Training Convnet with Backprop

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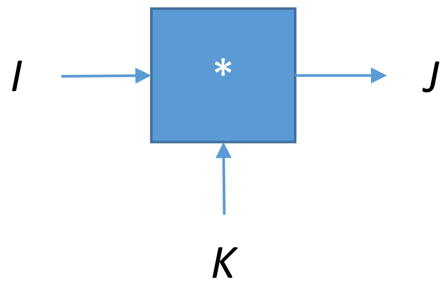
Backpropagation (BP) for a conv layer



$$J = I * K$$

Input to convolution layer: I, a H-by-W matrix

Parameter of the layer: K, a h-by-w matrix

Output of the layer: J, a $(H-h+1)$-by-$(W-w+1)$ matrix

Assume: $H >= h$ and $W >= w$

Given the gradient of loss function δJ with respect to J, BP tries to find answers to the following:

(1) What is the gradient of the loss function with respect to K? Denote this gradient by δK.

(2) What is the gradient of the loss function with respect to I? Denote this gradient by δI.

Why do we need δK? Because, we want to adjust the parameter K by gradient descent: $K = K$ – (learning rate)δK

Why do we need δI? Because, we want to apply BP to the layer that precedes this conv layer.

Derivation of δK

$$J(i,j) = \sum_{l=1}^{h}\sum_{m=1}^{w} I(i+l-1, j+m-1)K(l,m) \quad \Rightarrow \quad \frac{\partial J(i,j)}{\partial K(p,q)} = I(i+p-1, j+q-1)$$

Using chain rule of derivative:

$$\delta K(p,q) = \sum_{i=1}^{H-h+1}\sum_{j=1}^{W-w+1} \frac{\partial J(i,j)}{\partial K(p,q)} \delta J(i,j) = \sum_{i=1}^{H-h+1}\sum_{j=1}^{W-w+1} I(i+p-1, j+q-1)\delta J(i,j)$$

Thus, $\boxed{\delta K = I * \delta J}$

Derivation of δI

$$J(i,j) = \sum_{l=1}^{h}\sum_{m=1}^{w} I(i+l-1, j+m-1)K(l,m) \longrightarrow$$

$$\frac{\partial J(i,j)}{\partial I(p,q)} = \begin{cases} K(p-i+1, q-j+1), & \text{if } 0 \le p-i \le h-1 \text{ and } 0 \le q-j \le w-1, \\ 0, & \text{otherwise.} \end{cases}$$

Using chain rule of derivative:

$$\delta I(p,q) = \sum_{i=1}^{H-h+1}\sum_{j=1}^{W-w+1} \frac{\partial J(i,j)}{\partial I(p,q)} \delta J(i,j) = \sum_{i=\max(1,p-h+1)}^{\min(p,H-h+1)} \sum_{j=\max(1,q-w+1)}^{\min(q,W-w+1)} K(p-i+1, q-j+1)\delta J(i,j)$$

$$= \sum_{l=\max(1,p+h-H)}^{\min(p,h)} \sum_{\max(1,q+w-W)}^{\min(q,w)} K(l,m)\delta J(p-l+1, q-m+1)$$

Thus, $\boxed{\delta I = \text{pad}(\delta J) * \text{flip}(K)}$

“pad” function adds (h-1) 0 rows at the top and bottom and also adds (w-1) 0 columns at the left and at the right of a matrix.

Size of pad(δJ) is (H+h-1)-by-(W+w-1).

flip(K) is best understood by an example:

$$K = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \qquad \text{flip}(K) = \begin{bmatrix} 6 & 4 & 2 \\ 5 & 3 & 1 \end{bmatrix}$$

BP for a max pooling layer

$$\{x_i\}_{i=1}^{n} \longrightarrow \boxed{\text{Max pool}} \longrightarrow y = \max(\{x_i\}_{i=1}^{n})$$

Note that in case of a tie, only a single index is chosen for the following operation:

$$i = \mathrm{argmax}_k \{x_k\}_{k=1}^{n}$$

By chain rule: $$\delta x_i = \frac{\partial y}{\partial x_i}\delta y = \begin{cases} \delta y, & \text{if } \quad i = \mathrm{argmax}_k\{x_k\}_{k=1}^{n}, \\ 0, & \text{otherwise.} \end{cases}$$

Example of a 2-by-2, stride 2 max pooling:

$$x = \begin{bmatrix} 2 & -3 & 6 & 5 \\ 4 & 5 & 6 & -7 \\ 0 & -1 & 3 & -4 \\ -2 & -6 & 6 & 8 \end{bmatrix} \longrightarrow \boxed{\text{Max pool}} \longrightarrow y = \begin{bmatrix} 5 & 6 \\ 0 & 8 \end{bmatrix}$$

