This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2024.3520676

Optimal Adaptive Parameter Estimation with Online Varying Learning Gain

Siyu Chen, Jing Na, *Member, IEEE*, Yingbo Huang, *Member, IEEE*, Yashan Xing, *Member, IEEE*, Jing Zhao and Pak Kin Wong

***Abstract*—It has been well known that the learning gain plays a crucial role in the adaptive parameter estimation (APE) for guaranteeing fast convergence and robustness. However, the tuning of learning gains in the existing methods is generally empirical and time consuming. To address this issue, this paper presents a novel APE approach to explore the optimality principle in the design of adaptive laws to obtain an online updated *optimal* learning gain, which is derived to minimize a cost function of estimation error with two weighting matrices. For this purpose, the estimation error is first reconstructed by using filter operations on the system and introducing several auxiliary variables. Then, two constructive APE algorithms driven by the extracted estimation error are developed. Finally, inspired by the duality of control and estimation, the estimation error dynamics are reformulated as a closed-loop system with an analogous control action to minimize the cost function. By using the Hamiltonian function, a Differential Riccati Equation (DRE) is derived to find an optimal solution of learning gain to construct an optimal adaptive law. The robustness of the proposed adaptive laws subjected to disturbances is also analyzed. Comparative numerical simulations are given to illustrate the superiority of the proposed method.**

***Index Terms*—Adaptive parameter estimation, Optimality principle, Optimal estimation, Persistent excitation (PE)**

I. INTRODUCTION

ADAPTIVE parameter estimation (APE) has been well studied in system identification and adaptive control owing to its online learning ability to reconstruct the unknown model parameters via the measurable input and output [1]–[3]. During the past decades, considerable research on APE has been carried out, among which the gradient descent method and recursive least-squares (RLS) method are the most prevalent and well-known ones. The key merit is to design an observer emulating the system dynamics but with the estimated parameters, then an adaptive law can be derived based on the error between the measured output and the observer output to online update the estimated parameters. In this line, many recent works have also been proposed to analyze and enhance the convergence of least-squares methods [4], [5], where time-varying learning gains are adopted. On the other hand, although choosing large learning gains in the adaptive laws can help to accelerate the estimation convergence, the resulted high-gain learning could cause high-frequency oscillations in the estimated parameters. In fact, a well-known issue of the gradient-based APE methods is that they may suffer from the bursting phenomenon when the system is subject to disturbances or measurement noise [6].

To retain the robustness of adaptive laws, several modifications have been proposed, such as σ-modification [7] and e-modification [8], etc. The key idea is to introduce a leakage term with the estimated parameters to the original gradient algorithm to guarantee the boundedness of the estimated parameters, while these modifications could prevent the estimated parameters from converging to the true values, limiting their ability to achieve accurate parameter estimation. Nevertheless, in the gradient-based APE framework, the estimation error is coupled with the observer error, and thus its convergence response is influenced by the observer dynamics.

Motivated by the Kreisselmeier's filter [9] and the parameterization of filtered regression [2], an advanced APE approach [10] driven by the parameter estimation error was presented to retain both the convergence and robustness of parameter estimation, where several constructive adaptive laws were proposed. More specifically, proper filter operations and auxiliary variables are introduced in [10], where all variables involved in the APE implementation are bounded and the online calculation of inverse matrix in [11] is avoided. Owing to its salient merit, the idea was also extended to identify the characteristic parameters of sinusoidal signals in [12]. Recently, the method was further tailored by incorporating the prescribed performance boundary into the APE [13] to predefine the transient error convergence. Although satisfactory convergence can be retained via the APE schemes in [10]–[13], the dynamics of regressor are involved in the auxiliary variables used to derive the adaptive laws, such that the learning gains should be carefully tuned by the designers to account for the influence of regressor (i.e., the amplitude of each element in the regressor) on the error convergence, which is not a trivial task. To address this issue, a novel method named as dynamic regressor extension and mixing (DREM) [14], [15] has recently been proposed, which can achieve the decoupling of multi-parameters, so as to retain a very nice monotonous convergence property and simplify the tuning of learning gains. It was further shown in [16] that the Lion's and the Kreisselmeier's filters can be represented in a special form of DREM. However, the learning gains of DREM schemes should be manually set to be a large ones to retain the transient response.

Following the above discussions and inspired by the fact that time-varying learning gains may benefit the convergence of adaptive laws [17], this paper presents a new optimal APE method driven by the extracted estimation error, where the effects of regressor can be compensated via a new auxiliary variable (as inspired by the RLS scheme [1]) to enhance the convergence, and the learning gains can be online updated by incorporating the optimality principle [18], [19] into the

This work was supported by National Natural Science Foundation of China under grants 62273169, 62433012 and 62203194, and partially supported by the Science and Technology Development Fund of Macau SAR under the grant 0091/2023/AMJ. (Corresponding author: Jing Na)

Siyu Chen, Jing Na, Yingbo Huang and Yashan Xing are with the Faculty of Mechanical and Electrical Engineering, and Yunnan Key Laboratory of Intelligent Control and Application, Kunming University of Science and Technology, Kunming, 650500, China. (e-mail: chen_siyu2020@163.com; najing25@kust.edu.cn; Yingbo_Huang@126.com; yashan.xing@outlook.com).

Jing Zhao and Pak Kin Wong are with the Department of Electromechanical Engineering, University of Macau, Taipa, Macau (e-mail: jzhao@um.edu.mo;fstpkw@um.edu.mo).

design of adaptive laws. Note although the DREM studied the same problem and provided a monotonous property, this paper introduces a semi-automated selection for the adaptation gains. Thus, an auxiliary matrix is constructed by using the parameter estimation error to modify the adaptive laws originally proposed in [10], such that the influence of regressor can be eliminated, making the tuning of learning gains more straightforward. Moreover, aiming at achieving automatic online update of learning gains in the optimal sense, we recall the duality of control and estimation, and then incorporate the optimality principle into the APE synthesis. In this line, a closed-loop estimation system is suggested by taking the estimation error feedback in the adaptive law as an analogous control action. Then, a cost function of the estimation error with two weighting matrices is constructed and the associated Hamiltonian function is used to derive a Differential Riccati Equation (DRE), whose solution is obtained to set the optimal learning gain and minimize the cost function. Moreover, the external disturbances are also taken into account to exemplify the robustness of the proposed adaptive laws. Finally, comparative simulations are conducted to evaluate the theoretical studies.

This paper is organized as follows. The problem formulation and motivations are given in Section II. Section III provides the derivations of three estimation error based adaptive laws and analyzes their convergence properties. Robustness against bounded disturbances is exemplified in Section IV. Numerical simulation results are given in Section V. Conclusions are stated in Section VI.

II. Problem Formulation and Motivations

Consider a system with unknown parameters given by:

$$\dot{x} = \varphi(x,u) + \Phi(x,u)\Theta, \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the system output with $x(t_0) = x_0$, $u \in \mathbb{R}$ denotes the system input, $\Theta \in \mathbb{R}^q$ is the unknown constant parameter vector, $\Phi(x) \in \mathbb{R}^{n\times q}$ is the regressor, and $\varphi(x,u) \in \mathbb{R}^n$ is a known function vector. The studied problem is to online estimate the unknown parameter Θ by using measurable system output x and input u.

The following preliminaries are used:

Definition 1 ([20]): The regressor vector Φ in (1) satisfies the persistent excitation (PE) condition provided that there exist positive constants T, $\mu > 0$, such that $\int_t^{t+T} \Phi(\tau)\Phi^\top(\tau)\mathrm{d}\tau \geq \mu I$ holds for all $t \geq t_0$.

