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Nonlinear theory of drift-cyclotron kinetics and the possible breakdown of gyro-kinetics

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A nonlinear theory of drift-cyclotron kinetics (termed *cyclo-kinetics* here) is formulated to test the breakdown of the *gyro-kinetic* approximations. Six dimensional *cyclo-kinetics* can be regarded as an extension of five dimensional *gyro-kinetics* to include high-frequency cyclotron waves, which can interrupt the low-frequency gyro-averaging in the (sixth velocity grid) gyro-phase angle. Nonlinear cyclo-kinetics has no limit on the amplitude of the perturbations. Formally, there is no gyro-averaging when all cyclotron (gyro-phase angle) harmonics of the perturbed distribution function (delta-f) are retained. Retaining only the (low frequency) zeroth cyclotron harmonic in *cyclo-kinetics* recovers both linear and nonlinear *gyro-kinetics*. Simple recipes are given for converting continuum nonlinear delta-f *gyro-kinetic* transport simulation codes to *cyclo-kinetics* codes by retaining (at least some) higher cyclotron harmonics. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4773039]

I. INTRODUCTION AND SUMMARY

This paper formulates a nonlinear theory of drift-cyclotron kinetics (termed *cyclo-kinetics* here) for testing the breakdown of the *gyro-kinetic* approximations. Six dimensional *cyclo-kinetics* can be regarded as an extension of five dimensional *gyro-kinetics* to include high-frequency ion cyclotron waves, which can interrupt the low-frequency gyro-averaging in the (sixth velocity grid) gyro-phase angle. Formally, there is no gyro-averaging when all cyclotron (gyro-phase angle) harmonics of the perturbed distribution function (delta-f) are retained. *Cyclo-kinetics* recovers both linear and nonlinear *gyro-kinetics* when only the zeroth cyclotron harmonic is retained. Nonlinear *cyclo-kinetic* has no limit on the amplitude of the perturbations. Simple recipes are given for converting continuum nonlinear delta-f *gyro-kinetic* turbulent transport simulation codes to *cyclo-kinetic* codes by retaining (at least some) higher cyclotron harmonics.

Ion gyro-kinetics can be described as an extension of drift-kinetics treating cross-field wave lengths down to the ion gyro-radius (ρ_i) in which the distribution of gyro-centers is advanced with gyro-averaged fields. [$\rho_i = v_i^{th}/\Omega = \sqrt{2T_i/T_e}\rho_s$, where $\rho_s = c_s/\Omega$ is the "standard" ion gyroradius, $c_s = \sqrt{T_e/m_i}$ is the ion sound speed, and $\Omega = eB/m_i c$ is the ion cyclotron frequency.] The electrostatic formulations of the late 1960s[1,2] were generalized to include the electromagnetic fields in the early 1980s[3,4] and quickly extended from linear to nonlinear.[5] Gyro-kinetics is an approximation to cyclo-kinetics. As well known,[6] gyro-kinetics is limited to low-frequency drift waves excluding ion cyclotron waves ($\omega \ll \Omega$). It is less appreciated that gyro-kinetics is limited to weak turbulence with perturbed $E \times B$ velocities less than the ion thermal velocity ($\delta v_\perp^E \ll v_i^{th}$) so as not to interrupt or perturb the gyro-averaging orbits. (In essence without any nonlinear coupling to the higher cyclotron harmonics, the orbits remain unperturbed.) The nonlinear cyclo-kinetics has no such limitations. $\delta v_\perp^E \sim v_i^{th}$ is often referred to as the MHD ordering (Ref. 6, p. 112). As a kinetic theory, nonlinear gyro-kinetics is likely to breakdown first for the low-energy and low gyro-radius part of velocity space where $\delta v_\perp^E \sim v_\perp^i$ even at moderate turbulence levels. The delta-f formulations of gyro-kinetics and cyclo-kinetics, as presented here, are limited to *local* micro-turbulence on the cross field wavelength scale of turbulent eddies [$(1-10)\rho_s$], which are much less than the length scales of the density and temperature profiles represented by a (the typical minor radius of a tokamak); hence the "rho-star" parameter $\rho_* = \rho_s/a$ is typically less than a few percent. (An often-cited limitation to small parallel wavenumbers $k_{||}/k_\perp \sim O(\rho_*)$ is not essential, and the small effects from the so-called "parallel nonlinearity" may be retained to maintain strict conservation of energy if desired).

Linear cyclo-kinetics predates linear gyro-kinetics to the early 1960s. In fact, some of the earliest work on "anomalous diffusion arising from microinstabilities in a plasma" of fusion interest seemed to have focused first (1962) on high-frequency current drift[7] and grad-B drift[8] driven ion-cyclotron instabilities leading to Bohm size and scaled diffusivities $D_B = c_s\rho_s = cT_e/eB$. Only later (1965), did the work focus on the low-frequency "universal" (density gradient) diamagnetic drift waves,[9] which we now associate with gyro-kinetic gyro-Bohm size and scaled diffusion $D_{gB} = D_B\rho_* = [c_s/a]\rho_s^2 = [c_s a]\rho_*^2$. These examples of early work followed the time honored [Landau (1946)] procedure neglecting the nonlinear term and integrating along unperturbed particle orbits to arrive at the linear dispersion relation as a (possibly truncated) sum of cyclotron harmonics using quasi-neutrality. While unstable high-frequency ion cyclotron modes with low wave numbers ($k_\perp\rho_i \leq 1$) are not generally thought be the main cause of or even make a significant contribution to turbulent transport in tokamaks, the current drift driven ion cyclotron modes[7] were experimentally observed in the TFR

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tokamak[10] in 1978 and shortly, thereafter treated in sheared-toroidal tokamak geometry.[11] Of course the importance of ion (and electron) cyclotron waves in tokamak plasma heating and current drive is well known.

