量子信息与量子计算习题解答

quantum computation and quantum information 是量子计算领域的圣书，但书中的习题没有官方答案，互联网上的答案也是众说纷纭，缺乏**自我纠错**的特性。现将该书中的基础部分（第 1-2 章）习题解答如下。

1 Chapter 1

1.1

经典最好情形 **2** 次。当且仅当测量两次结果相同时无法确定平衡函数和常函数，其容错概率

$$p=\frac{\binom{2^{n-1}}{2}\cdot 2}{\binom{2^n}{2}}=\frac{2^{n-1}-1}{2^n-1}<\frac{1}{2}$$

而量子算法仅需 **1** 次完全确定结果（$\epsilon=0$），说明量子计算在该问题上的优越性。

1.2

若状态可区分，发送的状态可以确定，设计合适的哈密顿设计器，在相同的状态下建立第二个系统；相反，准备量子态的多个副本，计算可观测值的平均值加以区分量子态。

2 Chapter 2

2.1

$$\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}-\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

2.2

$$A=\begin{bmatrix}0&1\\1&0\end{bmatrix},input:\{|0\rangle,|1\rangle\},output:\{|1\rangle,|0\rangle\},A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

2.3

$$\begin{aligned}BA\left|v_i\right\rangle&=B\left(\sum_j A_{ji}\left|w_j\right\rangle\right)=\sum_j A_{ji}B\left|w_j\right\rangle=\sum_{j,k}A_{ji}B_{kj}\left|x_k\right\rangle\\&=\sum_k\left(\sum_j B_{kj}A_{ji}\right)\left|x_k\right\rangle=\sum_k(BA)_{ki}\left|x_k\right\rangle\end{aligned}$$

2.4

$$I_{ij} = \delta_{ij}$$

2.5

$$\begin{aligned}
&(1)\ \left((y_1,\cdots,y_n),\sum_i \lambda_i\,(z_{i1},\cdots,z_{in})\right) = \sum_{i,j} y_i^*\lambda_j z_{ji} = \sum_j \lambda_j\left(\sum_i y_i^* z_{ji}\right)\\
&\quad = \sum_j \lambda_j\,((y_1,\cdots,y_n),(z_{j1},\cdots,z_{jn})) = \sum_i \lambda_i\,((y_1,\cdots,y_n),(z_{i1},\cdots,z_{in}))\\
&(2)\ ((y_1,\cdots,y_n),(z_1,\cdots,z_n))^* = \left(\sum_i z_i^* y_i\right) = ((z_1,\cdots,z_n),(y_1,\cdots,y_n))\\
&(3)\ ((y_1,\cdots,y_n),(y_1,\cdots,y_n)) = \sum_i y_i^* y_i = \sum_i |y_i|^2 \ge 0\\
&\quad ((y_1,\cdots,y_n),(y_1,\cdots,y_n)) = 0 \Longleftrightarrow (y_1,\cdots,y_n) = 0
\end{aligned}$$

2.6

$$\left(\sum_i \lambda_i\,|w_i\rangle,|v\rangle\right) = \left(|v\rangle,\sum_i \lambda_i\,|w_i\rangle\right)^* = \left(\sum_i \lambda_i\,(|v\rangle,|w_i\rangle)\right)^* = \sum_i \lambda_i^*\,(|w_i\rangle,|v\rangle)$$

2.7

$$\langle w \mid v\rangle = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 0, \frac{|w\rangle}{\||w\rangle\|} = \frac{|w\rangle}{\sqrt{\langle w|w\rangle}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \frac{|v\rangle}{\||v\rangle\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

2.8

数学归纳法， $k=1$ 时 $\langle v_1|v_2\rangle = \langle v_1|\left(\frac{|w_2\rangle - \langle v_1|w_2\rangle\,|v_1\rangle}{\|\,|w_2\rangle - \langle v_1|w_2\rangle\,|v_1\rangle\|}\right) = 0$

