浙江大学 2002-2003 学年第 2 学期期末考试

《离散数学》课程试卷答案

开课学院：计算机学院　任课教师：＿＿＿＿　考试时间：120 分钟

专业：＿＿＿＿　班级：＿＿＿＿　姓名：＿＿＿＿　学号：＿＿＿＿

题序	1	2	3	4	5	6	7	8	总分	评卷
得分										

Zhejiang University
Discrete Mathematics, Spring 2003
Key to Final Exam

1. **(20%) Determine whether the following statements are true or false. If it is true write a ✓ otherwise a × in the bracket before the statement.**

   (a) (×) The recurrence relation $D_n = (n-1)(D_{n-1}+D_{n-2})$ is a linear homogeneous recurrence with constant coefficients.

   (b) (✓) There are 2^n-1 terms in the formula of the principle of inclusion- exclusion for the union of n sets.

   (c) (×) The function $f(n) = \lceil n/2 \rceil$ is a one-to-one function (injection) from $\mathbf{Z}$ to $\mathbf{Z}$, where $\mathbf{Z}$ is the set of integers.

   (d) (✓) The formulas $\forall x P(x) \wedge \exists x Q(x)$ and $\forall x \exists y (P(x) \wedge Q(y))$ are logically equivalent.

   (e) (✓) Each complete bipartite graph $K_{n,n}$ has a Hamilton circuit whenever $n > 1$.

   (f) (✓) There are two different equivalence relations on a set with two elements.

   (g) (✓) Let R be a relation on the set A. R equals its transitive closure if and only if R is transitive.

   (h) (✓) There are 20 students in a class. If each student has 0 or more of the other students in this class as friends, then there are at least 2 students have the same number of friends.

   (i) (✓) Let A, B be any sets. If $P(A) \in P(B)$, then $A \in B$, where $P(S)$ is the power set of S.

   (j) (×) There is an undirected tree with 2 vertices of 4 degrees, 3 vertices of 3 degrees. The remaining vertices are leaves. Then it contains 8 leaves.

2. **(20%) Fill in the blanks.**

   (a) Suppose that $|A| = 5$ and $|B| = 3$. The number of onto functions (surjections) $f : A \to B$ is $\underline{150}$.

   (b) The number of reflexive and symmetric relations on a set A with 5 elements is $\underline{2^{10}}$.

   (c) The value of extended binomial coefficient $\binom{-3}{3}$ is $\underline{-10}$.

(d) Suppose that there are 8 internal nodes (not leaves) in a binary tree. How many leaves can there be at most ? $\underline{9}$.

(e) Let $A = \{2,3,4\}$, $B = \{1,2\}$, $C = \{4,5,6\}$. $(A \oplus B) \oplus (B \oplus C)$ equals to $\underline{\{2,3,5,6\}}$, where $\oplus$ is the symmetric difference of two sets.

3. **(10%)** Let proposition formula $G = p \wedge (q \leftrightarrow r)$.

   (a) Find the full disjunctive normal form of G.

   (b) Write all the assignments that make this formula false.

   Solution:

   (a) $G = (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge r) = \sum(4,7) = \prod(0,1,2,3,5,6) \cdots\cdots$ (4 marks)

   (b) (0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,1,0),(1,0,1)$\cdots\cdots$ (6 marks)

4. **(10%)** Suppose that $A = \{2,4,6\}$. Let R be the relation defined on $A \times A$ where $((a,b),(c,d)) \in R$ means $a \leqslant c$ and $b \mid d$.

   (a) Show that R is a partial order relation.

   (b) Draw the Hasse diagram for the relation.

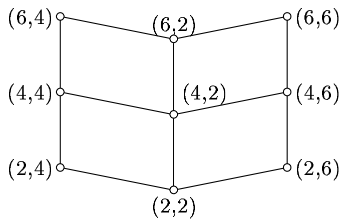
   (c) Find the maximal, minimal, greatest, and least elements of the poset $(A \times A, R)$.

   (d) Let $B = \{(2,4),(4,2)\}$. Find the upper bounds and lower bounds of the set B. Find the least upper bound and the greatest lower bound of the set B.