The very formal and generally electromagnetic work of Qin *et al.*[12,13] in the late 1990s extended gyro-kinetics to high frequency via Lagrangian Lie algebra perturbation methods (termed "gyrokinetic perpendicular dynamics" or "gyrocenter-gauge kinetic theory") based on gyrocenter co-ordinates in which "[slow] gyrocenter motion is decoupled from [fast] gyrophase motion" of perturbed orbits in separate kinetic equations for the gyrophase independent and dependent perturbed distribution functions. Still the focus of this work was entirely on a linear formulation with possible extensions of the method to treat nonlinearity apparently restricted to small amplitude perturbations. Reference 14 developed very efficient particle-in-cell algorithms for the faster but approximate gyrocenter-gauge kinetic theory. The method was illustrated with an application to linear electrostatic ion Bernstein waves in comparison to "exact" 6D Lorentz-force simulations. In contrast, the more pedestrian approach taken here for local cyclo-kinetics starts from the 6-dimensional Vlasov equation in Cartesian co-ordinates directly converting velocity space to gyro-phase (α), magnetic moment (μ) [and parallel velocity (u)] co-ordinates.[15] There is no attempt to separate slow from fast times scales; there is only one perturbed distribution function (delta-f) advanced in time making numerical implimentation very expensive. Most importantly, there is no restriction on the intensity of the turbulence (i.e., perturbed amplitudes). Apart from the restriction to locality and possible truncation in the number of high frequency ion cyclotron harmonics retained, there are no explicit approximations (excepting the artificial physical simplifications of the 4D test problem described below.) The novel feature of the work here is the focus on a nonlinear formulation (with perturbed orbits) summing over cyclotron harmonics and how it may differ with nonlinear gyro-kinetics when only the zeroth cyclotron harmonic is retained. The high and low frequency linear mode dispersion relations for the "gyrocenter-gauge kinetic" and cyclo-kinetics approaches are expected to have no practically significant difference in principle.

The formulation here is a prelude to nonlinear simulations of the cyclo-kinetic equations to quantify the breakdown of gyro-kinetics (especially at high turbulence levels) by comparison with a nonlinear simulation of the gyro-kinetic equations for a physically simple but identical problem. The cyclo-kinetics equations when expressed in gyroBohm units [macro-lengths scaled to a, cross-field microturbulence lengths to ρ_s, velocities to c_s, and rates to (c_s/a)] contain the normalized cyclotron frequency $\Omega_* = \Omega/[c_s/a] = 1/\rho_* \sim 100$, which cannot be scaled away in the non-zero cyclotron harmonic equations; thus cyclo-kinetic diffusivity breaks local gyro-Bohm diffusivity scaling. Since the unstable high-frequency cyclotron harmonics can provide an additional channel to tap the free energy in the plasma gradient, we might expect cyclo-kinetics to have larger (perhaps some way toward Bohm-sized) diffusion. The positive and negative cyclotron modes will beat together driving (or possibly damping) the low-frequency drift wave turbulence. Of course, cyclo-kinetic simulations will likely be more than 100-fold more expensive, since an Ω_* times smaller time step is required to follow even the lowest cyclotron-harmonic.

For simplicity of exposition (and to set up the physically simplest test problem), the derivation of nonlinear drift-cyclotron kinetics is given for electrostatic local radial (x-) transport and local homogeneous 2-dimensional (x,y) turbulence in the ion channel with adiabatic (or near adiabatic) electrons for a straight shearless magnetic field in the z-direction and with constant x-logarithmic gradients in density and ion temperature. Parallel field motion and variation in the z-direction are suppressed (ignored), and the focus is on the cross-field perturbed $E \times B$ turbulent motion. Quasi-neutrality is enforced. A radial electric field sets up an easily accounted $E \times B$ (Doppler rotation) drift in the y-direction. A radially decreasing magnetic field sets up a grad-B drift in the y-direction. The grad-B drift combines with the ion-temperature gradient to provide a simple model of the "toroidal" ion temperature gradient (ITG) instability. Formulation on a gyro-phase angle grid is given in Sec. II, and in terms of cyclotron harmonics in Sec. III. Straightforward generalizations are described in a concluding Sec. IV.

II. FORMULATION IN TERMS OF THE GYRO-PHASE ANGLE

The fundamental equation for the 6-dimensional Cartesian space $[\vec{x}, \vec{v}]$ distribution function f is

$$\partial f/\partial t + \vec{v} \cdot \vec{\nabla} f + (e/m)[\vec{E} + \vec{v}/c \times \vec{B}] \cdot \partial f/\partial \vec{v} = C + S, \tag{1}$$

where C represents a collision term and S is the source. In the presence of turbulent electrostatic field fluctuations $\vec{E} \Rightarrow E_x^0 \hat{\varepsilon}_x + \delta\vec{E}_\perp$, we decompose $f = F + \delta f$ where the statistical (or time) average $\delta\vec{E}$ and δf vanish with $\partial F/\partial t \sim 0$. Collisions are likely need to keep the equilibrium background F close to the assumed Maxwellian but otherwise ignored henceforth. Time averaging Eq. (1), we have the transport equation

$$\begin{aligned}&\partial F/\partial t + \vec{\nabla} \cdot (\vec{v}F) + (e/m)[E_x^0 \hat{\varepsilon}_x + \vec{v}/c \times \vec{B}] \cdot \partial F/\partial \vec{v}\\ &\quad = D + S,\end{aligned} \tag{2}$$

where

$$D = -(e/m)\delta\vec{E}_\perp \cdot \partial \delta f/\partial \vec{v}_\perp, \tag{3}$$

with a time average on D implicit in Eq. (2). Multiplying Eq. (2) by $\int dx^3/V \int dv^3 v_y [1, mv^2/2]$ and integrating by parts, we obtain as usual the box (volume V or flux surface) average turbulent electrostatic $E \times B$ radial particle (energy) transport flux

$$[\Gamma_x, Q_x] = \int dx^3/V \int dv^3 [1, mv^2/2](c\delta E_y/B)\delta f, \tag{4}$$

as well as a turbulent heating source (see, Ref. 15), which does not concern us here.

Unless otherwise noted, we will follow the notation and analysis of Ref. 15 converting the co-ordinates $[\vec{x}, \vec{v}]$ to $[\vec{x}', u, \mu, \alpha]$, where $\vec{x} = \vec{x}'$, $u = \hat{b} \cdot \vec{v}$ the parallel velocity with $\hat{b} = \vec{B}/B \Rightarrow \hat{\varepsilon}_z$, $\mu = v_\perp^2/2B$ the magnetic moment with the cross field velocity $\vec{v}_\perp$. What will become the *gyro-phase angle* α is defined by $\vec{v} = u\hat{b} + \vec{v}_\perp$ and $\vec{v}_\perp = v_\perp(\hat{\varepsilon}_x \cos\alpha + \hat{\varepsilon}_y \sin\alpha)$ with $\hat{\varepsilon}_y = \hat{b} \times \hat{\varepsilon}_x$, $mv^2/2 = m\mu B + mu^2/2$. Henceforth, the velocity of light $c \Rightarrow 1$. Equation (1) can be re-written (again ignoring collisions C) as

$$\partial f/\partial t + \Lambda f - \Omega \partial f/\partial \alpha = S, \tag{5}$$

