rCYCLO Technical Guide

Zhao Deng and R. E. Waltz

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Introduction of rCYCLO

Background

Magnetic confinement fusion is a historic scientific challenge which may provide a significant contribution to the world long term energy needs. The main idea of which is to constrain the high temperature plasma in a designed magnetic field. The energy and particle transportation largely determines the quality of the confinement. In recent years, the large-scale simulation by Gyrokinetic code GYRO of low core (0< r/a<0.7) transport and turbulence intensity in typical DIII D L-modes seems to be good agreement with experiments where turbulence levels are low; however, there is a 5 to 10-fold short fall in gyrokinetic based low-k turbulence and transport levels in the cold L-mode edge where turbulence levels are high, which was discovered first by C. Holland and A E White who compared the transport level of experimental DIIID shot 128913 and GYRO-simulation at r/a=0.75 [1]. It can be inferred that the turbulence transport near edge of L-mode is missed. Another evidence is that R.E. Waltz and J. Candy suppress the kinetic effect in GYRO simulation intending to return to the drift kinetic ion simulation which have no gyro-averaging but can recover much of the experimental levels [2], this could be a demonstration of edge transport missing and suggest that transport missing may be caused by gyrokinetics breaking down.

Motivated by exploring whether these transport missing is caused by gyrokinetics breaking down, we plan to build a gyrokinetic model and a cyclokinetic model, and to develop a corresponding suit of codes (named rCYCLO) for an identical problem, which could run the Gyrokinetics and Cyclokinetics at the same time for the same case. We will be using this code to comparing and contrast the gyrokinetics and gyclo-kinetics to find out if this is the key factor causing the Missing transport near L-mode Edge.

Cyclokinetics can be regarded as an extension of gyrokinetics to include the high-frequency ion cyclotron waves, and can recover both linear and nonlinear gyrokinetics when only the first cyclotron harmonic is retained. Gyrokinetics could be broken down if these unstable high-frequency ion cyclotron modes interrupt the gyro-averaging or drive the turbulence levels higher. Nonlinear cyclokinetics has no limit on the amplitude of the perturbations, however, gyrokinetic is limited to weak turbulence with perturbed $\vec{E} \times \vec{B}$ velocity less than the ion thermal velocity $\left(\delta\upsilon_{\perp}^{E} << \upsilon_{i}^{th}\right)$ so as not to interrupt or perturb the gyro-averaging orbits.

Since the 6 dimensional cyclokinetic simulation is too expensive, for simplifying the exposition and improving the simulation efficiency, a 4D [kx, ky, μ, α] nonlinear cyclokinetic model is developed, where we have suppressed parallel variations $\nabla_{\|} \Rightarrow 0$, and parallel velocity $u \Rightarrow 0$, which means the derivation of gyrokinetic and cyclokinetic is given in 2-dimensional spatial space (x, y) (most of the time expressed in its Fourier space (kx, ky)), and 2-dimensional velocity space (μ,α), where μ represents magnetic moment and α is gyro-phase; The associated curvature drifts (absent in a straight B-field) are killing, and we retain only the grad-B drift with $-\nabla_x B / B = 1/R$, the simulation is located in a box (about 100ρ_s size) of an infinite aspect ratio circular s-αgeometry, which is the same as in the slab geometry with a constant magnetic gradient.

This rCYCLO Technical Guide documents the cyclokinetic formulations, the derivations of the corresponding equations, and the numerical methods used in rCYCLO.

Cyclokinetic equation in gyro-phase angle form

Starting with the 6 dimensional Vlasov equation, rCYCLO assumes a two dimensional local flux tube geometry (*x*, *y*) perpendicular to magnetic field $\vec{B}$. Parallel variations and parallel velocity are suppressed $\partial / \partial_z \Rightarrow 0$, $u_z \Rightarrow 0$. The simulation box (*x*, *y*) is a small symmetric rectangle of side *L* with $\rho_s << L << a$. ρ_s is the ion gyro-radius, and *a* is the tokamak minor radius. The straight and shearless magnetic field varies only in the *x* direction $B(x) = B[1-(x+r)/R]$. *x* represents the radial position with *x*=0 at center of the box, *R* is the tokamak major radius, and *r* is the minor radius of the flux tube. Since no trapped particles are considered, the value of the local inverse aspect ratio *r*/*R* does not enter in this work. rCYCLO transforms real space (*x*, *y*) to Fourier wave number space (k_x, k_y). Following from Eq. (9b) of Ref. [1], the nonlinear non-conservative form of the normalized ion cyclokinetic equation in gyro-phase angle space is given by

$$\begin{aligned} &D\delta \hat{f}_k / D\hat{t} - i\hat{\omega}_k^d \delta\hat{g}_k + i\vec{\hat{\upsilon}}_\perp \cdot \vec{\hat{k}}\Omega_* \delta\hat{g}_k - \Omega_* \partial_\alpha \delta\hat{g}_k = -i\hat{\omega}_{*k}^{nT} \delta\hat{\phi}_k n_0 F_M(\hat{\mu}) \\ &\quad + \sum_{k1} [(T_e / T_i)\delta\hat{\phi}_{k1} i\vec{\hat{k}}_1 \cdot \vec{\hat{\upsilon}}_\perp \partial_{\hat{\mu}} \delta\hat{f}_{k2} + \delta\hat{\phi}_{k1} i\vec{\hat{k}}_1 \cdot \hat{b} \times \vec{\hat{\upsilon}}_\perp / \hat{\upsilon}_\perp^2 \partial_\alpha \delta\hat{f}_{k2}] \end{aligned}, \tag{1}$$

where $\delta\hat{\phi}_k$ is the perturbed electric potential. $\delta\hat{f}_k$ is the perturbed ion distribution function and $\delta\hat{g}_k(\hat{\mu},\alpha) = \delta\hat{f}_k(\hat{\mu},\alpha) + (T_e/T_i)\delta\hat{\phi}_k n_0 F_M(\hat{\mu})$ represents the non-adiabatic part of the distribution function. α is the gyro-phase angle. $\mu = m_i \upsilon_\perp^2 / 2B$ is the ion magnetic moment with the perpendicular ion velocity $\upsilon_\perp$ and the ion mass m_i. T_e (T_i) is the electron (ion) temperature. n_0 is the unperturbed ion density. $F_M(\hat{\mu})$ is the two dimensional Maxwell ion distribution function. In rCYCLO, variables are normalized to gyroBohm units. Any macro-lengths are scaled to the tokamak minor radius *a*, cross-field micro-turbulence lengths to the sound speed ion gyro-radius ρ_s, velocities to the ion sound speed c_s, time to a/c_s, and ion velocity to the ion thermal speed $\upsilon_i^{th} = \sqrt{2T_i / m_i}$. The ion sound speed is defined by $c_s = \sqrt{T_e / m_i}$, with $\rho_s = c_s / \Omega$ where $\Omega = eB / m_i c$ is the ion gyro frequency. $\vec{k}_\perp = k_\perp[\cos(\beta)\hat{\varepsilon}_x + \sin(\beta)\hat{\varepsilon}_y]$ is the cross field wave number vector in Cartesian coordinates with $\hat{\varepsilon}_y = \hat{\varepsilon}_z \times \hat{\varepsilon}_x$, $\hat{\varepsilon}_z = \hat{b} = \vec{B}/B$, and β is the wave angle.

The three-wave interaction requires $\vec{k}_2 = \vec{k} - \vec{k}_1$. The perpendicular ion velocity can be expressed as $\vec{\upsilon}_\perp = \upsilon_\perp(\hat{\varepsilon}_x \cos\alpha + \hat{\varepsilon}_y \sin\alpha)$ with $0 \le \alpha < 2\pi$. We normalize the wave number, time, and magnetic moment as $\hat{k} = k_\perp \rho_s$, $\hat{t} = t(c_s / a)$, and $\hat{\mu} = (\upsilon_\perp / \upsilon_i^{th})^2$ respectively. The perturbed ion distribution function, electric potential, and Maxwell distribution function are normalized as $\delta \hat{f}_k = (\delta f_k T_i / m_i) / \rho^*$, $\delta\hat{\phi}_k = (e\delta\phi_k / T_e) / \rho^*$, and $F_M(\hat{\mu}) = e^{-\hat{\mu}} / 2\pi$ respectively, where $\rho^* = \rho_s / a$ is the relative ion gyro radius. $\hat{\omega}_k^d = 2\hat{k}_y (T_i / T_e)\hat{\mu}(a / R)$ is the grad-*B* drift frequency. We have added the factor 2 here to compensate for the curvature drive from the suppressed parallel direction. $\hat{\omega}_*^{nT} = \hat{k}_y[(a / L_n) + (a / L_{Ti})(\hat{\mu} - 1)]$ is the density and temperature gradient drive frequency for the drift waves. In rCYCLO, artificial damping is included in the time derivative operator $D / D\hat{t} = \partial / \partial\hat{t} + \mu_{HK}\hat{k}^4 + \mu_{LK} / \hat{k}^2$. μ_{HK} and μ_{LK} are constants. $\mu_{HK}\hat{k}^4$ is the damping of high-*k* modes which allows turbulence energy to escape to short wave length modes where it is physically damped by collisions. $\mu_{LK} / \hat{k}^2$ is the damping of low-*k* modes. Physically it represents the neglected magnetic shear damping which makes the low-*k* modes slightly stable. This allows the low-*k* modes to saturate the inverse cascade energy from the higher *k* driving modes. The large factor $\Omega^* = 1 / \rho^*$ is the relative ion cyclotron frequency, which ranges from 10 to 1000 with 100 being a typical physical value of interest. We set $n_0 \equiv 1$ in units of density n_e, which means n_i=n_e. The velocity space integration of the two dimensional Maxwell ion distribution function is $\oint d\alpha \int_0^\infty d\hat{\mu} F_M(\hat{\mu}) = 1$ where the parallel velocity has been suppressed. The perturbed potential and distribution function satisfy the conjugate property $\delta\hat{\phi}_{-k} = \delta\hat{\phi}_k^*$, $\delta \hat{f}_{-k} = \delta \hat{f}_k^*$.

The Poisson equation is given as

$$(\hat{\lambda}_D^2 \hat{k}^2)\delta\hat{\phi}_k = \oint d\alpha \int_0^\infty d\hat{\mu}\delta \hat{f}_k(\hat{\mu}, \alpha) / n_0 - \delta\hat{n}_k^e, \tag{2}$$

where $\hat{\lambda}_D << 1$ is the Debye length in units of ρ_s. $\delta\hat{n}_k^e = (\delta n_k^e / n_0) / \rho^*$ is the perturbed electron density, and $\delta\hat{n}_k^i = \oint d\alpha \int_0^\infty d\hat{\mu}\delta \hat{f}_k(\hat{\mu}, \alpha) / n_0$ is the perturbed ion density. The electron descriptions are given in Sec. II.D.

The radial ion particle and energy fluxes in gyroBohm units of $n_0 c_s \rho^{*2}$ and $T_e n_0 c_s \rho^{*2}$ are

$$[\hat{\Gamma}, \hat{Q}^{\perp}] = Re \oint d\alpha \int_0^{\infty} d\hat{\mu} \sum_k \; [1, (T_i / T_e)\hat{\mu}] i\hat{k}_y \delta\hat{\phi}_k^* [\delta\hat{g}_k(\hat{\mu}, \alpha) / n_0] \, , \tag{3}$$

where *Re* means only the real part is retained. Eq. (3) indicates that transport is proportional to the non-adiabatic perturbed distribution function. The diffusivities are defined as fluxes divided by gradients $[\hat{D}, \hat{\chi}] = [\hat{\Gamma}, \hat{Q}^{\perp}] / [a / L_n, a / L_{Ti}]$.

We do not attempt to simulate Eq. (1) since the stiff α-derivative term $\Omega_* \partial_\alpha \delta\hat{g}_k$ is difficult to time advance accurately. It is better to transform α to a harmonic space so as to replace the derivative operator by the harmonic number identity $\partial_\alpha \Rightarrow in$. There are two kinds of harmonic transforms: Fourier harmonics and cyclotron harmonics. Nonlinear cyclokinetic simulations with Fourier harmonics (CKinFH) were presented in Ref. [3] where the numerical methods were documented and grid convergences were demonstrated. The formulation of CKinFH is given in the Appendix. The simulations in this paper were done in the cyclotron harmonic representation (CKinCH) to which we now turn.

Cyclokinetic equation in Fourier harmonic form (CKinFH)

After straightforwardly expanding $\delta \hat{f}_k(\hat{\mu},\alpha)$ in Eq. (1) with finite (or discrete) Fourier harmonics $\delta \hat{f}_k(\hat{\mu},\alpha) = \sum_{n=-N_\alpha+1}^{n=N_\alpha-1} \delta\hat{F}_k^n(\hat{\mu})\exp(in\alpha)$, we obtain the 4D cyclokinetic equation in Fourier harmonic form given by Eq. (4) where $n = -N_\alpha+1,...0...,N_\alpha-1$ is the harmonic number with respect to α.

$$\begin{gathered} D\delta\hat{F}_k^n / D\hat{t} - i\hat{\omega}_k^d \delta\hat{G}_k^n + i(\hat{\upsilon}_\perp \hat{k}/2)\Omega_*[\exp(-i\beta)\delta\hat{G}_k^{n-1} + \exp(+i\beta)\delta\hat{G}_k^{n+1}] - \Omega_* in\hat{G}_k^n = \\ -i\hat{\omega}_{*k}^{nT}\delta\hat{\phi}_k n_0 F_M(\hat{\mu})\delta_0^n + {}^{NL}S(\hat{k},\hat{\mu},n) \end{gathered}. \tag{4}$$

The non-adiabatic part of the distribution function is $\delta\hat{G}_k^n = \delta\hat{F}_k^n + (T_e/T_i)\delta\hat{\phi}_k n_0 F_M(\hat{\mu})\delta_0^n$ with $\delta_0^n = [0,1]$. The perturbed distribution functions are linearly coupled and satisfy the conjugate property $\delta\hat{F}_{-k}^{-n} = [\delta\hat{F}_k^n]^*$. ${}^{NL}S(\hat{k},\hat{\mu},n)$ represents the nonlinear dynamics.

Since gyro phase angle space is cyclic $\delta \hat{f}_k(\hat{\mu},\alpha+2\pi) = \delta \hat{f}_k(\hat{\mu},\alpha)$ on $0 \le \alpha < 2\pi$, its Fourier form $\delta\hat{F}_k^n$ is also cyclic. For finite harmonic number $n = -N_\alpha+1,...0...,N_\alpha-1$, cyclic condition at $\alpha = [0,2\pi]$ is provided by $\delta\hat{F}_k^{-N_\alpha} = \delta\hat{F}_k^{N_\alpha-1}$, $\delta\hat{F}_k^{N_\alpha} = \delta\hat{F}_k^{-N_\alpha+1}$. The derivative operator also needs a Fourier transform $\partial_\alpha \delta \hat{f}_k(\hat{\mu},\alpha) \Rightarrow in\delta\hat{F}_k^n(\hat{\mu})$ and we must be careful to apply the cyclic boundary condition to the derivative harmonic identity *in*: $i(-N_\alpha)\delta\hat{F}_k^{-N_\alpha} = i(N_\alpha-1)\delta\hat{F}_k^{N_\alpha-1}$ and $i(N_\alpha)\delta\hat{F}_k^{N_\alpha} = i(-N_\alpha+1)\delta\hat{F}_k^{-N_\alpha+1}$.

Eq. (1) is non-conservative form, and we can obtain its conservative form by applying the crucial identity Eq. (5) to Eq. (1). Proved in Appendix A.

$$(T_e/T_i)\partial_{\hat{\mu}}\vec{\hat{k}}_1 \cdot \vec{\hat{\upsilon}}_\perp + \partial_\alpha \vec{\hat{k}}_1 \cdot \hat{b} \times \vec{\hat{\upsilon}}_\perp / \hat{\upsilon}_\perp^2 = 0. \tag{5}$$

After the discrete Fourier harmonic transformation, we get both the conservative and non-conservative form of the nonlinear dynamics. The conservative form is

$$\begin{gathered} {}^{NL}_{Con}S(\hat{k},\hat{\mu},n) = \partial_{\hat{\mu}}\sum_{k1} (T_e/T_i)\delta\hat{\phi}_{k1} i(\hat{\upsilon}_\perp \hat{k}_1/2)[\exp(-i\beta_1)\delta\hat{F}_{k2}^{n-1} + \exp(+i\beta_1)\delta\hat{F}_{k2}^{n+1}] \\ -\sum_{k1} \delta\hat{\phi}_{k1} i\hat{k}_1/(2\hat{\upsilon}_\perp) n[\exp(-i\beta_1)\delta\hat{F}_{k2}^{n-1} - \exp(+i\beta_1)\delta\hat{F}_{k2}^{n+1}] \end{gathered}, \tag{6}$$

and the non-Conservative form is

$$
\begin{aligned}
{}^{NL}_{NCon}S(\hat{k},\hat{\mu},n) = & \sum_{k1} (T_e/T_i)\delta\hat{\phi}_{k1} i(\hat{\upsilon}_\perp \hat{k}_1/2)[\exp(-i\beta_1)\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n-1} + \exp(+i\beta_1)\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n+1}] \\
& -\sum_{k1} \delta\hat{\phi}_{k1} i\hat{k}_1/(2\hat{\upsilon}_\perp)[\exp(-i\beta_1)(n-1)\delta\hat{F}_{k2}^{n-1} - \exp(+i\beta_1)(n+1)\delta\hat{F}_{k2}^{n+1}]
\end{aligned}
\quad . \qquad (7)
$$

Derive the $\partial_{\hat{\mu}}$ into the bracket in the first term of Eq. (6), then the conservative form Eq. (6) and non-conservative form Eq. (7) are found to be equivalent as required. Notice that the nonlinear coupling coefficient is enhanced by $1/\hat{\upsilon}_\perp$ at low velocities.

In the rCYCLO code, the nonlinear term is calculated by $[{}^{NL}_{Con}S(\hat{k},\hat{\mu},n) + {}^{NL}_{NCon}S(\hat{k},\hat{\mu},n)]/2$ in order to cancel any numerical error in the nonlinear conservation of incremental entropy. The calculations of nonlinear dynamics Eq. (6), (7) are parallelized in the α and μ directions. The incremental entropy of CKinFH is defined by

$$
E = \sum_k \oint d\alpha \int_0^\infty d\hat{\mu}(\delta\hat{f}_k^* \delta\hat{f}_k) = 2\pi\int_0^\infty d\hat{\mu}\sum_k \sum_{n=-N_\alpha+1}^{n=N_\alpha-1} (\delta\hat{F}_k^n)^* \delta\hat{F}_k^n \, . \qquad (8)
$$

With the linear terms in Eq. (1) and Eq. (4) deleted, it is easy to show in the gyro-phase form, and with some pains-taking algebra in the discrete Fourier transform form that the total E is conserved. Numerical nonlinear E conservation is a crucial test of any turbulence simulation code (e.g. see Ref. [4] in the case of the GYRO code). Without accurate conservation there may be no nonlinear saturation. The discretization of the derivative operator $\partial_{\hat{\mu}}$ and numerical incremental entropy conservation are discussed in detail in section 3.2.

The Poisson equation for CKinFH is given as

$$
\delta\hat{\phi}_k = \frac{2\pi\int d\hat{\mu}\delta\hat{F}_k^0(\hat{\mu})/n_0 - C_{DW}\cdot\delta\hat{n}_k^e}{(1-C_{DW})R_k + C_{DW}\cdot(\hat{\lambda}_D^2\hat{k}^2)} \, . \qquad (9)
$$

Eq. (9) indicates that only the zero-th Fourier harmonic $\delta\hat{F}_k^0$ contributes to the electrostatic potential for CKinFH. Whether the Debye length $\hat{\lambda}_D$ is a very small number or even zero is not so important for low-*k* ITG gyrokinetics (unless we are looking at high-*k* electromagnetic ETG). However it is very important to cyclokinetics, since Eq. (9) is numerically close to “0/0” when C_{DW}=1. Moreover, there is a non-zero polarization term in the denominator of the gyrokinetic Poisson equation, while there is no such polarization term in the denominator of CKinFK Poisson equation Eq. (9). If the simulation were in real space (x, y), it would likely be impractical to resolve such small scales while on the ion gyro-scale. We would need to extrapolate the physically small $\hat{\lambda}_D^2$ from some larger and numerically practical $\hat{\lambda}_D^2$. However $\hat{\lambda}_D^2$ should not be too large, since an unphysically large $\hat{\lambda}_D^2$ will produce spurious high-*k* ITG modes in both gyrokinetics and cyclokinetics. In the rCYCLO simulations, the linear growth rates and frequencies do not significantly depend on the size of $\hat{\lambda}_D^2$ if it is small enough e.g. 0.01.

The particle and energy fluxes are given by

$$[\hat{\Gamma}, \hat{Q}^{\perp}] = Re\, 2\pi \int_0^{\infty} d\hat{\mu} \sum_k [1, (T_i / T_e)\hat{\mu}] i\hat{k}_y \delta\hat{\phi}_k^* [\delta\hat{G}_k^0(\hat{\mu}) / n_0], \tag{10}$$

where only the non-adiabatic distribution function for the n=0 harmonic appears.

Numerical methods of CKinFH

Coordinate in the code

Spatial-space:

The simulation box is chosen as a symmetric rectangle, and we transform (x, y) to (kx, ky), $-k_y^{\max} \le k_y \le k_y^{\max}$, $-k_x^{\max} \le k_x \le k_x^{\max}$.

Since $\delta F_{-\vec{k}}^{n} = (\delta F_{\vec{k}}^{-n})^*$, $\delta\hat{\phi}_{-\vec{k}} = \delta\hat{\phi}_{\vec{k}}^*$, ${}^{NL}S_{-\vec{k},j}^{n} = ({}^{NL}S_{\vec{k},j}^{-n})^*$, so we only need field solves for upper half k-plane: $0 \le k_y \le k_y^{\max}$, $-k_x^{\max} \le k_x \le k_x^{\max}$ but we need the lower half k-plane c.c. projection to get $-k_y^{\max} \le k_y < 0$

Velocity-Space:

The abscissae and weight $(\hat{\mu}_j, W_j)$ are given by either of the three methods. The first is equal mod weight grid; the second is Gauss-Legendre grid; the third is equal Chi weight grid. There comes the difference: the mod weight of *equal mod weight grid* is equal to each $\hat{\mu}$ box ; the mod weight of *Gauss-Legendre grid* concentrate on $\hat{\mu}$ close to 0 part; the mod weight of *equal Chi grid* is concentrate on the tail of the $\hat{\mu}$ axis. We can choose either of these three type of grid while running rCYCLO code, however, *Equal weight grid* is the one we usually use, and also is the one GYRO use, since there is a same number of particles for each $\hat{\mu}$ grid box. Applying

Gauss-Legendre grid would usually make a large numerical instability for the Full kinetic cases, and blow up δF_k to infinite when $\hat{\mu}$ is small (for example 0.1).

Equal mod weight grid

Velocity space integration is numerically expressed as:

$$\oint d\alpha \int_0^\infty d\hat{\mu}\delta \hat{f}_k(\hat{\mu},\alpha) = 2\pi \sum_{n=-N_\alpha+1}^{n=N_\alpha-1} \int_0^\infty d\hat{\mu}\delta\hat{F}_k^n(\hat{\mu}) = \sum_{n=-N_\alpha+1}^{n=N_\alpha-1}\sum_{j=1}^{j=N_\mu}\delta\hat{F}_k^n(\hat{\mu}_j)W_j \ . \quad (11)$$

We split the interval of integration in $\hat{\mu}$ into two regions: $[0,\hat{\mu}_{max})$ and $[\hat{\mu}_{max},\infty)$. The first region is described by $\hat{\mu}_j$ (j=1, 2,…, N_μ-1) , and each of the μ-grid points represents the center of a “bin” with “bin weight” W_j. The corresponding modified bin weights $\bar{W}_j = W_j F_M(\hat{\mu}_j)$ satisfy $\bar{W}_1 = \bar{W}_2 = ... = \bar{W}_{N_\mu - 1}$, which means the weights W_j are chosen such that for a Maxwellian distribution each μ-grid bin is distributed with the same percentage of the distribution. The second region is represented by the last grid point $\hat{\mu}_{N_\mu}$, and we set $\hat{\mu}_{N_\mu} = \hat{\mu}_{max}$. The weight of last grid point evaluates the infinite integral according to $2\pi\int_{\hat{\mu}_{N_\mu}}^\infty d\hat{\mu}F_M(\hat{\mu}) = \bar{W}_{N_\mu}$, and we let $\bar{W}_{N_\mu} = \bar{W}_j$ (j=1, 2,…, N_μ-1). Thus the weights of all the grid points have the desirable property that $2\pi\int_0^\infty d\hat{\mu}F_M(\hat{\mu}) = \sum_{j=1}^{j=N_\mu} W_j F_M(\hat{\mu}_j) = \sum_{j=1}^{j=N_\mu} \bar{W}_j = 1$. An eight μ-grid example is listed in Table 1.