$n=k$ 成立，对于 $n=k+1$ 有，$\forall\ 1 \le j \le n$

$$\begin{aligned}
\langle v_j|v_{n+1}\rangle &= \langle v_j|\left(\frac{|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle\,|v_i\rangle}{\|\,|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle\,|v_i\rangle\|}\right)\\
&= \frac{\langle v_j|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle\,\langle v_j|v_i\rangle}{\|\,|w_{n+1}\rangle - \sum_{i=1}^n \langle v_i|w_{n+1}\rangle\,|v_i\rangle\|} = 0
\end{aligned}$$

2.9

根据完备性关系，$A = I_W A I_V = \sum_{i,j} |\omega_j\rangle\langle\omega_j|A|v_i\rangle\langle v_i| = \sum_{i,j}\langle\omega_j|A|v_i\rangle|\omega_j\rangle\langle v_i|$

$$X=|0\rangle\langle 1|+|1\rangle\langle 0|, Y=-i|0\rangle\langle 1|+i|1\rangle\langle 0|, Z=|0\rangle\langle 0|-|1\rangle\langle 1|$$

2.10

$$\left(|v_j\rangle\langle v_k|\right)_{pq}=\delta_{pj}\delta_{kq}$$

2.11

$$\lambda_{X,Y,Z}=\pm 1,\begin{cases} X:|\lambda=1\rangle=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix},|\lambda=-1\rangle=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}\\ Y:|\lambda=1\rangle=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix},|\lambda=-1\rangle=\frac{1}{\sqrt{2}}\begin{bmatrix}i\\1\end{bmatrix}\\ Z:|\lambda=1\rangle=\begin{bmatrix}1\\0\end{bmatrix},|\lambda=-1\rangle=\begin{bmatrix}0\\1\end{bmatrix}\end{cases}$$

$$\text{对角表示为}\begin{cases} X=\begin{pmatrix}0&1\\1&0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1&1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1&-1\end{pmatrix}\\ Y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\i\end{pmatrix}\begin{pmatrix}1&-i\end{pmatrix}-\frac{1}{2}\begin{pmatrix}i\\1\end{pmatrix}\begin{pmatrix}-i&1\end{pmatrix}\\ Z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1&0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0&1\end{pmatrix}\end{cases}$$

2.12

$$\lambda=1,|\lambda=1\rangle=\begin{bmatrix}0\\1\end{bmatrix},A\neq a\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

2.13

伴随算子的抽象定义：$\forall\, |v\rangle,|\omega\rangle,(|v\rangle,A|\omega\rangle)=(A^\dagger|v\rangle,|\omega\rangle)$

$$\begin{aligned}\forall\, |x\rangle,|y\rangle,\langle x|(|\omega\rangle\langle v|)^\dagger|y\rangle&=\big(|x\rangle,(|\omega\rangle\langle v|)^\dagger|y\rangle\big)=\big((|\omega\rangle\langle v|)^\dagger|y\rangle,|x\rangle\big)^*\\&=(|y\rangle,|\omega\rangle\langle v|x\rangle)^*=(\langle y|\omega\rangle\langle v|x\rangle)^*=\langle v|x\rangle^*\langle y|\omega\rangle^*=\langle x|v\rangle\langle\omega|y\rangle\end{aligned}$$

$$\langle x|(|\omega\rangle\langle v|)^\dagger|y\rangle=\langle x|v\rangle\langle\omega|y\rangle\Longrightarrow(|w\rangle\langle v|)^\dagger=|v\rangle\langle w|$$

2.14

$$(a_iA_i)^\dagger=(a_iIA_i)^\dagger=A_i^\dagger(a_iI)^\dagger=a_i^*A^\dagger$$

2.15

$$\left(\left(A^\dagger\right)^\dagger|\psi\rangle,|\phi\rangle\right)=\big(|\psi\rangle,A^\dagger|\phi\rangle\big)=\big(A^\dagger|\phi\rangle,|\psi\rangle\big)^*=(|\phi\rangle,A|\psi\rangle)^*=(A|\psi\rangle,|\phi\rangle)$$