   Solution:

   (a) $\forall a, b \in A$, we have $a \leqslant a$ and $b \mid b$, therefore $((a,b),(a,b)) \in R$. R is reflexive. $\forall a,b,c,d \in A$, if we have $a \leqslant c$ and $b \mid d$, then we don't have $c \leqslant a$ and $d \mid b$. R is anti-symmetric. $\forall a,b,c,d,e,f \in A$, if we have $a \leqslant c$ and $b \mid d$ and $c \leqslant e$ and $d \mid f$, then we have $a \leqslant e$ and $b \mid f$. R is transitive. Therefore, R is a partial order relation.$\cdots\cdots$ (3 marks)

   (b)$\cdots\cdots$ (3 marks)



   (c) maximal: $\{(6,6),(6,4)\}$; minimal: $\{(2,2)\}$; greatest: ϕ; least: $\{(2,2)\}$
   $\cdots\cdots$ (2 marks)

   (d) upper bounds of B: $\{(4,4),(6,4)\}$, LUB: $\{(4,4)\}$
   lower bounds of B: $\{(2,2)\}$, GLB: $\{(2,2)\}$
   $\cdots\cdots$ (2 marks)

5. **(10%)** How many solutions are there to the equation $x_1 + x_2 + x_3 = 12$, where x_1, x_2, and x_3 are nonnegative integers with:

   (a) $x_1 > 1$, $x_2 < 3$, $x_3 > 4$.

   (b) x_1 and x_2 being odd numbers, and $x_3 > 5$.

   Solution:

   (a) The generation function for this equation with problem (a) is:

   $$G(x) = (x^2 + x^7 + x^9 + \cdots)(1 + x + x^2)(x^5 + x^6 + \cdots)$$
   $$= x^7 + \cdots + 15x^{12} + \cdots$$

   We need the coefficient of term x^{12}, therefore there are 15 solutions to the above equation.$\cdots\cdots$(5 marks)

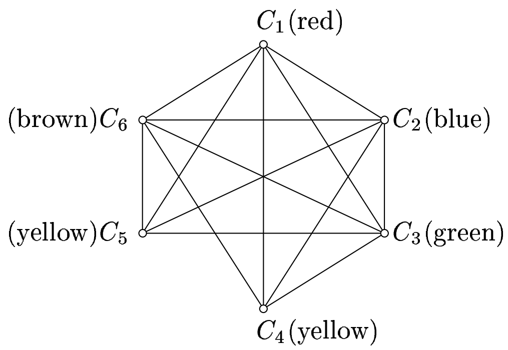
   (b) The generation function for this equation with problem (b) is:

   $$G(x) = (x + x^3 + x^5 + \cdots)(x^6 + x^7 + \cdots)$$
   $$= x^7 + \cdots + 6x^{12} + \cdots$$

   We need the coefficient of term x^{12}, therefore there are 6 solutions to the above equation.$\cdots\cdots$(5 marks)

6. **(10%)** The Computer Department has 6 committees and each committee meet once a month. Find the least meeting times to assure that no one is scheduled to be at 2 meetings at the same time within one month, if committees and their members are: C_1 = {Allen,Brooks,Marg}, C_2 = {Brooks,Jones,Morton}, C_3 = {Allen,Marg,Morton}, C_4 = {Jones,Marg,Morton}, C_5 = {Allen,Brooks}, C_6 = {Brooks,Marg,Morton}.

   Solution:



   We use graph model to solve this problem. Construct a graph by assigning a vertex to each committee. Two vertices are connected by an edge if two committees have the same member. $\cdots\cdots$(6 marks)

   An assignment of meeting times corresponds to a coloring of a graph, where each color represents a different meeting time. Therefore, it needs at least 5 meeting times to assure that no one is scheduled to be at 2 meetings at the same time within one month. $\cdots\cdots$(4 marks)

7. **(10%)** A regular polyhedron (正多面体) is a polyhedron in which all faces are regular polygons (正多边形) of the same size and shape, with the configuration at each vertex being the same. For example, there are cube (正方体), dodecahedron (正十二面体) etc. Prove that there are only 5 different types of regular polyhedrons. (Hint: every regular polyhedron is isomorphic to a planar graph.)

Solution:

Let n, e and r be the number of vertices, edges and faces, l is the degree of each vertex, k is the degree of each face. Then

(1) $n - e + r = 2$ (Euler Formula)$\cdots\cdots$(3 marks)

(2) $\sum_{i=1}^{n} \deg(v_i) = 2e, \quad l \cdot n = 2e \cdots\cdots$(1 mark)

(3) $\sum_{i=1}^{r} \deg(R_i) = 2e, \quad k \cdot r = 2e\cdots\cdots$(1 mark)

It is obvious that $l \geqslant 3$ and $k \geqslant 3$. From (1),(2) and (3) we obtain $(\frac{2}{l} + \frac{2}{k} - 1)e = 2$, hence

(4) $(\frac{2}{l} + \frac{2}{k} - 1) > 0\cdots\cdots$(1 mark)

If $k \geqslant 6$, the an internal angle of a regular k polygon is at least $120°$, the sum of three adjacent angles is greater than $360°$, hence the degree of face on a regular polyhedron must be less than 6. See the Fig. Problem 7.

