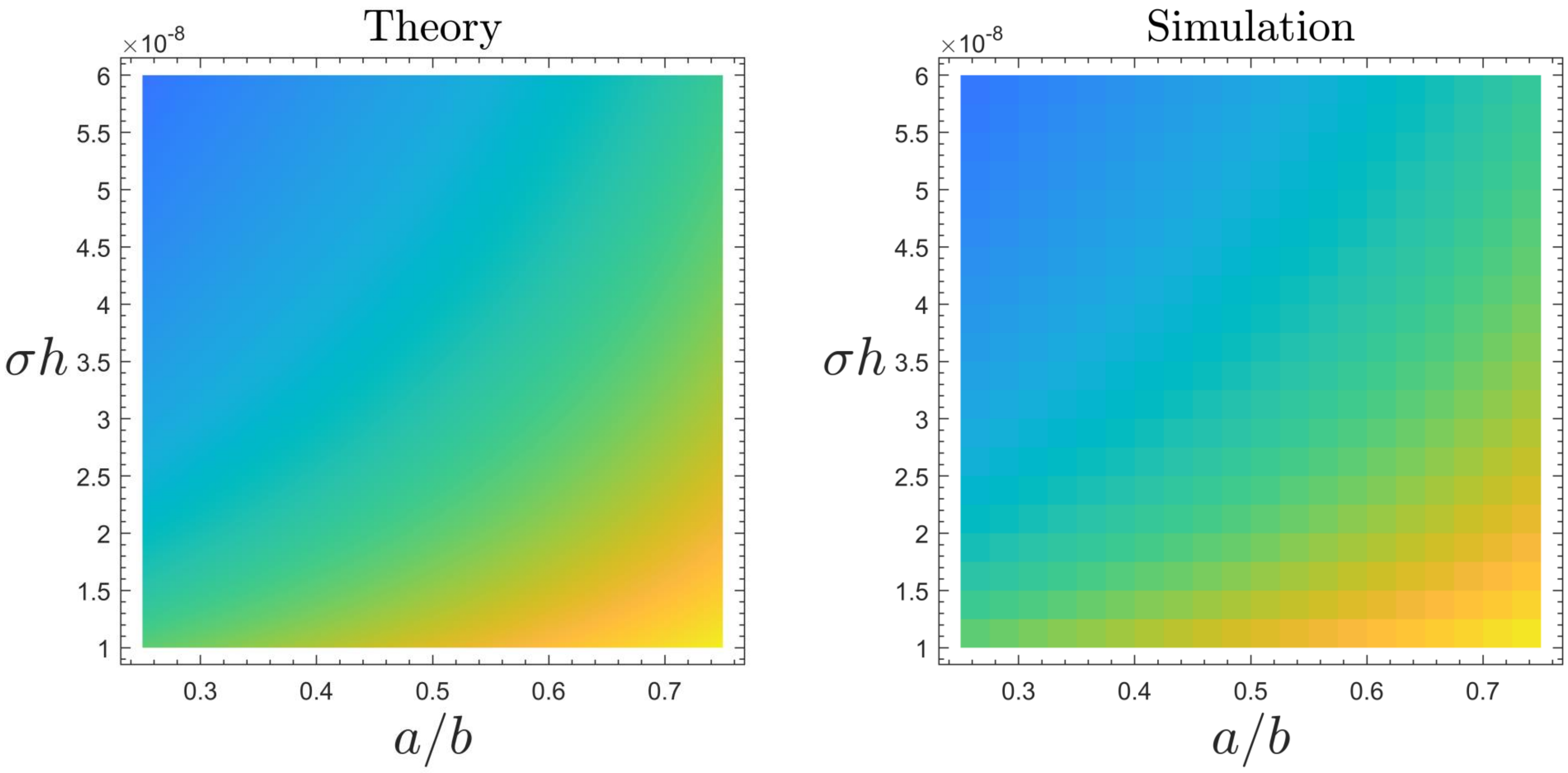
$$\Rightarrow I = \int_S \vec{J}\,\mathrm{d}\vec{S} = \int_a^b \frac{2V_0\sigma}{\pi\rho} h\,\mathrm{d}\rho = \frac{2V_0\sigma h\ln(b/a)}{\pi}$$