2.16

$$P=\sum_{i=1}^{k}|i\rangle\langle i|, P^2=\left(\sum_{i=1}^{k}|i\rangle\langle i|\right)^2=\sum_{i=1}^{k}|i\rangle\langle i|i\rangle|i\rangle+2\sum_{i<j}|i\rangle\langle i|j\rangle|j\rangle=\sum_{i=1}^{k}|i\rangle\langle i|=P$$

2.17

$$\Longrightarrow: A|\lambda\rangle=\lambda|\lambda\rangle, \langle\lambda|A|\lambda\rangle=\lambda\langle\lambda|\lambda\rangle=\langle\lambda|A^\dagger|\lambda\rangle=(A|\lambda\rangle)^\dagger|\lambda\rangle=\lambda^*\langle\lambda|\lambda\rangle, \lambda=\lambda^*$$
$$\Longleftarrow: AA^\dagger=A^\dagger A, A=\sum_i\lambda_i|\lambda_i\rangle\langle\lambda_i|, A^\dagger=\sum_i\lambda_i^*|\lambda_i\rangle\langle\lambda_i|=A^\dagger$$

2.18

设 λ 为 U 特征值， $|\lambda\rangle$ 为 U 在 λ 本征空间的特征向量， $U|\lambda\rangle=\lambda|\lambda\rangle, \langle\lambda|U^\dagger=\langle\lambda|\lambda^*$

相乘得 $\langle\lambda|U^\dagger U|\lambda\rangle=\lambda\lambda^*\langle\lambda|\lambda\rangle=\|\lambda\|^2\langle\lambda|\lambda\rangle=\langle\lambda|\lambda\rangle\Longrightarrow\|\lambda\|=1$

2.19

$$X=X^\dagger, Y=Y^\dagger, Z=Z^\dagger, XX^\dagger=X^2=I, YY^\dagger=Y^2=I, ZZ^\dagger=Z^2=I,$$

2.20

$$\langle v_i|A|v_j\rangle=\sum_{kl}\langle v_i\mid w_k\rangle\langle w_k|A|w_l\rangle\langle w_l\mid v_j\rangle$$

2.21

厄米算子 A 的对角化：与正规算子相似，其中 QMQ 的正规性易得

$$A=IAI=(P+Q)A(P+Q)=PAP+QAP+PAQ+QAQ=\lambda P+QAQ$$

$$QAQ(QAQ)^\dagger=QAQQA^\dagger Q=QA^\dagger QQAQ=(QAQ)^\dagger QAQ$$

将 A 的对角化化解为 P 和 Q 的，从而由归纳法，任何厄米算子在标准正交基下可对角化

2.22

$$A|\lambda_1\rangle=\lambda_1|\lambda_1\rangle, A|\lambda_2\rangle=\lambda_2|\lambda_2\rangle, \langle\lambda_2|A^\dagger=\langle\lambda_2|\lambda_2^*\Longrightarrow\langle\lambda_2|A=\langle\lambda_2|\lambda_2$$
$$\langle\lambda_2|A|\lambda_1\rangle=\lambda_2\langle\lambda_2|\lambda_1\rangle=\lambda_1\langle\lambda_2|\lambda_1\rangle, \lambda_1\neq\lambda_2\Longrightarrow\langle\lambda_2|\lambda_1\rangle=0$$

2.23

$$P|\lambda\rangle=\lambda|\lambda\rangle=P^2|\lambda\rangle=P\lambda|\lambda\rangle=\lambda^2|\lambda\rangle, |\lambda\rangle\neq 0, \lambda^2=\lambda, \lambda=0,1$$

2.24

仿造实矩阵可分解为实对称矩阵和实反对称矩阵，将 A 拆解为

$$A=\frac{A+A^\dagger}{2}+i\frac{A-A^\dagger}{2i}=B+iC$$

其中 B 和 C 都为厄米矩阵，则 $\langle v|A|v\rangle=\langle v|B|v\rangle+i\langle v|C|v\rangle\geq 0$