Suppose, $$\delta y = \begin{bmatrix} \delta y_1 & \delta y_3 \\ \delta y_2 & \delta y_4 \end{bmatrix},$$

then, $$\delta x = \begin{bmatrix} 0 & 0 & \delta y_3 & 0 \\ 0 & \delta y_1 & 0 & 0 \\ \delta y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta y_4 \end{bmatrix}.$$

BP for a ReLU layer

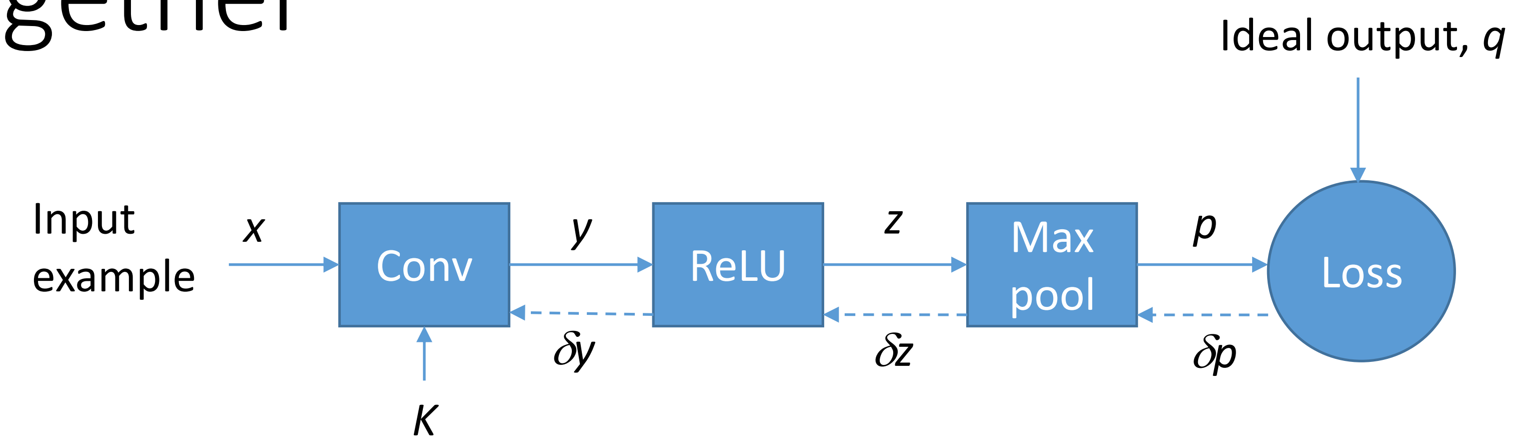
$$x \longrightarrow \boxed{\text{ReLU}} \longrightarrow y = \max(0, x)$$

By chain rule: $$\delta x = \frac{\partial y}{\partial x}\,\delta y = \begin{cases} \delta y, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Example: $$x = \begin{bmatrix} -5 & 4 \\ 0 & 2 \end{bmatrix} \longrightarrow \boxed{\text{ReLU}} \longrightarrow y = \begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix}$$

Suppose, $\delta y = \begin{bmatrix} \delta y_1 & \delta y_3 \\ \delta y_2 & \delta y_4 \end{bmatrix}$, then $\delta x = \begin{bmatrix} 0 & \delta y_3 \\ 0 & \delta y_4 \end{bmatrix}$.

Putting it all together



An example network with 3 layers and a loss function

<u>Convnet training algorithm:</u>

Initialize parameter K.

Iterate:

Step 1: (Forward pass)

Step 1a: Randomly choose a training example x and its corresponding ideal output q.

Step 1b: Pass x through "Conv" to get y; pass y though ReLU to get z; pass z through Max pool to get p.

Step 2: Compute "Loss" function for diagnostic purposes. /* Loss function measures deviation of p from q. */

Step 3: (Backward pass aka backpropagation)

Step 3a: Compute gradient of Loss function with respect to p. Denote this gradient by δp.

Step 3b: Compute δz given δp. /* Look at "BP for Max pooling." */

Step 3c: Compute δy given δz. /* Look at "BP for ReLU." */

Step 4c: Compute δK given δy. /* Look at "BP for Conv." */

Step 4: (Update parameter K by gradient descent) $K = K -$ (learning rate) δK.

Note: We don't have to compute δx, because there is no preceding layer before conv in the example above.