Ohm's Law

$$\Rightarrow R = \frac{V_0}{I} = \frac{\pi}{2\sigma h\ln(b/a)}$$

Problem 1

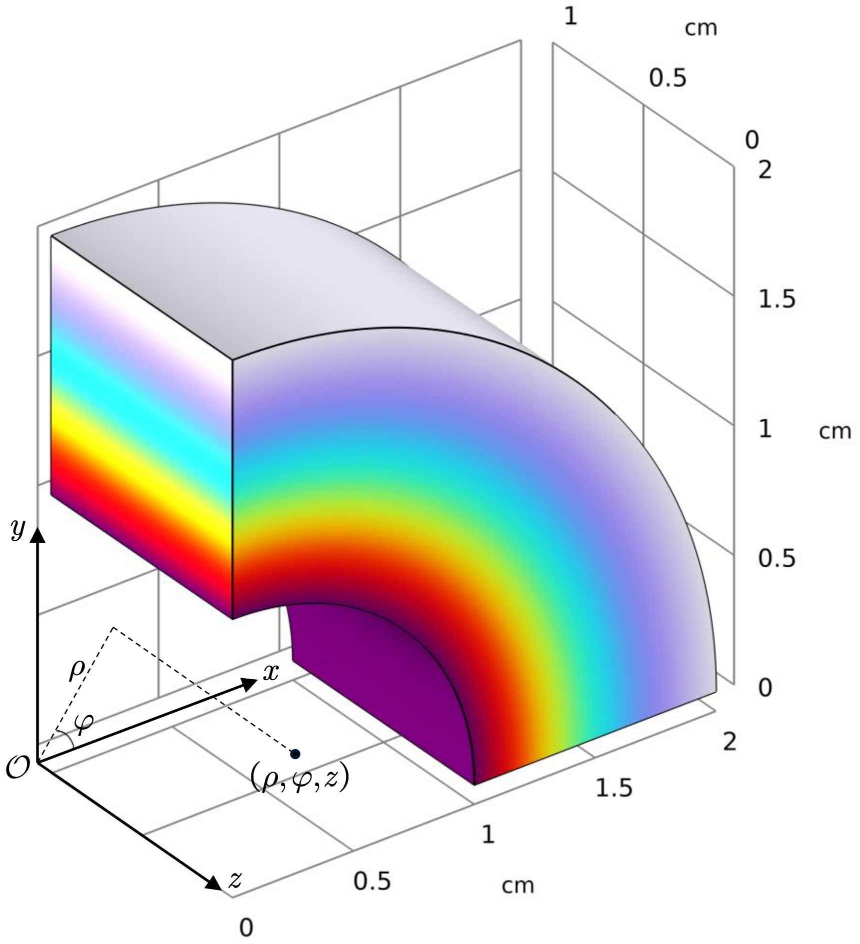
- The resistances versus b/a and σh for case 1



$$R = \pi/(2\sigma h \ln(b/a)) \qquad a/b\uparrow \Rightarrow \ln(b/a)\downarrow \Rightarrow R\uparrow \quad \sigma h\uparrow \Rightarrow R\downarrow$$

Problem 1

- Analytical solution for case 2



Laplace's equation (cylindrical coordinate system)

$$\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

$$\frac{\partial\phi}{\partial\varphi} = 0 \ \& \ \frac{\partial\phi}{\partial z} = 0$$

$$\Rightarrow \quad \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) = 0$$

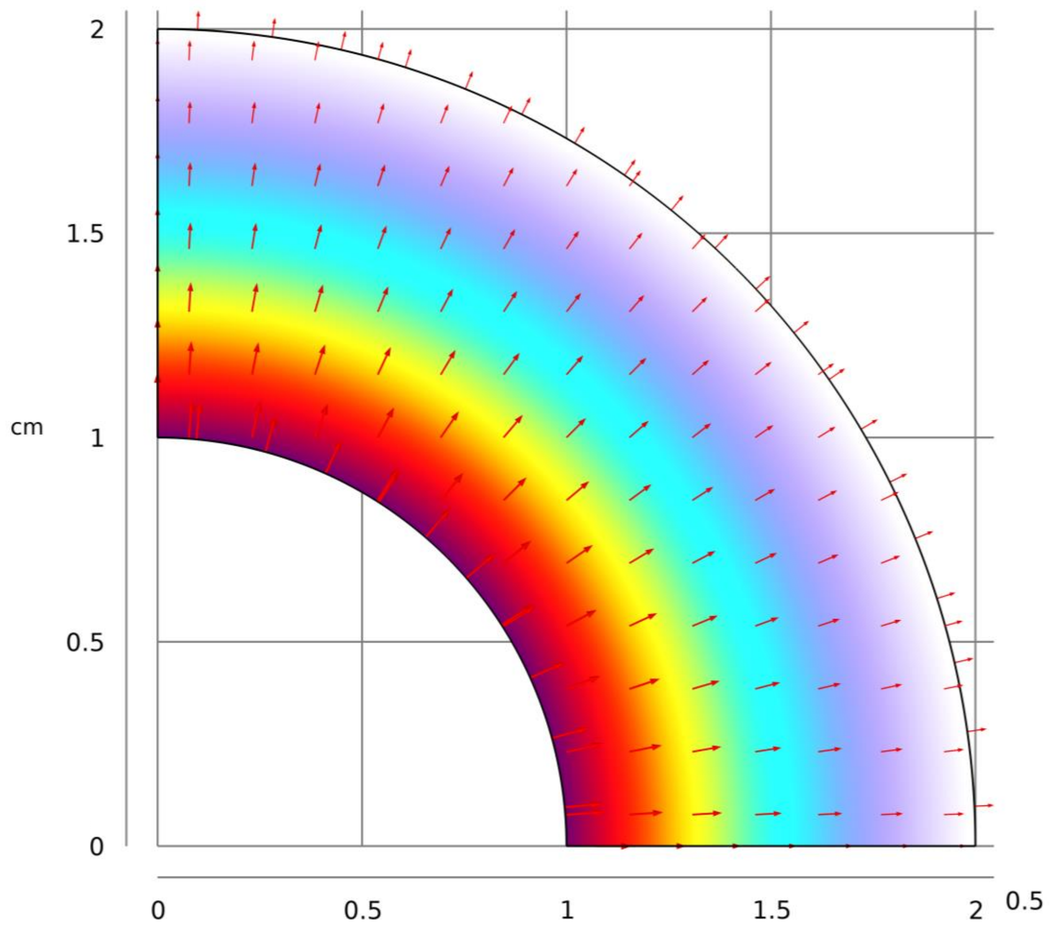
$$\Rightarrow \quad \phi(\rho) = C_1\ln(\rho) + C_2$$