而由厄米矩阵可以谱分解为 $\sum\limits_i\lambda_i|i\rangle\langle i|$，其中 $|i\rangle$ 为标准正交基，且 λ_i 为实数

故 $i\langle v|C|v\rangle$ 为虚数，从而 $\langle v|C|v\rangle=0\Longrightarrow C=O$，故 $A=B=\dfrac{A+A^\dagger}{2}$ 为厄米算子

引理：$\forall\,|v\rangle,\langle v|C|v\rangle=0\Longleftrightarrow C=O$。

$\forall\,|u\rangle,|v\rangle$，可以构造出下列等式使得 $\langle u|C|v\rangle=0$：

$$\begin{aligned}(|u\rangle,C|v\rangle)=\frac{1}{4}\Big[&(|u+v\rangle,C|u+v\rangle)-(|u-v\rangle,C|u-v\rangle)+\frac{1}{i}(|u+iv\rangle,C|u+iv\rangle)\\&-\frac{1}{i}(|u-iv\rangle,C|u-iv\rangle)\Big]\end{aligned}$$

上式右侧化简 $\dfrac{2(|v\rangle,C|u\rangle)+2(|u\rangle,C|v\rangle)}{4}+\dfrac{2(|iv\rangle,C|u\rangle)+2(|u\rangle,iC|v\rangle)}{4i}$

由复空间内积定义

$(|iv\rangle,C|u\rangle)=\overline{i}\,\langle v|C|u\rangle=-i\langle v|C|u\rangle,(|u\rangle,iC|u\rangle)=i\langle u|C|v\rangle$

$$RHS=\frac{(|v\rangle,C|u\rangle)+(|u\rangle,C|v\rangle)}{2}+\frac{-i(|v\rangle,C|u\rangle)+i(|u\rangle,C|v\rangle)}{2i}=(|u\rangle,C|v\rangle)$$

故 $\forall\,|v\rangle,\langle v|C|v\rangle=0\Longleftrightarrow\forall\,|u\rangle,|v\rangle,\langle u|C|v\rangle=0$，而取 $|u\rangle=C|v\rangle$ ，由内积正定性

$(C|v\rangle,C|v\rangle)=0\Longrightarrow C|v\rangle=0,\forall\,|v\rangle\Longrightarrow C=O$

2.25

$$\forall\,|v\rangle,\langle v|A^\dagger A|v\rangle=(A|v\rangle,A|v\rangle)\geq 0$$

2.26

$$\begin{aligned}|\psi\rangle^{\otimes 2}&=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)=\frac{1}{2}(|00\rangle+|01\rangle+|10\rangle+|11\rangle)\\|\psi\rangle^{\otimes 3}&=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\\&=\frac{\sqrt{2}}{4}(|000\rangle+|001\rangle+|010\rangle+|011\rangle+|100\rangle+|101\rangle+|110\rangle+|111\rangle)\end{aligned}$$

2.27

$$X \otimes Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$I \otimes X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$X \otimes I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

一般情形下，张量积不可交换。

2.28

$$(A \otimes B)^* = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^* = \begin{bmatrix} A_{11}^*B^* & \cdots & A_{1n}^*B^* \\ \vdots & \ddots & \vdots \\ A_{m1}^*B^* & \cdots & A_{mn}^*B^* \end{bmatrix} = A^* \otimes B^*$$

$$(A \otimes B)^T = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}^T = \begin{bmatrix} A_{11}B^T & \cdots & A_{m1}B^T \\ \vdots & \ddots & \vdots \\ A_{1n}B^T & \cdots & A_{mn}B^T \end{bmatrix} = A^T \otimes B^T$$

$$(A \otimes B)^\dagger = ((A \otimes B)^*)^T = (A^* \otimes B^*)^T = (A^*)^T \otimes (B^*)^T = A^\dagger \otimes B^\dagger$$

2.29

$$(U_1 \otimes U_2)(U_1 \otimes U_2)^\dagger = U_1 U_1^\dagger \otimes U_2 U_2^\dagger = I \otimes I = I$$