Table 1

Sample magnetic moment μ-grids and weights

j	$\hat{\mu}_j$	$\bar{W}_j$
1	0.066765696	0.1249072
2	0.2106067	0.1248763
3	0.3788429	0.1248270
4	0.5815754	0.1247410
5	0.8369882	0.1245700
6	1.183562	0.1241478
7	1.732868	0.1225323
8	2.079442	0.1250000

The details of equal mod weights μ grid abscissae and weight set up:

$$\int_0^{\infty} e^{-\mu} d\mu = 1$$

$$\int_0^{\bar{\mu}_1} e^{-\mu} d\mu = 1 - e^{-\bar{\mu}_1} = (1 - e^{-\mu_{\max}})/(J-1)$$

$$\int_{\bar{\mu}_1}^{\bar{\mu}_2} e^{-\mu} d\mu = e^{-\bar{\mu}_1} - e^{-\bar{\mu}_2} = (1 - e^{-\mu_{\max}})/(J-1)$$

$$\int_{\bar{\mu}_2}^{\bar{\mu}_3} e^{-\mu} d\mu = e^{-\bar{\mu}_2} - e^{-\bar{\mu}_3} = (1 - e^{-\mu_{\max}})/(J-1)$$

...

$$\int_{\bar{\mu}_{J-2}}^{\bar{\mu}_{J-1}} e^{-\mu} d\mu = e^{-\bar{\mu}_{J-2}} - e^{-\bar{\mu}_{J-1}} = (1 - e^{-\mu_{\max}})/(J-1)$$

$$(1 - e^{-\mu_{\max}})/(J-1) = e^{-\mu_{\max}} \rightarrow e^{-\mu_{\max}} = 1/J \qquad solve \quad for \quad \mu_{\max} = -\ln(1/J) \Rightarrow 1.79175...@J = 6$$

$$\bar{\mu}_{J-1} = \mu_{\max}$$

$$e^{-\bar{\mu}_{J-2}} - e^{-\bar{\mu}_{J-1}} = (1 - e^{-\mu_{\max}})/(J-1) = (1-1/J)/(J-1) \qquad solve \quad for \quad \bar{\mu}_{J-2}$$

...

$$finally \quad solve \quad \bar{\mu}_1$$

$$Now \quad set \quad the \quad grids:$$

$$\mu_J = \mu_{\max}$$

$$\mu_1 = (\bar{\mu}_1 + 0)/2$$

$$\mu_2 = (\bar{\mu}_2 + \bar{\mu}_1)/2$$

$$\mu_3 = (\bar{\mu}_3 + \bar{\mu}_2)/2$$

...

$$\mu_{J-1} = (\bar{\mu}_{J-1} + \bar{\mu}_{J-2})/2$$

$$Set\ their\ weight:$$

$$W_j^{\text{mod}} = (1-1/J)/(J-1) \quad \& \quad JW_j^{\text{mod}} = 1$$

$$W_j = W_j^{\text{mod}} / F_M(\mu_j)$$

Gauss-Legendre weights mu-grid

1 to N-1 point are decide by *Gaussian quadrature rule* referring to Wikipedia, the weight at last point $\hat{\mu}^*$ is $\mathrm{W}_{j*} = 2\pi$. Abscissas and weights of the Gauss-Legendre n-point quadrature formula in μ-direction

The integral over **μ** is divided in to two parts.

$$2\pi\int_0^{\infty} \delta\hat{F}(\hat{\mu}) d\hat{\mu} = 2\pi \int_0^{\hat{\mu}^*} d\hat{\mu}\delta\hat{F}(\hat{\mu}) + 2\pi\int_{\hat{\mu}^*}^{\infty} d\hat{\mu}\delta\hat{F}(\hat{\mu})$$

Discretization:

$$2\pi\int_0^{\hat{\mu}^*} d\hat{\mu}\delta\hat{F}(\hat{\mu}) = \sum_{j=1}^{N-1} W_j \delta\hat{F}(\hat{\mu}_j) = \sum_{j=1}^{N-1}\left(W_j^{\mathrm{mod}} / F_M^j\right)\delta\hat{F}(\hat{\mu}_j)$$

Equal Chi weights mu-grid set-up

Since $\chi \propto \mu$, and $\int_0^{\infty} \mu e^{-\mu} d\mu = 1$, which will be divided by J parts.

$$\int_0^{\bar{\mu}_1} \mu e^{-\mu} d\mu = -(\bar{\mu}_1 + 1)e^{-\bar{\mu}_1} + 1 = 1/J \quad \Rightarrow -(\bar{\mu}_1 + 1)e^{-\bar{\mu}_1} = 1/J - 1$$
$$\int_{\bar{\mu}_1}^{\bar{\mu}_2} \mu e^{-\mu} d\mu = -(\bar{\mu}_2 + 1)e^{-\bar{\mu}_2} + (\bar{\mu}_1 + 1)e^{-\bar{\mu}_1} = 1/J \quad \Rightarrow -(\bar{\mu}_2 + 1)e^{-\bar{\mu}_2} = 2/J - 1$$
$$\int_{\bar{\mu}_2}^{\bar{\mu}_3} \mu e^{-\mu} d\mu = -(\bar{\mu}_3 + 1)e^{-\bar{\mu}_3} + (\bar{\mu}_2 + 1)e^{-\bar{\mu}_2} = 1/J \quad \Rightarrow -(\bar{\mu}_3 + 1)e^{-\bar{\mu}_3} = 3/J - 1 \quad .$$
$$\ldots$$
$$\int_{\bar{\mu}_{J-2}}^{\bar{\mu}_{J-1}} \mu e^{-\mu} d\mu = -(\bar{\mu}_{J-1} + 1)e^{-\bar{\mu}_{J-1}} + (\bar{\mu}_{J-2} + 1)e^{-\bar{\mu}_{J-2}} = 1/J \quad \Rightarrow -(\bar{\mu}_{J-1} + 1)e^{-\bar{\mu}_{J-1}} = (\mathrm{J}-1)/J - 1$$

Solve $\bar{\mu}_1, ..., \bar{\mu}_{J-1}$ by iterative method.

$$\begin{aligned}
&\textit{Now set the grids}: \\
&\mu_1 = (\bar{\mu}_1 + 0)/2 \\
&\mu_2 = (\bar{\mu}_2 + \bar{\mu}_1)/2 \\
&\mu_3 = (\bar{\mu}_3 + \bar{\mu}_2)/2 \\
&\ldots \\
&\mu_{J-1} = (\bar{\mu}_{J-1} + \bar{\mu}_{J-2})/2 \\
&\mu_J = \mu_{\max} = \bar{\mu}_{J-1}
\end{aligned}$$

$$\begin{aligned}
&\textit{Set their weight}: \\
&W_1 = 2\pi(\bar{\mu}_1 - 0) \\
&W_2 = 2\pi(\bar{\mu}_2 - \bar{\mu}_1) \\
&\ldots \\
&W_{j-1} = 2\pi(\bar{\mu}_{J-1} - \bar{\mu}_{J-2}) \\
&W_j = 2\pi
\end{aligned}$$

For a total of N gird points. We remark that this method has the desirable property:

$$\sum_j W_j^{\mathrm{mod}} = 1, \quad j = 1,2,3....N \ \ \textit{such that} \ \forall_j, W_j^{\mathrm{mod}} > 0$$

using $2\pi\int_0^{\infty} d\hat{\mu}[\ ] = \sum_j W_j[\ ]_j = \sum_j \left(W_j^{\mathrm{mod}} / F_M^j\right)[\ ]_j$ where

$$\oint d\alpha \int_0^\infty d\hat{\mu} F_M = 2\pi \int_0^\infty d\hat{\mu} F_M = \int_0^\infty d\hat{\mu} \exp(-\hat{\mu}) = 1$$

$$= \sum_j \left(W_j^{\mathrm{mod}} / F_M^j \right) [F_M]_j = \sum_j W_j [F_M]_j$$

The μ-derivative operator

The μ-derivative operator $D_{jj'}^{\mu}$ is defined as

$$\partial_{\hat{\mu}} V(\hat{\mu})|_j = \sum_{j'=1}^{j'=N_\mu} D_{jj'}^{\mu} V(\hat{\mu}_{j'}) , \tag{12}$$

where $V(\hat{\mu})$ is an arbitrary function of $\hat{\mu}$. $D_{jj'}^{\mu}$ is chosen to preserve the integration by parts required for incremental entropy conservation. For arbitrary functions $U(\hat{\mu})$ and $V(\hat{\mu})$, the discretized integration by parts is expressed as:

$$\sum_j W_j U(\hat{\mu}_j) \sum_{j'} D_{jj'}^{\mu} V(\hat{\mu}_{j'}) = U(\hat{\mu}) V(\hat{\mu}) |_{\hat{\mu}_1}^{\hat{\mu}_{N_\mu}} - \sum_j W_j V(\hat{\mu}_j) \sum_{j'} D_{jj'}^{\mu} U(\hat{\mu}_{j'}) .$$

The boundary term $U(\hat{\mu}) V(\hat{\mu}) |_{\hat{\mu}_1}^{\hat{\mu}_{N_\mu}}$ appears to be non-zero, since numerically $\hat{\mu}_1$ and $\hat{\mu}_{N_\mu}$ cannot get to 0 and ∞ as required by the μ-continuum equations. In order to get rid of the non-zero boundary term, the μ-derivative operator $D_{jj'}^{\mu}$ must satisfy the remaining part of the above equation: $\sum_j W_j U(\hat{\mu}_j) \sum_{j'} D_{jj'}^{\mu} V(\hat{\mu}_{j'}) = -\sum_j W_j V(\hat{\mu}_j) \sum_{j'} D_{jj'}^{\mu} U(\hat{\mu}_{j'})$ which then can be reduced to:

$$W_j D_{jj'}^{\mu} = -W_{j'} D_{j'j}^{\mu} . \tag{13}$$

Next we define the tridiagonal derivative matrix $H_{jj'}$ from a two point differential $\partial_{\hat{\mu}} \delta \hat{F}_k(\hat{\mu}_j) = [\delta \hat{F}_k(\hat{\mu}_{j+1}) - \delta \hat{F}_k(\hat{\mu}_{j-1})] / (\hat{\mu}_{j+1} - \hat{\mu}_{j-1}) = \sum_{j'=1}^{j'=N_\mu} H_{jj'} \delta \hat{F}_k(\hat{\mu}_{j'})$ as following:

$$H_{j,j'} = 1/(\hat{\mu}_{j+1} - \hat{\mu}_{j-1}) \quad \text{for} \quad j' = j+1,$$

$$H_{j,j'} = -1/(\hat{\mu}_{j+1} - \hat{\mu}_{j-1}) \quad \text{for} \quad j' = j-1, \tag{14}$$

$$H_{j,j'} = 0 \quad \text{for other } j, j'.$$

We finally obtain the μ-derivative operator Eq. (21) by combining Eq. (13) for $D_{jj'}^{\mu}$ with Eq. (14) for $H_{jj'}$.

$$D^{\mu}_{jj'} = [H_{jj'} - (W_{j'} / W_j) H_{j'j}] / 2 \,. \tag{15}$$

It should be clear that if the μ-grid is closely spaced, then $W_j \sim W_{j\pm1}$ and $D^{\mu}_{jj'} \sim H_{jj'} \sim -H_{j'j}$. As might be expected from the compromised "zero" μ-boundary conditions, the μ-derivative matrix operation becomes rather inaccurate near $j \to 1$ and $j \to N_{\mu}$. However, since the μ-derivative operator only appears in the nonlinear terms, as long as the incremental entropy is exactly conserved, a certain level of inaccuracy in the μ-derivative operation is just an error in nonlinear incremental entropy transfer and "scrambling". This may not be so important. The important point is that with increasing N_{μ} (and in particular $\hat{\mu}_1$ getting closer to zero) the μ-grid convergence is easily achieved as shown in Section 4.2.

Linear eigenvalue solver

CKinFH is taken as an example to illustrate the rCYCLO eigenvalue solver. The Poisson equation Eq. (9) indicates that the electric field $\delta\hat{\phi}_k$ is functions of $\delta\hat{F}^n_k(\hat{\mu})$. Then we can obtain the dispersion matrix $M^{nn'}_{jj'}$ and the linear eigenvalue equation Eq. (16) through combining the linear parts of Eq. (4) and electron motion equation with Eq. (9).

$$-i\hat{\omega}\delta\hat{F}^n_k(\hat{\mu}_j) = \sum\nolimits_{n'=-N_\alpha+1}^{n'=N_\alpha-1} \sum\nolimits_{j'=0}^{j'=N_\mu} M^{nn'}_{jj'}(\hat{k})\delta\hat{F}^{n'}_k(\hat{\mu}_{j'}) \,, \tag{16}$$

where $j = 1,2,3...N_{\mu}$ represent the ion, and $j = 0$ represents the electron (for example: $\delta\hat{F}^0_k(\hat{\mu}_0) = \delta\hat{n}^e_k$). The linear frequency and growth rate are just the eigenvalues (solved by the numerical library LAPACK) of the dispersion matrix $M^{nn'}_{jj'}$. It is the same way for gyrokinetics and CKinCH to solve their linear frequency and growth rate as CKinFH.

The CKinFH dispersion matrix $M^{nn'}_{jj'}$ in rCYCLO is given as following. Note that for ion $\delta\hat{F}^n_k(\mu_j)$, $j = 1,2,3...N_{\mu}$; for electron $\delta\hat{F}_k(\mu_0) = \delta\hat{n}^e_k$, $j = 0$

(ion) for $j = 1,2,3...N_{\mu}$

$$D\delta\hat{F}^n_k / D\hat{t} - i\hat{\omega}^d_k\delta\hat{G}^n_k + ik\rho\Omega_* / 2\left[\exp(-i\beta)\delta\hat{G}^{n-1}_k + \exp(i\beta)\delta\hat{G}^{n+1}_k\right]$$
$$-\Omega_* in\delta\hat{G}^n_k + i\hat{\omega}^{nT}_{*k}\delta\hat{\phi}_k(n_0F_M)\delta^n_0 = 0$$

$$\Rightarrow$$

$$-i\omega\delta\hat{F}_k^n - i\hat{\omega}_k^d\delta\hat{F}_k^n + i(\hat{\upsilon}_\perp\hat{k}_\perp/2)\Omega_*\{\exp(-i\beta_k)\delta\hat{F}_k^{n-1} + \exp(+i\beta_k)\delta\hat{F}_k^{n+1}\} - \Omega_* in\hat{F}_k^n$$
$$+(T_e/T_i)[-i\hat{\omega}_k^d\delta_0^n + i(\hat{\upsilon}_\perp\hat{k}_\perp/2)\Omega_*\{\exp(-i\beta_k)\delta_0^{n-1} + \exp(+i\beta_k)\delta_0^{n+1}\} + (T_i/T_e)i\hat{\omega}_{*k}^{nT}\delta_0^n]F_M^j\delta\hat{\phi}_k = 0$$

$$\Rightarrow$$

$$-i\omega\delta\hat{F}_k^n - i\hat{\omega}_k^d\delta\hat{F}_k^n + i(\hat{\upsilon}_\perp\hat{k}_\perp/2)\Omega_*\{\exp(-i\beta_k)\delta\hat{F}_k^{n-1} + \exp(+i\beta_k)\delta\hat{F}_k^{n+1}\} - \Omega_* in\hat{F}_k^n$$
$$+(T_e/T_i)[-i\hat{\omega}_k^d\delta_0^n + i(\hat{\upsilon}_\perp\hat{k}_\perp/2)\Omega_*\{\exp(-i\beta_k)\delta_0^{n-1} + \exp(+i\beta_k)\delta_0^{n+1}\} + (T_i/T_e)i\hat{\omega}_{*k}^{nT}\delta_0^n]F_M^j$$
$$\otimes\frac{\sum_{j'=1}^{N_\mu} W_{j'}\delta\hat{F}_k(\hat{\mu}_{j'})/n_0 - CWD\cdot\delta\hat{F}_k(\hat{\mu}_0)}{(1-CDW)\cdot(\lambda_k - i\delta_k) + CDW\cdot(\hat{\lambda}_D^2\hat{k}_\perp^2)} = 0$$

$(electron)\ for\ j=0$

$$D\delta\hat{n}_k^e/Dt + i\hat{\omega}_d^e\delta\hat{n}_k^e + \alpha_A(\delta\hat{n}_k^e - \delta\hat{\phi}_k) = -i(\hat{\omega}_*^n - \hat{\omega}_d^e)\delta\hat{\phi}_k$$
$$\Rightarrow$$
$$D\delta\hat{F}_k(\hat{\mu}_0)/Dt = -i\hat{\omega}_d^e\delta\hat{F}_k(\hat{\mu}_0) - \alpha_A\delta\hat{F}_k(\hat{\mu}_0) + \left[\alpha_A + i(\hat{\omega}_d^e - \hat{\omega}_*^n)\right]$$
$$\otimes\frac{\sum_{j'=1}^{N_\mu} W_{j'}\delta\hat{F}_k(\hat{\mu}_{j'})/n_0 - CWD\cdot\delta\hat{F}_k(\hat{\mu}_0)}{(1-CDW)\cdot(\lambda_k - i\delta_k) + CDW\cdot(\hat{\lambda}_D^2\hat{k}_\perp^2)}$$

For the linear dispersion matrix:

$$-i\hat{\omega}\delta\hat{F}_k^n(\hat{\mu}_j) = \sum_{n'=-N_\alpha+1}^{n'=N_\alpha-1}\ \sum_{j'=0}^{j'=N_\mu}\ M_{jj'}^{nn'}(\hat{k})\delta\hat{F}_k^{n'}(\hat{\mu}_{j'})$$

$For\ j=1,...,N_\mu$

$$M_{jj'}^{nn'}(\vec{k}) = \delta_{n'}^n[i\hat{\omega}_{k,j}^d + in\Omega_*]\delta_{jj'}$$
$$-i(\hat{\upsilon}_\perp\hat{k}_\perp/2)\Omega_*\exp(+i\beta_k)\delta_{jj'}\delta_{n'-1}^n$$
$$-i(\hat{\upsilon}_\perp\hat{k}_\perp/2)\Omega_*\exp(-i\beta_k)\delta_{jj'}\delta_{n'+1}^n$$
$$-(T_e/T_i)[-i\hat{\omega}_{k,j}^d + i(T_i/T_e)\hat{\omega}_{*k}^{nT}](F_M^j)\delta_0^n\delta_0^{n'}\frac{[(1-\delta_{j'}^0)W_{j'}/n_0 - CWD\cdot\delta_{j'}^0]}{(1-CDW)\cdot(\lambda_k - i\delta_k) + CDW\cdot(\hat{\lambda}_D^2\hat{k}_\perp^2)}$$
$$-(T_e/T_i)[i(\hat{\upsilon}_\perp\hat{k}_\perp/2)\Omega_*\{\exp(-i\beta_k)\delta_0^{n-1} + \exp(+i\beta_k)\delta_0^{n+1}\}F_M^j\delta_0^{n'}\frac{[(1-\delta_{j'}^0)W_{j'}/n_0 - CWD\cdot\delta_{j'}^0]}{(1-CDW)\cdot(\lambda_k - i\delta_k) + CDW\cdot(\hat{\lambda}_D^2\hat{k}_\perp^2)}$$

$For\ j=0$

$$M_{jj'}^{nn'} = -(i\hat{\omega}_d^e + \alpha_A)\delta_j^0\delta_{j'}^0\delta_n^0\delta_{n'}^0$$
$$+\left[\alpha_A + i(\hat{\omega}_d^e - \hat{\omega}_*^n)\right]\delta_j^0\delta_n^0\delta_{n'}^0\frac{[(1-\delta_{j'}^0)W_{j'}/n_0 - CWD\cdot\delta_{j'}^0]}{(1-CDW)\cdot(\lambda_k - i\delta_k) + CDW\cdot(\hat{\lambda}_D^2\hat{k}_\perp^2)}$$

Obviously one needs to add the diagonal damping elements

$$M_{jj'}^{nn'}(\vec{\mathrm{k}}) \Rightarrow M_{jj'}^{nn'}(\vec{\mathrm{k}}) - \delta_{nn'}\delta_{jj'}\left[\mu_{HK}\hat{k}_\perp^2 + \nu_{LK} / \hat{k}_\perp^2\right]$$

Cyclic condition should be applied here:

$$-i(\hat{\upsilon}_\perp \hat{k}_\perp / 2)\Omega_* \exp(+i\beta_k)\delta_{jj'}\delta_{n'-1}^{n} \quad @n = N-1 \Rightarrow N-1 = n'-1 \Rightarrow n' = N$$

does not exist, which means we need to add the cyclic coupling matrix element

$$-i(\hat{\upsilon}_\perp \hat{k}_\perp / 2)\Omega_* \exp(+i\beta_k)\delta_{jj'}[\delta_{N-1}^{n}\delta_{-N+1}^{n'}]$$

likewise $-i(\hat{\upsilon}_\perp \hat{k}_\perp / 2)\Omega_* \exp(-i\beta_k)\delta_{jj'}\delta_{n'+1}^{n} \quad @n = -N+1 \Rightarrow -N+1 = n'+1 \Rightarrow n' = -N$

does not exist, which means we need to add the cyclic coupling matrix element

$$-i(\hat{\upsilon}_\perp \hat{k}_\perp / 2)\Omega_* \exp(+i\beta_k)\delta_{jj'}[\delta_{-N+1}^{n}\delta_{N-1}^{n'}]$$

Time advance method

Again taking CKinFH as an example, the nonlinear time evolution equation corresponding to Eq. (16) is:

$$D\delta\hat{F}_{k,j}^{n} / D\hat{t} = \sum_{n'=-N_\alpha+1}^{n'=N_\alpha-1} \sum_{j'=0}^{j'=N_\mu} M_{jj'}^{nn'}(\hat{k})\delta\hat{F}_{k,j'}^{n'} + {}^{NL}S_{k,j}^{n} . \tag{17}$$

The Ω^* terms result in the large entries in $M_{jj'}^{nn'}$ with a wide range of eigenvalues making it a stiff matrix. The stiff matrix requires an implicit time advance. Thus rCYCLO has a time centered implicit linear advance and an explicit nonlinear advance. The perturbed distribution function of next time step $\delta\bar{\hat{F}}_{k,j}^{n}$ can be obtained by multiplying the invers matrix $[R^{-1}]_{jj'}^{nn'}$ on $\bar{S}_{k,j}^{n}$.

$$\delta\bar{\hat{F}}_{k,j}^{n} = \sum_{n'=-N_\alpha+1}^{n'=N_\alpha-1} \sum_{j'=0}^{j'=N_\mu} [R^{-1}]_{jj'}^{nn'}(\hat{k})\, \bar{S}_{k,j'}^{n'} , \tag{18a}$$

where $\bar{S}_{k,j}^{n}$ and $R_{jj'}^{nn'}$ are given by

$$\bar{S}_{k,j}^{n} = \delta\hat{F}_{k,j}^{n} + (d\hat{t} / 2)\sum_{n'=-N_\alpha+1}^{n'=N_\alpha-1} \sum_{j'=0}^{j'=N_\mu} M_{jj'}^{nn'}(\hat{k})\delta\hat{F}_{k,j'}^{n'} + (d\hat{t})\ {}^{NL}S_{k,j}^{n} \tag{18b}$$

$$R_{jj'}^{nn'}(\hat{k}) = \delta_{n'}^{n}\delta_{j'}^{j} - (d\hat{t} / 2)M_{jj'}^{nn'}(\hat{k}) . \tag{18c}$$

Here $\delta\hat{F}_{k,j}^{n}$ is the perturbed distribution function at the previous time step. ${}^{NL}S_{k,j}^{n}$ is the value of the nonlinear terms on the previous time step. The Poisson equation Eq. (9) is used to update

$\delta\hat{\phi}_k$ at the end of each time step. The calculations of Eq. (18a) and (18b) are parallelized in k_x, k_y direction.

Subroutine nonlinear of CKinFH in rCYCLO

Conservative form:

$$
\begin{aligned}
&{}_{C}^{NL}S_{k,j}^{n} = {}_{C}^{NL}S_{n,k,j}^{\hat{\mu}} + {}_{C}^{NL}S_{n,k,j}^{\alpha} = \\
&+\partial_{\hat{\mu}}\Sigma k1\ (T_e/T_i)\delta\hat{\phi}_{k1} i(\hat{\upsilon}_{\perp}\hat{k}_1/2)\{\exp(-i\beta_1)\delta\hat{F}_{k2}^{n-1} + \exp(+i\beta_1)\delta\hat{F}_{k2}^{n+1}\} \\
&-\Sigma k1\ \delta\hat{\phi}_{k1} i\hat{k}_1/(2\hat{\upsilon}_{\perp}) n\{\exp(-i\beta_1)\delta\hat{F}_{k2}^{n-1} - \exp(+i\beta_1)\delta\hat{F}_{k2}^{n+1}\}
\end{aligned}
$$

Non-Conservative terms

$$
\begin{aligned}
&{}_{NC}^{NL}S_{k,j}^{n} = {}_{NC}^{NL}S_{n,k,j}^{\hat{\mu}} + {}_{NC}^{NL}S_{n,k,j}^{\alpha} = \\
&+\sum_{k1} (T_e/T_i)\delta\hat{\phi}_{k1} i(\hat{\upsilon}_{\perp}\hat{k}_1/2)\{\exp(-i\beta_1)\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n-1} + \exp(+i\beta_1)\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n+1}\} \\
&-\sum_{k1} \delta\hat{\phi}_{k1} i\hat{k}_1/(2\hat{\upsilon}_{\perp})\{\exp(-i\beta_1)(n-1)\delta\hat{F}_{k2}^{n-1} - \exp(+i\beta_1)(n+1)\delta\hat{F}_{k2}^{n+1}\}
\end{aligned}
$$

Discretization of Conservative terms:

Nonlinear source k-space convolution inputs:

$$\delta\hat{\phi}_k^{(\pm)} \equiv [(T_e/T_i)(i\hat{k}_{\perp}/2)\delta\hat{\phi}_k \exp(\pm i\beta_k)]$$

$${}_{C}^{\mu}\delta\hat{F}_{k,j}^{n(\pm)} = \sum j'\,{}^{\mu}M_{jj'}[\hat{\upsilon}_{\perp j'}\delta\hat{F}_{k,j'}^{n\pm1}]$$

$${}_{C}^{\alpha}\delta\hat{F}_{k,j}^{n(\pm)} = (T_i/T_e)(n/\hat{\upsilon}_{\perp j})\delta\hat{F}_{k,j}^{n\pm1}$$

Cyclic condition:

$$
\begin{aligned}
\delta\hat{F}_{-N} &= \delta\hat{F}_{N-1} \\
\delta\hat{F}_{N} &= \delta\hat{F}_{-N+1}
\end{aligned}
$$

Discretization of Non-Conservative terms:

$$\delta\hat{\phi}_k^{(\pm)} \equiv [(T_e/T_i)(i\hat{k}_{\perp}/2)\delta\hat{\phi}_k \exp(\pm i\beta_k)]\ \text{(the same)}$$

$$ {}^{\mu}_{NC}\delta\hat{F}^{n(\pm)}_{k,j} = \hat{\upsilon}_{\perp j}\sum_{j'} {}^{\mu}M_{jj'}\delta\hat{F}^{n\pm1}_{k,j'} $$

$$ {}^{\alpha}_{NC}\delta\hat{F}^{n(\pm)}_{k,j} = (T_i / T_e)(n\pm1)/\hat{\upsilon}_{\perp j}\delta\hat{F}^{n\pm1}_{k,j} $$

Pay attention to the operator ∂_α is cyclic, which means its corresponding frequency $i(\mathrm{n}\pm1)$ is cyclic too.

$$ i(-N)\delta\hat{F}_{-N} = i(N-1)\delta\hat{F}_{N-1} $$
$$ i(N)\delta\hat{F}_{N} = i(-N+1)\delta\hat{F}_{-N+1} $$

Nonlinear source k-space convolution:

$$ {}^{NL}_{C}S^{n}_{k,j} = \sum k1\{\delta\phi_{k1}^{(-)}[{}^{\mu}_{C}\delta\hat{F}^{n(-)}_{k_2,j}] + \delta\hat{\phi}_{k1}^{(+)}[{}^{\mu}_{C}\delta\hat{F}^{n(+)}_{k_2,j}] - \delta\hat{\phi}_{k1}^{(-)}[{}^{\alpha}_{C}\delta\hat{F}^{n(-)}_{k_2,j}] + \delta\hat{\phi}_{k1}^{(+)}[{}^{\alpha}_{C}\delta\hat{F}^{n(+)}_{k_2,j}]\} $$

$$ {}^{NL}_{NC}S^{n}_{k,j} = \sum k1\{\delta\phi_{k1}^{(-)}[{}^{\mu}_{NC}\delta\hat{F}^{n(-)}_{k_2,j}] + \delta\hat{\phi}_{k1}^{(+)}[{}^{\mu}_{NC}\delta\hat{F}^{n(+)}_{k_2,j}] - \delta\hat{\phi}_{k1}^{(-)}[{}^{\alpha}_{NC}\delta\hat{F}^{n(-)}_{k_2,j}] + \delta\hat{\phi}_{k1}^{(+)}[{}^{\alpha}_{NC}\delta\hat{F}^{n(+)}_{k_2,j}]\} $$

$$ @\vec{k}_{\perp2} = \vec{k}_{\perp} - \vec{k}_{\perp1} $$

In order to conserve entropy (Proved in Appendix B), the nonlinear term are chosen as

$$ {}^{NL}S^{n}_{k,j} = \frac{1}{2}\left({}^{NL}_{C}S^{n}_{k,j} + {}^{NL}_{NC}S^{n}_{k,j}\right). $$

In the code:

$$ \exp(+i\beta_k) = \cos\beta_k + i\sin\beta_k = k_x / \sqrt{k_x^2+k_y^2} + ik_y / \sqrt{k_x^2+k_y^2} $$

$$ \exp(-i\beta_k) = \cos\beta_k - i\sin\beta_k = k_x / \sqrt{k_x^2+k_y^2} - ik_y / \sqrt{k_x^2+k_y^2} $$

Cyclic condition:

$$ \delta f_k = \delta f(\alpha_k) $$
$$ \alpha_k = k\Delta \qquad k = 1,2,3....2N-1 $$
$$ n = -N+1,...0...,N-1 $$
$$ \alpha_0 = 0 \qquad \alpha_{2N-1} = 2\pi $$
$$ f_k = f_{k\pm(2N-1)} $$
$$ \delta f_0 = \delta f_{2N-1} $$
$$ \delta f_1 = \delta f_{2N} $$

.....

since $\delta f(\alpha)$ is cyclic, likewise δF_n is cyclic, and can prove

$$ \delta\hat{F}_n = \delta\hat{F}_{n\pm(2N-1)} $$

$$\delta\hat{F}_{-N} = \delta\hat{F}_{N-1}$$
$$\delta\hat{F}_{N} = \delta\hat{F}_{-N+1}$$
.....

$$\partial_\alpha \delta f(\alpha) \Rightarrow in\delta F_n$$

Pay attention that the derivative operator ∂_α and the corresponding frequency in in the non-conservative nonlinear term should also be cyclic!

$$\left[\partial_\alpha \delta f(\alpha)\right]|_0 = \left[\partial_\alpha \delta f(\alpha)\right]|_{2N-1}$$
$$\left[\partial_\alpha \delta f(\alpha)\right]|_1 = \left[\partial_\alpha \delta f(\alpha)\right]|_{2N}$$

$$i(-N)\delta\hat{F}_{-N} = i(N-1)\delta\hat{F}_{N-1}$$
$$i(N)\delta\hat{F}_{N} = i(-N+1)\delta\hat{F}_{-N+1}$$

Also we have Parseval's theorem:

$$\sum_{k=1}^{2N-1} |\delta\hat{f}_k|^2 = \frac{1}{2N-1}\sum_{n=-N+1}^{N-1} |\delta\hat{F}_n|^2$$

Discretization of diagnose variables:

Transport fluxes: in gyro-Bohm units

$$[\hat{\Gamma}_i, \hat{Q}_i^{\perp}] = \mathrm{Re}\oint d\alpha \int_0^\infty d\hat{\mu} \sum_{\vec{k}} [1, (T_i/T_e)\hat{\mu}] i\hat{k}_y \delta\hat{\phi}_k^* [\delta\hat{g}_k(\hat{\mu},\alpha)/n_0]$$
$$[\hat{\Gamma}_i, \hat{Q}_i^{\perp}] = \mathrm{Re}\, 2\pi \int_0^\infty d\hat{\mu} \sum_{\vec{k}} [1, (T_i/T_e)\hat{\mu}] i\hat{k}_y \delta\hat{\phi}_k^* [\delta\hat{G}_k^0(\hat{\mu})/n_0]$$
$$\left[\hat{D}_i, \hat{\chi}_i\right] = \mathrm{Re}\sum_{\vec{k}} \sum_j W_j [1, (T_i/T_e)\hat{\mu}_j] i\hat{k}_y \delta\hat{\phi}_k^* \delta\hat{G}_k^0(\hat{\mu}_j)/n_0 / [a/L_n, a/L_T]$$

Turbulence strength:

Density: $<(\delta n^e(\vec{r})/n_0)^2>_{\vec{r}} = \sum_k \delta\hat{n}_k^{e*}\delta\hat{n}_k^{e}\rho^{*2}$

Potential: $<(e\delta\phi(\vec{r})/T_e)^2>_{\vec{r}} = \sum_k \delta\hat{\phi}_k^*\delta\hat{\phi}_k\rho^{*2}$

Distribution function over mu:

$$\delta\hat{F}(\hat{\mu}) = \left|\oint d\alpha \sum_{\vec{k}} \delta\hat{f}_k(\hat{\mu},\alpha)\right|$$
$$= 2\pi\left|\sum_{\vec{k}} \delta\hat{F}_k^0(\hat{\mu})\right|$$
$$\delta\hat{F}^{Tot}(\hat{\mu}) = 2\pi\left|\sum_{\vec{k}} \sum_{n=-N+1}^{N-1} \delta\hat{F}_k^n(\hat{\mu})\right|$$

Entropy:

$$entropyft = 2\pi \int d\hat{\mu} \sum_{\vec{k}} \sum_{n} \left[\left(\delta \hat{F}_{\vec{k}}^{n}(\hat{\mu}) \right)^{*} \delta \hat{F}_{\vec{k}}^{n}(\hat{\mu}) \right]$$
$$= \sum_{j} W_j \left[\left(\delta \hat{F}_{\vec{k}}^{n}(\hat{\mu}_j) \right)^{*} \delta \hat{F}_{\vec{k}}^{n}(\hat{\mu}_j) \right] / n_0$$

Integration by parts test of mu-derivative operator:

The mu-derivative operator should make sure integration by parts works.

$$\int_0^\infty d\hat{\mu} \delta U(\hat{\mu}) \sqrt{\hat{\mu}} \partial_{\hat{\mu}} \delta V(\hat{\mu}) = \sqrt{\hat{\mu}} \delta U(\hat{\mu}) \delta V(\hat{\mu}) |_0^\infty - \int_0^\infty d\hat{\mu} \delta V(\hat{\mu}) \partial_{\hat{\mu}} [\sqrt{\hat{\mu}} \delta U(\hat{\mu})]$$

The discretized formula gives:

$$\sum_{j'} \sum_{j} W_j \delta U_j \delta V_{j'} [\sqrt{\hat{\mu}_j} M_{jj'}] = -\sum_{j'} \sum_{j} W_j \delta V_j \delta U_{j'} M_{jj'} \sqrt{\hat{\mu}_{j'}}$$
$$= -\sum_{j'} \sum_{j} W_{j'} \delta V_{j'} \delta U_j M_{j'j} \sqrt{\hat{\mu}_j}$$

$$\Rightarrow$$
$$\sum_{j} \sum_{j'} \left[W_j M_{jj'} + W_{j'} M_{j'j} \right] \sqrt{\hat{\mu}_j} \delta U_j \delta V_{j'} = 0$$

This formula is equal to 0 for arbitrary $\sqrt{\hat{\mu}_j} \delta U_j \delta V_{j'}$, which bring the Integration by parts test of mu-derivative operator in the code:

$$\frac{\sum_{j} \sum_{j'} \left| W_j M_{jj'} + W_{j'} M_{j'j} \right|}{\sum_{j} \sum_{j'} \left| W_j M_{jj'} - W_{j'} M_{j'j} \right|} = 0$$

The derivation of CKinFH nonlinear terms and the nonlinear conservation law proof are shown in Appendix B.