where {from Ref. 15 and Eqs. (12) and (24)}

$$\begin{aligned}\Lambda f &= \vec{v} \cdot \vec{\nabla}' f + \dot{u} \partial f/\partial u + \dot{\mu} \partial f/\partial \mu + \dot{\alpha} \partial f/\partial \alpha \\ &= (1/B)\vec{\nabla}' \cdot (B\vec{v} f) + \partial(\dot{u} f)/\partial u + \partial(\dot{\mu} f)/\partial \mu + \partial(\dot{\alpha} f)/\partial \alpha,\end{aligned} \tag{6}$$

and [from Ref. 15 and Eqs. (16)–(18)], we have

$$\dot{u} = \vec{v} \cdot (\vec{\nabla}\hat{b}) \cdot \vec{v}_\perp + (e/m)E_{||} \Rightarrow 0, \tag{7a}$$

$$\begin{aligned}B\dot{\mu} &= -\vec{v} \cdot [\mu \vec{\nabla} B + u(\vec{\nabla}\hat{b}) \cdot \vec{v}_\perp] + (e/m)\vec{E}_\perp \cdot \vec{v}_\perp \\ &\Rightarrow \mu B/R v_x + (e/m)E_x^0 v_x + (e/m)\delta\vec{E}_\perp \cdot \vec{v}_\perp,\end{aligned} \tag{7b}$$

$$\begin{aligned}\dot{\alpha} &= -\vec{v} \cdot [(\vec{\nabla}\hat{\varepsilon}_x) \cdot \vec{\nabla}\hat{\varepsilon}_y + u/v_\perp^2 (\vec{\nabla}\hat{b}) \times \hat{b} \cdot \vec{v}_\perp] \\ &\quad + (e/m)\vec{E}_\perp \cdot \hat{b} \times \vec{v}_\perp/v_\perp^2 \\ &\Rightarrow -(e/m)E_x^0 v_y/v_\perp^2 + (e/m)\delta\vec{E}_\perp \cdot \hat{b} \times \vec{v}_\perp/v_\perp^2,\end{aligned} \tag{7c}$$

where we have suppressed parallel variations $\nabla_{||} \Rightarrow 0$, and parallel velocity $u \Rightarrow 0$ killing the associated curvature drifts (absent in a straight B-field). We retain only the grad-B drift with $-\nabla_x B/B = 1/R$, a radial electric field E_x^0, and the turbulent fluctuating field $\delta\vec{E}_\perp$ as described in Sec. I. The reduced 4-dimensional model has the phase space volume $\int dx_\perp^2 \int_0^{2\pi} d\alpha \int_0^\infty B d\mu$.

Subtracting the time average of Eq. (5) from itself, we have the equation for the perturbed distribution function

$$\begin{aligned}&\partial\delta f/\partial t + \vec{v}_\perp \cdot \vec{\nabla}\delta f + [\mu B/R + (e/m)E_x^0] v_x \partial_{B\mu}\delta f \\ &\quad -\{[\mu B/R + (e/m)E_x^0] v_y/v_\perp^2 - \mu B/R(v_y/v_\perp^2) + \Omega\}\partial_\alpha \delta f \\ &= -(e/m)\delta\vec{E}_\perp \cdot \vec{v}_\perp \partial_{B\mu} F - (e/m)\delta\vec{E}_\perp \cdot \hat{b} \times \vec{v}_\perp/v_\perp^2 \partial_\alpha F \\ &\quad -(e/m)\delta\vec{E}_\perp \cdot \vec{v}_\perp \partial_{B\mu}\delta f - (e/m)\delta\vec{E}_\perp \cdot \hat{b} \times \vec{v}_\perp/v_\perp^2 \partial_\alpha \delta f.\end{aligned} \tag{8}$$

In order for the background equilibrium F to be the stationary solution of Eq. (1), it must be a function of the three invariants of Eq. (1), which are $[mv_\perp^2/2,\ x + v_y/\Omega,\ y - v_x/\Omega]$ with the third precluded so as to have no y-dependence in the radial flux surface. However, F should have radial x-gradients, which brings in the second invariant. Hence, F is a Maxwellian drifting in the y-direction. We take $F = P_{nT} P_T P_B n_0 F_M$ where we choose $F_M = 1/(\pi v_{th}^2)\exp(-m\mu B/T)$, $P_{nT} = \exp[-(1/L_n - 1/L_T)(x + v_y/\Omega)]$, $P_T = \exp[-(m\mu B/TL_T)(x + v_y/\Omega)]$, and $P_B = \exp[-(m\mu B/TR)(x + v_y/\Omega)]$. Equation (8) is not clearly Lorenz frame invariant with respect to $E_x^0 \hat{\varepsilon}_x \times B\hat{\varepsilon}_z$ drifts in the y-direction. However, a simple shift of the velocity space variable $\vec{v}_\perp = v_\perp(\hat{\varepsilon}_x \cos\alpha + \hat{\varepsilon}_y \sin\alpha) \Rightarrow v_E\hat{\varepsilon}_y + v_\perp\ (\hat{\varepsilon}_x \cos\alpha + \hat{\varepsilon}_y \sin\alpha)$ where $v_E = E_x^0/B$ removes E_x^0 in both Eqs. (1) and (2), and Eq. (8) transforms $\partial\delta f/\partial t \Rightarrow [\partial\delta f/\partial t - v_E \nabla_y \delta f]$, which displays the Doppler rotation Lorentz invariance. We can subtract out the grad-B drift in the y-direction by the same manipulation on Eq. (1), namely substituting $\vec{v}_\perp = v_\perp(\hat{\varepsilon}_x\cos\alpha + \hat{\varepsilon}_y\sin\alpha) \Rightarrow v_d\hat{\varepsilon}_y + v_\perp(\hat{\varepsilon}_x\cos\alpha + \hat{\varepsilon}_y\sin\alpha)$ into $\vec{v} \times \vec{B}$ where $v_d = (m\mu B)/eBR$ and replacing $\vec{v}_\perp \cdot \vec{\nabla}\delta f \Rightarrow -v_d \nabla_y \delta f + \vec{v}_\perp \cdot \vec{\nabla}\delta f$ and $E_x^0 \Rightarrow \Delta E_x^0 = -m\mu B/eR$ in both Eqs. (1) and (8). In Eq. (8), the combination $[\mu B/R + (e/m)E_x^0] \Rightarrow [\mu B/R + (e/m)\Delta E_x^0] = 0$ and the remaining $\{-(\mu B/R)(v_y/v_\perp^2) + \Omega\}\partial_\alpha \delta f = \{1 - (\rho/R)\sin(\alpha)/2\}\Omega\partial_\alpha \delta f \sim \Omega\partial_\alpha\delta f$ since $\rho/R = (v_\perp/\Omega)/R \sim O(\rho_*)$. Fourier transforming in the cross field direction $[x,y]$ to $[k_x,k_y]$, the k-component of the fluctuating electric field is $\delta\vec{E}_{\perp k} = -i\vec{k}\delta\phi_k$. After straightforward manipulation of the linear F term on the RHS of Eq. (8), we have the nonlinear equation for δf_k