Lemma 1 ([21]): For the system $\dot{x} = F(t)x$ where $F(t)$ is bounded, let $\Psi(t,t_0)$ denote the transition matrix of $F(t)$, then it has a unique solution $\Psi(t,t_0)x_0$ with $x(t_0) = x_0$, where $\Psi(t,t_0)$ satisfies the following conditions: 1) $\frac{d\Psi}{dt} = F(t)\Psi$, $\Psi(t_0,t_0) = I$; 2) $\Psi(t,\tau)\Psi(\tau,t_0) = \Psi(t,t_0)$; 3) $\Psi^{-1}(t,\tau) = \Psi(\tau,t)$.

It is noted that most of existing parameter estimation algorithms stem from the well-recognized gradient descent algorithm, which adopts an observer for system (1) as

$$\dot{\hat{x}} \triangleq \varphi(x,u) + \Phi(x,u)\hat{\Theta} + K(x-\hat{x}), \tag{2}$$

where $\hat{x}$ represents the observer output, $\hat{\Theta}$ denotes the estimation of unknown parameter Θ, and $K > 0$ is the observer gain. Then, the gradient descent based adaptive law [3] can be designed as

$$\dot{\hat{\Theta}} \triangleq -\mathit{\Gamma}\Phi^\top(x,u)e, \tag{3}$$

where $e \triangleq \hat{x} - x$ represents the observe error and $\mathit{\Gamma} \in \mathbb{R}^{q\times q}$ is a manually set adaptive learning gain.

As shown in (3), the gradient algorithm is designed using the observed error e. Although the estimation error $\tilde{\Theta} \triangleq \Theta - \hat{\Theta}$ is related to the observer error e and thus the asymptotic convergence of estimation error e can be proved under the PE condition of regressor Φ as shown in [3], its robustness and convergence response may be influenced by the measurement noise. Although robust adaptive laws [3], [7], [8] have been proposed to retain the robustness of (3), the asymptotic convergence property may be lost owing to the induced damping effects. Nevertheless, our recent work [10] presented new adaptive laws driven by the derived estimation error $\tilde{\Theta}$, so that exponential or even finite-time convergence can be strictly proved. However, the learning gains thereby should also be manually set by the designers, which is not a trivial task.

Hence, the motivation of this paper is to propose a new solution for APE of system (1), which is driven by the estimation error $\tilde{\Theta}$ as [10] but with an online updated optimal learning gain, which is derived by using the optimality principle, so that further enhanced convergence response can be retained.

III. Optimal Adaptive Parameter Estimation

The developed optimal APE mainly contains three constructive steps: 1) apply filter operations on system (1) to derive intermediate variables including the estimation error; 2) use the optimality principle to construct a closed-loop estimation system; 3) propose a DRE-based optimal solution to online update the learning gain for adaptive laws. The detailed design of three adaptive laws will be given in this section.

A. Extraction of Parameter Estimation Error

Different from the gradient descent based APE in (2) which uses the observer error to drive the adaptive law, we will propose alternative adaptive laws driven by the estimation error $\tilde{\Theta}$ derived from the measurable variables as [10]. For the purpose, a low-pass filter $\mathcal{L}(\cdot) \triangleq 1/(ks+1)$ is first used to obtain the variables x_f, φ_f, Φ_f as:

$$\begin{cases} k\dot{x}_f + x_f = x, & x_f(t_0) = 0, \\ k\dot{\varphi}_f + \varphi_f = \varphi, & \varphi_f(t_0) = 0, \\ k\dot{\Phi}_f + \Phi_f = \Phi, & \Phi_f(t_0) = 0, \end{cases} \tag{4}$$

where the filter coefficient is set as a positive constant $k > 0$.

Moreover, the auxiliary matrix $\mathcal{M} \in \mathbb{R}^{q\times q}$ and vector $\mathcal{N} \in \mathbb{R}^q$ are defined as follows[1]:

$$\begin{cases} \dot{\mathcal{M}} \triangleq -l\mathcal{M} + \dfrac{\Phi_f^T\Phi_f}{1+\|\Phi_f\|^2}, & \mathcal{M}(t_0) = 0, \\ \dot{\mathcal{N}} \triangleq -l\mathcal{N} + \dfrac{\Phi_f^T}{1+\|\Phi_f\|^2}\left(\dfrac{x-x_f}{k} - \varphi_f\right), & \mathcal{N}(t_0) = 0, \end{cases} \tag{5}$$

[1]The normalization operation is used to guarantee the boundedness of auxiliary matrix $\mathcal{M}$ and vector $\mathcal{N}$ even with unbounded regressor Φ.

where $l > 0$ is a positive constant to ensure the boundedness of $\mathcal{M}$ and $\mathcal{N}$.

A benefit of constructing the auxiliary matrix $\mathcal{M}$ is that the PE condition can be reformulated as the positive definiteness of $\mathcal{M}$ and thus can be online verified, which is given as:

Lemma 2: [10] If the regressor $\Phi(x, u)$ satisfies the PE condition, then there exists a positive constant m_0 such that the matrix $\mathcal{M}$ is bounded by $m_0 I \leq \mathcal{M} \leq l^{-1} I$ for $\forall t \geq t_0 + T$.

Proof: The proof of $m_0 I \leq \mathcal{M}$ can be found in [10], which will not be repeated here. On the other hand, it is known that $\frac{\Phi_f^T \Phi_f}{1+\|\Phi_f\|^2} \leq I$ and $\int_{t_0}^t e^{-l(t-\tau)} \mathrm{d}\tau \leq l^{-1}$ for $t \geq t_0 + T$, thus we can obtain $\mathcal{M} \leq l^{-1} I$.

Furthermore, an auxiliary matrix $\mathcal{H} \in \mathbb{R}^q$ containing the estimation error $\tilde{\Theta}$ is defined using $\mathcal{M}$ and $\mathcal{N}$ given in (4) and (5) as

$$\mathcal{H} \triangleq \mathcal{M}\hat{\Theta} - \mathcal{N}. \tag{6}$$

Lemma 3: The above defined matrix $\mathcal{H}$ fulfills the fact:

$$\mathcal{H} = -\mathcal{M}\tilde{\Theta}. \tag{7}$$

Proof: According to system (1) and filter operation (4), one can deduce that[2]

$$\dot{x}_f = \frac{x - x_f}{k} = \varPhi_f \Theta + \varphi_f. \tag{8}$$

Then, by integrating both sides of Eq. (5), the solutions of $\mathcal{M}$ and $\mathcal{N}$ are given as

$$\begin{cases} \mathcal{M}(t) = \displaystyle\int_{t_0}^t e^{-l(t-\tau)} \frac{\Phi_f^\top(\tau)\Phi_f(\tau)}{1 + \|\Phi_f(\tau)\|^2} \mathrm{d}\tau, \\ \mathcal{N}(t) = \displaystyle\int_{t_0}^t e^{-l(t-\tau)} \frac{\Phi_f^\top(\tau)}{1 + \|\Phi_f(\tau)\|^2} \bigg(\frac{x(\tau) - x_f(\tau)}{k} \\ \qquad - \varphi_f(\tau) \bigg) \mathrm{d}\tau. \end{cases} \tag{9}$$

From (8) and (9), it can be verified that

$$\mathcal{N}(t) = \int_{t_0}^t e^{-l(t-\tau)} \frac{\Phi_f^\top(\tau)\Phi_f(\tau)}{1 + \|\Phi_f(\tau)\|^2} \mathrm{d}\tau \Theta = \mathcal{M}(t)\Theta. \tag{10}$$

Substituting (10) into (6) and recalling the definition of $\tilde{\Theta}$, the fact given in (7) can be validated.