Cyclokinetic equation in cyclotron harmonic form (CKinCH)

For the convenience of the readers we will follow Ref. [1] to provide the nonlinear cyclokinetic equations in the cyclotron harmonic form (CKinCH). The cyclotron harmonic transform is defined by

$$\delta\hat{f}_k(\hat{\mu},\alpha)\exp[-ik\rho\sin(\alpha-\beta)]=\sum\nolimits_{n=-N_\alpha+1}^{n=N_\alpha-1}\delta\hat{F}_k^n(\hat{\mu})\exp[in(\alpha-\beta)], \tag{19}$$

where $\rho=\upsilon_\perp/\Omega$ is ion gyro radius. [The Eq. (12) of Ref. [1] used $ik\rho\sin(\alpha-\beta)$, when it should have been $-ik\rho\sin(\alpha-\beta)$]. In order to formulate numerically practical equations, we apply finite expansion in gyro-phase space here instead of infinite expansion in Ref. [1]. $n=-N_\alpha+1,...0...,N_\alpha-1$ is the harmonic number with respect to *α*. Unlike the Fourier harmonics, the cyclotron harmonics are the linearly uncoupled "normal modes" of cyclokinetics. The CKinCH nonlinear simulations are considerably more expensive than the CKinFH, but CKinCH is more easily identified with gyrokinetics analytically.

Eq. (1) is non-conservative form, and we can obtain its conservative form by applying the crucial identity Eq. (20) to its nonlinear term:

$$(T_e/T_i)\partial_{\hat{\mu}}\vec{\hat{k}}_1\cdot\vec{\upsilon}_\perp+\partial_\alpha\vec{\hat{k}}_1\cdot\hat{b}\times\vec{\upsilon}_\perp/\hat{\upsilon}_\perp^2=0\,. \tag{20}$$

Applying the cyclotron harmonic transform Eq. (19), we obtain the cyclokinetic equation in cyclotron harmonic representation (CKinCH). The CKinCH equation with the non-conservative nonlinear expression is

$$\begin{aligned}&D\delta\hat{F}_k^n/D\hat{t}-i\hat{\omega}_k^d\delta\hat{G}_k^n-in\Omega_*\delta\hat{G}_k^n+J_n(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT}n_0F_M(\hat{\mu})\\&=\sum\nolimits_{n'}\sum\nolimits_{k1}\hat{b}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{J_{n-n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)\\&+\sum\nolimits_{n'}\sum\nolimits_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho)-J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(n'\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)\\&+\sum\nolimits_{n'}\sum\nolimits_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho)+J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(2\hat{\mu}\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)\end{aligned}\quad, \tag{21}$$

and with conservative nonlinear expression written compactly as

$$
\begin{aligned}
& D\delta\hat{F}_k^n / D\hat{t} - i\hat{\omega}_k^d \delta\hat{G}_k^n - in\Omega_* \delta\hat{G}_k^n + J_n(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT} n_0 F_M(\hat{\mu}) \\
& = \sum_{n'} \sum_{k1} \hat{b}\cdot\vec{\hat{k}}_1 \times \vec{\hat{k}}_2 \{J_{n-n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2) \\
& + \sum_{n'} \sum_{k1} (\hat{k}_1 / \hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho) - J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(n\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2) \\
& + \sum_{n'} \sum_{k1} \delta\hat{\phi}_{k1}(\mathrm{T}_e/\mathrm{T}_i)\partial_{\hat{\mu}}\left\{[J_{n-n'+1}(k_1\rho) + J_{n-n'-1}(k_1\rho)](\hat{k}_1\hat{\upsilon}_\perp)\delta\hat{F}_{k2}^{n'}/2i\right\}\Delta_n^{n'}(\beta,\beta_1,\beta_2)
\end{aligned} \quad . (22)
$$

where $\delta\hat{F}_k^n(\hat{\mu}) = \delta\hat{G}_k^n(\hat{\mu}) - J_n(k\rho)\delta\hat{\phi}_k(T_e/T_i)n_0 F_M(\hat{\mu})$, and $J_n(k_1\rho)$ is Bessel function. $\Delta_n^{n'}(\beta,\beta_1,\beta_2) \equiv \exp[-in(\beta_1-\beta)+in'(\beta_1-\beta_2)]$ is a phase factor between different wave angles. $\sum_{n'}$ represents $\sum_{n'=-N_\alpha+1}^{n'=N_\alpha-1}$. The perturbed distribution function satisfies the conjugate property $\delta\hat{F}_{-k}^{-n} = (-1)^n[\delta\hat{F}_k^n]^*$. After some careful algebra, the conservative and non-conservative form are found to be equivalent as required. Notice that the nonlinear coupling coefficient in the last two lines of nonlinear term is enhanced by $1/\hat{\upsilon}_\perp$ at low velocities.

In rCYCLO code, the nonlinear term is calculated by adding the conservative form with the equivalent non-conservative form of nonlinear terms and divided by 2. This operation cancels any numerical errors from the μ-derivative operator (described in Ref. [3]) in the nonlinear conservation of incremental entropy. The incremental entropy of CKinCH is defined by

$$
E = \sum_k \oint d\alpha \int_0^\infty d\hat{\mu}(\delta\hat{f}_k^*\delta\hat{f}_k) = 2\pi\int_0^\infty d\hat{\mu}\sum_k\sum_n(\delta\hat{F}_k^n)^*\delta\hat{F}_k^n . \quad (23)
$$

Using the compact form of the conservative nonlinear expression Eq. (21) together with the non-conservative expression Eq. (22), it is straightforward to prove CKinCH nonlinear entropy conservation. Numerical nonlinear incremental entropy conservation is a crucial test of any turbulence simulation code (e.g. see Ref. [4] in the case of the GYRO code). The discretization of the derivative operator $\partial_{\hat{\mu}}$ is designed to ensure the conservation of the incremental entropy which has been discussed in Ref. [3]. All the symbols $\sum_n$ represent $\sum_{n=-N_\alpha+1}^{n=N_\alpha-1}$ in this paper.

The Poisson equation for CKinCH is given as

$$
\delta\hat{\phi}_k = \frac{\sum_n 2\pi\int d\hat{\mu}J_n(k\rho)\delta\hat{F}_k^n(\hat{\mu})/n_0 - \delta\hat{n}_k^e}{(T_e/T_i)[1-\sum_n 2\pi\int d\hat{\mu}F_M(\hat{\mu})J_n^2(k\rho)] + \hat{\lambda}_D^2\hat{k}^2} , \quad (24)
$$

The polarization density formally vanishes when all cyclotron harmonics are retained: $[1-\sum_n 2\pi\int d\hat{\mu}F_M(\hat{\mu})J_n^2(k\rho)] \to 0|_{n\to\pm\infty}$.

The radial ion particle and energy fluxes are

$$[\hat{\Gamma},\hat{Q}^{\perp}] = Re\, 2\pi \int_0^{\infty} d\hat{\mu} \sum_k \sum_n [1,(T_i/T_e)\hat{\mu}] i\hat{k}_y \delta\hat{\phi}_k^* J_n(k\rho)\delta\hat{G}_k^n(\hat{\mu})/n_0 \,. \tag{25}$$

Numerical methods of CKinCH

Linear motion equation:

Note that for ion $\delta\hat{F}_k^n(\mu_j)$, $j = 1,2,3...N_\mu$; for electron $\delta\hat{F}_k(\mu_0) = \delta\hat{n}_k^e$, $j = 0$

(ion) for $j = 1,2,3...N_\mu$

$$D\delta\hat{F}_k^n / D\hat{t} - i\hat{\omega}_k^d \delta\hat{G}_k^n - in\Omega_* \delta\hat{G}_k^n + J_n(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT} n_0 \hat{F}_M = 0$$

$$\Rightarrow$$

$$-i\omega\delta\hat{F}_k^n - i\hat{\omega}_k^d \delta\hat{F}_k^n - in\Omega_* \delta\hat{F}_k^n - i\left[(\hat{\omega}_k^d + n\Omega_*)T_e/T_i - \hat{\omega}_{*k}^{nT}\right] J_n(k\rho) n_0 \hat{F}_M \delta\hat{\phi}_k = 0$$

$$\Rightarrow$$

$$-i\omega\delta\hat{F}_k^n - i\hat{\omega}_k^d \delta\hat{F}_k^n - in\Omega_* \delta\hat{F}_k^n - i\left[(\hat{\omega}_k^d + n\Omega_*)T_e/T_i - \hat{\omega}_{*k}^{nT}\right] J_n(k\rho) n_0 \hat{F}_M$$
$$\otimes \frac{\sum_{n'} 2\pi \int d\hat{\mu} J_{n'}(k_\perp\rho)\delta\hat{F}_k^{n'}(\mu)/n_0 - CWD\cdot\delta\hat{F}_k(\mu_0)}{(T_0^e/T_0^i)\left[1 - \sum_n 2\pi\int d\hat{\mu} F_M J_n^2(\hat{k}_\perp\rho)\right] + (1-CDW)\cdot(\lambda_k - i\delta_k) + CDW\cdot(\hat{\lambda}_D^2 \hat{k}_\perp^2)} = 0$$

$(electron)$ for $j = 0$

$$D\delta\hat{n}_k^e / Dt + i\hat{\omega}_d^e \delta\hat{n}_k^e + \alpha_A(\delta\hat{n}_k^e - \delta\hat{\phi}_k) = -i(\hat{\omega}_*^n - \hat{\omega}_d^e)\delta\hat{\phi}_k$$

$$\Rightarrow$$

$$D\delta\hat{F}_k(\hat{\mu}_0)/Dt = -i\hat{\omega}_d^e \delta\hat{F}_k(\hat{\mu}_0) - \alpha_A \delta\hat{F}_k(\hat{\mu}_0) + \left[\alpha_A + i(\hat{\omega}_d^e - \hat{\omega}_*^n)\right]$$
$$\otimes \frac{\sum_{n'} 2\pi \int d\hat{\mu} J_{n'}(k_\perp\rho)\delta\hat{F}_k^{n'}(\mu)/n_0 - CWD\cdot\delta\hat{F}_k(\mu_0)}{(T_0^e/T_0^i)\left[1 - \sum_n 2\pi\int d\hat{\mu} F_M J_n^2(\hat{k}_\perp\rho)\right] + (1-CDW)\cdot(\lambda_k - i\delta_k) + CDW\cdot(\hat{\lambda}_D^2 \hat{k}_\perp^2)}$$

For linear dispersion matrix:

$$-i\omega\delta\hat{F}_{\vec{k},j}^{n} = \sum_{n'}\sum_{j'} M_{jj'}^{nn'}(\vec{k})\delta\hat{F}_{\vec{k},j'}^{n'}$$

For $j = 1,...,N_\mu$

$$M_{jj'}^{nn'}(\vec{k}) = \delta_{n'}^{n}[i\hat{\omega}_{k,j}^{d} + in\Omega_*]\delta_{jj'}$$
$$+ i(T_e / T_i)[\hat{\omega}_{k,j}^{d} + n\Omega_* - (T_i / T_e)\hat{\omega}_{*k}^{nT}](F_M^{j})J_n^{j}(k\rho)n_0$$
$$\otimes \frac{\sum_{n'} 2\pi\int d\hat{\mu}J_{n'}(k_\perp\rho)\delta\hat{F}_k^{n'}(\mu)/n_0 - CWD\cdot\delta\hat{F}_k(\mu_0)\delta_{n'}^0\delta_{j'}^0}{(T_0^e/T_0^i)\left[1-\sum_n 2\pi\int d\hat{\mu}F_M J_n^2(\hat{k}_\perp\rho)\right] + (1-CDW)\cdot(\lambda_k - i\delta_k) + CDW\cdot(\hat{\lambda}_D^2\hat{k}_\perp^2)}$$

$$For\ j = 0$$
$$M_{jj'}^{nn'} = -(i\hat{\omega}_d^e + \alpha_A)\delta_j^0\delta_{j'}^0\delta_n^0\delta_{n'}^0$$
$$+\left[\alpha_A + i(\hat{\omega}_d^e - \hat{\omega}_*^n)\right]\delta_j^0\delta_n^0$$
$$\otimes \frac{\sum_{n'} 2\pi\int d\hat{\mu}J_{n'}(k_\perp\rho)\delta\hat{F}_k^{n'}(\mu)/n_0 - CWD\cdot\delta\hat{F}_k(\mu_0)\delta_{n'}^0\delta_{j'}^0}{(T_0^e/T_0^i)\left[1-\sum_n 2\pi\int d\hat{\mu}F_M J_n^2(\hat{k}_\perp\rho)\right] + (1-CDW)\cdot(\lambda_k - i\delta_k) + CDW\cdot(\hat{\lambda}_D^2\hat{k}_\perp^2)}$$

Obviously one needs to add the diagonal damping elements

$$M_{jj'}^{nn'}(\vec{k}) \Rightarrow M_{jj'}^{nn'}(\vec{k}) - \delta_{n'}^{n}\delta_{jj'}[\mu_{HK}k_\perp^4 + \mu_{LK}/k_\perp^2]$$

Subroutine Motion

Since there exist stiff Ω^* term, we apply the implicit time advance method to avoid numerical errors.

$$\partial_t\delta\hat{F}_{k,j}^{n} = \sum_{n'}\sum_{j'}M_{jj'}^{nn'}(\vec{k})\delta\hat{F}_{k,j'}^{n'} + {}^{NL}S_{k,j}^{n}$$
$$\Rightarrow$$
$$\delta\bar{\hat{F}}_{k,j}^{n} - dt/2\sum_{n'}\sum_{j'}M_{jj'}^{nn'}(\vec{k})\delta\bar{\hat{F}}_{k,j'}^{n'} = \delta\hat{F}_{k,j}^{n} + dt/2\sum_{n'}\sum_{j'}M_{jj'}^{nn'}(\vec{k})\delta\hat{F}_{k,j'}^{n'} + dt\,{}^{NL}S_{k,j}^{n}$$
$$\sum_{n'}\sum_{j'}R_{jj'}^{nn'}(\vec{k})\delta\bar{\hat{F}}_{k,j'}^{n'} = S_{k,j}^{n}$$
$$S_{k,j}^{n} = \delta\hat{F}_{k,j}^{n} + dt/2\sum_{n'}\sum_{j'}M_{jj'}^{nn'}(\vec{k})\delta\hat{F}_{k,j'}^{n'} + dt\,{}^{NL}S_{k,j}^{n}$$
$$R_{jj'}^{nn'}(\vec{k}) = \delta_{n'}^{n}\delta_{j'}^{j} - (dt/2)M_{jj'}^{nn'}(\vec{k})$$

$$\Rightarrow$$
$$\delta\bar{\hat{F}}_{k,j}^{n} = \sum_{n'}\sum_{j'}[R^{-1}]_{jj'}^{nn'}(\vec{k})S_{k,j'}^{n'}$$

Subroutine diagnose

Transport fluxes: in gyro-Bohm units

$$[\hat{\Gamma}_i, \hat{Q}_i^{\perp}] = \mathrm{Re} \oint d\alpha \int_0^{\infty} d\hat{\mu} \sum_{\vec{k}} \; [1, (T_i / T_e)\hat{\mu}] i\hat{k}_y \delta\hat{\phi}_k^* [\delta\hat{g}_k(\hat{\mu}, \alpha) / n_0]$$

$$[\hat{\Gamma}_i, \hat{Q}_i^{\perp}] = \mathrm{Re}\; 2\pi \int_0^{\infty} d\hat{\mu} \sum_{\vec{k}} \; \sum_n \; [1, (T_i / T_e)\hat{\mu}] i\hat{k}_y \delta\hat{\phi}_k^* J_n(k\rho) \delta\hat{G}_k^n(\hat{\mu}) / n_0$$

$$\left[\hat{D}_i, \hat{\chi}_i\right] = \mathrm{Re}\; 2\pi \int_0^{\infty} d\hat{\mu} \sum_{\vec{k}} \; \sum_n \; [1, (T_i / T_e)\hat{\mu}] i\hat{k}_y \delta\hat{\phi}_k^* J_n(k\rho) \delta\hat{G}_k^n(\hat{\mu}) / n_0 / [a / L_n, a / L_T]$$

Turbulence strength:

Density: $< (\delta n^e(\vec{r}) / n_0)^2 >_{\vec{r}} = \sum_k \; \delta\hat{n}_k^{e*} \delta\hat{n}_k^{e} \rho^{*2}$

Potential: $< (e\delta\phi(\vec{r}) / T_e)^2 >_{\vec{r}} = \sum_k \; \delta\hat{\phi}_k^* \delta\hat{\phi}_k \, \rho^{*2}$

The derivation of CKinCH nonlinear terms and the nonlinear conservation law proof are attached in Appendix B.

Gyrokinetics

The CKinCH equation entirely recovers the gyrokinetic equation when truncated at the 0-th cyclotron harmonic. The 0-th cyclotron harmonic describes the low frequency drift motion while other harmonics represent the high frequency ion cyclotron motions. Keeping only the 0-th harmonic (by setting $n = n' = 0$) in Eq. (21) [or the equivalent Eq. (22)], we obtain the gyrokinetic equation:

$$D\delta\hat{F}_k / D\hat{t} - i\hat{\omega}_k^d \delta\hat{G}_k = -i\hat{\omega}_*^{nT} J_0(k\rho)\delta\hat{\phi}_k n_0 F_M(\hat{\mu}) + \sum_{k1} \hat{b}\cdot(\vec{\hat{k}}_1 \times \vec{\hat{k}}_2) J_0(k_1\rho)\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}, \tag{26}$$

where $\delta\hat{F}_k(\hat{\mu}) = \delta\hat{G}_k(\hat{\mu}) - (T_e / T_i)\delta\hat{\phi}_k n_0 F_M(\hat{\mu}) J_0(k\rho)$.

The Poisson equation of gyrokinetics is given by

$$\delta\hat{\phi}_k = \frac{2\pi\int d\hat{\mu} J_0(k\rho)\delta\hat{F}_k(\hat{\mu}) / n_0 - \delta\hat{n}_k^e}{(T_e / T_i)2\pi\int d\hat{\mu} F_M(\hat{\mu})[1 - J_0^2(k\rho)] + \hat{\lambda}_D^2 \hat{k}^2}, \tag{27}$$

where $2\pi\int d\hat{\mu} F_M(\hat{\mu})[1 - J_0^2(k\rho)]$ represents the ion polarization. Gyrokinetics nonlinearly conserves the incremental entropy defined as $E = 2\pi\sum_k \int_0^\infty d\hat{\mu}(\delta\hat{F}_k^* \delta\hat{F}_k)$.

The radical ion particle and energy fluxes are

$$[\hat{\Gamma}, \hat{Q}^\perp] = Re\, 2\pi\int_0^\infty d\hat{\mu} \sum_k\; [1, (T_i / T_e)\hat{\mu}] i\hat{k}_y \delta\hat{\phi}_k^* [J_0(k\rho)\delta\hat{G}_k(\hat{\mu}) / n_0]. \tag{28}$$

$$\begin{aligned}\hat{\lambda}_D^2 &= (\lambda_D^2 / a^2) / \rho_*^2 = [T_0^e / 4\pi n_0 e^2] / [\sqrt{T_0^e / m_i}) / (eB / cm_i)]^2 \\ &= [(7.43\times 10^2)^2 / (10^{13} n_{19})] / [(1.02\times 10^2\sqrt{2})^2 / (10^4 B_T)^2 \\ &= 2.65\times 10^{-4} / (n_{19} B_T^2)\end{aligned}$$

at $B_T = 2.1$ *Tesla* and $n_{19} = 0.1$, $\hat{\lambda}_D^2 = 0.6\times 10^{-3}$.

Obviously $\hat{\lambda}_D^2$ is a very smaller number of no importance of GK (unless we are looking at high-k electromagnetic ETG), but for FK (and keeping all cyclotron harmonics in CK where $(T_0^e / T_0^i)\sum_n 2\pi\int d\hat{\mu} F_M([1 - J_n^2(\hat{k}_\perp\rho)] \to 0$), it is very important. We have always know that if we do FK (or CK) we can only do the field solve by keeping some non-zero value of relative DeBye length, $\hat{\lambda}_D^2 \neq 0$. If the simulation code were in real space [x,y], it would likely be impractical to resolve such small scales while on the ion gyro-scale, so we would need to extrapolate to the smaller $\hat{\lambda}_D^2 \neq 0$ from some larger and numerically practical $\hat{\lambda}_D^2 \neq 0$ …We expect to show the result does not actually depend on size of $\hat{\lambda}_D^2$ if it is *small enough*. (This is analogous to the $\lambda_0 = 0.1 \rightarrow 0.01$ extrapolation).

Numerical methods of GK

Physical variables used in normalization:

$$\rho_s = c_s / \Omega \quad \rho^* = \frac{\rho_s}{a} \quad \rho_i = v_i^{th} / \Omega = \sqrt{2T_i / T_e}\rho_s \quad \Omega = eB / m_i c \quad c_s = \sqrt{T_e / m_i}$$

$$n_0 = 1 \quad \hat{t} = t\frac{c_s}{a} \qquad \hat{k} = k\rho_s \qquad \hat{\mu} = (\frac{v_\perp}{v_{th}})^2 = \mu\left(\frac{2B}{v_{th}^2}\right) \qquad \hat{\rho} = \sqrt{2\frac{T_i}{T_e}\hat{\mu}}$$

In Gyro-Bohm unite:

$$F_k = f_k \frac{T_i}{m_i \rho^*} \qquad \delta\hat{\phi}_k = \frac{e\delta\phi_k}{T_e \rho^*} \qquad F_M(\hat{\mu}) = \frac{1}{2\pi}e^{-\hat{\mu}} \qquad \delta\hat{n}_k^e = \frac{\delta n_k^e}{n_0 \rho^*}$$

Linear motion equation:

Assume in mu-grid for ion $\delta\hat{F}_k(\mu_j)$, $j = 1,2,3...N_\mu$; for electron $\delta\hat{F}_k(\mu_0) = \delta\hat{n}_k^e$, $j = 0$

(ion) for $j = 1,2,3...N_\mu$

$$D\delta\hat{F}_k / D\hat{t} = i\hat{\omega}_k^d \delta\hat{G}_k - i\hat{\omega}_*^{nT} J_0(k_\perp \rho)\delta\hat{\phi}_k n_0 F_M$$

$$D\delta\hat{F}_k / D\hat{t} = i\hat{\omega}_k^d \delta\hat{F}_k + i(\hat{\omega}_k^d T_e / T_i - \hat{\omega}_*^{nT})J_0(k_\perp \rho)n_0 F_M$$
$$\otimes \frac{\sum_{j'=1}^{N_\mu} W_{j'} J_0^{j'} \delta\hat{F}_k(\hat{\mu}_{j'}) / n_0 - CWD \cdot \delta\hat{F}(\hat{\mu}_0)}{(T_0^e / T_0^i)\sum_{j'=1}^{N_\mu} W_{j'} F_M [1 - J_0^2] + (1 - CDW)\cdot(\lambda_k - i\delta_k) + CDW \cdot (\hat{\lambda}_D^2 \hat{k}_\perp^2)}$$

$(electron)$ for $j = 0$

$$D\delta\hat{n}_k^e / Dt + i\hat{\omega}_d^e \delta\hat{n}_k^e + \alpha_A(\delta\hat{n}_k^e - \delta\hat{\phi}_k) = -i(\hat{\omega}_*^n - \hat{\omega}_d^e)\delta\hat{\phi}_k$$

$$D\delta\hat{F}_k(\hat{\mu}_0) / Dt = -i\hat{\omega}_d^e \delta\hat{F}_k(\hat{\mu}_0) - \alpha_A \delta\hat{F}_k(\hat{\mu}_0) + \left[\alpha_A + i(\hat{\omega}_d^e - \hat{\omega}_*^n)\right]$$
$$\otimes \frac{\sum_{j'=1}^{N_\mu} W_{j'} J_0^{j'} \delta\hat{F}_k(\hat{\mu}_{j'}) / n_0 - CWD \cdot \delta\hat{F}_k(\hat{\mu}_0)}{(T_0^e / T_0^i)\sum_{j'=1}^{N_\mu} W_{j'} F_M [1 - J_0^2] + (1 - CDW)\cdot(\lambda_k - i\delta_k) + CDW \cdot (\hat{\lambda}_D^2 \hat{k}_\perp^2)}$$