$$\phi(\rho = b) = 0 \ \& \ \phi(\rho = a) = V_0$$

$$\Rightarrow \quad \phi(\varphi) = \left(\frac{V_0}{\ln(a/b)} - \frac{V_0}{\ln(a/b)}\right)\ln b$$

Problem 1

- Analytical solution for case 2



The relationship between electric field and electric potential

$$\vec{E} = -\nabla\phi = -\frac{1}{\rho}\frac{V_0}{\ln(a/b)}\vec{e}_\rho$$

$$\nabla = \vec{e}_\rho\frac{\partial}{\partial\rho} + \vec{e}_\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \vec{e}_z\frac{\partial}{\partial z}$$

$$\Rightarrow \quad \vec{J} = \sigma\vec{E} = -\frac{1}{\rho}\frac{\sigma V_0}{\ln(a/b)}\vec{e}_\rho$$

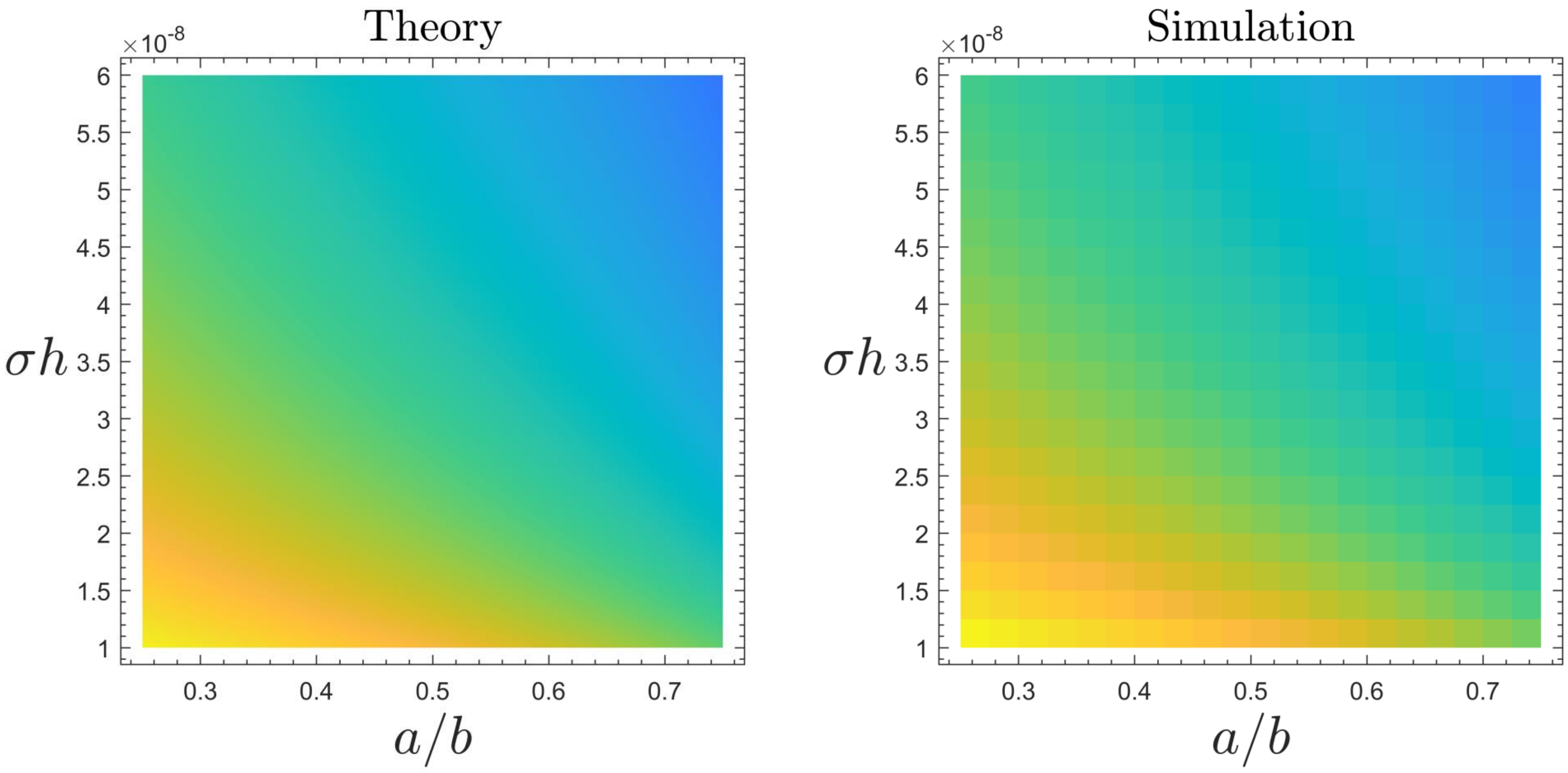
$$\Rightarrow \quad I = \int_S \vec{J}\,\mathrm{d}\vec{S} = -\frac{\sigma\pi h V_0}{2\ln(a/b)}$$