B. Design of Adaptive Law with Estimation Error

As shown in Lemma 3, the vector $\mathcal{H}$, deduced from $\mathcal{M}$ and $\mathcal{N}$, contains the information of estimation error $\tilde{\Theta}$ coupled with the matrix $\mathcal{M}$. Hence, our previous work [10] presented an adaptive law as:

$$\dot{\hat{\Theta}} \triangleq -\varGamma \mathcal{H} = \varGamma \mathcal{M}\tilde{\Theta}, \tag{11}$$

where $\varGamma > 0$ is a manually set constant learning gain.

This leads to the following results:

[2]Note the vanishing effect stemming from nonzero initial conditions x_0 in the filter $\mathcal{L}(\cdot) \triangleq 1/(ks + 1)$ is omitted as shown in [22], which does not violate the main conclusions.

Theorem 1: [10] The adaptive law (11) for system (1) can ensure the exponential convergence of $\tilde{\Theta}$ provided that the regressor Φ fulfills the PE condition.

Proof: Define a Lyapunov function $V_1(\tilde{\Theta}) \triangleq \tilde{\Theta}^\top \varGamma^{-1} \tilde{\Theta}$. Since the estimation error of (11) fulfills

$$\dot{\tilde{\Theta}} = -\varGamma \mathcal{M}\tilde{\Theta}, \tag{12}$$

it follows

$$\dot{V}_1(\tilde{\Theta}) = -2\tilde{\Theta}^\top \mathcal{M}\tilde{\Theta} \leq -\sigma_1 V_1(\tilde{\Theta}), \tag{13}$$

where $\sigma_1 \triangleq 2m_0$ is a positive constant under the PE condition and Lemma 2. Hence, according to the Lyapunov theorem, it can be claimed that $V_1(\tilde{\Theta})$ and thus the estimation error $\tilde{\Theta}$ will converge to zero exponentially.

Remark 1: Although the adaptive law (11) driven by the estimation error $\tilde{\Theta}$ in $\mathcal{H}$ can guarantee the exponential convergence of $\tilde{\Theta}$, which is superior to the gradient algorithm, there are certain limitations involved. The estimation error $\tilde{\Theta}$ in $\mathcal{H}$ is coupled with the matrix $\mathcal{M}$, which denotes the direction of gradient descent of $\mathcal{H}$ as a whole. Hence, the existence of $\mathcal{M}$ unavoidably influences the transient convergence response of error dynamics given in (12), which also makes the tuning of learning gains $\varGamma$ (to eliminate the effects of $\mathcal{M}$) a tedious and time-consuming procedure. To address this issue, a further tailored adaptive law will be introduced, where the effects of $\mathcal{M}$ can be compensated.

According to (7), the estimation error is represented as $\tilde{\Theta} = \mathcal{M}^{-1}\mathcal{H}$, and thus the unknown parameter can be calculated as $\Theta = \mathcal{M}^{-1}\mathcal{N}$ by multiplying the inverse matrix $\mathcal{M}^{-1}$ on both sides of (10). However, the online calculation of inverse matrix $\mathcal{M}^{-1}$ is not practically feasible due to the computational complexity and potential singularity issue. To avoid computing the inverse matrix $\mathcal{M}^{-1}$, an auxiliary matrix $\mathcal{K} \in \mathbb{R}^{q\times q}$ is designed as:

$$\dot{\mathcal{K}} \triangleq l\mathcal{K} - \mathcal{K}\frac{\Phi_f^T \Phi_f}{1 + \|\Phi_f\|^2}\mathcal{K}, \tag{14}$$

where $\mathcal{K}^{-1}(t_0) = \mathcal{K}_0 = k_0 I$ with a positive constant $k_0 > 0$.

The design of matrix $\mathcal{K}$ is partially inspired by the RLS algorithm [1], where $\mathcal{K}$ can approach to the inverse of $\mathcal{M}$.

Lemma 4: Considering the auxiliary matrices $\mathcal{M}$ and $\mathcal{K}$ given in (5) and (14), the following fact holds:

$$\lim_{t\to\infty} = \mathcal{K}(t)\mathcal{M}(t) = I. \tag{15}$$

Moreover, it is known that $0 \leq \mathcal{K}(t)\mathcal{M}(t) \leq I$ for any $t \geq t_0$, and if the regressor $\Phi(x)$ satisfies the PE condition then $\mathcal{K}(t)\mathcal{M}(t) \geq m_0/(m_0 + k_0)$ for $t \geq t_0 + T$ with a positive constant m_0 denoting the minimum eigenvalue of $\mathcal{M}$.

Proof: Based on the matrix equality $\frac{\mathrm{d}}{\mathrm{dt}}(\mathcal{K}\mathcal{K}^{-1}) = \dot{\mathcal{K}}\mathcal{K}^{-1} + \mathcal{K}\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{K}^{-1} = 0$ and (9), the solution of (14) can be deduced as

$$\begin{aligned} \mathcal{K}(t) &= \left[e^{-lt}\mathcal{K}_0 + \int_{t_0}^t e^{-l(t-\tau)} \frac{\Phi_f^\top(\tau)\Phi_f(\tau)}{1 + \|\Phi_f^T(\tau)\|^2} \mathrm{d}\tau \right]^{-1} \\ &= \left[e^{-lt}\mathcal{K}_0 + \mathcal{M}(t) \right]^{-1}. \end{aligned} \tag{16}$$

Since the matrix $\mathcal{M}$ is symmetric and non-negative, it can be reformulated via the singular value decomposition (SVD) as

$$\mathcal{M}(t)=\int_{t_0}^{t}e^{-l(t-\tau)}\frac{\Phi_f^\top(\tau)\Phi_f(\tau)}{1+\|\Phi_f(\tau)\|^2}\mathrm{d}\tau=U(t)S(t)U^\top(t), \tag{17}$$

where $S=\operatorname{diag}(s_1,\cdots,s_q)\in\mathbb{R}^{q\times q}$ is a diagonal matrix with s_i $(i=1,\cdots,q)$ the singular values of matrix $\mathcal{M}$ and $U\in\mathbb{R}^{q\times q}$ is the unitary matrix.

By substituting (17) into (16), it is further obtained that

$$\begin{aligned}\mathcal{K}(t)&=\left[e^{-lt}\mathcal{K}_0+\mathcal{M}(t)\right]^{-1}\\&=U(t)(S(t)+e^{-lt}\mathcal{K}_0)^{-1}U^\top(t).\end{aligned} \tag{18}$$

Then, the matrix $\mathcal{K}\mathcal{M}$ with $\mathcal{K}_0=k_0I$ can be derived as

$$\mathcal{K}\mathcal{M}=U\bar{\mathcal{S}}U^\top, \tag{19}$$

with

$$\begin{aligned}\bar{\mathcal{S}}&=\left[S+e^{-lt}\mathcal{K}_0\right]^{-1}S\\&=\operatorname{diag}(\frac{s_1}{s_1+e^{-lt}k_0},\cdots,\frac{s_q}{s_q+e^{-lt}k_0}).\end{aligned} \tag{20}$$

From (9), it is clear that the matrix $\mathcal{M}$ is symmetric and non-negative, so we have $0\le\mathcal{K}(t)\mathcal{M}(t)\le I$ for any $t\ge t_0$. Furthermore, according to Lemma 2, when the PE condition is satisfied, it is obtained that $\mathcal{K}(t)\mathcal{M}(t)\ge m_0/(m_0+e^{-lt}k_0)\ge m_0/(m_0+k_0)$ for $t\ge t_0+T$. Moreover, since $\lim\limits_{t\to\infty}s_i/(s_i+e^{-lt}k_0)=1,\ i=1,2,...,q$ is true for any positive l and k_0, we can verify that $\lim\limits_{t\to\infty}=\mathcal{K}(t)\mathcal{M}(t)=I$ holds.