2.30

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B$$

2.31

$$(\langle u| \otimes \langle v|)(A \otimes B)|(u\rangle \otimes |v\rangle) = \langle u|A|u\rangle\langle v|B|v\rangle \geq 0$$

2.32

$$(P_1 \otimes P_2)^2 = P_1^2 \otimes P_2^2 = P_1 \otimes P_2$$

2.33

$$
\begin{aligned}
H &= \frac{1}{\sqrt{2}}[|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|] = \frac{1}{\sqrt{2}}\sum_{x,y}(-1)^{x\cdot y}|x\rangle\langle y| \\
H^{\otimes n} &= \frac{1}{\sqrt{2^n}}\sum_{x1,y1}(-1)^{x_1\cdot y_1}|x_1\rangle\langle y_1| \otimes \cdots \otimes \sum_{x_n,y_n}(-1)^{x_n\cdot y_n}|x_n\rangle\langle y_n| \\
&= \frac{1}{\sqrt{2^n}}\sum_{\boldsymbol{x},\boldsymbol{y}}(-1)^{\boldsymbol{x}\cdot\boldsymbol{y}}|\boldsymbol{x}\rangle\langle\boldsymbol{y}| \\
H^{\otimes 2} &= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\1 & -1 & 1 & -1\\1 & 1 & -1 & -1\\1 & -1 & -1 & 1\end{bmatrix}
\end{aligned}
$$

2.34

正规算子 A （正交）谱分解 $A = \sum_i a|a\rangle\langle a|$ ，定义算子函数 $f(A) = \sum_i f(a)|a\rangle\langle a|$

将 $A = \begin{bmatrix}4 & 3\\3 & 4\end{bmatrix}$ 谱分解 $A = 1\cdot\frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} + 7\cdot\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}$，从而

$$
\sqrt{A} = 1\cdot\frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} + \sqrt{7}\cdot\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1+\sqrt{7} & -1+\sqrt{7}\\-1+\sqrt{7} & 1+\sqrt{7}\end{bmatrix}
$$
$$
\log(A) = 0\cdot\frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} + \log 7\cdot\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} = \frac{\log(7)}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}
$$

2.35

$$
\vec{v}\cdot\vec{\sigma} = \sum_{i=1}^{3} v_i\sigma_i = v_1\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + v_2\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} + v_3\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = \begin{bmatrix}v_3 & v_1 - iv_2\\v_1 + iv_2 & -v_3\end{bmatrix}
$$

计算特征值 $\det(\vec{v}\cdot\vec{\sigma} - \lambda I) = (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) = \lambda^2 - 1$

由其为厄米矩阵，则可以（正交）谱分解 $\vec{v}\cdot\vec{\sigma} = |\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|$

其中 $|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}| = I$，而代入指数函数分别作用在 $\pm i\theta$ 上，展开得到

$$
\begin{aligned}
e^{i\theta\vec{v}\cdot\vec{\sigma}} &= e^{i\theta}|\lambda_1\rangle\langle\lambda_1| + e^{-i\theta}|\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= (\cos\theta + i\sin\theta)|\lambda_1\rangle\langle\lambda_1| + (\cos\theta - i\sin\theta)|\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= \cos\theta(|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}|) + i\sin\theta(|\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|) \\
&= \cos(\theta)I + i\sin(\theta)\vec{v}\cdot\vec{\sigma}
\end{aligned}
$$

2.36

$$
\mathrm{Tr}(\sigma_1) = 0 + 0 = \mathrm{Tr}(\sigma_2) = 0 + 0 = \mathrm{Tr}(\sigma_3) = 1 + (-1) = 0
$$

2.37

$$
\begin{aligned}
\operatorname{Tr}(AB) &= \sum_i \langle i|AB|i\rangle = \sum_{i,j}\langle i|A|j\rangle\langle j|B|i\rangle = \sum_{i,j}\langle j|B|i\rangle\langle i|A|j\rangle \\
&= \sum_j \langle j|BA|j\rangle = \operatorname{Tr}(BA)
\end{aligned}
$$