Ohm's Law

$$\Rightarrow \quad R = \frac{V_0}{I} = \frac{2\ln(b/a)}{\sigma\pi h}$$

Problem 1

- The resistances versus b/a and σh for case 2



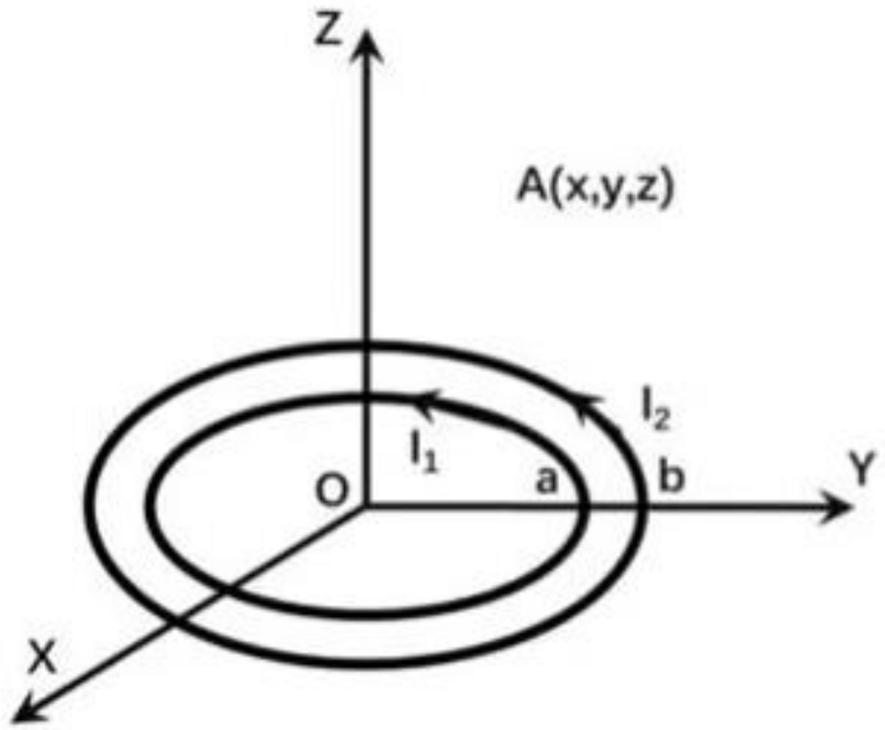
$$R = 2\ln(b/a)/(\sigma\pi h) \qquad a/b\uparrow \Rightarrow \ln(b/a)\downarrow \Rightarrow R\downarrow \quad \sigma h\uparrow \Rightarrow R\downarrow$$