According to the aforementioned derivations, the matrix $\mathcal{K}\mathcal{M}$ will converge to an identity matrix I as $t\to\infty$. Owing to this property, one can find that the online updated matrix $\mathcal{K}$ can be used to replace $\mathcal{M}^{-1}$ in the design of adaptive laws to handle the effect of $\mathcal{M}$. Therefore, a new adaptive law with a time-varying compensation $\mathcal{K}$ is established as:

$$\dot{\hat{\Theta}}\triangleq-\mathit{\Gamma}\mathcal{K}\mathcal{H}, \tag{21}$$

where $\mathit{\Gamma}>0$ is a manually set constant learning gain.

Clearly, the use of matrix $\mathcal{K}$ in (14) that approaches to $\mathcal{M}^{-1}$ avoids the calculation of inverse of matrix $\mathcal{M}$, and compensates its effects on the error dynamics. Specifically, it can be found from Lemma 2 that $m_0I\le\mathcal{M}\le l^{-1}I$. Hence, to obtain a fast convergence of $\mathcal{K}\mathcal{M}$, k_0 should not be significantly larger than l^{-1}. Nevertheless, selecting a too small k_0 is also not preferable in order to avoid the potential singularity in the numerical calculation.

Theorem 2: Considering the adaptive law (21) for system (1), the estimation error $\tilde{\Theta}$ will exponentially converge to zero if the regressor Φ fulfills the PE condition.

Proof: For the Lyapunov function $V_1(\tilde{\Theta})\triangleq\tilde{\Theta}^\top\mathit{\Gamma}^{-1}\tilde{\Theta}$, the derivative of $V_1(\tilde{\Theta})$ is derived from (7) and (21) as

$$\dot{V}_1(\tilde{\Theta})=-2\tilde{\Theta}^\top\mathcal{K}\mathcal{M}\tilde{\Theta}\le-\sigma_2V_1(\tilde{\Theta}), \tag{22}$$

where $\sigma_2\triangleq 2\min\left(\operatorname{eig}(\mathcal{K}\mathcal{M})\right)$ is a positive constant under the PE condition (recalling Lemma 4 that $\mathcal{K}(t)\mathcal{M}(t)\ge m_0/(m_0+k_0)$ holds for $t\to\infty$). Therefore, based on the Lyapunov theorem, we can obtain that $\tilde{\Theta}$ will exponentially converge to zero.

Remark 2: The essential difference between adaptive laws (11) and (21) is that their convergence rates σ_1 and σ_2 are determined by the minimum eigenvalues of $\mathcal{M}$ and $\mathcal{K}\mathcal{M}$, respectively. Clearly, as shown in Lemma 4, by using the online updated matrix $\mathcal{K}$, the effects of $\mathcal{M}$ on the convergence rate can be eliminated (due to the fact $\lim\limits_{t\to\infty}=\mathcal{K}(t)\mathcal{M}(t)=I$ holds under the PE condition, and we do not need to know the exact value of m_0), such that better convergence response can be obtained with (21). However, in these schemes, the learning gain $\mathit{\Gamma}$ should be manually tuned, which is critical to determine their transient convergence response. Hence, it remains as an open problem to design an online learning algorithm to refine the convergence of estimation error and simplify the tuning of learning gain $\mathit{\Gamma}$.

C. Design of Optimal Adaptive Law

In this subsection, the optimal control method will be used to further tailor adaptive law (21), by which the learning gain $\mathit{\Gamma}$ can be automatically updated online in an optimal manner, such that the tedious parameter tuning procedure is simplified. In order to apply the optimality principle [18], [19], we first recall adaptive law (11), whose error dynamics are described by (12). Note that Eq. (12) without the learning gain $\mathit{\Gamma}$ (i.e., $\dot{\tilde{\Theta}}=-\mathcal{M}\tilde{\Theta}$) can be regarded as an open-loop system with the state $\tilde{\Theta}$ and state matrix $\mathcal{M}$ given in Definition 1. Then, based on the control theory and duality [18], the open-loop estimation system (12) can be transformed into a closed-loop estimation system by designing an analogous controller to minimize a constructive cost function as exploited in [23] and further enhance the convergence. In this sense, we redefine the error dynamics (12) as follows:

$$\dot{\tilde{\Theta}}\triangleq-\mathcal{M}\tilde{\Theta}+u_{\mathrm{est}}, \tag{23}$$

where $u_{\mathrm{est}}\in\mathbb{R}^q$ is an analogous control action to be designed via the optimal control method.

In order to design an analogous optimal control variable u_{est}, by recalling adaptive law (21), one can take the estimation error $\mathit{\Gamma}\mathcal{K}\mathcal{H}$ as the control variable, i.e., $u_{\mathrm{est}}\triangleq\mathit{\Gamma}\mathcal{K}\mathcal{H}$. It should be noted that although the analogous state $\tilde{\Theta}$ in system (23) is unknown, from the fact $\mathcal{K}\mathcal{M}\to I$ in Lemma 4, the optimality principle can still be applied to derive the optimal gain $\mathit{\Gamma}$ through $u_{\mathrm{est}}\triangleq\mathit{\Gamma}\mathcal{K}\mathcal{H}\approx-\mathit{\Gamma}\tilde{\Theta}$. Hence, the desired error dynamics (23) can be defined as

$$\dot{\tilde{\Theta}}\triangleq\underbrace{-\mathcal{M}\tilde{\Theta}}_{\mathcal{H}}+\underbrace{\mathit{\Gamma}\mathcal{K}\mathcal{H}}_{u_{\mathrm{est}}}. \tag{24}$$

It is shown that (24) can be regarded as an analogous closed-loop estimation system, where the learning gain $\mathit{\Gamma}$ is taken as the control gain, which is critical to guarantee the error convergence. Hence, we will use the optimal control design based on the variational method to obtain the optimal gain $\mathit{\Gamma}$.

To use the optimality principle, a cost function is established as follows:

$$J \triangleq \phi(\tilde{\Theta},t)|_{t=t_f} + \lim_{t_f\to\infty}\frac{1}{2}\int_{t_0}^{t_f}(\tilde{\Theta}^\top Q\tilde{\Theta} + u_{\text{est}}^\top Ru_{\text{est}})\mathrm{d}t, \tag{25}$$

where $\phi(\tilde{\Theta},t)|_{t=t_f} \triangleq \tilde{\Theta}^\top(t_f)P(t_f)\tilde{\Theta}(f_f)$ represents the constraint of terminal state and $Q, R \in \mathbb{R}^{q\times q}$ are positive definite symmetric matrices governing the convergence performance. Without loss of generality, we assume that the final state is zero, i.e., $\tilde{\Theta}(t_f) = 0$, thus $\phi(\tilde{\Theta},t)|_{t=t_f}$ is ignored in the subsequent analysis.

Remark 3: As shown in (25), the quadratic term $\tilde{\Theta}^\top Q\tilde{\Theta}$ determines the estimation error convergence and the other term $u_{\text{est}}^\top Ru_{\text{est}}$ introduces the effects of analogous control action of error feedback $\mathcal{KH}$ with a gain Γ, which can be set as the optimal value $\Gamma = R^{-1}P$ with a positive definite matrix P defined later. Hence, the weighting matrices Q and R are tuned to balance the convergence of $\tilde{\Theta}$ and the effect of u_{est}, respectively. In general, increasing Q can lead to a fast convergence of $\tilde{\Theta}$. Conversely, increasing R indicates a more conservative learning where the convergence of parameter estimation will be smoother and more sluggish. This LQR based design simplifies the tuning of learning gain Γ.

To minimize the cost function (25) via the analogous control u_{est}, a Hamiltonian function H can be defined as

$$H(\tilde{\Theta},u_{\text{est}}) \triangleq \frac{1}{2}[\tilde{\Theta}^\top Q\tilde{\Theta} + u_{\text{est}}^\top Ru_{\text{est}}] + \lambda^\top[-\mathcal{M}\tilde{\Theta} + u_{\text{est}}], \tag{26}$$

where $\lambda \in \mathbb{R}^q$ is an adjoint variable.