$$\begin{aligned}&\partial\delta f_k/\partial t - iv_E k_y \delta f_k - iv_d k_y \delta g_k + i\vec{v}_\perp \cdot \vec{k}_\perp \delta g_k - \Omega \partial_\alpha \delta g_k \\ &\quad = -i\omega_*^{nT} e\delta\phi/T_e(n_0 F_M) + \sum\nolimits_{k1} (e/m)[\delta\phi_{k1} i\vec{k}_1 \cdot \vec{v}_\perp \partial_{B\mu}\delta f_{k2} \\ &\qquad + \delta\phi_{k1} i\vec{k}_1 \cdot \hat{b} \times \vec{v}_\perp/v_\perp^2 \partial_\alpha \delta f_{k2}],\end{aligned} \tag{9a}$$

where $\vec{k}_2 = \vec{k} - \vec{k}_1$, $\delta f_k = \delta g_k - e\delta\phi_k/T(n_0 F_M)$ with δg_k the non-adiabatic part, and $\omega_*^{nT} = (T_e/eB)k_y\{1/L_n + [(m\mu B/T) -1]/L_T\}$ the drift frequency (with the parallel motion $\{\ldots +[(mu^2/2)/T - 1/2]/L_T\}$ suppressed in the 4-dimensional description here). The $P_{nT}P_T P_B$ $= \exp\{-[(1/L_n - 1/L_T) +(m\mu B/T)(1/L_T + 1/R)](x + v_y/\Omega)\}$ factor has been set to unity (after taking the $[\partial_{B\mu}, \partial_\alpha]$ velocity derivatives) to suppress any exponential radial profile variation, which would be inconsistent with local homogeneous turbulence and the k_x- transform. [Note the "radial box" is such that $x/L_{n,T,B}$ ranges over $\sim O(10's\rho_*)$, whereas $(v_y/\Omega)/L_{n,T,B} \sim O(\rho_*)$.]

The reader may not be entirely convinced by the possibly less than rigorous conversion of the Eq. (7b) linear $\dot{\mu}\partial_{\mu B}\delta f$ components in Eq. (8) to linear "drifts" terms in Eq. (9a) [while retaining the Eq. (7b) nonlinear $\dot{\mu}(\delta\vec{E})\partial_\mu \delta f$ component]. Thus, it is interesting to note that Qin *et al.* end up with the same linear gyro-center drifts terms ($\dot{X}\partial_X$) [e.g., linear system Eqs. (34)–(35) in Ref. 12 and Eqs. (21)–(22) in Ref. 13], with (in the notation here) a linear $\dot{\alpha}\partial_\alpha$ term, but no linear $\dot{\mu}\partial_{\mu B}$ term.

At this point (and henceforth), it is convenient to work in gyro-Bohm units (described in Sec. I). Our primary cyclokinetic Eq. (9a) can be re-written as

$$\begin{aligned}&\partial\delta\hat{f}_k/\partial\hat{t} - i\hat{\omega}_k^E \delta\hat{f}_k - i\hat{\omega}_k^d \delta\hat{g}_k + i\hat{\vec{v}}_\perp \cdot \hat{\vec{k}}\Omega_* \delta\hat{g}_k - \Omega_* \partial_\alpha \delta\hat{g}_k \\ &\quad = -i\hat{\omega}_{*k}^{nT} \delta\hat{\phi}_k (n_0 F_M) + \sum\nolimits_{k1} [(T_e/T_i)\delta\hat{\phi}_{k1} i\hat{\vec{k}}_1 \cdot \hat{\vec{v}}_\perp \partial_{\hat{\mu}} \delta\hat{f}_{k2} \\ &\qquad + \delta\hat{\phi}_{k1} i\hat{\vec{k}}_1 \cdot \hat{b} \times \hat{\vec{v}}_\perp/\hat{v}_\perp^2 \partial_\alpha \delta\hat{f}_{k2}],\end{aligned} \tag{9b}$$

where $\delta\hat{f} = \delta\hat{g} - (T_e/T_i)\delta\hat{\phi}(n_0 F_M Z) = \delta f/\rho_*$, $\delta\hat{\phi} = (e\delta\phi/T_e)/\rho_*$, $\hat{k} = \vec{k}_\perp \rho_s$, $\hat{t} = t[c_s/a]$, with $F_M(\hat{\mu}) = 1/2\pi\exp(-\hat{\mu})$ and $\hat{\mu} = m\mu B/T_i$. In units of $[c_s/a]$, $\hat{\omega}_k^E = \hat{v}^E \hat{k}_y$,

$\hat{\omega}_k^d = \hat{k}_y (T_i/T_e)\hat{\mu}(a/R)/Z$, $\hat{\omega}_{*k}^{nT} = \hat{k}_y\{(a/L_n)+(a/L_{Ti})[\hat{\mu}-1]\}$, and in units of $[c_s]$, $\hat{v}_\perp = \sqrt{2T_i/T_e \hat{\mu}}$. Again, parallel velocity $\hat{u} = u/v_{th}^i$ and parallel energy $\hat{u}^2$ with thermal average 1/2 have been suppressed. The radial transport fluxes in units of $[c_s n_0, c_s n_0 T_e]\rho_*^2$ are

$$[\hat{\Gamma}, \hat{Q}_i^\perp] = \oint d\alpha \int_0^\infty d\hat{\mu} \sum\nolimits_{\vec{k}} [1, (T_i/T_e)\hat{\mu}] i\hat{k}_y \delta\hat{\phi}_k^* [\delta\hat{g}_k(\hat{\mu},\alpha)/n_0]. \tag{10}$$

The quasi-neutrality relation determining $\delta\hat{\phi}_k$ with the electrons nearly adiabatic is

$$\oint d\alpha \int_0^\infty d\hat{\mu}\,\delta\hat{f}_k(\hat{\mu},\alpha)/n_0 = \delta\hat{\phi}_k[\lambda_k - i\hat{k}_y\delta_1], \tag{11}$$

where $\lambda_k = [1, 0]$ @ $[\hat{k}_y \neq 0, \hat{k}_y = 0]$, and $\delta_1 > 0$ will add unstable electron drift waves to the unstable "toroidal" ITG modes. [Note that physically δ_1 is a strong function of frequency, so the nearly adiabatic electron response (with a fixed δ_1) assumed here should be seen as an illustrative test model. In more physically realistic tokamak turbulence simulations, electrons are usually treated with simple drift-kinetics in the low-wave microturbulence considered here $\hat{k} < 1$.] Adding high-$\hat{k}^2$ dissipation to get a nonlinearly saturated state of ITG and ion cyclotron mode turbulence, this 4-dimensional system can be simulated on a $[\hat{k}_x, \hat{k}_y, \hat{\mu}, \alpha]$ grid. If a cyclic grid $\alpha_n = n2\pi/N$ with $n = 0, 1, 2, ..., N-1$ is used, N cyclotron frequency modes will be retained (e.g., $N = 7 \Rightarrow [-3, -2, -1, 0, +1, +2, +3]\Omega_*$).