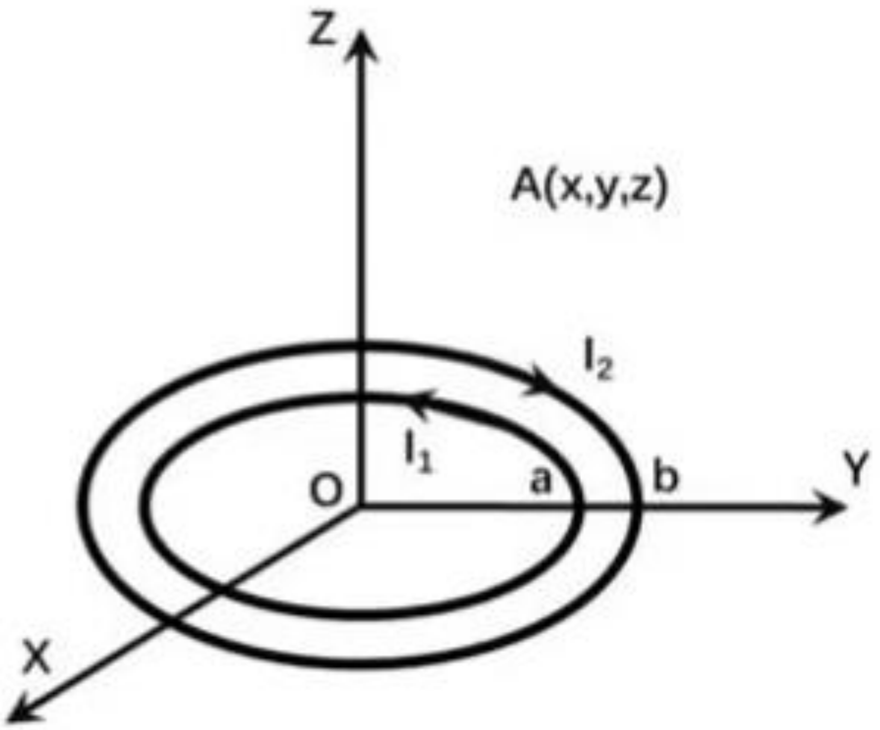
Thanks to Shuaiyang Wei !

Problem 2

Problem 2

- Calculate the magnetic flux density at an arbitrary point A(x, y, z) of two steady electric currents, I_1 and I_2 , forming concentric circular loops





Problem 2

- 1) Calculate the magnetic flux density when I_1 and I_2 in the same direction

According to **the Biot-Savart Law**:

Solution:
$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\oint_{l'} \frac{dl'}{R_1}$$

Current unit of coil vector :

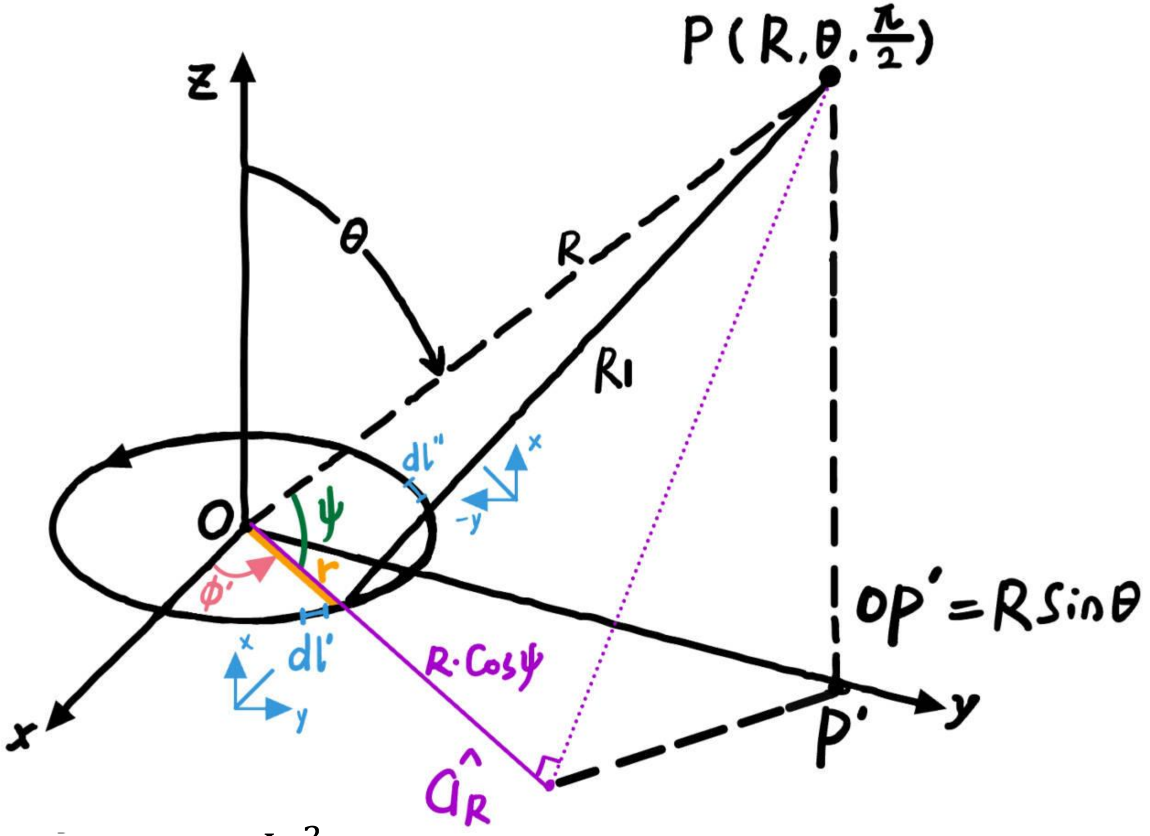
$$\mathrm{d}\boldsymbol{l}' = r d\emptyset' \boldsymbol{a}_{\emptyset} = r d\emptyset'(-\boldsymbol{a}_x \sin\emptyset' + \boldsymbol{a}_y \cos\emptyset')$$