For the GK linear dispersion matrix:

$$D\delta\hat{F}_{k,j}/D_{\hat{t}} = \sum_{j'=0}^{N_{\hat{\mu}}} M_{jj'}(\vec{k})\delta\hat{F}_{k,j'}$$

$$\begin{aligned}
&For\ \ j=1,...,N_{\mu} \\
&M_{jj'} = i\hat{\omega}_k^d\delta_{jj'} \\
&\quad + i(\hat{\omega}_{k,j}^d T_e/T_i - \hat{\omega}_*^{nT}) J_0^j n_0 F_M^j \\
&\quad\quad \otimes \frac{(1-\delta_{j'}^0)W_{j'}J_0^{j'}/n_0 - CWD\cdot\delta_{j'}^0}{(T_0^e/T_0^i)\sum_{j'=1}^{N_\mu} W_{j'}F_M[1-J_0^2] + (1-CDW)\cdot(\lambda_k - i\delta_k) + CDW\cdot(\hat{\lambda}_D^2\hat{k}_\perp^2)} \\
&For\ \ j=0 \\
&M_{jj'} = -(i\hat{\omega}_d^e + \alpha_A)\delta_j^0\delta_{j'}^0 \\
&\quad + \left[\alpha_A + i(\hat{\omega}_d^e - \hat{\omega}_*^n)\right]\delta_j^0 \frac{(1-\delta_{j'}^0)W_{j'}J_0^{j'}/n_0 - CWD\cdot\delta_{j'}^0}{(T_0^e/T_0^i)\sum_{j'=1}^{N_\mu} W_{j'}F_M[1-J_0^2] + (1-CDW)\cdot(\lambda_k - i\delta_k) + CDW\cdot(\hat{\lambda}_D^2\hat{k}_\perp^2)}
\end{aligned}$$

The diagonal damping elements need to be added.

$$M_{jj'}(\vec{\mathrm{k}}) \Rightarrow M_{jj'}(\vec{\mathrm{k}}) - \delta_{jj'}\left[\mu_{HK}\hat{k}_\perp^2 + \nu_{LK}/\hat{k}_\perp^2\right]$$

Subroutine Motion

Since there exist stiff Ω^* term, we apply the implicit time advance method to avoid numerical errors.

$$\begin{aligned}
&\partial_t\delta\hat{F}_{k,j} = \sum_{j'} M_{jj'}(\vec{k})\delta\hat{F}_{k,j'} + {}^{NL}S_{k,j} \\
&\Rightarrow \\
&\delta\bar{\hat{F}}_{k,j} - dt/2\sum_{j'} M_{jj'}(\vec{k})\delta\bar{\hat{F}}_{k,j'} = \delta\hat{F}_{k,j} + dt/2\sum_{j'} M_{jj'}(\vec{k})\delta\hat{F}_{k,j'} + dt\,{}^{NL}S_{k,j} \\
&\sum_{j'} R_{jj'}(\vec{k})\delta\bar{\hat{F}}_{k,j'} = S_{k,j} \\
&S_{k,j} = \delta\hat{F}_{k,j} + dt/2\sum_{j'} M_{jj'}(\vec{k})\delta\hat{F}_{k,j'} + dt\,{}^{NL}S_{k,j} \\
&R_{jj'}(\vec{k}) = \delta_{j'}^{j} - (dt/2)M_{jj'}(\vec{k}) \\
&\Rightarrow \\
&\delta\bar{\hat{F}}_{k,j} = \sum_{j'} [R^{-1}]_{jj'}(\vec{k})S_{k,j'}
\end{aligned}$$

3D gyrokinetic equation with GAM

For enriching the physics, Ky=0 geodesic acoustic modes (GAMs) $\delta\hat{f}_{kx}^0$ would be added to gyrokinetic equations, since $\hat{\upsilon}_{Ex} = -\delta\hat{\phi}_k\hat{k}_y\times\hat{z}$ GAMs have no contribution to transport.:

$$D\delta\hat{F}_{k} / D\hat{t} - i\hat{\omega}_k^d \delta\hat{G}_{k} = -i\hat{\omega}_*^{nT} F_M J_0(\hat{k}\hat{\rho})\delta\hat{\phi}_k$$

$$-\sum_{k1} \hat{b}\cdot(\hat{\vec{k}}_1 \times \hat{\vec{k}}_2) J_0(\hat{k}_{\perp 1}\hat{\rho})\delta\hat{\phi}_{k1}(\delta\hat{F}_{k2} + \delta^0_{ky_2}\delta\hat{f}^0_{\vec{k}_{x2}})$$

The Poisson equation becomes:

$$2\pi\int d\hat{\mu}[J_0(\hat{k}_\perp\hat{\rho})(\delta\hat{F}_{k} + \delta^0_{k_y}\delta\hat{f}^0_{k_x}) - \frac{T_{e0}}{T_{i0}} F_M (1 - J_0^2(\hat{k}_\perp\hat{\rho}))\delta\hat{\phi}_k] = \delta\hat{\phi}_k(\lambda_k - i\delta_k)$$

The motion equation of GAM is given by:

$$\partial_{\hat{t}}\delta\hat{f}^0_{k_x}(\hat{\mu}) = -i\hat{\omega}_{GAM}\delta\hat{f}^0_{k_x}(\hat{\mu}) - \hat{\gamma}_{GAM}\delta\hat{f}^0_{k_x}(\hat{\mu}) - \delta^0_{ky}\sum_{k1}\hat{b}\cdot(\hat{\vec{k}}_1\times\hat{\vec{k}}_2)J_0(\hat{\vec{k}}_{\perp 1}\hat{\rho})\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}(\hat{\mu})$$

Where the frequency of GAM is given by $\omega_{GAM} = \sqrt{7/2 + T_e/T_i}(v^i_{th}/R)$; Referring to: H. Sugama, T.H. Watanabe(2006) [HT(2006)],F. Hinton, M. Rosenbluth (1999)[HR(1999)] . Using the parameter we use in this paper $T_e/T_i = 1$ and $\varepsilon = a/R = 1/3$, then we get $\hat{\omega}_{GAM} = 1.0$ Actually, in the simulation, γ_{GAM} are given a certain value directly.

3D Drift-Kinetic equations

Drift-Kinetic equations could be obtain by adding parameter G into the Bessel function of Gyrokinetic equations $J_0(G\hat{k}_\perp\hat{\rho})$ (Bessel function could be simplified: $\lim_{G\to 0} J_0(G\hat{k}_\perp\hat{\rho}) = 1$), adjusting G close to 0.

$$D\delta\hat{F}_{k} / D\hat{t} - i\hat{\omega}_k^d\delta\hat{G}_{k} = -i\hat{\omega}_*^{nT} F_M \delta\hat{\phi}_k$$

$$-\sum_{k1}\hat{b}\cdot(\hat{\vec{k}}_1\times\hat{\vec{k}}_2)\delta\hat{\phi}_{k1}(\delta\hat{F}_{k2} + \delta^0_{ky_2}\delta\hat{f}^0_{k_{x2}})$$

We keep the polarization term $\hat{k}^2\delta\hat{\phi}_{\hat{\vec{k}}}$ in 3D gyrokinetic equation. By dividing G2 to $[1 - J_0^2(\hat{k}_\perp\hat{\rho})]$ term of Poisson equation. Then the Poisson equation becomes:

$$2\pi\int d\hat{\mu}(\delta\hat{F}_{k} + \delta^0_{k_y}\delta\hat{f}^0_{k_x}) - \hat{k}^2\delta\hat{\phi}_k = \delta\hat{\phi}_k(\lambda_k - i\delta_k)$$

The motion equation of GAM is given by:

$$\partial_{\hat{t}}\delta\hat{f}^0_{k_x}(\hat{\mu}) = -i\hat{\omega}_{GAM}\delta\hat{f}^0_{k_x}(\hat{\mu}) - \hat{\gamma}_{GAM}\delta\hat{f}^0_{k_x}(\hat{\mu}) - \delta^0_{ky}\sum_{k1}\hat{b}\cdot(\hat{\vec{k}}_1\times\hat{\vec{k}}_2)\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}(\hat{\mu})$$

Transport coefficient

$$[\hat{\Gamma}_i, \hat{Q}_i^{\perp}] = \mathrm{Re}\oint d\alpha \int_0^{\infty} d\hat{\mu} \sum_{\vec{k}} \ [1, (T_i / T_e)\hat{\mu}] i\hat{k}_y \delta\hat{\phi}_k^* [\delta \hat{f}_k(\hat{\mu}) / n_0]$$

$$[\hat{\Gamma}_i, \hat{Q}_i^{\perp}] = \mathrm{Re}\, 2\pi \int_0^{\infty} d\hat{\mu} \sum_{\vec{k}} \ [1, (T_i / T_e)\hat{\mu}] i\hat{k}_y \delta\hat{\phi}_k^* [J_0 \delta\hat{F}_k(\hat{\mu}) / n_0]$$

$$\left[\hat{D}_i, \hat{\chi}_i\right] = \mathrm{Re} \sum_{\vec{k}} \ \sum_j W_j [1, (T_i / T_e)\hat{\mu}_j] i\hat{k}_y \delta\hat{\phi}_k^* J_0 \delta\hat{F}_k(\hat{\mu}_j) / n_0 / [a / L_n, a / L_T]$$

Another formula of D is given here, and these two formula should give the same result.

$$D_{i\delta}^{phy} = \mathrm{Re} \sum_{\vec{k}} \delta_k \hat{k}_y \delta\hat{\phi}_k^* \delta\hat{\phi}_k \ [L_n c_s \rho_*^2]$$

Note： $D_{i\delta}$ *equal to* D_i analiticly, see when you utilize $2\pi\int d\hat{\mu} J_0(\hat{k}_\perp \hat{\rho}) \delta\hat{F}_k$ from Poisson equation to $D_i = \mathrm{Re} \sum_k 2\pi \int d\hat{\mu} \ i\hat{k}_y J_0(\hat{k}_\perp \hat{\rho}) \delta\hat{\phi}_k^* \delta\hat{F}_k$, then keep the real part which equals to $D_{i\delta}$.

Turbulence strength:

Density: $< (\delta n^e(\vec{r}) / n_0)^2 >_{\vec{r}} = \sum_k \ \delta\hat{n}_k^{e*} \delta\hat{n}_k^{e} \rho^{*2}$

Potential: $< (e\delta\phi(\vec{r}) / T_e)^2 >_{\vec{r}} = \sum_k \ \delta\hat{\phi}_k^* \delta\hat{\phi}_k \, \rho^{*2}$

Conservation Laws: (The prove see the Appendix C)

Energy is defined as:

$$\hat{E}(\hat{\mu}) = \frac{2\pi}{2} \sum_k \left[\delta\hat{F}_k J_0 \delta\hat{\phi}_k^* + \delta\hat{F}_k^* J_0 \delta\hat{\phi}_k \right]$$

Entropy is defined as:

$$S(\mu) = \sum_{\vec{k}} \delta\hat{F}_{\vec{k}} \delta\hat{F}_{\vec{k}}^* / F_M$$

Applying the Nonlinear conservation Verify conjugation in Appendix A, we can easily obtain two conservation laws:

1. Conservation law of total entropy

$$\frac{\partial}{\partial \hat{t}} \int d\hat{\mu} \hat{S}(\hat{\mu}) = 0$$

2. conservation law of total energy：

$$\frac{\partial}{\partial \hat{t}} \int d\hat{\mu} \hat{E}(\hat{\mu}) = 0$$

The derivation of GK nonlinear terms and the nonlinear conservation law proof are shown in Appendix C.

Electron descriptions

Near adiabatic electron:

$$\frac{\delta n_e}{n_0} = \frac{e\delta\hat{\phi}_k}{T_e}(\lambda_{\hat{k}} - \delta_k)$$

Where $\lambda_{\vec{k}} = [1,0]$ *for* $[k_y \neq 0, k_y = 0]$. $\delta_{\vec{k}} = \delta_1 k_y / [1 + \eta k_x^2]$ where δ_1 and η **(**as well as μ and (a/L_n) drive) are adjusted to give in linear instability rate peaked on the (a/L_n) -axis.

Collisional Drift Wave (CDW) electrons

$$D\delta\hat{n}_k^e / Dt + i\hat{\omega}_d^e \delta\hat{n}_k^e + \alpha_A(\delta\hat{n}_k^e - \delta\hat{\phi}_k) = -i(\hat{\omega}_*^n - \hat{\omega}_d^e)\delta\hat{\phi}_k + \sum \hat{b}\cdot(\hat{k}_1 \times \hat{k}_2)\delta\hat{\phi}_{k1}\delta\hat{n}_{k2}^e$$

$\hat{\omega}_*^n \Rightarrow \hat{k}_y(a/L_n)$ is the density gradient driving.

$\hat{\omega}_d^e \Rightarrow \hat{k}_y(a/R)$ is the curvature driving to electron.

$\alpha_A = \hat{k}_{\|}^2 / [\hat{\nu}_{ei}(m_e/m_i)]$ is the CDW adiabatic parameter, which comes from Ohm's law in parallel direction. $\hat{k}_{\|} = a/Rq$ is the parallel wave number. Keep $(a/R) = 0.33333$ and default $q = 2$ and $1/(m_e/m_i) = (\mu_{mass})^2$ with $\mu_{mass} = 60$ for deuterium. With these parameters and $\hat{\nu}_{ei} \sim 0.1$ typical of the core, $\alpha_A \sim 1000$ making the $\alpha_A(\hat{n}_k^e - \delta\hat{\phi}_k)$ term extremely stiff. If we go to the DIIID L-mode at r/a=0.9 in APS2012 talk $\alpha_A \sim [1/(2.922*3.12)]^2(60)^2/1.5 \sim 90$ which is still a very large…since rCYCLO GK and FK (as well as CK) are full implicit linearly so there should be absolutely no problem.

rCYCLO code instruction

PURPOSE:

We develop rCYCLO code in order to explore the missing transport near L-mode edge problems [6].

HOW TO START:

Before started, you should make sure there is an available lapack library.

1. Compile rCYCLO code, remember to connect the lapack lib. e.g.

For NERSC: ftn rCYCLO.f90

If install the lapack by yourself: mpif90 2014.5.14_rCYCLO.o -L../lib -llapack -ltmglib –lblas. lib is the directory where you put the '*.a' compiled lapack library source files.

2. Submit the job, the number of processors should follow the rules in the annotation of GK_FK_CK variable below.

INTRODUCE:

Gyrokinetic simulations of L-mode near edge tokamak plasmas with the GYRO code underpredict both the transport and the turbulence levels by 5 to 10 fold [6], which suggest either some important mechanism is missing from current gyrokinetic codes like GYRO or the gyrokinetic approximation itself is breaking down. It is known that GYRO drift-kinetic simulations with gyro-averaging suppressed recover most of the missing transport[2]. With these motivations, we developed a flux tube nonlinear cyclokinetic[1] code rCYCLO with the parallel motion and variation suppressed. rCYCLO dynamically follows the high frequency ion gyro-phase motion (with no averaging) which is nonlinearly coupled into the low frequency drift-waves thereby interrupting and possibly suppressing the gyro-averaging. By comparison with the corresponding gyrokinetic simulations, we can test the conditions for the breakdown of gyrokinetics. rCYCLO nonlinearly couples grad-B driven ion temperature gradient (ITG) modes and collisional fluid electron drift modes to ion cyclotron (IC) modes.

IMPORTANT CONTROL PARAMETERS

rCYCLO code includes four independent parts, controlled by the parameter GK_FK_CK=0, 1, 2, and -2. Since the parallelization of rCYCLO is entirely relied on the dimensions, the number of the job processors is already determined when the grid variables are fixed.

GK_FK_CK=0 means to choose gyrokinetics. The number of processors should be: N_mu+1.

GK_FK_CK=1 means to choose cyclokinetics in Fourier harmonic representation (CKinFH). The number of processors should be: (N_mu+1)*(2*N_FT-1).

GK_FK_CK=2 means to choose cyclokinetics in cyclotron harmonic representation (CKinCH). The number of processors should be: (N_mu+1)*(2*N_CY-1).

GK_FK_CK=-2 is designed for deep parallelized CKinCH, since CKinCH is extremely expensive. The number of processors should be: (2*pmax-1)*(N_mu+1)*(2*N_CY-1).

The input parameters should be set in rCYCLO input file: inputRCYCLO.txt .

The control variables are introduced below, other parameters are explained in input file or Ref [3].

restart rCYCLO code has the ability to restart. The code will back up the basic information at each backup point. After the case running finished or stopped, we can restart rCYCLO either from the last backup point or rollback to the point before last backup point.

restart=0 for run a case from time=0;

restart=1 for restart from the last backup point;

restart=2 for restart and rollback to the point before last backup point.

rCYCLO offers two electron descriptions which is controlled by CDW parameter:

CDW=0 for i*delta electron;

CDW=1 for collisional drift wave (CDW) electron.

muDtype is used for choosing mu-derivative operator.

muDtype=0 is recommended since it is the one been proved to conserve the incremental entropy [3].

mugridtype is used for choosing the mu grid and the corresponding weight. Mugridtype=1 (equal mod weight grid) is recommended since it is most efficient. 0 for Gauss-Legendre grid. 2 for equal chi grid...

output_step. Output interval of diagnosing.

backup_num. The number of backups.

Const_NL. The coefficient in front of NL term, normally Const_NL=1.

Const_L. The coefficient in front of Linear term, normally Const_L=1.

verify_L. verify_L=0 is defult value, for a high performence running; verify_L=1 for linear verification and time diagnose of each subroutine; verify_L=2 for time initial growth rate convergence test, after each time step all the variables will be divided by a certain number.

verify_NL. verify_NL=0 is defult value, for a high performence running and only output the basic information for plotting total Chi vs time, total_D vs time, and growth rate (and frequency) vs ky. verify_NL=1 nonlinear verification and outputing more variables; verify_NL=2 for give the output files Phi_kOMG.txt and GOMG.txt used for plotting freq spectrum.

The rCYCLO offers a visualized program in order to plot and analyze the data, named vuRCYCLO. The standard example cases of GK, CKinFH and CKinCH are given in the folder of vuRCYCLO.

rCYCLO excution process diagram

```
call load(myid)

do nt=1,ntmax
    call nonlinear(myid)
    call motion(myid)
enddo

call diagnose(myid)
```

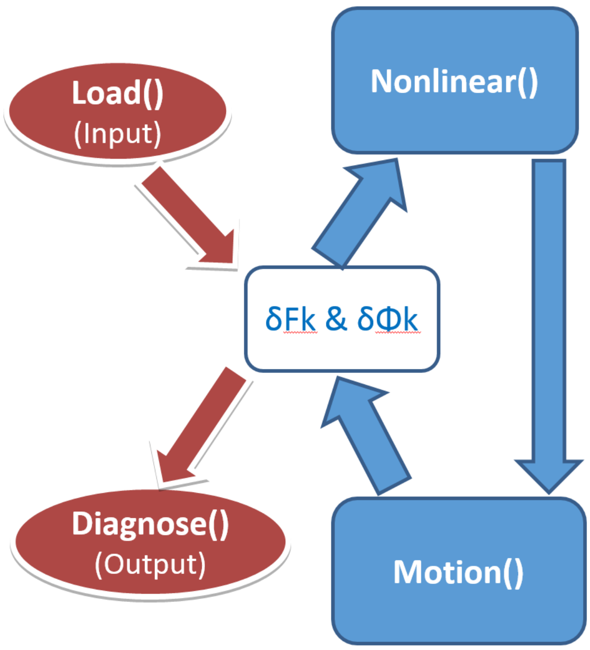

Fig. 1. Left is the main loop (fortran 90) of rCYCLO. Right is the excution process diagram of the rCYCLO code.

rCYCLO veriable dictionary:

In rCYCLO	In formulas	Descriptions
a_Ln	a/L_n	Density gradient
a_LTi	a/L_{Ti}	Ion temperature gradient
AlphaA	α_A	Collisional drift wave adiabaticity
CDW	C_{DW}	0 for i*delta electron; 1 for CDW electron
Chi_kcy	$\hat{\chi}^{CKinCH}$	CKinCH energy diffusivity
Chi_kft	$\hat{\chi}^{CKinFH}$	CKinFH energy diffusivity
Chi_kx_ky	$\hat{\chi}^{GK}$	GK energy diffusivity
Chi_mu	$\hat{\chi}^{GK}(\hat{\mu})$	GK energy diffusivity (function of mu) for output
Chi_mucy	$\hat{\chi}^{CKinCH}(\hat{\mu})$	CKinCH energy diffusivity (function of mu) for output
Chi_muft	$\hat{\chi}^{CKinFH}(\hat{\mu})$	CKinFH energy diffusivity (function of mu) for output
D_kcy	$\hat{D}^{CKinCH}$	CKinCH particle diffusivity
D_kx_ky	$\hat{D}^{GK}$	GK particle diffusivity
D_kft	$\hat{D}^{CKinFH}$	CKinFH particle diffusivity
delta_1	δ_1	Coefficient multiplied on δ_k
delta_k	δ_k	Electron non-adiabatic response for i*delta electron model
Delta_nnp	$\Delta_n^{n'}(\beta,\beta_1,\beta_2)$	Phase factor of CKinCH nonlinear terms for GK_FK_CK=2

Delta_nnpdp	$\Delta_n^{n'}(\beta,\beta_1,\beta_2)$	Phase factor of CKinCH nonlinear terms for GK_FK_CK==-2
Epsilon	a/R	Aspect ratio
expibm	$\exp(-\mathrm{i}\,\beta)$	Phase factor of mode k
Expibp	$\exp(+\mathrm{i}\,\beta)$	Phase factor of mode k
F_k	$\delta\hat{F}_k$	Ion distribution function of GK
F_k_int	$\delta\hat{F}(\mathrm{t}=0)$	Initial value of ion distribution function
F_kcy	$\delta\hat{F}_k^n$	Ion distribution function of CKinCH
F_kfta	$\delta\hat{F}_k^n$	Ion distribution function of CKinFH
FM	F_M	Maxwell ion distribution function
G	G	Drift-k Ion distribution function of CKinFH
G_kcy	$\delta\hat{G}_k^n$	Non adiabatic part of ion distribution function of CKinCH
G_m_delta	$(T_e/T_i)2\pi\int d\hat{\mu}F_M(\hat{\mu})$ $[1-J_0^2(k\rho)]+\hat{\lambda}_D^2\hat{k}^2$	The denominator of GK Poisson equation
GamFun	$2\pi\int d\hat{\mu}F_M(\hat{\mu})J_n^2(k\rho)$	Gamma function.
gamIC	γ_{IC}	The ion cyclotron (IC) growth rate.
gamIC_k	$(-1.778\hat{k}_y^2+$ $2.667\lvert\hat{k}_y\rvert)\gamma_{IC}$	The ion cyclotron (IC) growth rate modified function
Jn	J_n	Bessel function
k1x	$\hat{k}_{1x}$	The *x* direction component of wave vector
k1y	$\hat{k}_{1y}$	The *y* direction component of wave vector
krho	$k\rho$	
kxmax	$\hat{k}_{x\max}$	The maximum value of *kx*

kymax	$\hat{k}_{y\max}$	The maximum value of *ky*
lamb_m_delta	$(1-CDW)\cdot(\lambda_k - i\delta_k)$ $+CDW\cdot(\hat{\lambda}_D^2\hat{k}_\perp^2)$	Part of the denominator of Poisson equation
lambda_0	λ_0	Zonal flow inertia
lambda_D	$\hat{\lambda}_D$	Debye length
lambda_k	λ_k	Wave inertia
lambda_n	λ_n	Drift wave inertia
M_Matrix	$M_{jj'}$	Dispersion matrix of GK
M_Matrixft	$M_{jj'}^{nn'}$	Dispersion matrix of CKinFH
MDmu	$D_{jj'}^{\mu}$	μ-derivative operator
mu_HK	μ_{HK} μ_{HK}	Damping of high-k modes
mu_LK	μ_{LK} μ_{LK}	Damping of low-k modes
mu_Point	$\hat{\mu}_j$	μ-grid points
N_CY	N_α	Gyro phase harmonic number for CKinCH
N_FT	N_α	Gyro phase harmonic number for CKinFH
N_mu	N_μ	The number of μ-grid point
NLScy	${}^{NL}S_{k,j}$	The nonlinear term of CKinCH
NLSfta	${}^{NL}S_{k,j}$	The nonlinear term of CKinFH
nmax	$n_{\max}$	The number of *ky*-grid
nonlinear_term	${}^{NL}S_{k,j}$	The nonlinear term of GK
ntmax	$N_{t\max}$	The number of time steps.
Omega_d	$\hat{\omega}_k^d$	Grad-B drift frequency

Omega_matr	$\hat{\omega}$	Linear rate (frequency and growth rate) of GK
Omega_matrcy	$\hat{\omega}$	Linear rate (frequency and growth rate) of CKinCH
Omega_matrft	$\hat{\omega}$	Linear rate (frequency and growth rate) of CKinFH
Omega_nT	$\hat{\omega}_*^{nT}$	Density and temperature gradient drive frequency
Omega_star	Ω^*	Relative ion cyclotron frequency
phi_k	$\delta\hat{\phi}_k$	Perturbed electron potential for GK
phi_kcy	$\delta\hat{\phi}_k$	Perturbed electron potential for CKinCH
phi_kft	$\delta\hat{\phi}_k$	Perturbed electron potential for CKinFH
pmax	$p_{\max}$	The number of *kx*-grid
ratio_TiTe	$\mathrm{T}_i / \mathrm{T}_e$	Ratio of ion temperature versus electron termperature.
rho	ρ	
signk	$\mathrm{Sign}(\vec{\mathrm{k}})$	Sign of wave vector k
total_Chi	$\int d\hat{\mu}\sum_{\vec{k}}\hat{\chi}^{GK}(\vec{k},\hat{\mu})$	The total energy diffusivity of GK for output
total_D_3G	$\int d\hat{\mu}\sum_{\vec{k}}\hat{D}^{GK}(\vec{k},\hat{\mu})$	The total particle diffusivity of GK for output
tstep	Δt	Time step
w_point	W_j	μ-grid weight.