Then, Eq.(25) can be written along with (23) and (26) as

$$J = \lim_{t_f\to\infty}\frac{1}{2}\int_{t_0}^{t_f}\left[H(\tilde{\Theta},u_{\text{est}}) - \lambda^\top\dot{\tilde{\Theta}}\right]\mathrm{d}t. \tag{27}$$

By using the Leibniz rule, the cost function J can be considered as an increment function, such that the variation in J with respect to $\tilde{\Theta}, \lambda, u_{\text{est}}$ and t_f can be obtained with the initial state $\tilde{\Theta}(t_0)$ as

$$\begin{aligned}\delta J =&(H - \lambda^\top\dot{\tilde{\Theta}})\mathrm{d}t|_{t=t_f} + \int_{t_0}^{t_f}\left[\frac{\partial H^\top}{\partial\tilde{\Theta}}\delta\tilde{\Theta} - \lambda^\top\delta\dot{\tilde{\Theta}}\right.\\ &\left.+(\frac{\partial H^\top}{\partial\lambda} - \dot{\tilde{\Theta}}^\top)\delta\lambda + \frac{\partial H^\top}{\partial u_{\text{est}}}\delta u_{\text{est}}\right]\mathrm{d}t.\end{aligned} \tag{28}$$

To facilitate the subsequent analysis, the integration by parts is applied on $\delta\dot{\tilde{\Theta}}$, such that

$$-\int_{t_0}^{t_f}\lambda^\top\delta\dot{\tilde{\Theta}}\mathrm{d}t = -\lambda^\top\delta\tilde{\Theta}|_{t=t_f} + \lambda^\top\delta\tilde{\Theta}|_{t=t_0} + \int_{t_0}^{t_f}\dot{\lambda}^\top\delta\tilde{\Theta}\mathrm{d}t. \tag{29}$$

Substituting (29) into (28), the term $\delta\tilde{\Theta}(t_f)$ can be described as $\delta\tilde{\Theta}(t_f) = \mathrm{d}\tilde{\Theta}(t_f) - \dot{\tilde{\Theta}}(t_f)\mathrm{d}t_f$ according to the relationship [19] between the variation $\delta\tilde{\Theta}$ and the differential $\mathrm{d}\tilde{\Theta}$. Therefore, Eq.(28) can be further presented as

$$\begin{aligned}\delta J =&(H\mathrm{d}t - \lambda^\top\mathrm{d}\tilde{\Theta})|_{t=t_f} + \int_{t_0}^{t_f}\left[(\frac{\partial H^\top}{\partial\tilde{\Theta}} + \dot{\lambda}^\top)\delta\tilde{\Theta}\right.\\ &\left.+(\frac{\partial H^\top}{\partial\lambda} - \dot{\tilde{\Theta}}^\top)\delta\lambda + \frac{\partial H^\top}{\partial u_{\text{est}}}\delta u_{\text{est}}\right]\mathrm{d}t.\end{aligned} \tag{30}$$

Given the fact that the cost function J will retain a minimum value provided that the variation δJ is zero [24], the following adjoint equation and the stationarity condition [19] hold:

$$\dot{\lambda} = -\frac{\partial H}{\partial\tilde{\Theta}} = -Q\tilde{\Theta} + \mathcal{M}\lambda, \tag{31}$$

$$0 = \frac{\partial H}{\partial u_{\text{est}}} = Ru_{\text{est}} + \lambda, \tag{32}$$

In addition, the fact $\dot{\tilde{\Theta}} = -\mathcal{M}\Theta + u_{\text{est}} = \partial H/\partial\lambda$ is always satisfied, and the boundary condition can be obtained as

$$(H\mathrm{d}t - \lambda^\top\mathrm{d}\tilde{\Theta})|_{t=t_f} = 0. \tag{33}$$

Since the final state is set as zero, i.e., $\tilde{\Theta}(t_f) = \mathrm{d}\tilde{\Theta}(t_f) = 0$ is true, the boundary condition (33) will degenerate to zero when the final time is free, i.e., $\exists\mathrm{d}t_f \neq 0, H\mathrm{d}t|_{t=t_f} = 0$.

Multiplying the inverse matrix R^{-1} on both sides of (32), the analogous control action can be yielded as

$$u_{\text{est}}(t) = -R^{-1}\lambda. \tag{34}$$

As analyzed in [18], there exists an equation concerning the constrained terminal state for the continuous system with the specified terminal state and unspecified terminal time, which is given as

$$\lambda(t_f) = \frac{\partial\phi}{\partial\tilde{\Theta}}|_{t=t_f} = P(t_f)\tilde{\Theta}(f_f). \tag{35}$$

To obtain the adjoint variable $\lambda(t)$ with the specified terminal state (35), the sweep method [18] is applied by taking

$$\lambda = P\tilde{\Theta}, \tag{36}$$

for a positive definite matrix P.

To seek an appropriate P such that the adjoint equation (31) is satisfied, an DRE is deduced based on (36) as:

$$\dot{P} = \mathcal{M}P + P\mathcal{M} - PR^{-1}P + Q, \tag{37}$$

with $P(t_0) = (RQ)^{\frac{1}{2}}$. Here, it should be noted that to obtain the solution P of DRE (37), given an initial condition $P(t_0) > 0$, a direct forward integration can be used to calculate its evolution along with the time t. This simplification arises from the features of the considered problem, where the terminal state is fixed as $\tilde{\Theta}(t_f) = 0$ so that the terminal cost $\phi(\tilde{\Theta},t)|_{t=t_f}$ can be ignored for the cost function (25), and the terminal time t_f is unspecified. Consequently, any positive semi-definite $P(t_f)$ is permitted.

From (23), (34) and (36), the following optimal adaptive law can be designed via the solution of (37) as

$$\dot{\hat{\Theta}} \triangleq -\mathcal{H} - R^{-1}P\mathcal{KH}. \tag{38}$$

Remark 4: The motivation for introducing the optimality principle is to find a feasible solution u_{est} to minimize the cost function (25), where the necessary conditions include (31)-(33). In particular, the boundary condition (33) is related to the terminal state and time of systems. In this paper, the former is fixed at zero and the latter is free. Hence, these preconditions ensure the reasonability for applying the sweep method [18], where P can be directly computed by forward integration.

To guarantee the convergence of adaptive law (38), the properties of P are essential, which can be given as follows:

Lemma 5 ([21]): Consider $\Pi(t,t_0,P_0)$ as the solution of (37) with $\Pi(t_0,t_0,P_0)=P_0$, it is true for $t>t_0$ that:

$$0 \le \Pi(t,t_0,P_0) \le \Psi(t,t_0)P_0\Psi^\top(t,t_0) + \int_{t_0}^{t}\Psi(t,\tau)Q(\tau)\Psi^\top(t,\tau)\mathrm{d}\tau, \quad (39)$$

where $\Psi(t,t_0)$ is the transition matrix of $\mathcal{M}$.