Vector from A to d$\boldsymbol{l}$:

$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{-\boldsymbol{a}_x \sin\emptyset' + \boldsymbol{a}_y \cos\emptyset'}{R_1} r d\emptyset'$$
$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{-\boldsymbol{a}_x \sin\emptyset'}{R_1} r d\emptyset'$$

Laws of cosine:

$$R_1 = R^2 + r^2 - 2r\cos\varphi = R^2 + r^2 - 2r\underbrace{R\sin\theta}_{\mathrm{OP}'}\sin\emptyset'$$
$$\frac{1}{R_1} = \frac{1}{R}\left(1 + \frac{r^2}{R^2} - \frac{2r}{R}\sin\theta\sin\emptyset'\right)^{-\frac{1}{2}}$$
$$\mathbf{A} = -\boldsymbol{a}_x \frac{\mu_0 I r}{4\pi R}\int_0^{2\pi}\left(1 + \frac{r}{R}\sin\theta\sin\emptyset'\right)\sin\emptyset'\, d\emptyset'$$
$$= -\boldsymbol{a}_x \frac{\mu_0 I r}{4R}\frac{r}{R}\sin\theta$$



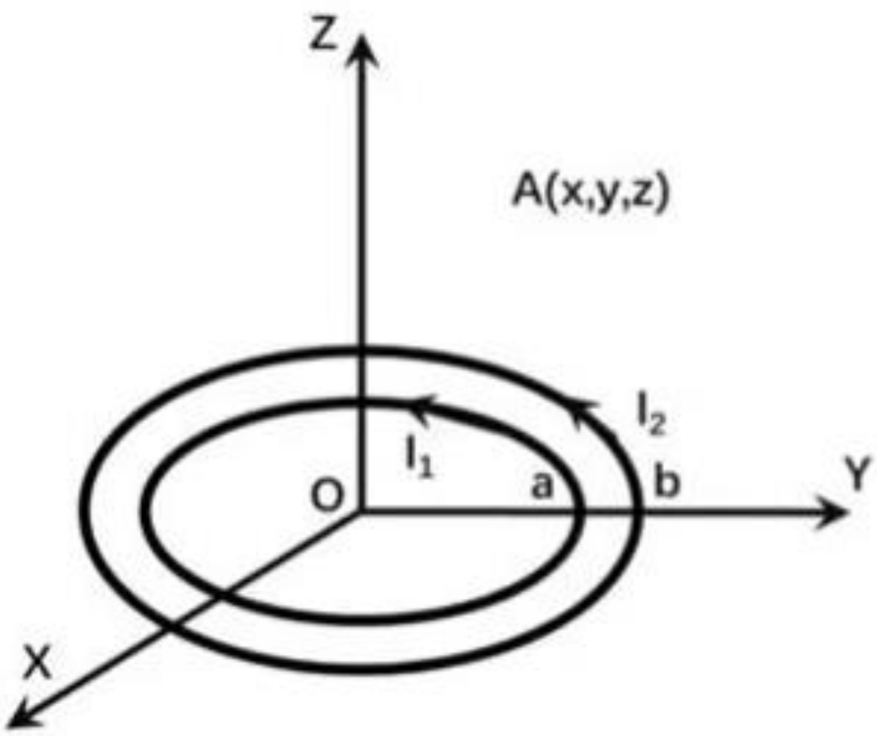
$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{\mu_0 I r^2}{4R^3}(\boldsymbol{a}_R 2\cos\theta + \boldsymbol{a}_\theta \sin\theta)$$