Appendix A

Cyclokinetics in Fourier harmonic form (CKinFH)

Some useful formulas:

$$\vec{\hat{\upsilon}}_\perp \cdot \vec{\hat{k}}_\perp = \hat{\upsilon}_\perp \hat{k}_\perp \cos(\alpha-\beta) = (\hat{\upsilon}_\perp \hat{k}_\perp / 2)\{\exp[i(\alpha-\beta)] + \exp[-i(\alpha-\beta)]\}$$

$$\vec{\hat{\upsilon}}_\perp \cdot \vec{\hat{k}}_{\perp 1} = \hat{\upsilon}_\perp \hat{k}_{\perp 1} \cos(\alpha-\beta_1) = (\hat{\upsilon}_\perp \hat{k}_{\perp 1} / 2)\{\exp[i(\alpha-\beta_1)] + \exp[-i(\alpha-\beta_1)]\}$$

$$\vec{\hat{k}}_1 \cdot \hat{b} \times \vec{\hat{\upsilon}}_\perp / \hat{\upsilon}_\perp^2 = -(\hat{k}_{1\perp} / \hat{\upsilon}_\perp)\sin(\alpha-\beta_1) = i(\hat{k}_{1\perp} / 2\hat{\upsilon}_\perp)\{\exp[i(\alpha-\beta_1)] - \exp[-i(\alpha-\beta_1)]\}$$

Prove the identity:

$$(T_e / T_i)\partial_{\hat{\mu}} \vec{\hat{k}}_1 \cdot \vec{\hat{\upsilon}}_\perp + \partial_\alpha \vec{\hat{k}}_1 \cdot \hat{b} \times \vec{\hat{\upsilon}}_\perp / \hat{\upsilon}_\perp^2 = 0$$

Since $\hat{\upsilon}_\perp = \hat{\rho} = \sqrt{2(T_i / T_e)\hat{\mu}}$

$$(T_e / T_i)\partial_{\hat{\mu}} \vec{\hat{k}}_1 \cdot \vec{\hat{\upsilon}}_\perp = (T_e / T_i)\hat{k}_1 \cos(\alpha-\beta_1)\partial_{\hat{\mu}}\hat{\upsilon}_\perp = \hat{k}_1 \cos(\alpha-\beta_1) / \hat{\upsilon}_\perp$$

$$\partial_\alpha \vec{\hat{k}}_1 \cdot \hat{b} \times \vec{\hat{\upsilon}}_\perp / \hat{\upsilon}_\perp^2 = -\hat{k}_1 \cos(\alpha-\beta_1) / \hat{\upsilon}_\perp$$

The derivation of cyclokinetics in Fourier harmonic (CKinFH) form:

Now expand $\delta \hat{f}_k(\hat{\mu},\alpha)$ with Fourier harmonics in Eq.(5) and sum over Eq. (9b) of Ref. [1] by

$$D\delta\hat{F}^n_k / D\hat{t} - i\hat{\omega}^d_k \delta\hat{G}^n_k$$

$$+\frac{ik\rho\Omega_* / 2}{2N-1}\sum_{k=1}^{2N-1} \left[\exp[i(\alpha-\beta)] + \exp[-i(\alpha-\beta)]\right]\left\{\sum_{n'=-N+1}^{N-1} \delta\hat{G}^{n'}_k(\hat{\mu})\exp[in'(\alpha_k)]\right\}\exp[-in\alpha_k]$$

$$-\frac{\Omega_*}{2N-1}\sum_{k=1}^{2N-1} \partial_\alpha \left\{\sum_{n'=-N+1}^{N-1} \delta\hat{G}^{n'}_k(\hat{\mu})\exp[in'(\alpha_k)]\right\}\exp[-in\alpha_k] + i\hat{\omega}^{nT}_{*k}\delta\hat{\phi}_k(n_0F_M)\delta^n_0$$

$$= \sum_{k=1}^{2N-1} \partial_\alpha NL \otimes \exp[-in\alpha_k] + \sum_{k=1}^{2N-1} \partial_{\hat{\mu}} NL \otimes \exp[-in\alpha_k]$$

$$\frac{1}{2N-1}\sum_{k=1}^{2N-1} eq.(9)\exp[-in\alpha_k]$$

$\Rightarrow$

$$D\delta\hat{F}^n_k / D\hat{t} - i\hat{\omega}^d_k \delta\hat{G}^n_k + ik\rho\Omega_* / 2\left[\exp(-i\beta)\delta\hat{G}^{n-1}_k + \exp(i\beta)\delta\hat{G}^{n+1}_k\right]$$

$$-\Omega_* in\delta\hat{G}^n_k + i\hat{\omega}^{nT}_{*k}\delta\hat{\phi}_k(n_0F_M)\delta^n_0$$

$$=\sum_{k=1}^{2N-1}\partial_\alpha NL\otimes\exp[-in\alpha_k]+\sum_{k=1}^{2N-1}\partial_{\hat{\mu}}NL\otimes\exp[-in\alpha_k]$$

Nonlinear terms derivation:

non-conservative form:

$$\frac{1}{2N-1}\sum_{k=1}^{2N-1}\partial_\alpha NL\otimes\exp[-in\alpha_k]$$

$$=\sum_{k1}\frac{1}{2N-1}\sum_{k=1}^{2N-1}\delta\hat{\phi}_{k1}i\hat{k}_1\cdot\hat{b}\times\hat{\upsilon}_\perp/\hat{\upsilon}^2_\perp\partial_\alpha\delta f_{k2}\otimes\exp[-in\alpha_k]$$

$$=\sum_{k1}\frac{1}{2N-1}\sum_{k=1}^{2N-1}\hat{k}_1\delta\hat{\phi}_{k1}i^2/(2\hat{\upsilon}_\perp)\left[\exp(i\alpha_k)\exp(-i\beta_1)-\exp(-i\alpha_k)\exp(i\beta_1)\right]\otimes$$

$$\partial_\alpha\left\{\sum_{n'=-N+1}^{N-1}\delta\hat{F}^{n'}_{k2}(\hat{\mu})\exp[in'(\alpha_k)]\right\}\exp[-in\alpha_k]$$

$$=\sum_{k1}\frac{1}{2N-1}\sum_{k=1}^{2N-1}\hat{k}_1\delta\hat{\phi}_{k1}i^2/(2\hat{\upsilon}_\perp)\left[\exp(i\alpha_k)(in')\exp(-i\beta_1)-\exp(-i\alpha_k)(in')\exp(i\beta_1)\right]\otimes$$

$$\left\{\sum_{n'=-N+1}^{N-1}\delta\hat{F}^{n'}_{k2}(\hat{\mu})\exp[in'(\alpha_k)]\right\}\exp[-in\alpha_k]$$

$$\sin ce: i\alpha+in'\alpha-in\alpha=0\Rightarrow n'=n-1$$

$$-i\alpha+in'\alpha-in\alpha=0\Rightarrow n'=n+1$$

$\Rightarrow$

$$=\sum_{k1}\hat{k}_1\delta\hat{\phi}_{k1}(-i)/(2\hat{\upsilon}_\perp)\left[(n-1)\exp(-i\beta_1)\delta\hat{F}^{n-1}_{k2}-(n+1)\exp(i\beta_1)\delta\hat{F}^{n+1}_{k2}\right]$$

$$\frac{1}{2N-1}\sum_{k=1}^{2N-1}\partial_{\hat{\mu}}NL\otimes\exp[-in\alpha_k]$$

$$=\sum_{k1}\frac{1}{2N-1}\sum_{k=1}^{2N-1}(T_e/T_i)\delta\hat{\phi}_{k1}i\hat{k}_1\cdot\hat{\upsilon}_\perp\partial_{\hat{\mu}}\delta\hat{f}_{k2}\otimes\exp[-in\alpha_k]$$

$$=\sum_{k1}\frac{1}{2N-1}\sum_{k=1}^{2N-1}(T_e/T_i)\delta\hat{\phi}_{k1}i\hat{k}_1\hat{\upsilon}_\perp/2\left[\exp(i\alpha_k)\exp(-i\beta_1)+\exp(-i\alpha_k)\exp(i\beta_1)\right]\otimes$$

$$\partial_{\hat{\mu}}\left\{\sum_{n'=-N+1}^{N-1}\delta\hat{F}^{n'}_{k2}(\hat{\mu})\exp[in'(\alpha_k)]\right\}\exp[-in\alpha_k]$$

$$=\sum_{k1}\frac{1}{2N-1}\sum_{k=1}^{2N-1}(T_e/T_i)\delta\hat{\phi}_{k1}i\hat{k}_1\hat{\upsilon}_\perp/2\left[\exp(i\alpha_k)\exp(-i\beta_1)+\exp(-i\alpha_k)\exp(i\beta_1)\right]\otimes$$

$$\left\{\sum_{n'=-N+1}^{N-1}\partial_{\hat{\mu}}\delta\hat{F}^{n'}_{k2}(\hat{\mu})\exp[in'(\alpha_k)]\right\}\exp[-in\alpha_k]$$

$$\sin ce: i\alpha + in'\alpha - in\alpha = 0 \Rightarrow n' = n-1$$

$$-i\alpha + in'\alpha - in\alpha = 0 \Rightarrow n' = n+1$$

$$\Rightarrow$$

$$= \sum_{k1} (T_e / T_i)\delta\hat{\phi}_{k1} i\hat{k}_1 \hat{\upsilon}_\perp / 2\left[\exp(-i\beta_1)\partial_{\hat{\mu}}\delta\hat{F}^{n-1}_{k2} + \exp(i\beta_1)\partial_{\hat{\mu}}\delta\hat{F}^{n+1}_{k2}\right]$$

We obtained the equation (non-conservative form):

$$D\delta\hat{F}^n_k / D\hat{t} - i\hat{\omega}^d_k \delta\hat{G}^n_k + ik\rho\Omega_* / 2\left[\exp(-i\beta)\delta\hat{G}^{n-1}_k + \exp(i\beta)\delta\hat{G}^{n+1}_k\right] - \Omega_* in\delta\hat{G}^n_k$$

$$+ i\hat{\omega}^{nT}_{*k}\delta\hat{\phi}_k (n_0 F_M)\delta^n_0$$

$$= \sum_{k1} \hat{k}_1 \delta\hat{\phi}_{k1}(-i)/(2\hat{\upsilon}_\perp)\left[(n-1)\exp(-i\beta_1)\delta\hat{F}^{n-1}_{k2} - (n+1)\exp(i\beta_1)\delta\hat{F}^{n+1}_{k2}\right]$$

$$+ \sum_{k1} (T_e / T_i)\delta\hat{\phi}_{k1} i\hat{k}_1 \hat{\upsilon}_\perp / 2\left[\exp(-i\beta_1)\partial_{\hat{\mu}}\delta\hat{F}^{n-1}_{k2} + \exp(i\beta_1)\partial_{\hat{\mu}}\delta\hat{F}^{n+1}_{k2}\right]$$

Conservative form:

$$\frac{1}{2N-1}\sum_{k=1}^{2N-1} \partial_\alpha NL \otimes \exp[-in\alpha_k]$$

$$= \sum_{k1}\frac{1}{2N-1}\sum_{k=1}^{2N-1} \partial_\alpha\left[\delta\hat{\phi}_{k1} i\hat{k}_1 \cdot \hat{b} \times \hat{\upsilon}_\perp / \hat{\upsilon}^2_\perp \delta\hat{f}_{k2}\right] \otimes \exp[-in\alpha_k]$$

$$= \sum_{k1}\frac{1}{2N-1}\sum_{k=1}^{2N-1} \partial_\alpha\{\hat{k}_1\delta\hat{\phi}_{k1} i^2 / (2\hat{\upsilon}_\perp)\left[\exp(i\alpha_k)\exp(-i\beta_1) - \exp(-i\alpha_k)\exp(i\beta_1)\right] \otimes$$

$$\sum_{n'=-N+1}^{N-1} \delta\hat{F}^{n'}_{k2}(\hat{\mu})\exp[in'(\alpha_k)]\}\exp[-in\alpha_k]$$

$$= \sum_{k1}\frac{1}{2N-1}\sum_{k=1}^{2N-1} \hat{k}_1\delta\hat{\phi}_{k1} i^2 / (2\hat{\upsilon}_\perp)\left[\exp(i\alpha_k)(in')\exp(-i\beta_1) - \exp(-i\alpha_k)(in')\exp(i\beta_1)\right] \otimes$$

$$\left\{\sum_{n'=-N+1}^{N-1} \delta\hat{F}^{n'}_{k2}(\hat{\mu})\exp[in'(\alpha_k)]\right\}\exp[-in\alpha_k]$$

$$+ \sum_{k1}\frac{1}{2N-1}\sum_{k=1}^{2N-1} \hat{k}_1\delta\hat{\phi}_{k1} i^2 / (2\hat{\upsilon}_\perp)\left[i\exp(i\alpha_k)\exp(-i\beta_1) + i\exp(-i\alpha_k)\exp(i\beta_1)\right] \otimes$$

$$\left\{\sum_{n'=-N+1}^{N-1} \delta\hat{F}^{n'}_{k2}(\hat{\mu})\exp[in'(\alpha_k)]\right\}\exp[-in\alpha_k]$$

$$=\sum_{k1}\frac{1}{2N-1}\sum_{k=1}^{2N-1}\hat{k}_1\delta\hat{\phi}_{k1}i^2/(2\hat{\upsilon}_\perp)\left[\exp(i\alpha_k)i(n'+1)\exp(-i\beta_1)-\exp(-i\alpha_k)i(n'-1)\exp(i\beta_1)\right]\otimes$$
$$\left\{\sum_{n'=-N+1}^{N-1}\delta\hat{F}_{k2}^{n'}(\hat{\mu})\exp[in'(\alpha_k)]\right\}\exp[-in\alpha_k]$$

$$\sin ce: i\alpha+in'\alpha-in\alpha=0\Rightarrow n'=n-1$$
$$-i\alpha+in'\alpha-in\alpha=0\Rightarrow n'=n+1$$
$$\Rightarrow$$
$$=\sum_{k1}\hat{k}_1\delta\hat{\phi}_{k1}(-i)/(2\hat{\upsilon}_\perp)\left[n\exp(-i\beta_1)\delta\hat{F}_{k2}^{n-1}-n\exp(i\beta_1)\delta\hat{F}_{k2}^{n+1}\right]$$

$$\frac{1}{2N-1}\sum_{k=1}^{2N-1}\partial_{\hat{\mu}}NL\otimes\exp[-in\alpha_k]$$

$$=\sum_{k1}\frac{1}{2N-1}\sum_{k=1}^{2N-1}\partial_{\hat{\mu}}\left[(T_e/T_i)\delta\hat{\phi}_{k1}i\hat{k}_1\cdot\hat{\upsilon}_\perp\delta\hat{f}_{k2}\right]\otimes\exp[-in\alpha_k]$$

$$=\sum_{k1}\frac{1}{2N-1}\sum_{k=1}^{2N-1}(T_e/T_i)\delta\hat{\phi}_{k1}i\hat{k}_1\hat{\upsilon}_\perp/2\left[\exp(i\alpha_k)\exp(-i\beta_1)+\exp(-i\alpha_k)\exp(i\beta_1)\right]\otimes$$
$$\left\{\sum_{n'=-N+1}^{N-1}\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'}(\hat{\mu})\exp[in'(\alpha_k)]\right\}\exp[-in\alpha_k]$$

$$+\sum_{k1}\frac{1}{2N-1}\sum_{k=1}^{2N-1}\delta\hat{\phi}_{k1}i\hat{k}_1/(2\hat{\upsilon}_\perp)\left[\exp(i\alpha_k)\exp(-i\beta_1)+\exp(-i\alpha_k)\exp(i\beta_1)\right]\otimes$$
$$\left\{\sum_{n'=-N+1}^{N-1}\delta\hat{F}_{k2}^{n'}(\hat{\mu})\exp[in'(\alpha_k)]\right\}\exp[-in\alpha_k]$$

$$\sin ce: i\alpha+in'\alpha-in\alpha=0\Rightarrow n'=n-1$$
$$-i\alpha+in'\alpha-in\alpha=0\Rightarrow n'=n+1$$
$$\Rightarrow$$
$$=\sum_{k1}(T_e/T_i)\delta\hat{\phi}_{k1}i\hat{k}_1\hat{\upsilon}_\perp/2\left[\exp(-i\beta_1)\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n-1}+\exp(i\beta_1)\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n+1}\right]$$
$$+\sum_{k1}\delta\hat{\phi}_{k1}i\hat{k}_1/(2\hat{\upsilon}_\perp)\left[\exp(-i\beta_1)\delta\hat{F}_{k2}^{n-1}+\exp(i\beta_1)\delta\hat{F}_{k2}^{n+1}\right]$$

$$=\sum_{k1}(T_e/T_i)\partial_{\hat{\mu}}\left\{\delta\hat{\phi}_{k1}i\hat{k}_1\hat{\upsilon}_\perp/2\left[\exp(-i\beta_1)\delta\hat{F}_{k2}^{n-1}+\exp(i\beta_1)\delta\hat{F}_{k2}^{n+1}\right]\right\}$$

Again, we obtained the equation (conservative form):

$$D\delta\hat{F}^{n}_{k} / D\hat{t} - i\hat{\omega}^{d}_{k}\delta\hat{G}^{n}_{k} + ik\rho\Omega_{*} / 2\left[\exp(-i\beta)\delta\hat{G}^{n-1}_{k} + \exp(i\beta)\delta\hat{G}^{n+1}_{k}\right] - \Omega_{*}in\delta\hat{G}^{n}_{k}$$
$$+ i\hat{\omega}^{nT}_{*k}\delta\hat{\phi}_{k}(n_0F_M)\delta^{n}_{0}$$
$$= \sum\nolimits_{k1} \hat{k}_1\delta\hat{\phi}_{k1}(-i) / (2\hat{\upsilon}_{\perp})\left[n\exp(-i\beta_1)\delta\hat{F}^{n-1}_{k2} - n\exp(i\beta_1)\delta\hat{F}^{n+1}_{k2}\right]$$
$$+ \sum\nolimits_{k1}(T_e / T_i)\partial_{\hat{\mu}}\left\{\delta\hat{\phi}_{k1}i\hat{k}_1\hat{\upsilon}_{\perp} / 2\left[\exp(-i\beta_1)\delta\hat{F}^{n-1}_{k2} + \exp(i\beta_1)\delta\hat{F}^{n+1}_{k2}\right]\right\}$$

Prove nonlinear conservation law:

$$\sum\nolimits_{\vec{k}}\oint d\alpha\int_0^{\infty} d\hat{\mu}(\delta\hat{f}^{*}_{k}\delta\hat{f}_{k})$$
$$\Rightarrow$$
$$\int_0^{\infty} d\hat{\mu}\sum\nolimits_{\vec{k}}\frac{2\pi}{2N-1}\sum_{k=1}^{2N-1}\ \{\sum\nolimits_{n=0}^{N-1}\ (\delta\hat{F}^{n}_{\vec{k}})^{*}\exp(-i2\pi nk)\}\{\sum\nolimits_{\bar{n}=0}^{N-1}\ \delta\hat{F}^{\bar{n}}_{\vec{k}}\exp(+i2\pi\bar{n}k)\}$$
$$= 2\pi\int_0^{\infty} d\hat{\mu}\sum\nolimits_{\vec{k}}\sum_{n=-N+1}^{N-1}\ (\delta\hat{F}^{n}_{\vec{k}})^{*}\delta\hat{F}^{n}_{\vec{k}}$$

conserved, need to prove $\int_0^{\infty} d\hat{\mu}\sum\nolimits_{\vec{k}}\sum\limits_{n=-N+1}^{N-1}\ (\delta\hat{F}^{n}_{\vec{k}})^{*}d\delta\hat{F}^{n}_{\vec{k}} / dt + c.c. = 0$

which is identical to $\int_0^{\infty} d\hat{\mu}\sum\nolimits_{\vec{k}}\sum\limits_{n=-N+1}^{N-1}\ (\delta\hat{F}^{n}_{\vec{k}})^{*NL}S^{n}_{k,j} + c.c. = 0$

Use $\delta\hat{F}^{n}_{-\vec{k}} = (\delta\hat{F}^{-n}_{\vec{k}})^{*}$

From conservative form:

$\partial_{\hat{\mu}}NL$ **term:**

$$\int_0^{\infty} d\hat{\mu}\sum\nolimits_{\vec{k}}\sum_{n=-N+1}^{N-1}\ (\delta\hat{F}^{n}_{\vec{k}})^{*}\otimes\partial_{\hat{\mu}}\{\sum\nolimits_{k1}\ (T_e / T_i)\delta\hat{\phi}_{k1}(i\hat{k}_1\hat{\upsilon}_{\perp} / 2)[\exp(-i\beta_1)\delta\hat{F}^{n-1}_{k2} + \exp(+i\beta_1)\delta\hat{F}^{n+1}_{k2}]\} + c.c.$$
$$k \to -k \quad \Rightarrow$$
$$(T_e / T_i)\int_0^{\infty} d\hat{\mu}\sum\nolimits_{\vec{k}}\sum\nolimits_{k1}\sum\nolimits_{k2}\delta(\vec{k} + \vec{k}_1 + \vec{k}_2)\sum_{n=-N+1}^{N-1}\ \delta\hat{F}^{-n}_{\vec{k}}\otimes$$
$$\partial_{\hat{\mu}}\left\{\delta\hat{\phi}_{k1}(i\hat{k}_1\hat{\upsilon}_{\perp} / 2)[\exp(-i\beta_1)\delta\hat{F}^{n-1}_{k2} + \exp(+i\beta_1)\delta\hat{F}^{n+1}_{k2}]\right\} + c.c.$$

Integration by parts

$$-(T_e / T_i)\int_0^{\infty} d\hat{\mu}\sum\nolimits_{\vec{k}}\sum\nolimits_{k1}\sum\nolimits_{k2}\delta(\vec{k} + \vec{k}_1 + \vec{k}_2)\sum_{n=-N+1}^{N-1}\ \otimes$$
$$\left\{\delta\hat{\phi}_{k1}(i\hat{k}_1\hat{\upsilon}_{\perp} / 2)[\exp(-i\beta_1)\delta\hat{F}^{n-1}_{k2} + \exp(+i\beta_1)\delta\hat{F}^{n+1}_{k2}]\right\}\partial_{\hat{\mu}}(\delta\hat{F}^{-n}_{\vec{k}}) + c.c.$$

While using the condition:

$$\left\{ \delta\hat{\phi}_{k1}(i\hat{k}_1\hat{\upsilon}_\perp / 2)[\exp(-i\beta_1)\delta\hat{F}_{k2}^{n-1} + \exp(+i\beta_1)\delta\hat{F}_{k2}^{n+1}](\delta\hat{F}_{\vec{k}}^{-n}) \right\} \Big|_0^\infty + c.c. \Big|_0^\infty = 0$$

From non-conservative form:

$$\int_0^\infty d\hat{\mu} \sum_{\vec{k}} \sum_{n=-N+1}^{N-1} (\delta\hat{F}_{\vec{k}}^n)^* \otimes \sum_{k1} (T_e / T_i)\delta\hat{\phi}_{k1}(i\hat{k}_1\hat{\upsilon}_\perp / 2)[\exp(-i\beta_1)\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n-1} + \exp(+i\beta_1)\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n+1}] + c.c.$$

$k \to -k \quad \Rightarrow$

$$(T_e / T_i)\int_0^\infty d\hat{\mu} \sum_{\vec{k}} \sum_{k1} \sum_{k2} \delta(\vec{k} + \vec{k}_1 + \vec{k}_2) \sum_{n=-N+1}^{N-1} \delta\hat{F}_{\vec{k}}^{-n} \otimes$$

$$\left\{ \delta\hat{\phi}_{k1}(i\hat{k}_1\hat{\upsilon}_\perp / 2)[\exp(-i\beta_1)\partial_{\hat{\mu}}(\delta\hat{F}_{k2}^{n-1}) + \exp(+i\beta_1)\partial_{\hat{\mu}}(\delta\hat{F}_{k2}^{n+1})] \right\} + c.c.$$

$\Rightarrow$

$$(T_e / T_i)\int_0^\infty d\hat{\mu} \sum_{\vec{k}} \sum_{k1} \sum_{k2} \delta(\vec{k} + \vec{k}_1 + \vec{k}_2) \sum_{n=-N+1}^{N-1} \otimes$$

$$\left\{ \delta\hat{\phi}_{k1}(i\hat{k}_1\hat{\upsilon}_\perp / 2)[\exp(-i\beta_1)\delta\hat{F}_{k2}^{n-1} + \exp(+i\beta_1)\delta\hat{F}_{k2}^{n+1}] \right\} \partial_{\hat{\mu}}(\delta\hat{F}_{\vec{k}}^{-n}) + c.c.$$

Where we used

$$\sum_{n=-N+1}^{N-1} \delta A^n \delta B^{-(n\pm1)} = \frac{1}{2N-1} \sum_{k=1}^{2N-1} \delta a^k \delta b^k \exp(\mp 2\pi ik / N) = \sum_{n=-N+1}^{N-1} \delta B^n \delta A^{-(n\pm1)} \qquad \text{(A-1)},$$

and $k_2 \leftrightarrow k$ makes negative of conservative form term

Next we need to prove (A-1) through nonlinear convolution.

$$c(\alpha) = a(\alpha) b(\alpha)$$

$$\delta f_k = \frac{1}{2N-1} \sum_{n=-N+1}^{N-1} \delta \hat{F}_n \exp(+2\pi i n k / N)$$

$$c(a) = \frac{1}{2N-1} \sum_{n=-N+1}^{N-1} \delta C_n \exp(+2\pi i n k / N)$$

$$= [\frac{1}{2N-1} \sum_{n_1=-N+1}^{N-1} \delta A_{n_1} \exp(+2\pi i n_1 k / N)] \frac{1}{2N-1} \sum_{n_2=-N+1}^{N-1} \delta B_{n_2} \exp(+2\pi i n_2 k / N)]$$

$$\delta C_n = \sum_{k=1}^{2N-1} c(a) \exp(-2\pi i \bar{n} k / N) \Rightarrow$$

$$\delta C_n = \sum_{k=1}^{2N-1} \exp(-2\pi i n k / N) \otimes$$

$$\{[\frac{1}{2N-1} \sum_{n_1=-N+1}^{N-1} \delta A_{n_1} \exp(+2\pi i n_1 k / N)][\frac{1}{2N-1} \sum_{n_2=-N+1}^{N-1} \delta B_{n_2} \exp(+2\pi i n_2 k / N)\}$$

$$\delta C_n = (\frac{1}{2N-1} \sum_{n_1=-N+1}^{N-1} \delta A_{n_1})(\frac{1}{2N-1} \sum_{n_2=-N+1}^{N-1} \delta B_{n_2} \sum_{k=1}^{2N-1} \exp[-2\pi i (n - n_1 - n_2) k / N]\}$$

$$\Rightarrow$$

$$\delta C_n = \frac{1}{2N-1} \sum_{n_1=-N+1}^{N-1} \delta A_{n_1} \delta B_{n-n_1} \quad for\ n_2 = n - n_1$$

$$\delta C_n = \frac{1}{2N-1} \sum_{n_2=-N+1}^{N-1} \delta A_{n-n_2} \delta B_{n_2} \quad for\ n_1 = n - n_2$$

So $\delta C_{\mp 1} = \sum_{n=-N+1}^{N-1} \delta A^n \delta B^{-(n\pm 1)} = \frac{1}{2N-1} \sum_{k=1}^{2N-1} \delta a^k \delta b^k \exp(\mp 2\pi i k / N) = \sum_{n=-N+1}^{N-1} \delta B^n \delta A^{-(n\pm 1)}$

$\partial_\alpha NL$ term, From conservative form:

$$\int_0^\infty d\hat{\mu} \sum_{\vec{k}} \sum_{n=-N+1}^{N-1} (\delta \hat{F}_{\vec{k}}^n)^* \otimes \sum_{k1} \delta \hat{\phi}_{k1} (-i)(\hat{k}_1 / 2\hat{\upsilon}_\perp) n \{\exp(-i\beta_1) \delta \hat{F}_{k2}^{n-1} - \exp(+i\beta_1) \delta \hat{F}_{k2}^{n+1}\} + c.c.$$

$$k \to -k \quad \Rightarrow$$

$$\int_0^\infty d\hat{\mu} \sum_{\vec{k}} \sum_{k1} \sum_{k2} \delta(\vec{k} + \vec{k}_1 + \vec{k}_2) \sum_{n=-N+1}^{N-1} n \delta \hat{F}_{\vec{k}}^{-n} \otimes$$

$$\delta \hat{\phi}_{k1} (-i)(\hat{k}_1 / 2\hat{\upsilon}_\perp) \{\exp(-i\beta_1) \delta \hat{F}_{k2}^{n-1} - \exp(+i\beta_1) \delta \hat{F}_{k2}^{n+1}\} + c.c$$

From non-conservative form:

$$\int_0^\infty d\hat{\mu}\sum_{\vec{k}}\sum_{n=-N+1}^{N-1}(\delta\hat{F}_{\vec{k}}^{n})^{*}\otimes\sum_{k1}\delta\hat{\phi}_{k1}(-i)(\hat{k}_1/2\hat{\upsilon}_\perp)\{\exp(-i\beta_1)(n-1)\delta\hat{F}_{k2}^{n-1}-\exp(+i\beta_1)(n+1)\delta\hat{F}_{k2}^{n+1}\}+c.c.$$

$$k\to -k\quad\Rightarrow$$

$$\int_0^\infty d\hat{\mu}\sum_{\vec{k}}\sum_{k1}\sum_{k2}\delta(\vec{k}+\vec{k}_1+\vec{k}_2)\sum_{n=-N+1}^{N-1}\delta\hat{F}_{\vec{k}}^{-n}\otimes$$

$$\delta\hat{\phi}_{k1}(-i)(\hat{k}_1/2\hat{\upsilon}_\perp)\{\exp(-i\beta_1)(n-1)\delta\hat{F}_{k2}^{n-1}-\exp(+i\beta_1)(n+1)\delta\hat{F}_{k2}^{n+1}\}+c.c.$$

$$=\int_0^\infty d\hat{\mu}\sum_{\vec{k}}\sum_{k1}\sum_{k2}\delta(\vec{k}+\vec{k}_1+\vec{k}_2)\sum_{n=-N+2}^{N}n\delta\hat{F}_{\vec{k}}^{-n}\otimes$$

$$\delta\hat{\phi}_{k1}(-i)(\hat{k}_1/2\hat{\upsilon}_\perp)\{\exp(-i\beta_1)\delta\hat{F}_{k2}^{n-1}-\exp(+i\beta_1)\delta\hat{F}_{k2}^{n+1}\}+c.c$$

$$=-\int_0^\infty d\hat{\mu}\sum_{\vec{k}}\sum_{k1}\sum_{k2}\delta(\vec{k}+\vec{k}_1+\vec{k}_2)\sum_{n=-N+1}^{N-1}n\delta\hat{F}_{\vec{k}}^{-n}\otimes$$

$$\delta\hat{\phi}_{k1}(-i)(\hat{k}_1/2\hat{\upsilon}_\perp)\{\exp(-i\beta_1)\delta\hat{F}_{k2}^{n-1}-\exp(+i\beta_1)\delta\hat{F}_{k2}^{n+1}\}+c.c$$

Where we used : $\sum_{\bar{n}=-N+2}^{\bar{n}=N}=\sum_{\bar{n}=-N+1}^{\bar{n}=N-1}$ and $k_2\leftrightarrow k$ makes negative of conservative form term

However, cyclic shift property $\sum_{\bar{n}=-N+2}^{\bar{n}=N}=\sum_{\bar{n}=-N+1}^{\bar{n}=N-1}$ is not correct. We know that $\delta\hat{F}_k^n$ is cyclic over n; however, since n is not cyclic, $n\delta\hat{F}_k^{-n}\delta\hat{F}_{k_2}^{(n\pm1)}$ is not cyclic. Thus, we could not apply $\sum_{\bar{n}=-N+2}^{\bar{n}=N}=\sum_{\bar{n}=-N+1}^{\bar{n}=N-1}$ here.