2.38

$$
\begin{aligned}
\operatorname{Tr}(A+B) &= \sum_i \langle i|A+B|i\rangle = \sum_i (\langle i|A|i\rangle + \langle i|B|i\rangle) = \operatorname{Tr}(A) + \operatorname{Tr}(B) \\
\operatorname{Tr}(zA) &= \sum_i \langle i|zA|i\rangle = z\sum_i \langle i|A|i\rangle = z\operatorname{Tr}(A)
\end{aligned}
$$

2.39

(1) 分别验证对第二个参数线性、交换共轭、自身正定性：

$$
\begin{aligned}
&(i)\ \left(A, \sum_i \lambda_i B_i\right) = \operatorname{Tr}\left[A^\dagger\left(\sum_i \lambda_i B_i\right)\right] = \sum_j \langle j|A^\dagger\left(\sum_i \lambda_i B_i\right)|j\rangle \\
&\quad = \sum_i \lambda_i \sum_j \langle j|A^\dagger B_i|j\rangle = \sum_i \lambda_i \operatorname{Tr}\left(A^\dagger B_i\right) \\
&(ii)\ (A,B)^* = \left(\sum_j \langle j|A^\dagger B|j\rangle\right)^* = \left(\sum_j \langle j|A^\dagger B|j\rangle\right)^\dagger = \sum_j \langle j|B^\dagger A|j\rangle = (B,A) \\
&(iii)\ (A,A) = \operatorname{tr}\left(A^\dagger A\right) = \sum_j \langle j|A^\dagger A|j\rangle \geq 0, (A,A) = 0 \Longleftrightarrow \forall\ |j\rangle, A|j\rangle = 0, A = O
\end{aligned}
$$

(2) $V \longmapsto V$ 中所有算子都可以用矩阵表示，维数为 $d \times d$，即 $\dim(L_V) = d^2$

(3) 对所有算子 A 的标准正交基为 $A_{ij} = |v_i\rangle\langle v_j|$，$|v_i\rangle$ 为标准正交基

由于对所有厄米算子 B 都可表为 $\dfrac{A+A^\dagger}{2}$，则构造正交基

$$
A_{ij} = \frac{A_{ij} + A_{ij}^\dagger}{2} = \frac{|v_i\rangle\langle v_j| + |v_j\rangle\langle v_i|}{2}
$$

对 $i=j$ 时为实数，$i \neq j$ 时可为虚数，两项相差负号，共 $d + 2\cdot\dfrac{d(d-1)}{2} = d^2$ 项

2.40

$$
\begin{aligned}
&[X,Y] = XY - YX = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} - \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}2i & 0\\0 & -2i\end{bmatrix} = 2iZ \\
&[Y,Z] = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} = \begin{bmatrix}0 & 2i\\2i & 0\end{bmatrix} = 2iX \\
&[Z,X] = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} - \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = 2i\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} = 2iY
\end{aligned}
$$

可简写为 $[\sigma_j,\sigma_k]=2i\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l$

2.41

$$\{\sigma_1,\sigma_2\}=\{\sigma_2,\sigma_3\}=\{\sigma_3,\sigma_1\}=0,\sigma_0^2=\sigma_1^2=\sigma_2^2=\sigma_3^2=I$$

2.42

$$\frac{[A,B]+\{A,B\}}{2}=\frac{AB-BA+AB+BA}{2}=AB$$

2.43

结合 $2.40-2.41$ 的计算结果可得

$$\sigma_j\sigma_k=\frac{[\sigma_j,\sigma_k]+\{\sigma_j,\sigma_k\}}{2}=\frac{2i\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l+2\delta_{jk}I}{2}=\delta_{jk}I+i\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l$$

2.44

$$[A,B]=0,\{A,B\}=0\Longrightarrow AB=0,A^{-1}AB=0=B$$

2.45

$$[A,B]^\dagger=(AB-BA)^\dagger=B^\dagger A^\dagger-A^\dagger B^\dagger=\left[B^\dagger,A^\dagger\right]$$