It is useful to note that the nonlinear coupling in Eq. (9b) does not (explicitly) have the familiar "$\vec{\hat{k}}_1 \times \vec{\hat{k}}_2$" form ($\sum_{k1} \hat{e}_z \cdot \vec{\hat{k}}_1 \times \vec{\hat{k}}_2 \delta\hat{\phi}_{k1}\delta\hat{f}_{k2}$) expected of simple nonlinear $E \times B$ motion in which self interaction $\vec{\hat{k}}_1 = \pm\vec{\hat{k}}_2$" in precluded. Nevertheless, because of the conservative property $\partial_{\hat{\mu}}\vec{\hat{k}}_1 \cdot \vec{\hat{v}}_\perp + \partial_\alpha \vec{\hat{k}}_1 \cdot \hat{b} \times \vec{\hat{v}}_\perp / \hat{v}_\perp^2 = 0$, the nonlinearity conserves the total incremental entropy $\sum_k \oint d\alpha \int_0^\infty d\hat{\mu} (\delta\hat{f}_k^* \delta\hat{f}_k)$.

It is useful to note that if we re-express Eq. (9b) in terms of gyro-phase *α-Fourier harmonics* $[\delta\hat{g}_k(\hat{\mu},\alpha) = \sum_{n=0}^{N-1} \delta\hat{g}_k^n(\hat{\mu})\exp(in\alpha)]$, (as opposed to *cyclotron harmonics* in Sec. III), the $i\vec{\hat{v}}_\perp \cdot \vec{\hat{k}}\Omega_*\delta\hat{g}_k$ term linearly couples $\delta\hat{g}_k^n$ to $\delta\hat{g}_k^{n\pm1}$, so the α-Fourier harmonics are not a diagonal basis. Furthermore, if only N = 3 such Fourier harmonics are retained $[\delta\hat{g}_k^0, \delta\hat{g}_k^1, \delta\hat{g}_k^2](\delta\hat{g}_k^{-1} = \delta\hat{g}_k^2, \delta\hat{g}_k^3 = \delta\hat{g}_k^0)$, then the low-frequency drift wave mode tracks the gyro-kinetic linear solution only at the very low-k drift-kinetic limit. This is one reason why we turn to the more efficient cyclotron harmonics in which the lowest frequency modes (zeroth cyclotron harmonic) linearly tracks gyro-kinetic modes to higher-k.

III. FORMULATION IN TERMS OF CYLOTRON HARMONICS

To make a clear connection between cyclo-kinetics and gyro-kinetics and a more easily quantified breakdown of gyro-kinetics, we reformulate Eqs. (11a) in terms of cyclotron harmonics. The cyclotron harmonic $\delta\hat{G}_k^n(\hat{\mu})$ is defined by introducing an integrating factor on $\delta\hat{g}_k(\hat{\mu},\alpha)$ [and similarly on $\delta\hat{f}_k(\hat{\mu},\alpha)$]

$$\begin{aligned}\delta\hat{G}_k(\hat{\mu},\alpha) &= \delta\hat{g}_k(\hat{\mu},\alpha)\exp[ik\rho\sin(\alpha-\beta)] \\ &= \sum\nolimits_{n=-\infty}^{n=+\infty} \delta\hat{G}_k^n(\hat{\mu})\exp[in(\alpha-\beta)],\end{aligned} \tag{12}$$

where $\vec{\hat{k}} = \hat{k}[\cos(\beta)\vec{\varepsilon}_x + \sin(\beta)\vec{\varepsilon}_y]$, $\vec{\hat{v}}_\perp \cdot \vec{\hat{k}} = k\rho\cos(\alpha-\beta)$, and $\delta\hat{G}_{-k}^{-n} = (-1)^n[\delta\hat{G}_k^n]^*$. Using $\oint d\alpha/2\pi \exp[ik\rho\sin(\alpha-\beta) - in(\alpha-\beta)] = J_n(k\rho)$, Eq. (9b) can be re-written in terms of cyclotron harmonics as (suppressing the Doppler $E \times B$ rotation frequency $\hat{\omega}_k^E$ henceforth)

$$\begin{aligned}&\partial\delta\hat{F}_k^n/\partial\hat{t} - i\hat{\omega}_k^d\delta\hat{G}_k^n - in\Omega_*\delta\hat{G}_k^n + J_n(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT} n_0 F_M \\ &= \sum\nolimits_{n'} \oint d\alpha/2\pi \exp[-in(\alpha-\beta) + ik\rho\sin(\alpha-\beta)] NL_k(\hat{\mu},\alpha) \\ &= {}^{\partial_\mu}NL_k^n(\hat{\mu}) + {}^{\partial_\alpha}NL_k^n(\hat{\mu}),\end{aligned} \tag{13a}$$

where

$$\begin{aligned}&{}^{\partial_\mu}NL_k^n(\hat{\mu}) \\ &= \sum\nolimits_{n'}\sum\nolimits_{k1}\delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[-in(\alpha-\beta) + ik\rho\sin(\alpha-\beta) \\ &\quad + in'(\alpha-\beta_2) - ik_2\rho\sin(\alpha-\beta_2)] \otimes [ik_1\rho\cos(\alpha-\beta_1)] \\ &\quad \otimes (T_e/T_i)[\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'} - \delta\hat{F}_{k2}^{n'} ik_2\rho\sin(\alpha-\beta_2)/2\hat{\mu}]\end{aligned} \tag{13b}$$