Proof: Given the matrices P,F,Y,Π_0, we first consider the following Riccati equation:

$$\dot{P} = FP + PF^\top + Y, \quad P(t_0)=\Pi_0, \quad (40)$$

where P,Y,Π_0 are symmetric matrices. Supposing $P=\Upsilon\xi\Upsilon^\top$ where $\Upsilon(t,t_0)$ is the transition matrix of F, we aim to find a general solution of (40) to satisfy this form. Substituting this solution in the left side to (40), we obtain

$$\dot{P} = \dot{\Upsilon}\xi\Upsilon^\top + \Upsilon\dot{\xi}\Upsilon^\top + \Upsilon\xi\dot{\Upsilon}^\top. \quad (41)$$

From Lemma 1, it is known that $\dot{\Upsilon}=F\Upsilon$, then it follows

$$\begin{aligned}\dot{P} &= F\Upsilon\xi\Upsilon^\top + \Upsilon\xi\Upsilon^\top F^\top + \Upsilon\dot{\xi}\Upsilon^\top \\ &= FP + PF^\top + \Upsilon\dot{\xi}\Upsilon^\top.\end{aligned} \quad (42)$$

It is obtained that $\Upsilon\dot{\xi}\Upsilon^\top = Y$, such that

$$\xi = \Pi_0 + \int_{t_0}^{t}\Upsilon^{-1}(\tau,t_0)Y(\tau)(\Upsilon^{-1}(\tau,t_0))^\top\mathrm{d}\tau. \quad (43)$$

Consequently, the general solution of (40) is given by

$$\begin{aligned}&P(t) = \Upsilon(t,t_0)\Pi_0\Upsilon^\top(t,t_0) \\ &+ \int_{t_0}^{t}\Upsilon(t,t_0)\Upsilon^{-1}(\tau,t_0)Y(\tau)(\Upsilon^{-1}(\tau,t_0))^\top\Upsilon^\top(t,t_0)\mathrm{d}\tau \\ &= \Upsilon(t,t_0)\Pi_0\Upsilon^\top(t,t_0) + \int_{t_0}^{t}\Upsilon(t,\tau)Y(\tau)\Upsilon^\top(t,\tau)\mathrm{d}\tau,\end{aligned} \quad (44)$$

where $\Upsilon(t,t_0)=\Upsilon(t,\tau)\Upsilon(\tau,t_0)$.

In view (37), it can be rewritten as

$$\dot{P} = (\mathcal{M}-PR^{-1})^\top P + P(\mathcal{M}-PR^{-1}) + PR^{-1}P + Q. \quad (45)$$

Define ψ as the transition matrix of $\mathcal{M}-PR^{-1}$, and according to (44), the general solution of P is derived as

$$\begin{aligned}\Pi(t,t_0,P_0) = \psi(t,t_0)P_0\psi^\top(t,t_0) + \int_{t_0}^{t}\psi(t,\tau)\left[Q(\tau)\right. \\ \left.+\Pi(t,t_0,P_0)R^{-1}\Pi(t,t_0,P_0)\right]\psi^\top(t,\tau)\mathrm{d}\tau \ge 0,\end{aligned} \quad (46)$$

where the matrices Q,R are positive definite.

Besides, define Ψ as the transition matrix of $\mathcal{M}$, it follows

$$\begin{aligned}&0 \le \Pi(t,t_0,P_0) = \Psi(t,t_0)P_0\Psi^\top(t,t_0) + \\ &\int_{t_0}^{t}\Psi(t,\tau)\left[Q - \Pi(\tau,t_0,P_0)R^{-1}\Pi(\tau,t_0,P_0)\right]\Psi^\top(t,\tau)\mathrm{d}\tau \\ &\le \Psi(t,t_0)P_0\Psi^\top(t,t_0) + \int_{t_0}^{t}\Psi(t,\tau)Q(\tau)\Psi^\top(t,\tau)\mathrm{d}\tau. \quad (47)\end{aligned}$$

According to Theorem 1, we know that the estimation error given in $\dot{\tilde{\Theta}} = -\mathcal{M}\tilde{\Theta}$ is exponentially stable, i.e., $\Psi \in \mathcal{L}_2$, so that $\int_{t_0}^{t}\Psi(t,\tau)Q(\tau)\Psi^\top(t,\tau)\mathrm{d}\tau$ is also bounded. Therefore, as shown in Lemma 5, the online updated matrix P is bounded and at least positive semi-definite. Then, the convergence of the proposed adaptive law (38) is summarized as:

Theorem 3: Considering the adaptive law (38) for system (1), the estimation error $\tilde{\Theta}$ will exponentially converge to zero provided that the regressor Φ fulfills the PE condition.

Proof: Choose the Lyapunov function $V_2(\tilde{\Theta}) \triangleq \tilde{\Theta}^\top\tilde{\Theta}$. The time derivative $\dot{V}_2(\tilde{\Theta})$ is derived from (38) as

$$\dot{V}_2(\tilde{\Theta}) = 2\tilde{\Theta}^\top\mathcal{H} + 2\tilde{\Theta}^\top R^{-1}P\mathcal{K}\mathcal{H}. \quad (48)$$

Based on (6) and (19), $\dot{V}_2(\tilde{\Theta})$ can be reformulated as

$$\dot{V}_2(\tilde{\Theta}) = -2\tilde{\Theta}^\top\mathcal{M}\tilde{\Theta} - 2\tilde{\Theta}^\top R^{-1}PU\bar{\mathcal{S}}U^\top\tilde{\Theta}. \quad (49)$$

From the fact $0 \le \mathrm{eig}(U\bar{\mathcal{S}}U^\top) \le 1$ and the aforementioned Lemma 5, Eq.(49) can be rewritten based on (19) and (20) as

$$\begin{aligned}\dot{V}_2(\tilde{\Theta}) &\le -2\tilde{\Theta}^\top\mathcal{M}U\bar{\mathcal{S}}U^\top\tilde{\Theta} - 2\tilde{\Theta}^\top R^{-1}PU\bar{\mathcal{S}}U^\top\tilde{\Theta} \\ &\le -2\tilde{\Theta}^\top(\mathcal{M}+R^{-1}P)U\bar{\mathcal{S}}U^\top\tilde{\Theta},\end{aligned} \quad (50)$$

where $(\mathcal{M}+R^{-1}P)>0$ is a symmetric positive matrix for $\mathcal{M},P,R>0$ when the PE condition is satisfied.

Consequently, we obtain

$$\dot{V}_2(\tilde{\Theta}) \le -2\tilde{\Theta}^\top(\mathcal{M}+R^{-1}P)U\bar{\mathcal{S}}U^\top\tilde{\Theta} \le -\sigma_3 V_2(\tilde{\Theta}), \quad (51)$$

where $\sigma_3 \triangleq 2\min\left(\mathrm{eig}((\mathcal{M}+R^{-1}P)U\bar{\mathcal{S}}U^\top)\right)$ is a positive constant. Therefore, from the Lyapunov theorem, one can conclude that $\tilde{\Theta}$ exponentially converges to zero.

Remark 5: Compared with adaptive law (21), the major contribution of adaptive law (38) is that the LQR design approach is used to derive an optimal learning gain as $\Gamma = R^{-1}P$, where not only the accurate parameter estimation can be guaranteed but also the tedious parameter tuning can be remedied. Furthermore, the adaptive law (38) is derived by constructing a closed-loop estimation system and introducing an analogous control to improve its robustness, where the cost function (25) is minimized under the conditions (34) and (37).

IV. Parameter Estimation under Disturbances

Consider system (1) subject to bounded disturbances, which is rewritten as

$$\dot{x} = \varphi(x,u) + \Phi(x,u)\Theta + \epsilon, \quad (52)$$

where ϵ denotes the lumped disturbances bounded by $\|\epsilon\| \le \bar{\epsilon}$, which can represent the modeling uncertainties, external disturbances and measurement noise.

Similar to the mathematical developments given in (4)-(10), an auxiliary matrix $\mathcal{H}$ containing the estimation error can be reformulated as

$$\mathcal{H} = -\mathcal{M}\tilde{\Theta} - \Delta, \quad (53)$$

where $\Delta(t) = \int_{t_0}^{t} e^{-l(t-\tau)}\frac{\Phi_f(\tau)\epsilon_f^\top(\tau)}{1+\|\Phi_f(\tau)\|^2}\mathrm{d}\tau$ stems from the bounded disturbances with ϵ_f given by $k\dot{\epsilon}_f + \epsilon_f = \epsilon$.