Another method to prove the last step, where we can use:

$$-\sum_{n=-N+1}^{N-1}in\delta A^{-n}\delta B^{(n\pm1)}=\frac{1}{2N-1}\sum_{k=1}^{2N-1}\partial_\alpha(\delta a^k)\delta b^k\exp(\pm2\pi ik/N)=\sum_{n=-N+1}^{N-1}i(n\pm1)\delta B^{-n}\delta A^{(n\pm1)}\tag{A-2}$$

and $k_2\leftrightarrow k$ makes negative of conservative form term. **From non-conservative form:**

$$\int_0^\infty d\hat{\mu}\sum_{\vec{k}}\sum_{n=-N+1}^{N-1}(\delta\hat{F}_{\vec{k}}^{n})^{*}\otimes\sum_{k1}\delta\hat{\phi}_{k1}(-i)(\hat{k}_1/2\hat{\upsilon}_\perp)\{\exp(-i\beta_1)(n-1)\delta\hat{F}_{k2}^{n-1}-\exp(+i\beta_1)(n+1)\delta\hat{F}_{k2}^{n+1}\}+c.c.$$

$$k\to -k\quad\Rightarrow$$

$$\int_0^\infty d\hat{\mu}\sum_{\vec{k}}\sum_{k1}\sum_{k2}\delta(\vec{k}+\vec{k}_1+\vec{k}_2)\sum_{n=-N+1}^{N-1}\delta\hat{F}_{\vec{k}}^{-n}\otimes$$

$$\delta\hat{\phi}_{k1}(-i)(\hat{k}_1/2\hat{\upsilon}_\perp)\{\exp(-i\beta_1)(n-1)\delta\hat{F}_{k2}^{n-1}-\exp(+i\beta_1)(n+1)\delta\hat{F}_{k2}^{n+1}\}+c.c.$$

Apply (A-2) and $k_2\leftrightarrow k$

$$=-\int_0^\infty d\hat{\mu}\sum_{\vec{k}}\sum_{k1}\sum_{k2}\delta(\vec{k}+\vec{k}_1+\vec{k}_2)\sum_{n=-N+1}^{N-1}n\delta\hat{F}_{\vec{k}}^{-n}\otimes$$

$$\delta\hat{\phi}_{k1}(-i)(\hat{k}_1/2\hat{\upsilon}_\perp)\{\exp(-i\beta_1)\delta\hat{F}_{k2}^{n-1}-\exp(+i\beta_1)\delta\hat{F}_{k2}^{n+1}\}+c.c$$

Next we need to prove (A-2).

$$c(\alpha)=\partial_\alpha\left[a(\alpha)\right]b(\alpha)$$

$$\delta\hat{f}_k=\frac{1}{2N-1}\sum_{n=-N+1}^{N-1}\delta\hat{F}_n\exp(+2\pi ink/N)$$

$$c(a)=\{\frac{1}{2N-1}\partial_\alpha\left[\sum_{n_1=-N+1}^{N-1}\delta A_{n_1}\exp(+2\pi in_1k/N)\right]\}[\frac{1}{2N-1}\sum_{n_2=-N+1}^{N-1}\delta B_{n_2}\exp(+2\pi in_2k/N)]$$

$$=[\frac{1}{2N-1}\sum_{n_1=-N+1}^{N-1}(in_1)\delta A_{n_1}\exp(+2\pi in_1k/N)][\frac{1}{2N-1}\sum_{n_2=-N+1}^{N-1}\delta B_{n_2}\exp(+2\pi in_2k/N)]$$

$$\delta C_n=\sum_{k=1}^{2N-1}c(a)\exp(-2\pi ink/N)\Rightarrow$$

$$\delta C_n=\sum_{k=1}^{2N-1}\exp(-2\pi ink/N)\otimes$$

$$\{[\frac{1}{2N-1}\sum_{n_1=-N+1}^{N-1}(in_1)\delta A_{n_1}\exp(+2\pi in_1k/N)][\frac{1}{2N-1}\sum_{n_2=-N+1}^{N-1}\delta B_{n_2}\exp(+2\pi in_2k/N)\}$$

$$\delta C_n=(\frac{1}{2N-1}\sum_{n_1=-N+1}^{N-1}(in_1)\delta A_{n_1})(\frac{1}{2N-1}\sum_{n_2=-N+1}^{N-1}\delta B_{n_2}\{\sum_{k=1}^{2N-1}\exp[-2\pi i(n-n_1-n_2)k/N]\}$$

$$\Rightarrow$$

$$\delta C_n=\frac{1}{2N-1}\sum_{n_1=-N+1}^{N-1}(in_1)\delta A_{n_1}\delta B_{n-n_1}\quad for\ n_2=n-n_1$$

$$\delta C_n=\frac{1}{2N-1}\sum_{n_2=-N+1}^{N-1}i(n-n_2)\delta A_{n-n_2}\delta B_{n_2}\quad for\ n_1=n-n_2$$

$$\delta C_{\mp1}=\sum_{n=-N+1}^{N-1}in\delta A^n\delta B^{-(n\pm1)}=\frac{1}{2N-1}\sum_{k=1}^{2N-1}\partial_\alpha(\delta a^k)\delta b^k\exp(\mp2\pi ik/N)=\sum_{n=-N+1}^{N-1}-i(n\pm1)\delta B^n\delta A^{-(n\pm1)}$$

Substitute -n to n, then we have:

$$-\sum_{n=-N+1}^{N-1}in\delta A^{-n}\delta B^{(n\pm1)}=\frac{1}{2N-1}\sum_{k=1}^{2N-1}\partial_\alpha(\delta a^k)\delta b^k\exp(\pm2\pi ik/N)=\sum_{n=-N+1}^{N-1}i(n\pm1)\delta B^{-n}\delta A^{(n\pm1)}$$

Now we have the two equations:

$$\int_0^\infty d\hat{\mu}\sum_{\vec{k}}\sum_{n=-N+1}^{N-1}(\delta\hat{F}_{\vec{k}}^n)^*\left({}^{NL}_{C}S^{\alpha}_{n,k,j}+{}^{NL}_{NC}S^{\alpha}_{n,k,j}\right)+c.c.=0$$

$$\int_0^\infty d\hat{\mu}\sum_{\vec{k}}\sum_{n=-N+1}^{N-1}(\delta\hat{F}_{\vec{k}}^n)^*\left({}^{NL}_{C}S^{\hat{\mu}}_{n,k,j}+{}^{NL}_{NC}S^{\hat{\mu}}_{n,k,j}\right)+c.c.=0$$

Which leads to:

$$\int_0^\infty d\hat{\mu}\sum_{\vec{k}}\sum_{n=-N+1}^{N-1}(\delta\hat{F}_{\vec{k}}^{n})^*\frac{1}{2}\left({}_{C}^{NL}S_{n,k,j}^{\alpha}+{}_{NC}^{NL}S_{n,k,j}^{\alpha}+{}_{C}^{NL}S_{n,k,j}^{\hat{\mu}}+{}_{NC}^{NL}S_{n,k,j}^{\hat{\mu}}\right)+c.c.=0$$

$$\Rightarrow$$

$$\int_0^\infty d\hat{\mu}\sum_{\vec{k}}\sum_{n=-N+1}^{N-1}(\delta\hat{F}_{\vec{k}}^{n})^{*NL}S_{k,j}^{n}+c.c.=0$$

The test of the identity $\sum_{n=-N+1}^{N-1} in\delta A^{-n}\delta B^{(n\pm1)}=-\sum_{n=-N+1}^{N-1} i(n\pm1)\delta B^{-n}\delta A^{(n\pm1)}$ will be:

$$\frac{\sum_{n=-N+1}^{N-1} i\left[n(\delta\hat{F}_{\vec{k}}^{-n})(\delta\hat{F}_{k2}^{(n+1)})+(n+1)(\delta\hat{F}_{\vec{k}2}^{-n})(\delta\hat{F}_{k}^{(n+1)})\right]}{\sum_{n=-N+1}^{N-1} i\left[n(\delta\hat{F}_{\vec{k}}^{-n})(\delta\hat{F}_{k2}^{(n+1)})-(n+1)(\delta\hat{F}_{\vec{k}2}^{-n})(\delta\hat{F}_{k}^{(n+1)})\right]}=0$$

$$\frac{\sum_{n=-N+1}^{N-1} i\left[n(\delta\hat{F}_{\vec{k}}^{-n})(\delta\hat{F}_{k2}^{(n-1)})+(n-1)(\delta\hat{F}_{\vec{k}2}^{-n})(\delta\hat{F}_{k}^{(n-1)})\right]}{\sum_{n=-N+1}^{N-1} i\left[n(\delta\hat{F}_{\vec{k}}^{-n})(\delta\hat{F}_{k2}^{(n-1)})-(n-1)(\delta\hat{F}_{\vec{k}2}^{-n})(\delta\hat{F}_{k}^{(n-1)})\right]}=0$$

Add an instability to high frequency IC modes for CKinFH

Instead of adding a high frequency modes instability $|n_c|\gamma_{IC}\delta\hat{F}_{k\mathbf{C}}^{n}$ directly, I pick up the added two terms from the CK expansion $\delta\hat{f}_k$:

$$\delta\hat{f}_k=\mathrm{e}^{-ik\rho\sin(\alpha-\beta)}\sum_{n=-\infty}^{+\infty}\delta\hat{F}_{k\mathbf{C}}^{n}(\mu)\mathrm{e}^{in(\alpha-\beta)}$$

So the two terms I need are (only n_c=+1, -1):

$$|\mathrm{n}_c|\gamma_{IC}\,\mathrm{e}^{-ik\rho\sin(\alpha-\beta)}\,\delta\hat{F}_{k\mathbf{C}}^{n_c}(\mu)\mathrm{e}^{in_c(\alpha-\beta)}$$

These two terms are part of the contribution to $\delta\hat{f}_k$ from **CK** harmonics. Then put this contribution to **FK** $1/2\pi\oint d\alpha e^{-in_F\alpha}\delta\hat{f}_k$, which gives:

$$\mathrm{d}\delta\hat{F}_{kF}^{n_F}/\mathrm{dt}=1/2\pi\oint d\alpha e^{-in_F\alpha}|\mathrm{n}_c|\gamma_{IC}\,\mathrm{e}^{-ik\rho\sin(\alpha-\beta)}\,\delta\hat{F}_{k\mathbf{C}}^{n_c}(\mu)\mathrm{e}^{in_c(\alpha-\beta)}$$

This is the term we want, what I need to do next, is just to do the transform:

$$\delta\hat{F}_{k\mathbf{C}}^{n_c}\ to\ \delta\hat{F}_{kF}^{n_F}$$

First, as the same way of last document, I can get :

$$|n_C|\gamma_{IC}\delta\hat{F}_{k\,\mathbf{C}}^{n_C} = |n_C|\gamma_{IC}\sum_{n'=-N+1}^{N-1}\delta\hat{F}_{k\,\mathbf{F}}^{n'}J_{n_C-n'}(\mathrm{k}\,\rho)\,\mathrm{e}^{in'\beta}$$

Put it into the equation:

$$\begin{aligned}\mathrm{d}\,\delta\hat{F}_{k\,F}^{n_F}\ /\ \mathrm{dt} = 1/2\pi\oint d\alpha e^{-in_F\alpha}\ \mathrm{e}^{-ik\rho\sin(\alpha-\beta)}\ \mathrm{e}^{in_c(\alpha-\beta)}\otimes \\ |n_C|\gamma_{IC}\sum_{n'=-N+1}^{N-1}\delta\hat{F}_{k\,\mathbf{F}}^{n'}J_{n_C-n'}(\mathrm{k}\,\rho)\,\mathrm{e}^{in'\beta}\end{aligned}$$

Then Substitute $e^{-ik\rho\sin(\alpha-\beta)} = \sum_{n_x=-\infty}^{+\infty} J_{n_x}(\mathrm{k}\,\rho)\,\mathrm{e}^{-in_x(\alpha-\beta)}$ into the first line of up formula and merge n_x then I obtain:

$$\mathrm{d}\,\delta\hat{F}_{k\,F}^{n_F}\ /\ \mathrm{dt} = J_{n_C-n_F}(\mathrm{k}\,\rho)\,\mathrm{e}^{-in_F\beta}\,|n_c|\gamma_{IC}\sum_{n'=-N+1}^{N-1}\delta\hat{F}_{k\,\mathbf{F}}^{n'}J_{n_C-n'}(\mathrm{k}\,\rho)\,\mathrm{e}^{in'\beta}$$

This new formula is quite different from the old one, to a certain $\delta\hat{F}_{k\,\mathbf{F}}^{n_F}$, now there are a n_F=-N+1 to N-1 coupling terms.

If only add two CY harmonic instability like n_c=+1, -1 , then:

$$\mathrm{d}\,\delta\hat{F}_{k\,F}^{n_F}\ /\ \mathrm{dt} = \mathrm{e}^{-in_F\beta}\,\gamma_{IC}\sum_{n'=-N+1}^{N-1}\delta\hat{F}_{k\,\mathbf{F}}^{n'}\,\mathrm{e}^{in'\beta}\left[J_{1-n'}(\mathrm{k}\,\rho)J_{1-n_F}(\mathrm{k}\,\rho) + J_{-1-n'}(\mathrm{k}\,\rho)J_{-1-n_F}(\mathrm{k}\,\rho)\right]$$

For the high frequency modes instability of all harmonics: $\sum_{n_c=-N_\alpha+1}^{n_c=N_\alpha-1} n_c\gamma_{IC}\delta\hat{F}_{k\,\mathbf{C}}^{n}$

$$\mathrm{d}\,\delta\hat{F}_{k\,F}^{n_F}\ /\ \mathrm{dt} = \mathrm{e}^{-in_F\beta}\,\gamma_{IC}\sum_{n'=-N_\alpha+1}^{n'=N_a-1}\delta\hat{F}_{k\,\mathbf{F}}^{n'}\,\mathrm{e}^{in'\beta}\sum_{n_c=-N_\alpha+1}^{n_c=N_\alpha-1} n_c J_{n_c-n_F}(\mathrm{k}\,\rho)J_{n_c-n'}(\mathrm{k}\,\rho)$$

Pay attention to that, in rCYCLO code, only half of the k plane $k_y \geq 0$ are calculated. The other half plane are considered to satisfy the conjugate condition. Thus, the actual function applied in the rCYCLO code are:

$$\begin{cases}\mathrm{d}\,\delta\hat{F}_{k\,F}^{n_F}\ /\ \mathrm{dt} = \sum_{n_c=-N_\alpha+1}^{n_c=N_\alpha-1} n_c\gamma_{IC}\delta\hat{F}_{k\,\mathbf{C}}^{n},\ for\ k_y \geq 0 \\ \mathrm{d}\,\delta\hat{F}_{k\,F}^{n_F}\ /\ \mathrm{dt} = -\sum_{n_c=-N_\alpha+1}^{n_c=N_\alpha-1} n_c\gamma_{IC}\delta\hat{F}_{k\,\mathbf{C}}^{n},\ for\ k_y < 0\end{cases}$$

Now this equation satisfies the conjugate condition.

Appendix B

Cyclokinetics in cyclotron harmonic form (CKinCH)

Derive the nonlinear terms:

More on cyclotron harmonic nonlinearities: closed forms with Bessel functions
REW 9.14.12 important corrections on 5.30.12 ZD corrections 9.17.12

Starting from the transform Eq. (12) of Ref. [1], in gyro-Bohm unit.

$\delta \hat{f}_k(\hat{\mu},\alpha)\exp[ik\rho\sin(\alpha-\beta)] = \sum_{n=-N_\alpha+1}^{n=N_\alpha-1}\delta\hat{F}_k^n(\hat{\mu})\exp[in(\alpha-\beta)]$,

the Eq. (18) of Ref. [1] are derived as following.

*For the $\partial H_k^n/\partial\hat{t}+...=\partial_\alpha - RHS$: $\sum_{k1}\delta\hat{\phi}_{k1}(i\hat{\vec{k}}_{\perp 1}\cdot\vec{b}\times\vec{\hat{\upsilon}}_\perp/\hat{\upsilon}_\perp^2)\partial_\alpha\delta\hat{f}_{k2}\Rightarrow$

$$\sum_{n'}\sum_{k1}\delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[-in(\alpha-\beta)+ik\rho\sin(\alpha-\beta)][i\hat{\vec{k}}_{\perp 1}\cdot\vec{b}\times\vec{\hat{\upsilon}}_\perp/\hat{\upsilon}_\perp^2]\otimes$$
$$\partial_\alpha\{\exp[in'(\alpha-\beta_2)-ik_2\rho\sin(\alpha-\beta_2)]\delta\hat{F}_{k2}^{n'}\}$$

$\Rightarrow$

$$\sum_{n'}\sum_{k1}\delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[-in(\alpha-\beta)+ik\rho\sin(\alpha-\beta)+in'(\alpha-\beta_2)-ik_2\rho\sin(\alpha-\beta_2)]\otimes$$
$$[i\hat{\vec{k}}_{\perp 1}\cdot\vec{b}\times\vec{\hat{\upsilon}}_\perp/\hat{\upsilon}_\perp^2]\otimes\{in'-ik_2\rho\cos(\alpha-\beta_2)\}\delta\hat{F}_{k2}^{n'}$$

Consider the n=0 & n'=0 case

$$\sum_{k1}\delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[ik\rho\sin(\alpha-\beta)][i\hat{\vec{k}}_{\perp 1}\cdot\vec{b}\times\vec{\hat{\upsilon}}_\perp/\hat{\upsilon}_\perp^2]\otimes\partial_\alpha\{\exp[-ik_2\rho\sin(\alpha-\beta_2)]\delta\hat{F}_{k2}^{0'}\}$$

$$=\sum_{k1}\delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[ik\rho\sin(\alpha-\beta)][i\hat{\vec{k}}_{\perp 1}\cdot\vec{b}\times\vec{\hat{\upsilon}}_\perp/\hat{\upsilon}_\perp^2][-ik_2\rho\cos(\alpha-\beta_2)]\exp[-ik_2\rho\sin(\alpha-\beta_2)]\delta\hat{F}_{k2}^{0'}$$

$\Rightarrow$

$$\sum_{k1}\delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[ik\rho\sin(\alpha-\beta)-ik_2\rho\sin(\alpha-\beta_2)][-ik_1\rho\sin(\alpha-\beta_1)][-ik_2\rho\cos(\alpha-\beta_2)]/\hat{\upsilon}_\perp^2\delta\hat{F}_{k2}^{0'}$$

$\Rightarrow$

$$-\sum_{k1}\delta\hat{\phi}_{k1}\hat{k}_{\perp 1}\hat{k}_{\perp 2}\delta\hat{F}_{k2}^{0'}\oint d\alpha/2\pi\exp[ik\rho\sin(\alpha-\beta)-ik_2\rho\sin(\alpha-\beta_2)][\sin(\alpha-\beta_1)\cos(\alpha-\beta_2)]$$

We have $\oint d\alpha/2\pi\exp[ik\rho\sin(\alpha-\beta)-in\alpha) = J_n(k\rho)\exp(-in\beta)$ &

$\hat{z}\cdot\hat{k}_1\times\hat{k}_2 = -\hat{k}_1\hat{k}_2\sin(\beta_1-\beta_2) = -\hat{k}_1\hat{k}_2\sin\Delta\beta$ so

$$\oint d\alpha/2\pi \exp[ik\rho\sin(\alpha-\beta)-ik_2\rho\sin(\alpha-\beta_2)][\sin(\alpha-\beta_1)\cos(\alpha-\beta_2)]$$

$$k\rho\sin(\alpha-\beta)-k_2\rho\sin(\alpha-\beta_2)=\hat{z}\cdot\hat{k}\times\hat{\upsilon}_\perp-\hat{z}\cdot\hat{k}_2\times\hat{\upsilon}_\perp=\hat{z}\cdot\hat{k}_1\times\hat{\upsilon}_\perp$$
$\Rightarrow$
$$\oint d\alpha/2\pi \exp[ik_1\rho\sin(\alpha-\beta_1)][\sin(\alpha-\beta_1)\cos(\alpha-\beta_2)]$$

$$=\oint d\alpha/2\pi \exp[ik_1\rho\sin(\alpha)][\sin(\alpha)\cos(\alpha+\beta_1-\beta_2)]$$

$$=\oint d\alpha/2\pi \exp[ik_1\rho\sin(\alpha)][-1/2\sin(\Delta\beta)+1/2\sin(2\alpha+\Delta\beta)] \qquad @\,\Delta\beta=\beta_1-\beta_2$$

$$=-1/2J_0(k_1\rho)\sin(\Delta\beta)+1/4i[J_{-2}(k_1\rho)\exp(+i\Delta\beta)-J_{+2}(k_1\rho)\exp(-i\Delta\beta)]$$

$$=-1/2[J_0(k_1\rho)-J_2(k_1\rho)]\sin(\Delta\beta)$$

hence

$$-\sum_{k1}\delta\hat{\phi}_{k1}\hat{k}_{\perp1}\hat{\hat{k}}_{\perp2}\delta\hat{G}^{0'}_{k2}\oint d\alpha/2\pi \exp[ik\rho\sin(\alpha-\beta)-ik_2\rho\sin(\alpha-\beta_2)][\sin(\alpha-\beta_1)\cos(\alpha-\beta_2)]$$
$\Rightarrow$
$$-\sum_{k1}\hat{z}\cdot\hat{k}_{\perp1}\times\hat{\hat{k}}_{\perp2}\{[1/2J_0(k_1\rho)-1/2J_2(k_1\rho)]\delta\hat{\phi}_{k1}\}\delta\hat{F}^{0'}_{k2}$$

the k1 x k2 property (a mode does not couple to itself) is preserved but we get a 1/2 factor?

Note that ignoring coupling to higher harmonics $\Omega_*\delta\hat{\phi}_{k1}\sim O(\rho_*^0)$ hence gyroBohm scaling is persevered.

Important algebra:

$$\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$$
$$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$$
$$\sin(x)\cos(y)=1/2[\sin(x-y)+\sin(x+y)]$$
$$\Delta\beta=\beta_1-\beta_2 \qquad x=\alpha \qquad y=\alpha+\Delta\beta$$
$$\sin(\alpha)\cos(\alpha+\Delta\beta)=1/2[-\sin(\Delta\beta)+\sin(2\alpha+\Delta\beta)]$$

---

*For the $\partial H^n_k/\partial\hat{t}+...=\partial_{\hat{\mu}}-RHS$: $\sum_{k1}\Omega_*\delta\hat{\phi}_{k1}\{(i\vec{\hat{k}}_{\perp1}\cdot\vec{\hat{\upsilon}}_\perp)(T_{e0}/T_{i0})\partial_{\hat{\mu}}\delta\hat{f}_{k2}\Rightarrow$

$$\sum_{n'}\sum_{k1}\delta\hat{\phi}_{k1}\oint d\alpha/2\pi \exp[-in(\alpha-\beta)+ik\rho\sin(\alpha-\beta)+in'(\alpha-\beta_2)-ik_2\rho\sin(\alpha-\beta_2)]\otimes$$
$$[ik_1\rho\cos(\alpha-\beta_1)](T_{e0}/T_{i0})\{\partial_{\hat{\mu}}\delta\hat{F}^{n'}_{k2}-\delta\hat{F}^{n'}_{k2}ik_2\rho\sin(\alpha-\beta_2)/2\hat{\mu}\}$$

using $[i\vec{\hat{k}}_{\perp1}\cdot\vec{\hat{\upsilon}}_\perp]=ik_1\rho\cos(\alpha-\beta_1)=ik_1\rho[e^{+i(\alpha-\beta_1)}+e^{-i(\alpha-\beta_1)}]/2$

Again consider n=0 & n'=0

$$\sum_{k1}\ \delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[+ik\rho\sin(\alpha-\beta)-ik_2\rho\sin(\alpha-\beta_2)]\otimes$$
$$[ik_1\rho\cos(\alpha-\beta_1)](T_{e0}/T_{i0})\{\partial_{\hat{\mu}}\delta\hat{F}^0_{k2}-\delta\hat{F}^0_{k2}ik_2\rho\sin(\alpha-\beta_2)/2\hat{\mu}\}$$
$\Rightarrow$
$$\sum_{k1}\ \delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[+ik_1\rho\sin(\alpha-\beta_1)][ik_1\rho\cos(\alpha-\beta_1)]\otimes$$
$$(T_{e0}/T_{i0})\{\partial_{\hat{\mu}}\delta\hat{F}^0_{k2}-\delta\hat{F}^0_{k2}ik_2\rho\sin(\alpha-\beta_2)/2\hat{\mu}\}$$
$\Rightarrow$
$$=\sum_{k1}\ \delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[+ik_1\rho\sin(\alpha)][ik_1\rho\cos(\alpha)]\otimes(T_{e0}/T_{i0})\{\partial_{\hat{\mu}}\delta\hat{F}^0_{k2}\}$$
$$\oplus\sum_{k1}\ \delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[+ik_1\rho\sin(\alpha)][ik_1\rho\cos(\alpha)]\otimes$$
$$(T_{e0}/T_{i0})\{-\delta\hat{F}^0_{k2}ik_2\rho\sin(\alpha+\beta_1-\beta_2)/2\hat{\mu}\}$$

First term has

$$\cos(\alpha)=[\exp(i\alpha)+\exp(-i\alpha)]/2$$
$\Rightarrow$
$$=\sum_{k1}\ \delta\hat{\phi}_{k1}[J_{-1}(k_1\rho)+J_{+1}(k_1\rho)]ik_1\rho(T_{e0}/T_{i0})\partial_{\hat{\mu}}\delta\hat{F}^{0'}_{k2}$$

$$\Rightarrow 0 \qquad\qquad @\,J_{-1}(k_1\rho)=-J_1(k_1\rho)$$

$$\oplus\sum_{k1}\ \delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[+ik_1\rho\sin(\alpha)][ik_1\rho ik_2\rho][\cos(\alpha)\sin(\alpha+\beta_1-\beta_2)](T_{e0}/T_{i0})\{-\delta\hat{F}^0_{k2}/2\hat{\mu}\}$$

$$[\cos(\alpha)\sin(\alpha+\beta_1-\beta_2)]=\{1/2+1/4[\exp(i2\alpha)+\exp(-i2\alpha)]\}\sin(\beta_1-\beta_2)$$
$$+1/4i[\exp(i2\alpha)-\exp(-i2\alpha)]\cos(\beta_1-\beta_2)$$
$$[ik_1\rho ik_2\rho]\sin(\beta_1-\beta_2)=[-\hat{k}_1\hat{k}_2\sin(\beta_1-\beta_2)](\rho/\rho_s)^2=[-\hat{k}_1\hat{k}_2\sin(\beta_1-\beta_2)]2\hat{\mu}=[\hat{z}\cdot\hat{k}_1\times\hat{k}_2]2\hat{\mu}(T_{i0}/T_{e0})$$
$$[ik_1\rho ik_2\rho]\cos(\beta_1-\beta_2)=-[\hat{k}_1\cdot\hat{k}_2]2\hat{\mu}(T_{i0}/T_{e0})$$

$\Rightarrow$
$$\oplus\sum_{k1}\ \delta\hat{\phi}_{k1}\{[\hat{z}\cdot\hat{k}_1\times\hat{k}_2][J_0(k_1\rho)/2+1/4[J_2(k_1\rho)+J_{-2}(k_1\rho)]](-\delta\hat{F}^0_{k2})$$
$$-[\hat{k}_1\cdot\hat{k}_2][1/4i[J_{-2}(k_1\rho)-J_{+2}(k_1\rho)]](-\delta\hat{F}^0_{k2})$$
$\Rightarrow$
$$\oplus\sum_{k1}\ \delta\hat{\phi}_{k1}\{[\hat{z}\cdot\hat{k}_1\times\hat{k}_2][J_0(k_1\rho)/2+1/2J_2(k_1\rho)](-\delta\hat{F}^0_{k2})$$

adding the n=n'=0 results gives

$$-\sum_{k1}\ \hat{z}\cdot\hat{k}_{\perp1}\times\hat{\bar{k}}_{\perp2}\{[1/2J_0(k_1\rho)-1/2J_2(k_1\rho)]\delta\hat{\phi}_{k1}\}\delta\hat{F}^{0'}_{k2}$$
$$+\sum_{k1}\ \delta\hat{\phi}_{k1}\{[\hat{z}\cdot\hat{k}_1\times\hat{k}_2][1/2J_0(k_1\rho)+1/2J_2(k_1\rho)](-\delta\hat{F}^0_{k2})$$
$\Rightarrow$
$$-\sum_{k1}\ \hat{z}\cdot\hat{k}_{\perp1}\times\hat{\bar{k}}_{\perp2}J_0(k_1\rho)\delta\hat{\phi}_{k1}\delta\hat{F}^{0'}_{k2}=-\sum_{k1}\Omega_*\hat{z}\cdot\hat{k}_{\perp1}\times\hat{\bar{k}}_{\perp2}J_0(k_1\rho)\delta\hat{\phi}_{k1}\delta\hat{G}^{0'}_{k2}$$

Which is the expected gyrokinetic result !!!! no more factors of 1/2 !!!!