and

$$\begin{aligned}{}^{\partial_\alpha}NL_k^n(\hat{\mu}) &= \sum\nolimits_{n'}\sum\nolimits_{k1}\delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[-in(\alpha-\beta) \\ &\quad + ik\rho\sin(\alpha-\beta) + in'(\alpha-\beta_2) - ik_2\rho\sin(\alpha-\beta_2)] \\ &\quad \otimes [-ik_1\rho\sin(\alpha-\beta_1)/\hat{v}_\perp^2] \otimes \{in' - ik_2\rho\cos(\alpha-\beta_2)\}\delta\hat{F}_{k2}^{n'},\end{aligned} \tag{13c}$$

where $\delta\hat{F}_k^n = [\delta\hat{G}_k^n - J_n(k\rho)\delta\hat{\phi}_k(T_e/T_i)n_0F_M]$ and $NL_k(\hat{\mu},\alpha)$ refers to the nonlinear terms on the RHS of Eq. (9b). ${}^{\partial_\alpha}NL_k^n(\hat{\mu}) + {}^{\partial_\alpha}NL_k^n(\hat{\mu})$ on the RHS of Eq. (13) correspond to the $\partial_{\hat{\mu}}-$ and $\partial_\alpha-$derivative terms on the RHS of Eq. (9b). $\delta\hat{F}_k^0$ is identical to the perturbed gyro-center density and $J_0(k\rho)\delta\hat{\phi}_k$ is the gyro-average potential perturbation. The $\delta\hat{F}_k^n(\hat{\mu}) \Longleftrightarrow \delta\hat{f}_k(\hat{\mu},\alpha)$ is analogous to $\delta\hat{G}_k^n(\hat{\mu}) \Longleftrightarrow \delta\hat{g}_k(\hat{\mu},\alpha)$ given by Eq. (12). Following Eqs. (10) and (12), the radial transport fluxes are

$$\begin{aligned}[\hat{\Gamma}, \hat{Q}_i^\perp] = 2\pi\int_0^\infty d\hat{\mu} \sum\nolimits_{\vec{k}}\sum\nolimits_n [1, (T_i/T_e)\hat{\mu}] i\hat{k}_y \delta\hat{\phi}_k^* \\ \otimes [J_n(k\rho)\delta\hat{G}_k^n(\hat{\mu})/n_0],\end{aligned} \tag{14}$$

which again recovers gyro-kinetics when truncated at $n = 0$.

Before unpacking the nonlinear terms, we explore the form of the dispersion relation. The quasi-neutrality relation Eq. (11) becomes

$$
\begin{aligned}
&n_0\delta\hat{\phi}_k[\lambda_k - i\hat{k}_y\delta_1] = -n_0\delta\hat{\phi}_k(T_e/T_i) + \oint d\alpha\int_0^\infty d\hat{\mu}\,\delta\hat{g}_k(\hat{\mu},\alpha)\\
&\Rightarrow\\
&= -n_0\delta\hat{\phi}_k(T_e/T_i) + \sum\nolimits_n 2\pi\int_0^\infty d\hat{\mu}J_n(k\rho)\delta\hat{G}_k^n(\hat{\mu})\\
&\Rightarrow\\
&= -n_0\delta\hat{\phi}_k(T_e/T_i)[1-\sum\nolimits_n\Gamma_n(b)] + \sum\nolimits_n 2\pi\int_0^\infty d\hat{\mu}J_n(k\rho)\delta\hat{F}_k^n(\hat{\mu}),
\end{aligned}
\tag{15}
$$

where we used $J_{-n} = (-1)^n J_n$, and Eq. 6.615, p. 710 of Ref. 16 to get the modified Bessel function (I_n) form: $\Gamma_n(b) = \exp(-b)I_n(b)$ where $b = (k\rho_i)^2/2 = (T_i/T_e)\hat{k}^2$. The first term on the RHS can be called the (perturbed) ion polarization density with the second called the density of gyro- (or cyclo-) centers. From the identity (Eq. 8.536, p. 980 of Ref. 16) $\sum_n J_n^2 = 1 \Rightarrow [1-\sum_n\Gamma_n(b)] = 0$, polarization density formally vanishes if all cyclotron harmonics $[-\infty < n < \infty]$ are retained (compare Eq. (11), which has no polarization density). From the linear LHS side of Eq. (13) $\delta\hat{G}_k^n = n_0F_MJ_n(k\rho)\delta\hat{\phi}_k[\hat{\omega}_{*k}^{nT} + \hat{\omega}(T_e/T_i)]/(\hat{\omega} + \hat{\omega}_k^d + n\Omega)$ and the intermediate Eq. (15), we can write the linear dispersion relation as

$$
\begin{aligned}
[\lambda_k - i\hat{k}_y\delta_1] = &-(T_e/T_i) + \sum\nolimits_n 2\pi\int_0^\infty d\hat{\mu}J_n^2(k\rho)F_M(\hat{\mu})\\
&\otimes[\hat{\omega}_{*k}^{nT}(\hat{\mu}) + \hat{\omega}(T_e/T_i)]/[\hat{\omega} + \hat{\omega}_k^d(\hat{\mu}) + n\Omega_*].
\end{aligned}
\tag{16}
$$

For the dissipative electron drive $i\delta_k = 0$, the $\hat{k}_y \neq 0$ dispersion relation has both low frequency drift wave and high frequency ion cyclotron reactive (non-dissipative) toroidal ITG modes driven by a/L_{T_i} and stabilized by a/L_{n_i}; however, the ITG ion cyclotron modes are neutrally stable without unphysically large values of a/L_{T_i}. The dispersion relation takes the more familiar drift-cyclotron mode form (e.g., Eq. 8.31, p. 170 of Ref. 17) with no charge separation $k^2\lambda_D^2 \to 0$ and $uk_{||} \to 0$) when the $\hat{\mu}$ dependence of $\hat{\omega}_{*k}^{nT}$ and $\hat{\omega}_k^d$ are suppressed so that $2\pi\int_0^\infty d\hat{\mu}J_n^2(k\rho)F_M(\hat{\mu}) \Rightarrow \Gamma_n(b)$, e.g., $a/L_{T_i} \Rightarrow 0$ and $a/R \Rightarrow 0$, which of course removes the toroidal ITG instability. For $\delta_k = \hat{k}_y\delta_1 > 0$ and expanding around the resonances $\hat{\omega}_k \sim -n\Omega_*$ we have

$$
\begin{aligned}
\hat{\omega}_k \sim -n\Omega_* + \Gamma_n(b)[\hat{\omega}_{*k}^n - n\Omega_*(T_e/T_i)]/[1 + (T_e/T_i)\\
\otimes(1-\Gamma_n(b)) - i\hat{k}_y\delta_1].
\end{aligned}
\tag{17}
$$