2.46

$$[A,B]=AB-BA=-(BA-AB)=-[B,A]$$

2.47

$$(i(BA-AB))^\dagger=-iA^\dagger B^\dagger+iB^\dagger A^\dagger=i(BA-AB)$$

2.48

$$P=UDU^\dagger(\lambda_i\geq 0),U=UII,H=U_1\sqrt{H^2},\sqrt{H^2}=U_2DU_2^\dagger,H=U_1U_2DU_2^\dagger$$

2.49

$$A=\sum_i\lambda_i|i\rangle\langle i|,\sqrt{A^\dagger A}=\sqrt{\sum_i|i\rangle\langle i|\lambda_i^*\sum_j\lambda_j|j\rangle\langle j|}=\sqrt{\sum_i\|\lambda_i\|^2|i\rangle\langle i|}=\sum_i\|\lambda_i\||i\rangle\langle i|$$

构造标准正交基 $|e_i\rangle$，对 $\lambda \neq 0$ 的所有 λ_i 取为 $\dfrac{\lambda_i|i\rangle}{\|\lambda_i\|}$，满足正交性，对 $\lambda = 0$ 扩充至全空间

令 $U = \sum_i |e_i\rangle\langle i|$，故 $A = U\sqrt{A^\dagger A} = \sum_i \dfrac{\lambda_i}{\|\lambda_i\|}\|\lambda_i\||e_i\rangle\langle i| = A$

2.50

$$A = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}, A^\dagger A = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$$

而对 $\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$ 计算特征值为 $\lambda_{1,2} = \dfrac{3 \pm \sqrt{5}}{2} = \left(\dfrac{\sqrt{5} \pm 1}{2}\right)^2$，对应特征向量为

$|\lambda = \dfrac{3+\sqrt{5}}{2}\rangle = \dfrac{1}{\sqrt{10-2\sqrt{5}}}\begin{bmatrix}2\\ -1+\sqrt{5}\end{bmatrix}, |\lambda = \dfrac{3-\sqrt{5}}{2}\rangle = \dfrac{1}{\sqrt{10+2\sqrt{5}}}\begin{bmatrix}2\\ -1-\sqrt{5}\end{bmatrix}$

$$J = \sqrt{A^\dagger A} = \sqrt{\lambda_+}|\lambda_+\rangle\langle\lambda_+| + \sqrt{\lambda_-}|\lambda_-\rangle\langle\lambda_-|$$

$$= \frac{\sqrt{5}+1}{2(10-2\sqrt{5})}\begin{bmatrix}2\\ -1+\sqrt{5}\end{bmatrix}\begin{bmatrix}2 & -1+\sqrt{5}\end{bmatrix} + \frac{\sqrt{5}-1}{2(10+2\sqrt{5})}\begin{bmatrix}2\\ -1-\sqrt{5}\end{bmatrix}\begin{bmatrix}2 & -1-\sqrt{5}\end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix}$$

从而 $J^{-1} = \dfrac{1}{\sqrt{5}}\begin{bmatrix}2 & -1\\ -1 & 3\end{bmatrix}$，计算 $U = AJ^{-1} = \dfrac{1}{\sqrt{5}}\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}$，则左极式分解为

$$A = U\sqrt{A^\dagger A} = \frac{1}{\sqrt{5}}\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}\sqrt{A^\dagger A}, \sqrt{A^\dagger A} = \frac{1}{\sqrt{5}}\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix}$$

而 $AA^\dagger = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$，同样经过繁杂计算后得到 $\sqrt{AA^\dagger} = \dfrac{1}{\sqrt{5}}\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}$，则右极式分解为

$$A = \sqrt{AA^\dagger}U = \sqrt{AA^\dagger}\frac{1}{\sqrt{5}}\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}, \sqrt{A^\dagger A} = \frac{1}{\sqrt{5}}\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}$$