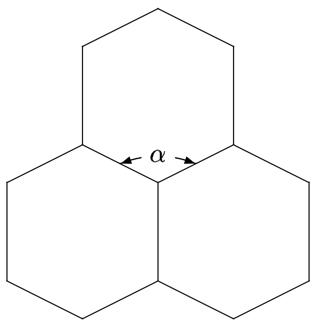


Fig. Problem 7

So we conclude that $3 \leqslant k \leqslant 5$.

(a) $k = 3$, since $\frac{2}{l} + \frac{2}{k} - 1 \geqslant 0$, hence $l < 6$, so it must be 3,4 or 5.

$(k, l) = (3, 3)$, $(r, n, e) = (4, 4, 6)$ 正四面体（三棱锥）

$(k, l) = (3, 4), (r, n, e) = (8, 6, 12)$ 正八面体

$(k, l) = (3, 5), (r, n, e) = (12, 20, 30)$ 正十二面体$\cdots\cdots$(2 marks)

(b) $k = 4$, since $\frac{2}{l} + \frac{2}{k} - 1 \geqslant 0$, hence $l < 4$, l must be only 3,

$(k, l) = (4, 3), (r, n, e) = (6, 8, 12)$ 正六面体，正方体$\cdots\cdots$(1 mark)

(c) $k = 5$, since $\frac{2}{l} + \frac{2}{k} - 1 \geqslant 0$, hence $l < \frac{10}{3}$, l must be only 3,

$(k, l) = (5, 3), (r, n, e) = (20, 12, 30)$ 正二十面体$\cdots\cdots$(1 mark)

8. **(10%)** A simple graph is shown in Fig. 1.

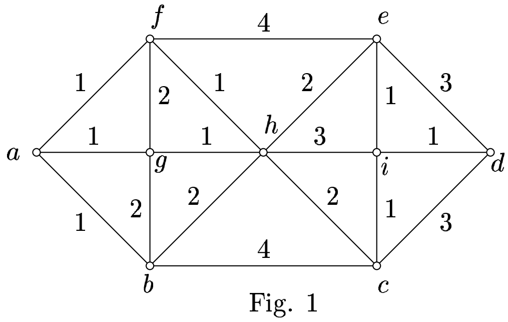


Fig. 1

(a) Suppose that b is chosen as the root of the spanning tree, and the vertices are ordered alphabetically. Use a depth-first search to produce a spanning tree for the graph.

(b) Suppose that b is chosen as the root of the spanning tree. Use a breadth-first search to produce a spanning tree for the graph.

(c) Find a minimum spanning tree for the graph.

Solution:

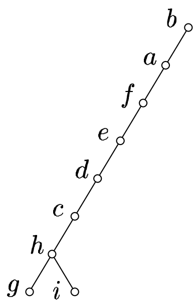
(a) Fig. (a) is the spanning tree produced by using a depth-first search of Fig.1

$\cdots\cdots$ (3 marks)

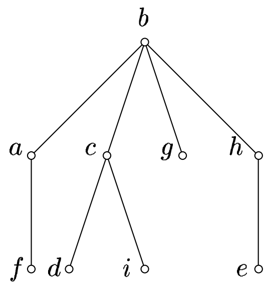
(b) Fig. (b) is the spanning tree produced by using a breadth-first search of Fig.1

$\cdots\cdots$ (3 marks)

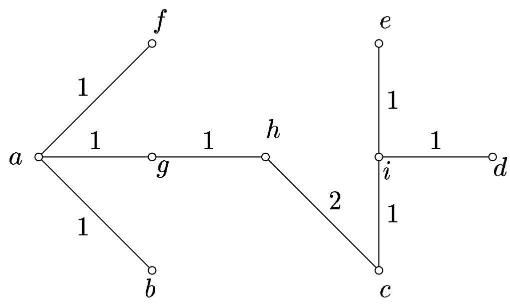
(c) Fig. (c) is the minimum spanning tree produced by using Kruskal's Algorithm.
$\cdots\cdots$ (4 marks)



(a)



(b)



(c)