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2024.3520676

According to (15) and (53), one can verify that

$$\mathcal{K}\mathcal{H} = -U\bar{\mathcal{S}}U^{\top}\tilde{\Theta} - \mathcal{K}\Delta. \tag{54}$$

Consequently, in the presence of ϵ, the estimation error of adaptive law (38) is modified as

$$\begin{aligned}\dot{\tilde{\Theta}} &= -\mathcal{M}\tilde{\Theta} - \Delta - R^{-1}P\left(U\bar{\mathcal{S}}U^{\top}\tilde{\Theta} + \mathcal{K}\Delta\right) \\ &= -\left(\mathcal{M} + R^{-1}PU\bar{\mathcal{S}}U^{\top}\right)\tilde{\Theta} - \left(I + R^{-1}P\mathcal{K}\right)\Delta.\end{aligned} \tag{55}$$

Theorem 4: For system (52) with the adaptive law (38) and the PE condition of regressor Φ, then $\tilde{\Theta}$ will converge to a small set Ω around zero.

Proof: Choose a Lyapunov function as $V_2(\tilde{\Theta}) = \tilde{\Theta}^{\top}\tilde{\Theta}$, then its derivative can be calculated along with (55) as

$$\begin{aligned}\dot{V}_2(\tilde{\Theta}) = &-2\Big[\tilde{\Theta}^{\top}\left(\mathcal{M} + R^{-1}PU\bar{\mathcal{S}}U^{\top}\right)\tilde{\Theta} \\ &+ \tilde{\Theta}^{\top}\left(I + R^{-1}P\mathcal{K}\right)\Delta\Big].\end{aligned} \tag{56}$$

According to (51), it can be derived that

$$\dot{V}_2(\tilde{\Theta}) \leq -\sigma_3 V_2(\tilde{\Theta}) - 2\tilde{\Theta}^{\top}\left(I + R^{-1}P\mathcal{K}\right)\Delta. \tag{57}$$

Based on the Young's inequality $\pm a^{\top}b \leq a^{\top}a/(2\eta) + \eta b^{\top}b/2, \eta > 0$, it follows

$$\begin{aligned}\dot{V}_2(\tilde{\Theta}) &\leq -\sigma_3 V_2(\tilde{\Theta}) + \frac{\tilde{\Theta}^{\top}\tilde{\Theta}}{\eta} + \eta\Xi^{\top}\Xi \\ &= -\sigma_4 V_2(\tilde{\Theta}) + \aleph,\end{aligned} \tag{58}$$

where $\aleph \triangleq \eta\Xi^{\top}\Xi$, $\Xi \triangleq \left(I + R^{-1}P\mathcal{K}\right)\Delta$ and $\sigma_4 \triangleq [\sigma_3 - 1/\eta]$ are all positive constants for a small constant η. Then, recalling the extended Lyapunov Theorem [2], it is known that $\tilde{\Theta}$ is bounded and ultimately converges to a compact set $\Omega \triangleq \left\{\tilde{\Theta}(t) | \|\tilde{\Theta}(t)\|^2 \leq e^{-\sigma_4 t}V_2(\tilde{\Theta}(t_0)) + \aleph/\sigma_4\right\}$.

It is shown that the proposed optimal adaptive law (38) can ensure the ultimate uniform boundedness of the estimation error in the presence of disturbances. Similar claims can be proved for adaptive laws (11) and (21), which are not repeated again herein.

V. Numerical Simulations

Consider a nonlinear mass-spring-damper system as [1]

$$m\ddot{y} + \varsigma\dot{y} + \kappa y + \zeta y^3 = u, \tag{59}$$

where $m = 1$ is the mass, $\varsigma = 3$ is the damping coefficient, $\kappa = 5$ is the spring constant and $\zeta = 0.5$ is the strength of the spring. In addition, u is the external force, and $\kappa y + \zeta y^3$ represents the nonlinear spring dynamics. Define $x_1 = y$, $x_2 = \dot{y}$ as the displacement and velocity, then system (59) can be rewritten as

$$\begin{cases}\dot{x}_1 = x_2, \\ \dot{x}_2 = -\dfrac{\kappa}{m}x_1 - \dfrac{\varsigma}{m}x_2 - \dfrac{\zeta}{m}x_1^3 + \dfrac{1}{m}u = \Phi\Theta,\end{cases} \tag{60}$$

with $\Phi = [-x_1, -x_2, -x_1^3, u]$ and $\Theta = [5, 3, 0.5, 1]^{\top}$. In the simulations, the initial condition is $x_0 = [-0.5, 1]^{\top}, \Theta(t_0) = [0, 0, 0, 0]^{\top}$ and the input $u = 10\sin t + 3\cos 5t$ is set to ensure the PE condition. For those adaptive laws (11), (21) and (38),

TABLE I
The Selection of Parameters

Adaptive laws	Parameters
Eq. (11)	$\Gamma = 10^4 I$
Eq. (21)	$\Gamma = 10I, \mathcal{K}(t_0) = 60I$
Eq. (38)	$Q = R = I, \mathcal{K}(t_0) = 60I, P(t_0) = I$
DREM [16, Eq. (26)]	$\gamma_i = 10^{25}, i = 1, 2, 3, 4$

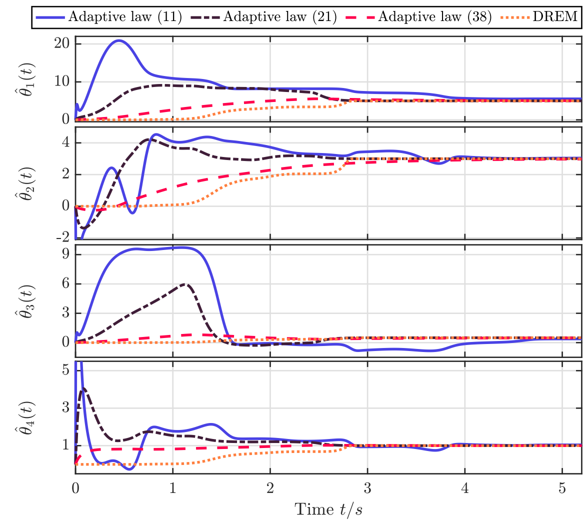

Fig. 1. Profile of estimated parameters.

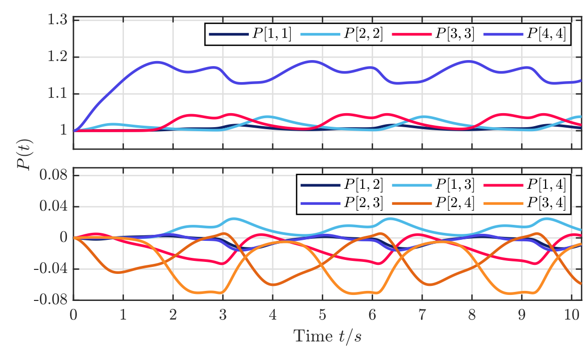

Fig. 2. Trajectory of matrix $P(t)$.

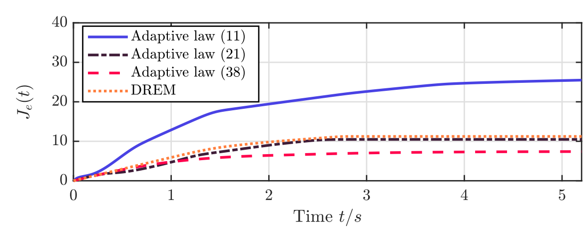

Fig. 3. Comparative error index $J_e = \int_{t_0}^{t} \|\tilde{\Theta}(\tau)\| \, \mathrm{d}\,\tau$.