$\Delta_n^n(\beta,\beta_1,\beta_2) \equiv \exp[-in(\beta_1-\beta)+in(\beta_1-\beta_2)]$

Considering all n's

*For the $\partial H_k^n / \partial\hat{t} + ... = \partial_\alpha - RHS$:

$$\sum_{k1} \delta\hat{\phi}_{k1}(i\vec{\hat{k}}_{\perp 1}\cdot\hat{b}\times\vec{\hat{\upsilon}}_\perp / \hat{\upsilon}_\perp^2)\partial_\alpha \delta\hat{f}_{k2} \Rightarrow$$

$$\sum_{n'}\sum_{k1} \delta\hat{\phi}_{k1}\oint d\alpha/2\pi \exp[-in(\alpha-\beta)+ik\rho\sin(\alpha-\beta)][i\vec{\hat{k}}_{\perp 1}\cdot\vec{b}\times\vec{\hat{\upsilon}}_\perp/\hat{\upsilon}_\perp^2]\otimes$$
$$\partial_\alpha\{\exp[in'(\alpha-\beta_2)-ik_2\rho\sin(\alpha-\beta_2)]\delta\hat{F}_{k2}^{n'}\}$$

$\Rightarrow$

$$\sum_{n'}\sum_{k1} \delta\hat{\phi}_{k1}\oint d\alpha/2\pi \exp[-in(\alpha-\beta)+ik\rho\sin(\alpha-\beta)+in'(\alpha-\beta_2)-ik_2\rho\sin(\alpha-\beta_2)]\otimes$$
$$[-ik_1\rho\sin(\alpha-\beta_1)/\hat{\upsilon}_\perp^2]\otimes\{in'-ik_2\rho\cos(\alpha-\beta_2)\}\delta\hat{F}_{k2}^{n'}$$

$\Rightarrow$

$$\sum_{n'}\sum_{k1} \delta\hat{\phi}_{k1}\oint d\alpha/2\pi \exp[-in(\alpha-\beta)+in'(\alpha-\beta_2)+ik_1\rho\sin(\alpha-\beta_1)]\otimes$$
$$[-ik_1\rho\sin(\alpha-\beta_1)/\hat{\upsilon}_\perp^2]\otimes\{in'-ik_2\rho\cos(\alpha-\beta_2)\}\delta\hat{F}_{k2}^{n'}$$

$\Rightarrow$

$$\sum_{n'}\sum_{k1} \delta\hat{\phi}_{k1}\oint d\alpha'/2\pi \exp[-in(\alpha'+\beta_1-\beta)+in'(\alpha'+\beta_1-\beta_2)+ik_1\rho\sin(\alpha')]\otimes$$
$$[-i\hat{k}_1\sin(\alpha')]\otimes\{in'/\hat{\upsilon}_\perp-i\hat{k}_2\cos(\alpha'+\beta_1-\beta_2)\}\delta\hat{F}_{k2}^{n'}$$

$\Rightarrow$

$$-\sum_{n'}\sum_{k1} \hat{k}_1\hat{k}_2\delta\hat{\phi}_{k1}\oint d\alpha'/2\pi \exp[+ik_1\rho\sin(\alpha')-i(n-n')\alpha']\exp[-in(\beta_1-\beta)+in'(\beta_1-\beta_2)]\otimes$$
$$[\sin(\alpha')\cos(\alpha'+\beta_1-\beta_2)]\delta\hat{F}_{k2}^{n'}$$
$$-\sum_{n'}\sum_{k1} \hat{k}_1\delta\hat{\phi}_{k1}\oint d\alpha'/2\pi \exp[+ik_1\rho\sin(\alpha')-i(n-n')\alpha']\exp[-in(\beta_1-\beta)+in'(\beta_1-\beta_2)][\sin(\alpha')][-n'/\hat{\upsilon}_\perp]\delta\hat{F}_{k2}^{n'}$$

$\Rightarrow$

$$-\sum_{n'}\sum_{k1} \hat{k}_1\hat{k}_2\delta\hat{\phi}_{k1}\oint d\alpha'/2\pi \exp[+ik_1\rho\sin(\alpha')-i(n-n')\alpha'][\sin(\alpha')\cos(\alpha'+\beta_1-\beta_2)]\otimes$$
$$\exp[-in(\beta_1-\beta)+in'(\beta_1-\beta_2)]\delta\hat{F}_{k2}^{n'}$$

$$-\sum_{n'}\sum_{k1} \hat{k}_1\delta\hat{\phi}_{k1}\oint d\alpha'/2\pi \exp[+ik_1\rho\sin(\alpha')-i(n-n')\alpha'][\sin(\alpha')]$$
$$[-n'/\hat{\upsilon}_\perp]\exp[-in(\beta_1-\beta)+in'(\beta_1-\beta_2)]\delta\hat{F}_{k2}^{n'}$$

First term is like the old n=0 n'=0 term

$$-\sum_{n'}\sum_{k1}\ \hat{k}_1\hat{k}_2\delta\hat{\phi}_{k1}\oint d\alpha'/2\pi\exp[+ik_1\rho\sin(\alpha')-i(n-n')\alpha'][\sin(\alpha')\cos(\alpha'+\beta_1-\beta_2)]\otimes$$
$$\exp[-in(\beta_1-\beta)+in'(\beta_1-\beta_2)]\delta\hat{F}_{k2}^{n'}$$
$$\Rightarrow$$
$$-\sum_{n'}\sum_{k1}\ \hat{k}_1\hat{k}_2\delta\hat{\phi}_{k1}\oint d\alpha'/2\pi\exp[+ik_1\rho\sin(\alpha')-i(n-n')\alpha']\otimes$$
$$\{1/2+1/4[\exp(i2\alpha)+\exp(-i2\alpha)]\}(-\sin(\beta_1-\beta_2)+1/4i[\exp(i(2\alpha+\beta_1-\beta_2)-\exp(-i2(\alpha+\beta_1-\beta_2)]\}\otimes$$
$$\exp[-in(\beta_1-\beta)+in'(\beta_1-\beta_2)]\delta\hat{F}_{k2}^{n'}$$
$$\Rightarrow$$
$$-\sum_{n'}\sum_{k1}\ \hat{k}_1\hat{k}_2\delta\hat{\phi}_{k1}\oint d\alpha'/2\pi\exp[+ik_1\rho\sin(\alpha')-i(n-n')\alpha']\otimes$$
$$\{1/2+1/4[\exp(i2\alpha)+\exp(-i2\alpha)]\}(-\sin(\beta_1-\beta_2)+1/4i[\exp(i2\alpha)\exp i(\beta_1-\beta_2)-\exp(-i2\alpha)\exp(-i(\beta_1-\beta_2)]\}\otimes$$
$$\exp[-in(\beta_1-\beta)+in'(\beta_1-\beta_2)]\delta\hat{F}_{k2}^{n'}$$
$$\Rightarrow$$
$$-\sum_{n'}\sum_{k1}\ \hat{k}_1\hat{k}_2\delta\hat{\phi}_{k1}\oint d\alpha'/2\pi\exp[+ik_1\rho\sin(\alpha')-i(n-n')\alpha']\otimes$$
$$\{1/2+1/4[\exp(i2\alpha)+\exp(-i2\alpha)]\}(-\sin(\Delta\beta)+1/4i[\exp(i2\alpha)(\cos\Delta\beta+i\sin\Delta\beta)-\exp(-i2\alpha)(\cos\Delta\beta+i\sin\Delta\beta)]\}\otimes$$
$$\exp[-in(\beta_1-\beta)+in'(\beta_1-\beta_2)]\delta\hat{F}_{k2}^{n'}$$

$$\Rightarrow$$
$$-\sum_{n'}\ \sum_{k1}\ \hat{z}\cdot\hat{k}_{\perp 1}\times\hat{\vec{k}}_{\perp 2}\{1/2J_{n-n'}(k_1\rho)-1/4[J_{2+n-n'}(k_1\rho)+J_{-2+n-n'}(k_1\rho)]\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$
$$+\sum_{n'}\ \sum_{k1}\ \hat{k}_{\perp 1}\cdot\hat{\vec{k}}_{\perp 2}\{1/4i[J_{2+n-n'}(k_1\rho)-J_{-2+n-n'}(k_1\rho)]\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$
The second new term n' NOT 0 term is
$$\Rightarrow$$
$$-\sum_{n'}\ \sum_{k1}(i\hat{k}_{\perp 1}n'/\hat{\upsilon}_\perp)\{1/2[J_{n-n'-1}(k_1\rho)-J_{n-n'+1}(k_1\rho)]\delta\hat{\phi}_{k1}\}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$
the n=0&n'=0 result is recovered and the 2nd n' NOT 0 admits k-space self-coupling

(for modes with $\beta=\beta_1=\beta_2$). Note that the 2nd term with $(ik_1n'/\hat{\upsilon}_\perp)$ contains the low-velocity

singularity

all n's

For the $\partial H_k^n/\partial\hat{t}+...=\partial_{\hat{\mu}}-RHS$: $\sum_{k1}\Omega_*\delta\hat{\phi}_{k1}\{(i\vec{\hat{k}}_{\perp 1}\cdot\vec{\hat{\upsilon}}_\perp)(T_{e0}/T_{i0})\partial_{\hat{\mu}}\delta\hat{f}_{k2}\Rightarrow$

$$\sum_{n'}\sum_{k1}\ \delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[-in(\alpha-\beta)+ik\rho\sin(\alpha-\beta)+in'(\alpha-\beta_2)-ik_2\rho\sin(\alpha-\beta_2)]\otimes$$
$$[ik_1\rho\cos(\alpha-\beta_1)](T_{e0}/T_{i0})[\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'}-\delta\hat{F}_{k2}^{n'}ik_2\rho\sin(\alpha-\beta_2)/2\hat{\mu}]$$
$$\Rightarrow$$
$$\sum_{n'}\sum_{k1}\ \delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[-in(\alpha-\beta)+in'(\alpha-\beta_2)+ik_1\rho\sin(\alpha-\beta_1)]\otimes$$
$$[ik_1\rho\cos(\alpha-\beta_1)](T_{e0}/T_{i0})[\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'}-\delta\hat{F}_{k2}^{n'}ik_2\rho\sin(\alpha-\beta_2)/2\hat{\mu}]$$
$$\Rightarrow$$
$$\sum_{n'}\sum_{k1}\ \delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[-in(\alpha'+\beta_1-\beta)+in'(\alpha'+\beta_1-\beta_2)+ik_1\rho\sin(\alpha')]\otimes$$
$$[ik_1\rho\cos(\alpha')](T_{e0}/T_{i0})[\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'}-\delta\hat{F}_{k2}^{n'}ik_2\rho\sin(\alpha'+\beta_1-\beta_2)/2\hat{\mu}]$$

The second term is

$$\sum_{n'}\sum_{k1}\ \delta\hat{\phi}_{k1}\oint d\alpha'/2\pi\exp[-in(\alpha'+\beta_1-\beta)+in'(\alpha'+\beta_1-\beta_2)+ik_1\rho\sin(\alpha')]\otimes$$
$$[ik_1\rho\cos(\alpha')](T_{e0}/T_{i0})[-\delta\hat{F}_{k2}^{n'}ik_2\rho\sin(\alpha'+\beta_1-\beta_2)/2\hat{\mu}]$$

$\Rightarrow$

$$\sum_{n'}\sum_{k1}\ \delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[-in(\alpha'+\beta_1-\beta)+in'(\alpha'+\beta_1-\beta_2)+ik_1\rho\sin(\alpha')]\otimes$$
$$(-\hat{k}_1\hat{k}_2)(-\delta\hat{F}_{k2}^{n'})\{1/2+1/4[\exp(i2\alpha')+\exp(-i2\alpha')]\sin(\beta_1-\beta_2)$$
$$+1/4i[\exp(i2\alpha')-\exp(-i2\alpha')]\cos(\beta_1-\beta_2)\}$$

$\Rightarrow$

$$\sum_{n'}\sum_{k1}\ \delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[-in(\alpha'+\beta_1-\beta)+in'(\alpha'+\beta_1-\beta_2)+ik_1\rho\sin(\alpha')]\otimes$$
$$(-\delta\hat{F}_{k2}^{n'})\{1/2+1/4[\exp(i2\alpha')+\exp(-i2\alpha')]\hat{z}\cdot\hat{k}_1\times\hat{k}_2$$
$$-1/4i[\exp(i2\alpha')-\exp(-i2\alpha')]\hat{k}_1\cdot\hat{k}_2\}$$

$\Rightarrow$

$$\sum_{n'}\sum_{k1}\ \delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[-i(n-n')\alpha'+ik_1\rho\sin(\alpha')]\otimes$$
$$(-\delta\hat{F}_{k2}^{n'})\{1/2+1/4[\exp(i2\alpha')+\exp(-i2\alpha')]\hat{z}\cdot\hat{k}_1\times\hat{k}_2$$
$$-1/4i[\exp(i2\alpha')-\exp(-i2\alpha')]\hat{k}_1\cdot\hat{k}_2\}\otimes$$
$$\exp[-in(\beta_1-\beta)+in'(\beta_1-\beta_2)]$$

$\Rightarrow$

$$-\sum_{n'}\sum_{k1}\{\hat{z}\cdot\hat{k}_1\times\hat{k}_2[J_{n-n'}/2+(J_{n-n'+2}+J_{n-n'-2})/4]-i\hat{k}_1\cdot\hat{k}_2[J_{n-n'+2}-J_{n-n'-2}]/4\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\otimes$$
$$\exp[-in(\beta_1-\beta)+in'(\beta_1-\beta_2)]$$

The first term is

$$\sum_{n'}\sum_{k1}\ \delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[-in(\alpha'+\beta_1-\beta)+in'(\alpha'+\beta_1-\beta_2)+ik_1\rho\sin(\alpha')]\otimes$$
$$[ik_1\rho\cos(\alpha')](T_{e0}/T_{i0})\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'}$$

$\Rightarrow$

$$\sum_{n'}\sum_{k1}\ \delta\hat{\phi}_{k1}\oint d\alpha/2\pi\exp[-in(\alpha'+\beta_1-\beta)+in'(\alpha'+\beta_1-\beta_2)+ik_1\rho\sin(\alpha')]\otimes$$
$$[i\hat{k}_1/\hat{\upsilon}_\perp]2\cos(\alpha')][\hat{\mu}\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'}]$$

$\Rightarrow$

$$\sum_{n'}\sum_{k1}\ \{i\hat{k}_1/\hat{\upsilon}_\perp[J_{n-n+1}+J_{n-n-1}]\}\delta\hat{\phi}_{k1}(\hat{\mu}\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

Define $\quad \Delta_n^{n'}(\beta,\beta_1,\beta_2)\equiv\exp[-in(\beta_1-\beta)+in'(\beta_1-\beta_2)]$

Non-conservative Form

Final form ${}^{\partial\alpha}NL_k^n(\hat{\mu})=$

$$-\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_{n-n'}(k_1\rho)-1/4[J_{n-n'+2}(k_1\rho)+J_{n-n'-2}(k_1\rho)]\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'}\sum_{k1}\vec{\hat{k}}_1\cdot\vec{\hat{k}}_2\{[J_{n-n'+2}(k_1\rho)-J_{n-n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho)-J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(n'\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$\Rightarrow$

$$-\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_0(k_1\rho)-1/2J_2(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{0'}$$

in the n=n'=0 case

Final form $^{\partial\mu}NL_k^n(\hat{\mu})=$

$$-\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_{n-n'}(k_1\rho)+1/4[J_{n-n'+2}(k_1\rho)+J_{n-n'-2}(k_1\rho)]\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1}\vec{\hat{k}}_1\cdot\vec{\hat{k}}_2\{[J_{n-n'+2}(k_1\rho)-J_{n-n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho)+J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(2\hat{\mu}\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$\Rightarrow$

$$-\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_0(k_1\rho)+1/2J_2(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}$$

in the n=n'=0 case

adding the n=n'=0 cases, recovers the gyrokinetic limit

$\Rightarrow$

$$-\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2J_0(k_1\rho)\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}=-\sum_{n'}\sum_{k1}\hat{z}\cdot\hat{k}_1\times\hat{k}_2J_0(k_1\rho)\delta\hat{\phi}_{k1}\delta\hat{G}_{k2}^{n'}$$

9.17.12 Adding the general n's together:

$$-\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{J_{n-n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$
$$-\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho)-J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(n'\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$
$$-\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho)+J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(2\hat{\mu}\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

Thus, we have got the Eq. (18) of Ref. [1] so far.

Conservative Form

The derivations of Conservative Form of CKinCH are in the manuscript. Here only list the final result:

Final form $^{\partial\alpha}NL_k^n(\hat{\mu})=$

$$-\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_{n-n'}(k_1\rho)-1/4[J_{n-n'+2}(k_1\rho)+J_{n-n'-2}(k_1\rho)]\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'}\sum_{k1}\vec{\hat{k}}_1\cdot\vec{\hat{k}}_2\{[J_{n-n'+2}(k_1\rho)-J_{n-n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[(n'-1)J_{n-n'+1}(k_1\rho)-(n'+1)J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$\Rightarrow -\sum_{n'} \sum_{k1} \hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_0(k_1\rho)-1/2J_2(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{0'}$$

in the n=n'=0 case

Final form ${}^{\partial\mu}NL_k^n(\hat{\mu})=$

$$=-\sum_{n'}\sum_{k1} \delta\hat{\phi}_{k1}(\mathrm{T}_e/\mathrm{T}_i)\partial_{\hat{\mu}}\left\{[J_{n-n+1}(k_1\rho)+J_{n-n-1}(k_1\rho)](\hat{k}_1\hat{\upsilon}_\perp)\delta\hat{F}_{k2}^{n'}/2i\right\}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1} \hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_{n-n'}(k_1\rho)+1/4[J_{n-n'+2}(k_1\rho)+J_{n-n'-2}(k_1\rho)]\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1} \vec{\hat{k}}_1\cdot\vec{\hat{k}}_2\{[J_{n-n'+2}(k_1\rho)-J_{n-n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1} \hat{k}_1^2\{[J_{n-n'+2}(k_1\rho)-J_{n-n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$\Rightarrow$$

$$-\sum_{n'}\sum_{k1} \hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_0(k_1\rho)+1/2J_2(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}$$

in the n=n'=0 case

adding the n=n'=0 cases, recovers the gyrokinetic limit

$$\Rightarrow\ -\sum_{n'}\sum_{k1} \hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2J_0(k_1\rho)\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}=-\sum_{n'}\sum_{k1} \hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2J_0(k_1\rho)\delta\hat{\phi}_{k1}\delta\hat{G}_{k2}^{n'}$$

3.6.2014 Adding the general n's together:

$$-\sum_{n'} \sum_{k1} \hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{J_{n-n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'} \sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho)-J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}n'\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'} \sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho)+J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1} \hat{k}_1^2\{[J_{n-n'+2}(k_1\rho)-J_{n-n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1} \delta\hat{\phi}_{k1}(\mathrm{T}_e/\mathrm{T}_i)\partial_{\hat{\mu}}\left\{[J_{n-n'+1}(k_1\rho)+J_{n-n'-1}(k_1\rho)](\hat{k}_1\hat{\upsilon}_\perp)\delta\hat{F}_{k2}^{n'}/2i\right\}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

Pay attention that when derive into the bracket of the last term, it will cancel the third term and part of the second term. The remains are exactly the same as non-conservative nonlinear terms. This is also a verification that non-conservative form should equal to conservative form.

Since the Eq. (12) of Ref. [1] used $ik\rho\sin(\alpha-\beta)$, when it should have been $-ik\rho\sin(\alpha-\beta)$, we correct the minus sign of Eq. (12) in Ref. [1].

$$\delta\hat{f}_k(\hat{\mu},\alpha)\exp[-ik\rho\sin(\alpha-\beta)]=\sum_{n=-N_\alpha+1}^{n=N_\alpha-1}\delta\hat{F}_k^n(\hat{\mu})\exp[in(\alpha-\beta)]$$

Define $\quad \Delta_n^{n'}(\beta,\beta_1,\beta_2)\equiv\exp[-in(\beta_1-\beta)+in'(\beta_1-\beta_2)]$

Non-conservative Form

Final form $^{\partial\alpha}NL_k^n(\hat{\mu}) =$

$$+\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_{-n+n'}(k_1\rho)-1/4[J_{-n+n'+2}(k_1\rho)+J_{-n+n'-2}(k_1\rho)]\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'}\sum_{k1}\vec{\hat{k}}_1\cdot\vec{\hat{k}}_2\{[J_{-n+n'+2}(k_1\rho)-J_{-n+n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)-J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(n'\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$\Rightarrow$

$$\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_0(k_1\rho)-1/2J_2(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{0'}$$

in the n=n'=0 case

Final form $^{\partial\mu}NL_k^n(\hat{\mu}) =$

$$+\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_{-n+n'}(k_1\rho)+1/4[J_{-n+n'+2}(k_1\rho)+J_{-n+n'-2}(k_1\rho)]\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1}\vec{\hat{k}}_1\cdot\vec{\hat{k}}_2\{[J_{-n+n'+2}(k_1\rho)-J_{-n+n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)+J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(2\hat{\mu}\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$\Rightarrow$

$$\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_0(k_1\rho)+1/2J_2(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}$$

in the n=n'=0 case

adding the n=n'=0 cases, recovers the gyrokinetic limit

$\Rightarrow$

$$\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2J_0(k_1\rho)\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}=\sum_{n'}\sum_{k1}\hat{z}\cdot\hat{k}_1\times\hat{k}_2J_0(k_1\rho)\delta\hat{\phi}_{k1}\delta\hat{G}_{k2}^{n'}$$

9.17.12 Adding the general n's together:

$$+\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{J_{-n+n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$
$$+\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)-J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(n'\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$
$$-\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)+J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(2\hat{\mu}\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

Conservative Form

Final form $^{\partial\alpha}NL_k^n(\hat{\mu}) =$

$$+\sum_{n'}\ \sum_{k1}\ \hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_{-n+n'}(k_1\rho)-1/4[J_{-n+n'+2}(k_1\rho)+J_{-n+n'-2}(k_1\rho)]\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'}\sum_{k1}\ \vec{\hat{k}}_1\cdot\vec{\hat{k}}_2\{[J_{-n+n'+2}(k_1\rho)-J_{-n+n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'}\ \sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)-J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(n'\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'}\ \sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)+J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$\Rightarrow\sum_{n'}\ \sum_{k1}\ \hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_0(k_1\rho)-1/2J_2(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{0'}$$

in the n=n'=0 case

Final form ${}^{\partial\mu}NL_k^n(\hat{\mu})=$

$$-\sum_{n'}\sum_{k1}\ \delta\hat{\phi}_{k1}(\mathrm{T}_e/\mathrm{T}_i)\partial_{\hat{\mu}}\left\{[J_{-n+n'+1}(k_1\rho)+J_{-n+n'-1}(k_1\rho)](\hat{k}_1\hat{\upsilon}_\perp)\delta\hat{F}_{k2}^{n'}/2i\right\}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'}\sum_{k1}\ \hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_{-n+n'}(k_1\rho)+1/4[J_{-n+n'+2}(k_1\rho)+J_{-n+n'-2}(k_1\rho)]\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1}\ \vec{\hat{k}}_1\cdot\vec{\hat{k}}_2\{[J_{-n+n'+2}(k_1\rho)-J_{-n+n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1}\ \hat{k}_1^2\{[J_{-n+n'+2}(k_1\rho)-J_{-n+n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$\Rightarrow\sum_{n'}\sum_{k1}\ \hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{1/2J_0(k_1\rho)+1/2J_2(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}$$

adding the n=n'=0 cases, recovers the gyrokinetic limit

$$\Rightarrow\ \sum_{n'}\sum_{k1}\ \hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2J_0(k_1\rho)\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}=\sum_{n'}\sum_{k1}\ \hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2J_0(k_1\rho)\delta\hat{\phi}_{k1}\delta\hat{G}_{k2}^{n'}$$

3.6.2014 Adding the general n's together:

$$+\sum_{n'}\ \sum_{k1}\ \hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{J_{-n+n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'}\ \sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)-J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}n'\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'}\ \sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)+J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1}\ \hat{k}_1^2\{[J_{-n+n'+2}(k_1\rho)-J_{-n+n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1}\ \delta\hat{\phi}_{k1}(\mathrm{T}_e/\mathrm{T}_i)\partial_{\hat{\mu}}\left\{[J_{-n+n'+1}(k_1\rho)+J_{-n+n'-1}(k_1\rho)](\hat{k}_1\hat{\upsilon}_\perp)\delta\hat{F}_{k2}^{n'}/2i\right\}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

Pay attention that when you derive the bracket in the last term, it will cancel the third term and part of the second term. The remains are exactly the same as non-conservative nonlinear terms.

$$\delta\hat{G}_k(\hat{\mu},\alpha)=\delta\hat{g}_k(\hat{\mu},\alpha)\exp[-ik\rho\sin(\alpha-\beta)]=\sum_{n=-\infty}^{n=+\infty}\delta\hat{G}_k^n(\hat{\mu})\exp[in(\alpha-\beta)]$$

$$\delta\hat{F}_k^n=\delta\hat{G}_k^n-J_n(k\rho)\delta\hat{\phi}_k(T_e/T_i)n_0F_M(-1)^n$$

Make a change: $\delta\hat{F}_k^n=\delta\hat{\bar{\hat{F}}}_k^n(-1)^n;\ \delta\hat{G}_k^n=\delta\hat{\bar{\hat{G}}}_k^n(-1)^n$

Then $\delta\hat{\bar{\hat{F}}}_k^n=\delta\hat{\bar{\hat{G}}}_k^n-J_n(k\rho)\delta\hat{\phi}_k(T_e/T_i)n_0F_M$

For non-conservative form we have

$$\begin{aligned}
&D\delta\hat{F}_k^n/D\hat{t}-i\hat{\omega}_k^d\delta\hat{G}_k^n-in\Omega_*\delta\hat{G}_k^n+J_n(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT}n_0F_M(-1)^n\\
&=\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{J_{-n+n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)\\
&+\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)-J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(n'\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)\\
&-\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)+J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(2\hat{\mu}\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)
\end{aligned}$$

After the change of $\delta\hat{F}_k^n=\delta\hat{\bar{\hat{F}}}_k^n(-1)^n;\ \delta\hat{G}_k^n=\delta\hat{\bar{\hat{G}}}_k^n(-1)^n$

$$\begin{aligned}
&D\delta\hat{\bar{\hat{F}}}_k^n/D\hat{t}-i\hat{\omega}_k^d\delta\hat{\bar{\hat{G}}}_k^n-in\Omega_*\delta\hat{\bar{\hat{G}}}_k^n+J_n(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT}n_0F_M\\
&=\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{J_{-n+n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{\bar{\hat{F}}}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)(-1)^{n-n'}\\
&+\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)-J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(n'\delta\hat{\bar{\hat{F}}}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)(-1)^{n-n'}\\
&-\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)+J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(2\hat{\mu}\partial_{\hat{\mu}}\delta\hat{\bar{\hat{F}}}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)(-1)^{n-n'}\\
&\Rightarrow\\
&D\delta\hat{\bar{\hat{F}}}_k^n/D\hat{t}-i\hat{\omega}_k^d\delta\hat{\bar{\hat{G}}}_k^n-in\Omega_*\delta\hat{\bar{\hat{G}}}_k^n+J_n(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT}n_0F_M\\
&=\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{J_{n-n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{\bar{\hat{F}}}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)\\
&+\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho)-J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(n'\delta\hat{\bar{\hat{F}}}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)\\
&+\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho)+J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(2\hat{\mu}\partial_{\hat{\mu}}\delta\hat{\bar{\hat{F}}}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)
\end{aligned}$$

For conservative form we have

$$\begin{aligned}
&D\delta\hat{F}_k^n/D\hat{t}-i\hat{\omega}_k^d\delta\hat{G}_k^n-in\Omega_*\delta\hat{G}_k^n+J_n(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT}n_0F_M(-1)^n\\
&=\sum_{n'}\sum_{k1}\hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{J_{-n+n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)\\
&+\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)-J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}n'\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)\\
&+\sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)+J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)\\
&-\sum_{n'}\sum_{k1}\hat{k}_1^2\{[J_{-n+n'+2}(k_1\rho)-J_{-n+n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)
\end{aligned}$$

$$-\sum_{n'}\sum_{k1}\ \delta\hat{\phi}_{k1}(\mathrm{T}_e/\mathrm{T}_i)\partial_{\hat{\mu}}\left\{[J_{-n+n'+1}(k_1\rho)+J_{-n+n'-1}(k_1\rho)](\hat{k}_1\hat{\upsilon}_\perp)\delta\hat{F}_{k2}^{n'}/2i\right\}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

After the change of $\delta\hat{F}_k^n=\delta\hat{\bar{F}}_k^n(-1)^n;\ \delta\hat{G}_k^n=\delta\hat{\bar{G}}_k^n(-1)^n$

$$D\delta\hat{\bar{F}}_k^n/D\hat{t}-i\hat{\omega}_k^d\delta\hat{\bar{G}}_k^n-in\Omega_*\delta\hat{\bar{G}}_k^n+J_n(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT}n_0F_M$$

$$=\sum_{n'}\ \sum_{k1}\ \hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{J_{n-n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{\bar{F}}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'}\ \sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho)-J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}n'\delta\hat{\bar{F}}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\ \sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho)+J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}\delta\hat{\bar{F}}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'}\sum_{k1}\ \hat{k}_1^2\{[J_{n-n'+2}(k_1\rho)-J_{n-n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{\bar{F}}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'}\sum_{k1}\ \delta\hat{\phi}_{k1}(\mathrm{T}_e/\mathrm{T}_i)\partial_{\hat{\mu}}\left\{[J_{n-n'+1}(k_1\rho)+J_{n-n'-1}(k_1\rho)](\hat{k}_1\hat{\upsilon}_\perp)\delta\hat{\bar{F}}_{k2}^{n'}/2i\right\}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

Compare both conservative and non-conservative form results with the Ref. [1] cyclokinetic equations. The only difference is a different sign on the total nonlinear terms. The result here is the correct one, since cyclokinetics recovers to gyrokinetic should be positive sign on nonlinear term. We are lucky that there is no difference when the total sign of nonlinear terms for both GK, CKinCH and CKinFH are flipped especially when we choose i*delta electron. Pay attention that the relative sign of electron nonlinear term and ion nonlinear terms should keep right when we choose CDW electron.