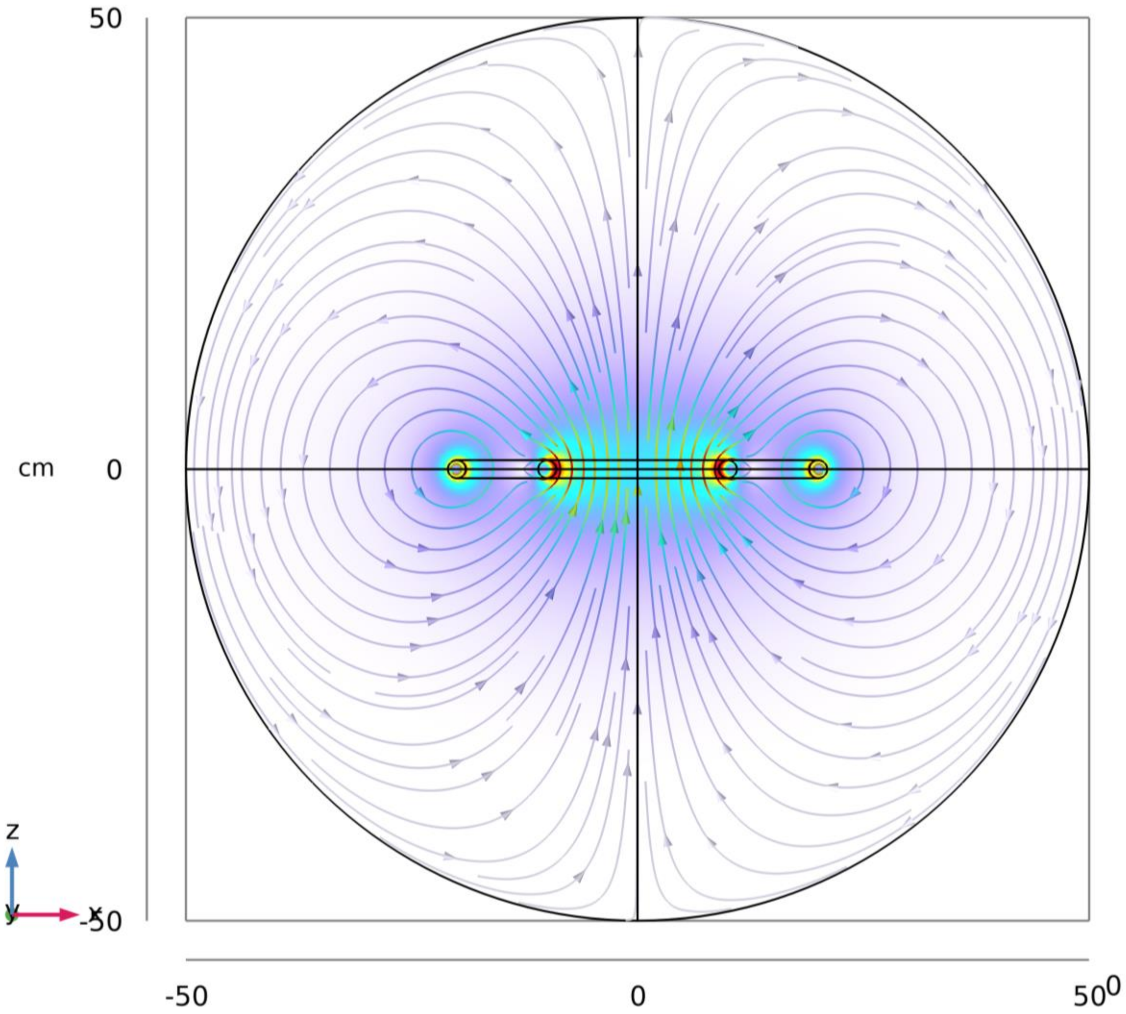
The total magnetic flux density :

$$\mathrm{B(A)=B1+B2} = \frac{\mu_0 I(a^2+b^2)}{4R^3}(\boldsymbol{a}_R 2\cos\theta + \boldsymbol{a}_\theta\sin\theta)$$

Problem 2

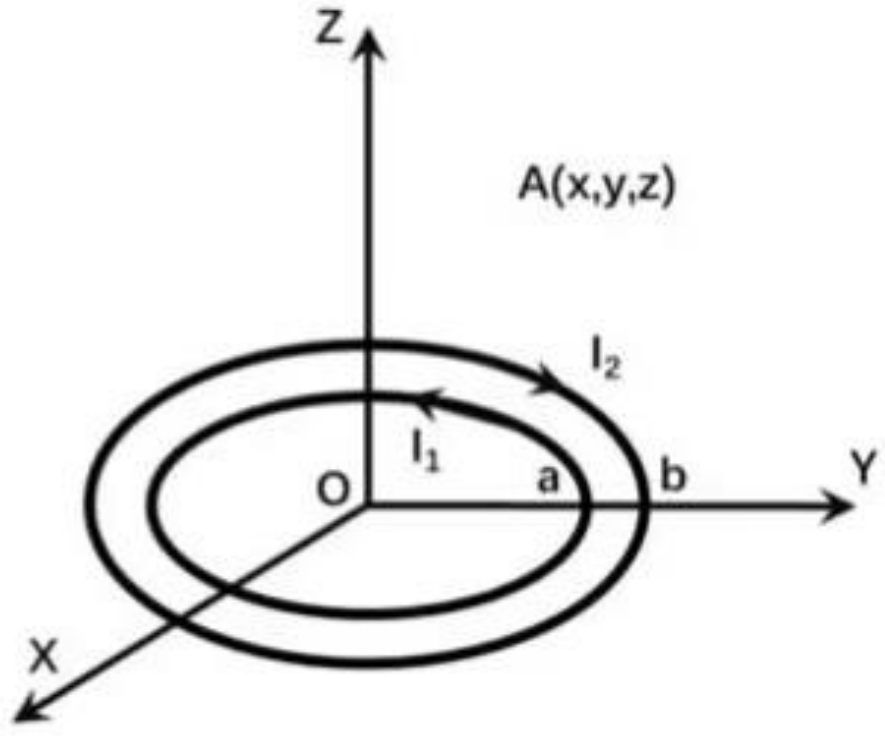
Magnetic field distribution:

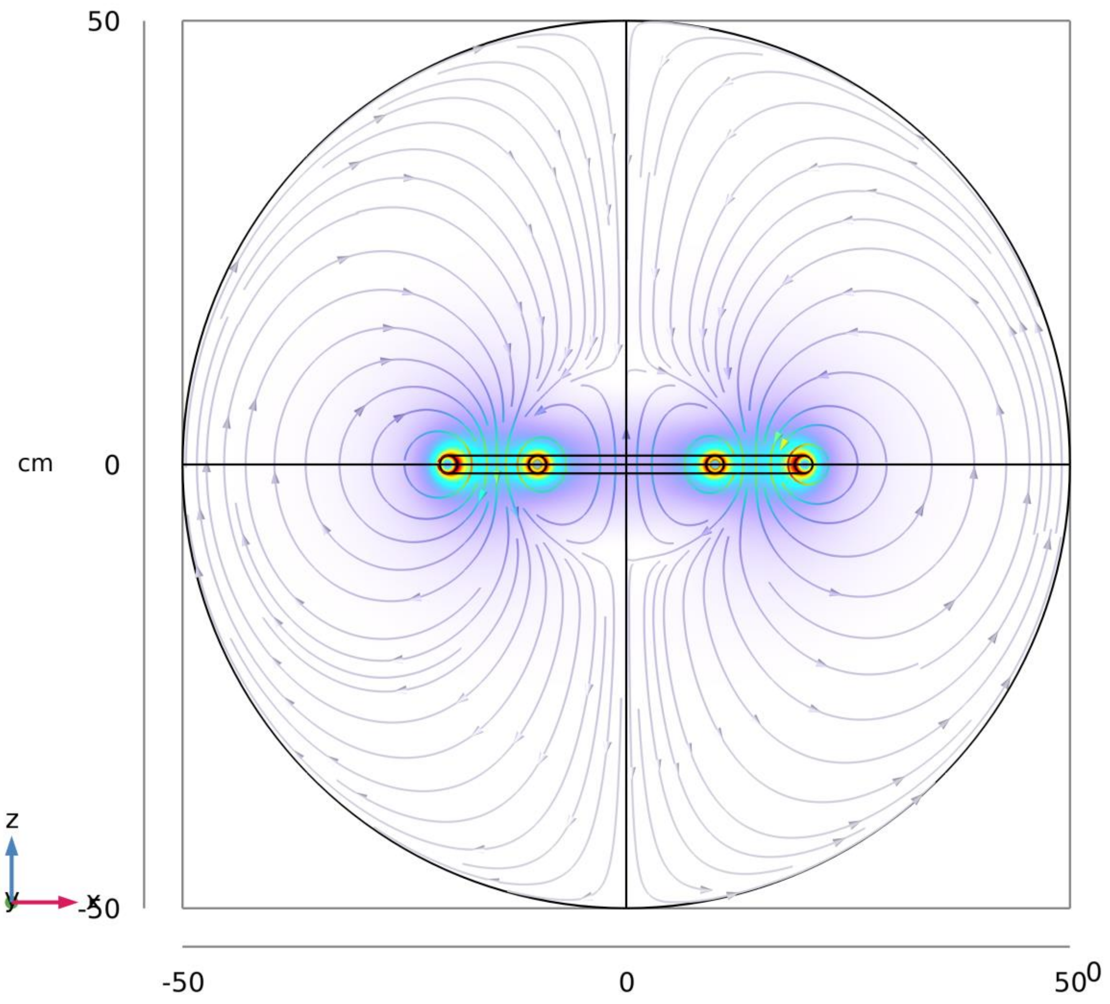




Problem 2

Magnetic field distribution:





THANK YOU

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