The well known low-frequency "i-delta" electron drift wave driven by density gradients $a/L_{n_i} > 0$ is recovered for $n = 0$. For $n \neq 0$, the $\hat{\omega}_{*k}^n$ term is not significant, and the ion cyclotron modes are driven/damped by $i\delta_k$ alone without a density gradient, as in the Drummond-Rosenbluth current drift ion cyclotron modes,[7] which has "kinetic" (i.e., nearly adiabatic) electrons (see, Ref. 11 for examples in shear slab and toroidal geometry). As we have noted, physically, the $i\delta_k$ depends on the frequency of the actual linear mode in question. It useful to note that in building a test model for nonlinear cyclo-kinetic simulations, the growth and damping rates of the low frequency drift waves can be separately controlled by the ITG temperature gradient a/L_{T_i} and the high frequency ion cyclotron modes by the δ_1 parameter. Future work with a 6D modified gyrokinetic code (as described in Sec. IV) exploring the stability of physically realistic ion cyclotron modes in sheared and toroidal magnetic field tokamak geometry with drift-kinetic electrons could prove fruitful. There is little linear interaction among the cyclotron harmonics, so they can be treated separately one n-harmonic at a time.

As noted at the end of Sec. II, the cyclo-kinetic nonlinearity [in Eq. (9)] does not (in an obvious way) have the familiar "$\vec{\hat{k}}_1 \times \vec{\hat{k}}_2$" form common to gyro-kinetics [and all previously explored 2-dimensional (planar) turbulence models] and which precludes nonlinear self-interaction. Hence, our focus in unpacking the nonlinear terms ${}^{\partial_\mu}NL_k^n(\hat{\mu}) + {}^{\partial_\alpha}NL_k^n(\hat{\mu})$ on the RHS of Eq. (13) is to isolate the "$\vec{\hat{k}}_1 \times \vec{\hat{k}}_2$" from the self interacting "$\vec{\hat{k}}_1 \cdot \vec{\hat{k}}_2$" components. Noting that

$$
\begin{aligned}
&k\rho\sin(\alpha-\beta) - ik_2\rho\sin(\alpha-\beta_2) = \hat{\varepsilon}_z\cdot\vec{\hat{k}}\times\hat{\upsilon}_\perp - \hat{\varepsilon}_z\cdot\vec{\hat{k}}_2\times\hat{\upsilon}_\perp\\
&= \hat{\varepsilon}_z\cdot\vec{\hat{k}}_1\times\hat{\upsilon}_\perp = k_1\rho\sin(\alpha-\beta_1)
\end{aligned}
$$

and $-\hat{k}_1\hat{k}_2\sin(\beta_1-\beta_2) = \hat{\varepsilon}_z\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2$ with $\hat{k}_1\hat{k}_2\cos(\beta_1-\beta_2) = \vec{\hat{k}}_1\cdot\vec{\hat{k}}_2$, and defining $\Delta_n^{n'}(\beta,\beta_1,\beta_2) \equiv \exp[-in(\beta_1-\beta) + in'(\beta_1-\beta_2)]$, we have after some manipulation of Bessel function identities

$$
\begin{aligned}
{}^{\partial_\alpha}NL_k^n(\hat{\mu}) = &-\sum\nolimits_{n'}\sum\nolimits_{k1}\hat{\varepsilon}_z\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_{n-n'}(k_1\rho) - 1/4[J_{n-n'+2}(k_1\rho) + J_{n-n'-2}(k_1\rho)]\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)\\
&+\sum\nolimits_{n'}\sum\nolimits_{k1}\vec{\hat{k}}_1\cdot\vec{\hat{k}}_2\{[J_{n-n'+2}(k_1\rho) - J_{n-n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)\\
&-\sum\nolimits_{n'}\sum\nolimits_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho) - J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(n'\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2),
\end{aligned}
\tag{18a}
$$

$$
\begin{aligned}
{}^{\partial_\mu}NL_k^n(\hat{\mu}) = &-\sum\nolimits_{n'}\sum\nolimits_{k1}\hat{\varepsilon}_z\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_{n-n'}(k_1\rho) + 1/4[J_{n-n'+2}(k_1\rho) + J_{n-n'-2}(k_1\rho)]\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)\\
&-\sum\nolimits_{n'}\sum\nolimits_{k1}\vec{\hat{k}}_1\cdot\vec{\hat{k}}_2\{[J_{n-n'+2}(k_1\rho) - J_{n-n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)\\
&-\sum\nolimits_{n'}\sum\nolimits_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho) + J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(2\hat{\mu}\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2).
\end{aligned}
\tag{18b}
$$

Adding Eqs. (18a) to (18b) greatly simplifies the result and the $\vec{k}_1 \cdot \vec{k}_2$ terms cancel

$$
\begin{aligned}
{}^{\partial_\alpha}NL_k^n(\hat{\mu}) + {}^{\partial_\mu}NL_k^n(\hat{\mu}) = &-\sum_{n'}\sum_{k1} \hat{\varepsilon}_z \cdot \vec{k}_1 \times \vec{k}_2 \{J_{n-n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2) \\
&-\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho) - J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(n'\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2) \\
&-\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho) + J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(2\hat{\mu}\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2). \qquad (18c)
\end{aligned}
$$

Equation (13) combined with Eq. (18c) is the penultimate result of Sec. III and is mathematically identical to Eq. (9b) of Sec II. The nonlinearity Eq. (18) again should preserve the incremental entropy: $\sum_k \oint d\alpha \int_0^\infty d\hat{\mu}(\delta\hat{f}_k^*\delta\hat{f}_k) = \sum_k\sum_n 2\pi\int_0^\infty d\hat{\mu}(\delta\hat{F}_k^{n*}\delta\hat{F}_k^n)$. This should prove an indispensable check on any coding.

Note that if *only* the low-frequency $\delta\hat{F}_k^0$ cyclotron harmonic is retained, $\rho_* = 1/\Omega_*$ drops from the equations and gyroBohm scaling of the fluxes results. *Otherwise,* the gyro-Bohm normalized fluxes $[\hat{\Gamma},\hat{Q}_i^\perp]$ (see Eq. (10)) will depend on ρ_* and gyroBohm scaling will be broken. Most importantly, cyclo-kinetics identically recovers linear and nonlinear gyrokinetics. The nonlinear interaction is simplified further. For $n = n' = 0$, only the first two terms in Eqs. (18a) and (18b) or essentially the first $\hat{\varepsilon}_z \cdot \vec{k}_1 \times \vec{k}_2$ term of (18c) survives with $\Delta_0^0(\beta,\beta_1,\beta_2) \equiv 1$

$$
\begin{aligned}
{}^{\partial_\mu}NL_k^n(\hat{\mu}) + {}^{\partial_\alpha}NL_k^n(\hat{\mu}) \Rightarrow &-\sum_{k1}\hat{\varepsilon}_z \cdot \vec{k}_1 \times \vec{k}_2 J_0(k_1\rho)\delta\hat{\phi}_{k1}\delta\hat{G}_{k2}^0 \\
&= -\sum_{k1} J_0(k_1\rho)\delta\hat{\vec{\upsilon}}_{k1}^E \cdot (i\vec{k}_2\delta\hat{G}_{k2}^0).
\end{aligned} \qquad (19)
$$

It is not surprising that the $\hat{\mu}\partial_{\hat{\mu}}\delta F$ term (last term in Eq. (18c)) in the nonlinearity has dropped away since gyrokinetics conserves magnetic moment.