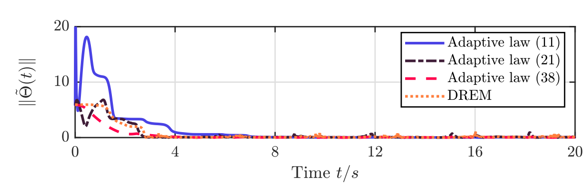

Fig. 4. Comparative estimation errors in the presence of disturbance.

$k = 0.001$ and $l = 5$ are used. Moreover, the DREM method proposed in [16, Eq. (26)] is also simulated for comparison. The other parameters are given in Table I.

Comparative simulation results of these four adaptive laws are given in Fig. 1 - Fig. 3. Clearly, all the three adaptive laws can achieve accurate parameter estimation for the studied system. Moreover, the proposed optimal adaptive law (38) has the best estimation performance in terms of both steady-state and transient convergence response, which is also verified by the estimation error index given in Fig.3. This is attributed to that an online updated optimal learning gain is used, whose profile is depicted in Fig. 2. Nevertheless, the adaptive law (21) with the regressor compensation $\mathcal{K}$ can achieve faster convergence rate and smaller overshoot than the adaptive law (11). However, the learning gains of (11) and (21) should be manually tuned, which influence the convergence response. This issue can be further remedied where the gain is online updated along with (37) and thus the overshoots in the estimated parameters can be suppressed. Nevertheless, the DREM can obtain a comparative response as (21) after a sluggish transient, while a large gain $\gamma = 10^{25}$ imposing a heavy computational cost has to be used.

Furthermore, a Band-Limited White Noise with power 0.0001 is generated as the bounded disturbance ϵ and injected into the system to test the robustness of the proposed adaptive laws. The norms of the corresponding estimation errors are provided in Fig. 4, which all converge to a small set around zero, implying that the estimated parameters with all adaptive laws can converge to the neighborhood around the true values as claimed in Theorem 4. For adaptive law (21) with the regressor compensation $\mathcal{K}$, the transient performance is improved, but an inappropriate Γ could undermine the robustness to disturbances. Nevertheless, with the help of the online varying learning gain (37), fast convergence and fair robustness can be achieved simultaneously.

VI. Conclusion

In this paper, an optimal APE is proposed for nonlinear systems with unknown constant parameters. The main idea is to establish an analogous closed-loop estimation system of the parameter estimation error via the optimality principle. Unlike the existing APE schemes, the learning gain for the proposed adaptive law can be online updated to minimize a cost function of estimation error, and the parameter tuning can be simplified as a semi-automated selection of two weighting matrices via the LQR design. In this line, filter operations and auxiliary matrices are first introduced to derive a formulation of estimation error information. Then, two constructive adaptive laws driven by the estimation error rather than the observer output error as the gradient algorithm are introduced. Finally, a cost function and a Hamiltonian function are designed to obtain an optimal solution, leading to a DRE to derive an optimal learning gain for the adaptive law. The convergence and robustness of the proposed APE algorithms are rigorously analyzed. Numerical simulations are given to verify the efficacy of the proposed methods. It is noted the PE condition is required to retain the error convergence. Hence, the design of optimal APE under weak excitation condition deserves further investigation.

References

[1] J. Slotine and W. Li, *Applied Nonlinear Control*. New Jersey: Prentice Hall, 1991.

[2] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos, *Nonlinear and Adaptive Control Design*. John Wiley & Sons, Inc., 1995.

[3] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Upper Saddle River, New Jersey: Prentice-Hall, Inc., 1996.

[4] W. Li and M. Krstic, "Stochastic Adaptive Nonlinear Control With Filterless Least Squares," *IEEE Transactions on Automatic Control*, vol. 66, no. 9, pp. 3893–3905, 2021.

[5] Y. Pan, T. Shi, and R. Ortega, "Comparative analysis of parameter convergence for several least-squares estimation schemes," *IEEE Transactions on Automatic Control*, vol. 69, no. 5, pp. 3341–3348, 2024.

[6] B. Anderson, "Failures of adaptive control theory and their resolution," *Communications in Information & Systems*, vol. 5, no. 1, pp. 1–20, 2005.

[7] P. A. Ioannou and K. Tsakalis, "A robust direct adaptive controller," *IEEE Transactions on Automatic Control*, vol. 31, no. 11, pp. 1033–1043, 1986.

[8] K. Narendra and A. Annaswamy, "A new adaptive law for robust adaptation without persistent excitation," *IEEE Transactions on Automatic Control*, vol. 32, no. 2, pp. 134–145, 1987.

[9] G. Kreisselmeier and B. Anderson, "Robust model reference adaptive control," *IEEE Transactions on Automatic Control*, vol. 31, no. 2, pp. 127–133, 1986.

[10] J. Na, M. Mahyuddin, G. Herrmann, X. Ren, and P. Barber, "Robust adaptive finite-time parameter estimation and control for robotic systems," *International Journal of Robust and Nonlinear Control*, vol. 25, no. 16, pp. 3045–3071, 2015.

[11] V. Adetola and M. Guay, "Performance improvement in adaptive control of linearly parameterized nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2182–2186, 2010.

[12] J. Na, J. Yang, X. Wu, and Y. Guo, "Robust adaptive parameter estimation of sinusoidal signals," *Automatica*, vol. 53, pp. 376–384, 2015.

[13] J. Na, Y. Huang, T. Liu, and Q. Zhu, "Reinforced adaptive parameter estimation with prescribed transient convergence performance," *Systems & Control Letters*, vol. 149, p. 104880, 2021.

[14] S. Aranovskiy, A. Bobtsov, R. Ortega, and A. Pyrkin, "Performance enhancement of parameter estimators via dynamic regressor extension and mixing," *IEEE Transactions on Automatic Control*, vol. 62, no. 7, pp. 3546–3550, 2017.

[15] R. Ortega, V. Nikiforov, and D. Gerasimov, "On modified parameter estimators for identification and adaptive control. A unified framework and some new schemes," *Annual Reviews in Control*, vol. 50, pp. 278–293, 2020.

[16] D. N. Gerasimov and V. O. Nikiforov, "On key properties of the Lion's and Kreisselmeier's adaptation algorithms," *International Journal of Adaptive Control and Signal Processing*, vol. 36, no. 6, pp. 1285–1304, 2022.

[17] J. E. Gaudio, A. M. Annaswamy, E. Lavretsky, and M. Bolender, "Parameter estimation in adaptive control of time-varying systems under a range of excitation conditions," *IEEE Transactions on Automatic Control*, vol. 67, no. 10, pp. 5440–5447, 2021.

[18] A. Bryson and Y.-C. Ho, *Applied Optimal Control: Optimization, Estimation and Control*. New York: Taylor & Francis Group, 1975.

[19] F. Lewis, D. Vrabie, and V. Syrmos, *Optimal Control*. John Wiley & Sons, Inc., 2012.

[20] B. Jenkins, A. Annaswamy, E. Lavretsky, and T. Gibson, "Convergence properties of adaptive systems and the definition of exponential stability," *SIAM Journal on Control and Optimization*, vol. 56, no. 4, pp. 2463–2484, 2018.

[21] R. Bucy and P. Joseph, *Filtering for Stochastic Processes with Applications to Guidance*. New York: John Wiley & Sons, Inc., 2005.

[22] R. Ortega, S. Aranovskiy, A. A. Pyrkin, A. Astolfi, and A. A. Bobtsov, "New Results on Parameter Estimation via Dynamic Regressor Extension and Mixing: Continuous and Discrete-Time Cases," *IEEE Transactions on Automatic Control*, vol. 66, no. 5, pp. 2265–2272, 2021.

[23] N. Nguyen, "Optimal control modification for robust adaptive control with large adaptive gain," *Systems & Control Letters*, vol. 61, no. 4, pp. 485–494, 2012.

[24] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge university press, 1998.