Any reasonable truncation of cyclo-kinetics [Eqs. (13) and (18)] must at least retain $\delta\hat{F}_k^{\pm1}$ since $\delta\hat{F}_k^1$ and $\delta\hat{F}_k^{-1}$, which can beat together to damp (or drive) $\delta\hat{F}_k^0$: $\hat{\omega}_{k1} + \hat{\omega}_{k2} \sim (\Omega_* \pm \Delta\hat{\omega}_{k1}) + (-\Omega_* \pm \Delta\hat{\omega}_{k2}) \sim (\hat{\omega}_{*k}^{nT} \pm \Delta\hat{\omega}_k)$. As noted in Sec I, nonlinear gyro-kinetics is expected to breakdown when the perturbed $E \times B$ motion is comparable to the ion thermal speed ($\delta\upsilon^E \sim \upsilon_{th}^i$). However, for $n = 0$ and $n' = \pm1$, the $(i\hat{k}_1/\hat{\upsilon}_\perp)$ terms in the last two lines of Eqs. (18c) are singular at low velocity $\hat{\upsilon}_\perp = \sqrt{2(T_i/T_e)\hat{\mu}} \to 0$, suggesting that a more accurate statement is that gyro-kinetics breaks down first for the more populous low velocity and small gyroradius ions where $\delta\upsilon^E \sim \upsilon_\perp$.

IV. RECIPES FOR GENERALIZATION AND CONVERSION OF CONTINUUM GYRO-KINETIC CODES TO CYCLO-KINETIC CODES

While the recovery of linear grad-B drift-kinetics in Sec. II was somewhat informal, the practical recipes for converting continuum δf–gyro-kinetic codes (like GYRO[18] or GS2[19]) appear to be straight-forward (and intuitively obvious). The linear grad-B drifts, curvature drifts, as well as parallel motion operations in real shaped geometry should remain unchanged. Following Eq. (13), in addition to the usual equation for gyro-kinetic perturbed density of gyro-centers or zeroth cyclotron harmonic $\partial\delta\hat{F}_k^0/\partial t = ..$, $\pm n$ symmetric equations for the higher harmonics $\partial\delta\hat{F}_k^n/\partial t \mp in\Omega_*\delta\hat{G}_k^n = ...$ are added with the gyro-averaged potential perturbation omega-star term generalized: $J_0(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT} n_0F_M \Rightarrow J_n(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT} n_0F_M$. The ion transport and quasi-neutrality equation follow Eqs. (14) and (15). Except when treating high-k $k_\perp\rho_i \gg 1$ electron temperature gradient (ETG) modes, drift-kinetic electrons are normally used in gyrokinetic codes in place of the i-delta model here. In addition, GYRO (and GS2) operate on a normalized energy $\hat{\varepsilon} = \hat{\mu}\hat{B} + \hat{u}^2$ and the pitch angle grid $\lambda = \hat{\mu}/\hat{\varepsilon}$ replaces the $[\hat{\mu},\hat{u}]$-grids here (poloidal variation in $\hat{B}$ accounts for the trapping). So the velocity space volume element becomes $\oint d\alpha\int_0^\infty \partial_{\hat{\mu}}\int_{-\infty}^\infty d\hat{u} \Rightarrow \sum_{\sigma=\pm1} 2\pi/\sqrt{2}\int_0^\infty \sqrt{\hat{\varepsilon}}d\hat{\varepsilon}\int_0^{1/\hat{B}} d\lambda/\sqrt{1-\lambda\hat{B}}$ and the $\partial_{\hat{\mu}}|_{\hat{u}} \Rightarrow \hat{B}[\partial_{\hat{\varepsilon}} + (1/\hat{\varepsilon}\hat{B})\partial_\lambda]$ gradient must be converted in the nonlinear terms Eqs. (18a) and (18b) replacing the gyro-kinetic nonlinearity Eq. (19). Generalization to electromagnetic perturbations and transport is straightforward. For example, adding $\delta\vec{B}_\perp$ perturbations by following the GYRO[18] notation and gyroBohm normalization, $\delta\hat{\phi}_k$ is replaced by $\delta\hat{U}_k = \delta\hat{\phi}_k - \sqrt{2T_i/T_e}\hat{u}\delta\hat{A}_k$ [with $\delta\hat{A} = (c_s/c)(e\delta A_{||}/T_e)/\rho_*$], $\delta\hat{H}_k^n = [\delta\hat{F}_k^n + J_n(k\rho)\sqrt{2T_i/T_e}\ \hat{u}\delta\hat{A}_k(T_e/T_i)n_0F_M]$ replaces $\delta\hat{F}_k^n$, and $\delta\hat{G}_k^n$ is replaced by $[\delta\hat{H}_k^n + J_n(k\rho)\delta\hat{U}_k(T_e/T_i)n_0F_M]$ in Eqs. (13)–(15) and (18).

The outstanding practical question for physically realistic codes like GYRO used to simulated tokamak experiments is whether or not truncation at the first significant $n = \pm1$ higher cyclotron harmonic is adequate. This question can only be answered with simulations of the highly reduced 4D cyclo-kinetic versus 3D gyro-kinetic test problems (suppressing the 2D corresponding to parallel variation and velocity) as described in the previous sections perhaps made more tractable by first reducing Ω_* from 100 to 10. Comparing numerical simulation experiments with cyclo-kinetic codes based on a gyro-phase finitely spaced α-grid using Eq. (9a) (in Sec. II) (or its n-truncated α-Fourier harmonic form) against codes based on the n-truncated cyclotron harmonics using Eqs. (18a) and (18c) (in Sec. III) should prove instructive; the two forms are "mathematically" equivalent with a continuous α and n extending to $\pm\infty$ but not necessarily "numerically" equivalent in terms of truncation efficiency.

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