Summary of CKinCH—version 1

Transform: $\delta\hat{G}_k(\hat{\mu},\alpha)=\delta\hat{g}_k(\hat{\mu},\alpha)\exp[-ik\rho\sin(\alpha-\beta)]=\sum_{n=-\infty}^{n=+\infty}\delta\hat{G}_k^n(\hat{\mu})\exp[in(\alpha-\beta)]$

Non-conservative form:

$$D\delta\hat{F}_k^n/D\hat{t}-i\hat{\omega}_k^d\delta\hat{G}_k^n-in\Omega_*\delta\hat{G}_k^n+J_n(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT}n_0F_M(-1)^n$$

$$=\sum_{n'}\ \sum_{k1}\ \hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{J_{-n+n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'}\ \sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)-J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(n'\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1}\ (\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)+J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(2\hat{\mu}\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

For conservative form:

$$D\delta\hat{F}_k^n/D\hat{t}-i\hat{\omega}_k^d\delta\hat{G}_k^n-in\Omega_*\delta\hat{G}_k^n+J_n(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT}n_0F_M(-1)^n$$

$$=\sum_{n'}\ \sum_{k1}\ \hat{z}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{J_{-n+n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'}\ \sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)-J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}n'\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$+\sum_{n'}\ \sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{-n+n'+1}(k_1\rho)+J_{-n+n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1}\ \hat{k}_1^2\{[J_{-n+n'+2}(k_1\rho)-J_{-n+n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2)$$

$$-\sum_{n'}\sum_{k1}\ \delta\hat{\phi}_{k1}(\mathrm{T}_e/\mathrm{T}_i)\partial_{\hat{\mu}}\left\{[J_{-n+n'+1}(k_1\rho)+J_{-n+n'-1}(k_1\rho)](\hat{k}_1\hat{\upsilon}_\perp)\delta\hat{F}_{k2}^{n'}/2i\right\}\Delta_n^{n'}(\beta,\beta_1,\beta_2)\ ,$$

Where $\delta\hat{F}_k^n=\delta\hat{G}_k^n-J_n(k\rho)\delta\hat{\phi}_k(T_e/T_i)n_0F_M(-1)^n$, $\delta\hat{F}_{-k}^{-n}=(-1)^n[\delta\hat{F}_k^n]^*$.

$\Delta_n^{n'}(\beta,\beta_1,\beta_2)\equiv\exp[-in(\beta_1-\beta)+in'(\beta_1-\beta_2)]$.

The Poisson equation is

$$\delta\hat{\phi}_k=\frac{\sum_n\ 2\pi\int d\hat{\mu}J_n(k\rho)\delta\hat{F}_k^n(\hat{\mu})(-1)^n/n_0-C_{DW}\cdot\delta\hat{n}_k^e}{(T_e/T_i)[1-\sum_n\ 2\pi\int d\hat{\mu}F_M(\hat{\mu})J_n^2(k\rho)]+(1-C_{DW})R_k+C_{DW}\cdot(\hat{\lambda}_D^2\hat{k}^2)}$$

The partile and energy fluxes are

$$[\hat{\Gamma},\hat{Q}^\perp]=Re\ 2\pi\int_0^\infty d\hat{\mu}\sum_k\ \sum_n\ [1,(T_i/T_e)\hat{\mu}]i\hat{k}_y\delta\hat{\phi}_k^*J_n(k\rho)\delta\hat{G}_k^n(\hat{\mu})(-1)^n/n_0$$

Summary of CKinCH—version 2

Cyclokinetic equation in cyclotron harmonic form (CKinCH)

Ref. [1] gives cyclokinetic equation in cyclotron harmonic form. The difference between Eq. (B.1) here and Eq. (18c) of Ref. [1] is that the sign of nonlinear terms is flipped, which is caused by the Eq. (19) of this paper corrected a minus sign to the Eq. (12) of Ref. [1].

Make a substitution of $\delta\hat{F}_k^n = \delta\hat{\hat{F}}_k^n(-1)^n;\ \delta\hat{G}_k^n = \delta\hat{\hat{G}}_k^n(-1)^n$ to **version 1,** finally we obtain:

The non-conservative form of CKinCH is

$$\begin{aligned}
&D\delta\hat{F}_k^n / D\hat{t} - i\hat{\omega}_k^d\delta\hat{G}_k^n - in\Omega_*\delta\hat{G}_k^n + J_n(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT} n_0 F_M(\hat{\mu}) \\
&= \sum_{n'}\sum_{k1} \hat{b}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{J_{n-n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2) \\
&\ + \sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho) - J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(n'\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2) \\
&\ + \sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho) + J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(2\hat{\mu}\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)
\end{aligned} \quad . \qquad \text{(B.1)}$$

The conservative form is

$$\begin{aligned}
&D\delta\hat{F}_k^n / D\hat{t} - i\hat{\omega}_k^d\delta\hat{G}_k^n - in\Omega_*\delta\hat{G}_k^n + J_n(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT} n_0 F_M(\hat{\mu}) \\
&= \sum_{n'}\sum_{k1} \hat{b}\cdot\vec{\hat{k}}_1\times\vec{\hat{k}}_2\{J_{n-n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2) \\
&\ + \sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho) - J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}n'\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2) \\
&\ - \sum_{n'}\sum_{k1}(\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho) + J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2) \\
&\ + \sum_{n'}\sum_{k1}\hat{k}_1^2\{[J_{n-n'+2}(k_1\rho) - J_{n-n'-2}(k_1\rho)]/4i\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2) \\
&\ + \sum_{n'}\sum_{k1}\delta\hat{\phi}_{k1}(\mathrm{T}_e/\mathrm{T}_i)\partial_{\hat{\mu}}\left\{[J_{n-n'+1}(k_1\rho) + J_{n-n'-1}(k_1\rho)](\hat{k}_1\hat{\upsilon}_\perp)\delta\hat{F}_{k2}^{n'}/2i\right\}\Delta_n^{n'}(\beta,\beta_1,\beta_2)
\end{aligned} \qquad \text{(B.2)}$$

where $\delta\hat{F}_k^n(\hat{\mu}) = \delta\hat{G}_k^n(\hat{\mu}) - J_n(k\rho)\delta\hat{\phi}_k(T_e/T_i)n_0F_M(\hat{\mu})$, and $J_n(k_1\rho)$ is Bessel function. $\Delta_n^{n'}(\beta,\beta_1,\beta_2) \equiv \exp[-in(\beta_1-\beta) + in'(\beta_1-\beta_2)]$ is a phase coefficient between different wave angles. The disturbed distribution function satisfies the conjugate property $\delta\hat{F}_{-k}^{-n} = (-1)^n[\delta\hat{F}_k^n]^*$

The line 3-5 of Eq. (B.2) could be combined after applying (see more details in Zhao Deng's onenote):

$$J_\nu(z) = \frac{z}{2\nu}\left(J_{\nu-1}(z) + J_{\nu+1}(z)\right)$$

$$J'_\nu(z) = \begin{cases} \frac{1}{2}\left(J_{\nu-1}(z) - J_{\nu+1}(z)\right) & \nu \neq 0 \\ -J_1(z) & \nu = 0 \end{cases}$$

Then we obtain the final version of CKinCH equations which is the same as the CKinCH equations in Ref. [3] the non-conservative nonlinear expression is:

$$
\begin{aligned}
& D\delta\hat{F}_k^n / D\hat{t} - i\hat{\omega}_k^d \delta\hat{G}_k^n - in\Omega_* \delta\hat{G}_k^n + J_n(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT} n_0 F_M(\hat{\mu}) \\
& = \sum_{n'} \sum_{k1} \hat{b}\cdot\vec{\hat{k}}_1 \times \vec{\hat{k}}_2 \{J_{n-n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2) \\
& + \sum_{n'} \sum_{k1} (\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho) - J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(n'\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2) \\
& + \sum_{n'} \sum_{k1} (\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho) + J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(2\hat{\mu}\partial_{\hat{\mu}}\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2)
\end{aligned}
\quad , \quad \text{(B.3)}
$$

and with conservative nonlinear expression written compactly as

$$
\begin{aligned}
& D\delta\hat{F}_k^n / D\hat{t} - i\hat{\omega}_k^d \delta\hat{G}_k^n - in\Omega_* \delta\hat{G}_k^n + J_n(k\rho)\delta\hat{\phi}_k i\hat{\omega}_{*k}^{nT} n_0 F_M(\hat{\mu}) \\
& = \sum_{n'} \sum_{k1} \hat{b}\cdot\vec{\hat{k}}_1 \times \vec{\hat{k}}_2 \{J_{n-n'}(k_1\rho)\}\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}^{n'}\Delta_n^{n'}(\beta,\beta_1,\beta_2) \\
& + \sum_{n'} \sum_{k1} (\hat{k}_1/\hat{\upsilon}_\perp)\{[J_{n-n'+1}(k_1\rho) - J_{n-n'-1}(k_1\rho)]/2i\}\delta\hat{\phi}_{k1}(n\delta\hat{F}_{k2}^{n'})\Delta_n^{n'}(\beta,\beta_1,\beta_2) \\
& + \sum_{n'} \sum_{k1} \delta\hat{\phi}_{k1}(\mathrm{T}_e/\mathrm{T}_i)\partial_{\hat{\mu}}\left\{[J_{n-n'+1}(k_1\rho) + J_{n-n'-1}(k_1\rho)](\hat{k}_1\hat{\upsilon}_\perp)\delta\hat{F}_{k2}^{n'}/2i\right\}\Delta_n^{n'}(\beta,\beta_1,\beta_2)
\end{aligned}
\quad \text{(B.4)}
$$

The Poisson equation of CKinCH is given as

$$
\delta\hat{\phi}_k = \frac{\sum_n 2\pi\int d\hat{\mu} J_n(k\rho)\delta\hat{F}_k^n(\hat{\mu})/n_0 - C_{DW}\cdot\delta\hat{n}_k^e}{(T_e/T_i)[1-\sum_n 2\pi\int d\hat{\mu} F_M(\hat{\mu})J_n^2(k\rho)] + (1-C_{DW})R_k + C_{DW}\cdot(\hat{\lambda}_D^2\hat{k}^2)} . \quad \text{(B.5)}
$$

The polarization density formally vanishes when all cyclotron harmonics are retained: $[1-\sum_n 2\pi\int d\hat{\mu} F_M(\hat{\mu})J_n^2(k\rho)] \to 0\big|_{n\to\pm\infty}$.

The ion particle and energy fluxes are

$$
[\hat{\Gamma}, \hat{Q}^\perp] = Re\, 2\pi\int_0^\infty d\hat{\mu}\sum_k \sum_n [1,(T_i/T_e)\hat{\mu}] i\hat{k}_y \delta\hat{\phi}_k^* J_n(k\rho)\delta\hat{G}_k^n(\hat{\mu})/n_0 . \quad \text{(B.6)}
$$

The proving of incremental entropy conservation law is in Zhao Deng’s OneNote.

Appendix C— Gyrokinetics

Derive 3D gyrokinetic equation

Starting from the drift kinetic equation in conservative form we replace the fields by the gyro-average fields and interpret F as the gyro-center distribution:

Ref. [5] Eq 55' and 65

$$\partial BF / \partial t + \vec{\nabla} \cdot [B(\hat{b}u + \vec{\upsilon}_D + \vec{u}_D)F] + \partial_u (Ba_u F) + \partial_\mu (a_\mu BF) = S$$

Where since $\vec{\kappa} = \vec{\nabla} B / B = 1/R, \quad u \Rightarrow 0, \quad \partial / \partial z \Rightarrow 0$

Ref. [5] Eq 53 and 54

$$\vec{\upsilon}_D = \hat{b} \times [\mu \vec{\nabla} B + u^2 \kappa - (e/m)\vec{E}_\perp / \Omega] \Rightarrow \hat{b} \times \mu \vec{\nabla} B + \vec{E}_\perp \times \hat{b} / B = \vec{v}_d + \vec{v}_E \qquad \text{(c=1)}$$

$$\vec{u}_D = (m/e)\mu \hat{b}\hat{b} \cdot \nabla \times \hat{b} \Rightarrow 0$$

Ref. [5] Eq 56

$$a_u = -\mu \hat{b} \cdot \vec{\nabla} B + (e/m)E_\parallel + u\vec{\upsilon}_D \cdot \vec{\kappa} - (\mu Bu / \Omega)\hat{b} \cdot \vec{\nabla} \times \vec{\kappa} \Rightarrow 0$$

Ref. [5] Eq 57

$$a_\mu = \mu [u^2 \hat{b} \cdot \vec{\nabla} \times \vec{\kappa} - (\hat{b} \cdot \vec{\nabla} \times \hat{b})\hat{b} \cdot (\mu \vec{\nabla} B - (e/m)\vec{E})] / \Omega \Rightarrow 0$$

and finally

$$\partial n / \partial t + \nabla_x \cdot [\int Bd\mu \delta \upsilon_{Ex} \delta f] = S$$

The 3D δF_k gyrokinetic equation is

[1] $\partial_t \delta f_k + v_d \cdot ik_\perp \delta g_k = -i(k_y / B)(1/L_n) J_0(k_\perp \rho)\delta\phi_k n_0 F_M + \sum_k \hat{b} \cdot (\vec{k}_1 / B \times \vec{k}_2) J_0(k_{\perp 1}\rho)\delta\phi_{k1}\delta f_{k2}$

$$\delta g_k = \delta f_k - (\mathrm{T}_e / \mathrm{T}_i)\delta\phi_k n_0 F_M$$

3D gyrokinetic equation normalization

The normalized 3D δF_k gyrokinetic equation, which added an artificial collision damping term,

$$\frac{D}{D_{\hat{t}}} \delta \hat{F}_k + \hat{v}_d \cdot i\hat{k}_\perp \delta \hat{G}_k = -i\hat{\omega}_*^{nT} J_0(k_\perp \rho)\delta\hat{\phi}_k n_0 F_M + \sum_k \hat{b} \cdot (\hat{k}_1 \times \hat{k}_2) J_0(k_{\perp 1}\rho)\delta\hat{\phi}_{k1}\delta\hat{F}_{k2}$$

$$\delta\hat{G}_k = \delta\hat{F}_k - (\mathrm{T}_e / \mathrm{T}_i)\delta\hat{\phi}_k n_0 F_M J_0$$

Poisson equation normalization

$$2\pi\int Bd\mu[J_0(k_\perp\rho)\delta f_k - n_0F_M e(1-J_0^2(k_\perp\rho))\delta\phi_k / T_{i0}] = n_0 e\delta\phi_k / T_{e0}(\lambda_k - i\delta_k)$$

Where $\lambda_{\vec{k}} = [1,0] \quad for \quad [k_y \neq 0, k_y = 0]$

$\delta_{\vec{k}} = \delta_1 k_y / [1+\eta k_x^2]$ where δ_1 and η **(**as well as μ and (a/L_n) drive) are adjusted to give in linear instability rate peaked on the k_y-axis.

$$2\pi\int d\hat{\mu}[J_0(k_\perp\rho)\delta\hat{F}_k - \frac{T_e}{T_i} n_0 F_M (1-J_0^2(k_\perp\rho))\delta\hat{\phi}_k] = n_0\delta\hat{\phi}_k(\lambda_k - i\delta_k)$$

$$\delta\hat{\phi}_k = \frac{2\pi\int d\hat{\mu} J_0(k_\perp\rho)\delta\hat{F}_k}{\frac{2\pi n_0 T_e}{T_i}\int d\hat{\mu} F_M(1-J_0^2(k_\perp\rho)) + n_0(\lambda_k - i\delta_k)}$$

Maxwellian background normalization

[3] $$F_M(\mu) = \frac{\exp[-m\mu B/T_{i0}]}{\int Bd\mu d\alpha \exp[-m\mu B/T_{i0}]} = \frac{m}{2\pi T_{i0}}\exp[\frac{-mB\mu}{T_{i0}}] \;;$$

After normalization:

[3]' $$F_M(\hat{\mu}) = \frac{1}{2\pi}\exp[-\hat{\mu}]$$

Derivation of conservation law of energy and entropy

The following is the derivation of the energy and entropy conservation laws under the condition of mu_CD=0 and delta_k=0.

We need the two following equations during the proving:

3D ion kinetic equation

$$\frac{\partial}{\partial t}\delta\hat{F}_{\vec{k}} = -i\omega^* J_0(k_\perp\rho)\delta\hat{\phi}_{\vec{k}} F_M + \sum\nolimits_k b\cdot(\vec{k}_1\times\vec{k}_2)J_0(k_{\perp 1}\rho)\delta\hat{\phi}_{\vec{k}_1}\delta\hat{F}_{\vec{k}_2}$$

Poisson equation:

$$\delta\hat{\phi}_{\vec{k}} = \frac{2\pi\int d\mu J_0(k_\perp\rho)\delta\hat{F}_{\vec{k}}}{\frac{T_{e0}}{T_{i0}}2\pi\int d\mu F_M[1-J_0^2(k_\perp\rho)] + \lambda_{\vec{k}}}$$

Energy is defined by:

$$E(\mu)=\sum_{\vec{k}}\frac{1}{2}\left[\delta F_{\vec{k}}J_0\delta\phi_{\vec{k}}^*+\delta F_{\vec{k}}^*J_0\delta\phi_{\vec{k}}\right]$$

The time derivative of Energy is:

$$\frac{\partial}{\partial t}E(\mu)=\sum_{\vec{k}}\left[\frac{\partial\delta F_{\vec{k}}}{\partial t}J_0\delta\phi_{\vec{k}}^*+\delta F_{\vec{k}}J_0\frac{\partial\delta\phi_{\vec{k}}^*}{\partial t}+\frac{\partial\delta F_{\vec{k}}^*}{\partial t}J_0\delta\phi_{\vec{k}}+\delta F_{\vec{k}}^*J_0\frac{\partial\delta\phi_{\vec{k}}}{\partial t}\right]\frac{1}{2}$$

$$=\frac{1}{2}\sum_{\vec{k}}\left[-i\omega^*J_0^2\delta\phi_{\vec{k}}^*F_M\delta\phi_{\vec{k}}+J_0\delta\phi_{\vec{k}}^*\sum_{\vec{k}=\vec{k}_1+\vec{k}_2}\vec{b}\cdot(\vec{k}_1\times\vec{k}_2)J_0(k_1\rho)\delta\phi_{\vec{k}_1}\delta F_{\vec{k}_2}\right]$$

$$+\frac{1}{2}\sum_{\vec{k}}\left[\frac{\delta F_{\vec{k}}J_0}{\frac{T_{e0}}{T_{i0}}\int d\mu F_M[1-J_0^2(k_\perp\rho)]+\lambda_{\vec{k}}}\int d\mu J_0\left(i\omega^*J_0F_M\delta\phi_{\vec{k}}^*+\sum_{\vec{k}=\vec{k}_1+\vec{k}_2}\vec{b}\cdot(\vec{k}_1\times\vec{k}_2)J_0(k_1\rho)\delta\phi_{\vec{k}_1}^*\delta F_{\vec{k}_2}^*\right)\right]$$

$$+\frac{1}{2}\sum_{\vec{k}}\left[i\omega^*J_0^2\delta\phi_{\vec{k}}F_M\delta\phi_{\vec{k}}^*+J_0\delta\phi_{\vec{k}}\sum_{\vec{k}=\vec{k}_1+\vec{k}_2}\vec{b}\cdot(\vec{k}_1\times\vec{k}_2)J_0(k_1\rho)\delta\phi_{\vec{k}_1}^*\delta F_{\vec{k}_2}^*\right]$$

$$+\frac{1}{2}\sum_{\vec{k}}\left[\frac{\delta F_{\vec{k}}^*J_0}{\frac{T_{e0}}{T_{i0}}\int d\mu F_M[1-J_0^2(k_\perp\rho)]+\lambda_{\vec{k}}}\int d\mu J_0\left(-i\omega^*J_0F_M\delta\phi_{\vec{k}}+\sum_{\vec{k}=\vec{k}_1+\vec{k}_2}\vec{b}\cdot(\vec{k}_1\times\vec{k}_2)J_0(k_1\rho)\delta\phi_{\vec{k}_1}\delta F_{\vec{k}_2}\right)\right]$$

In the summation, the first line add the third line go to 0 (use verify conjugation formula too). The integration of this equation by μ will get:

$$
\begin{aligned}
&\int \frac{\partial}{\partial t} E(\mu) d\mu \\
&= \frac{1}{2}\sum_{\vec{k}}\left[\frac{\int d\mu \delta F_{\vec{k}} J_0}{\frac{T_{e0}}{T_{i0}}\int d\mu F_M[1-J_0^2(k_\perp \rho)]+\lambda_{\vec{k}}}\int d\mu J_0\left(i\omega^* J_0 F_M \delta\phi_{\vec{k}}^* + \sum_{\vec{k}=\vec{k}_1+\vec{k}_2} \vec{b}\cdot(\vec{k}_1\times\vec{k}_2) J_0(k_1\rho)\delta\phi_{\vec{k}_1}^* \delta F_{\vec{k}_2}^*\right)\right] \\
&+\frac{1}{2}\sum_{\vec{k}}\left[\frac{\int d\mu \delta F_{\vec{k}}^* J_0}{\frac{T_{e0}}{T_{i0}}\int d\mu F_M[1-J_0^2(k_\perp \rho)]+\lambda_{\vec{k}}}\int d\mu J_0\left(-i\omega^* J_0 F_M \delta\phi_{\vec{k}} + \sum_{\vec{k}=\vec{k}_1+\vec{k}_2} \vec{b}\cdot(\vec{k}_1\times\vec{k}_2) J_0(k_1\rho)\delta\phi_{\vec{k}_1} \delta F_{\vec{k}_2}\right)\right] \\
&= \frac{1}{2}\sum_{\vec{k}}\left[\delta\phi_{\vec{k}}\int d\mu J_0\left(i\omega^* J_0 F_M \delta\phi_{\vec{k}}^* + \sum_{\vec{k}=\vec{k}_1+\vec{k}_2} \vec{b}\cdot(\vec{k}_1\times\vec{k}_2) J_0(k_1\rho)\delta\phi_{\vec{k}_1}^* \delta F_{\vec{k}_2}^*\right)\right] \\
&+\frac{1}{2}\sum_{\vec{k}}\left[\delta\phi_{\vec{k}}^*\int d\mu J_0\left(-i\omega^* J_0 F_M \delta\phi_{\vec{k}} + \sum_{\vec{k}=\vec{k}_1+\vec{k}_2} \vec{b}\cdot(\vec{k}_1\times\vec{k}_2) J_0(k_1\rho)\delta\phi_{\vec{k}_1} \delta F_{\vec{k}_2}\right)\right] \\
&= \frac{1}{2}\sum_{\vec{k}}\left[\left(\delta\phi_{\vec{k}}\delta\phi_{\vec{k}}^* - \delta\phi_{\vec{k}}^*\delta\phi_{\vec{k}}\right) i\omega^* \int d\mu J_0^2 F_M\right] \\
&+\frac{1}{2}\int d\mu \sum_{\vec{k}}\left[\delta\phi_{\vec{k}} J_0 \sum_{\vec{k}=\vec{k}_1+\vec{k}_2} \vec{b}\cdot(\vec{k}_1\times\vec{k}_2) J_0(k_1\rho)\delta\phi_{\vec{k}_1}^* \delta F_{\vec{k}_2}^* + \delta\phi_{\vec{k}}^* J_0 \sum_{\vec{k}=\vec{k}_1+\vec{k}_2} \vec{b}\cdot(\vec{k}_1\times\vec{k}_2) J_0(k_1\rho)\delta\phi_{\vec{k}_1} \delta F_{\vec{k}_2}\right] \\
&= 0
\end{aligned}
$$

So we get the conservation law of total energy:

$$\frac{\partial}{\partial t}\int d\mu E(\mu) = 0$$

Entropy is defined by:

$$S(\mu) = \sum_{\vec{k}} \delta F_{\vec{k}} \delta F_{\vec{k}}^* / F_M$$

Then we need to prove the Entropy does not change with time.

The time derivative of Entropy is:

$$
\begin{aligned}
\frac{\partial}{\partial t} S(\mu) &= \sum_{\vec{k}} \frac{1}{F_M}\left[\frac{\partial \delta F_{\vec{k}}}{\partial t}\delta F_{\vec{k}}^* + \delta F_{\vec{k}} \frac{\partial \delta F_{\vec{k}}^*}{\partial t}\right] \\
&= \sum_{\vec{k}}\left[-i(\omega^*) J_0 \delta\phi_{\vec{k}} \delta F_{\vec{k}}^* + \frac{\delta F_{\vec{k}}^*}{F_M}\sum_{\vec{k}=\vec{k}_1+\vec{k}_2} \vec{b}\cdot(\vec{k}_1\times\vec{k}_2) J_0(k_1\rho)\delta\phi_{\vec{k}_1} \delta F_{\vec{k}_2}\right] \\
&+ \sum_{\vec{k}}\left[i(\omega^*) J_0 \delta\phi_{\vec{k}}^* \delta F_{\vec{k}} + \frac{\delta F_{\vec{k}}}{F_{\mathrm{M}}}\sum_{\vec{k}=\vec{k}_1+\vec{k}_2} \vec{b}\cdot(\vec{k}_1\times\vec{k}_2) J_0(k_1\rho)\delta\phi_{\vec{k}_1}^* \delta F_{\vec{k}_2}^*\right]
\end{aligned}
$$

The second term and the forth term equal to 0 respectively, See the verify conjugation formula. Then we get:

$$\frac{\partial}{\partial t}S(\mu)=\sum_{\vec{k}}\left[-i\omega * J_0\delta\phi_{\vec{k}}\delta F_{\vec{k}}^{*}+i\omega * J_0\delta\phi_{\vec{k}}^{*}\delta F_{\vec{k}}\right]$$

The integration of this equation by μ will get:

$$\int\frac{\partial}{\partial t}S(\mu)\,d\mu=\sum_{\vec{k}}i\omega *\left[\delta\phi_{\vec{k}}^{*}\int d\mu J_0\delta F_{\vec{k}}-\delta\phi_{\vec{k}}\int d\mu J_0\delta F_{\vec{k}}^{*}\right]$$

From Poisson equation [A-2], we know:

$$\int d\mu J_0(k_\perp\rho)\delta F_{\vec{k}}=\delta\phi_{\vec{k}}\left[\frac{T_{e0}}{T_{i0}}\int d\mu F_M[1-J_0^2(k_\perp\rho)]+\lambda_{\vec{k}}\right]$$

So we get the conservation law of total entropy:

$$\frac{\partial}{\partial t}\int S(\mu)\,d\mu=\sum_{\vec{k}}i\omega *\left[\delta\phi_{\vec{k}}^{*}\delta\phi_{\vec{k}}-\delta\phi_{\vec{k}}\delta\phi_{\vec{k}}^{*}\right]\left[\frac{T_{e0}}{T_{i0}}\int d\mu F_M[1-J_0^2(k_\perp\rho)]+\lambda_{\vec{k}}\right]=0$$

Derive the growth rate when $i\delta_{\vec{k}}$ and μ_{CD} are not zero

We need to pick up the $i\delta_{\vec{k}}$ and μ_{CD} term in the following derivation:

$$\frac{\partial}{\partial t}\delta F_{\vec{k}}=-i\omega^{*}J_0(k_\perp\rho)\delta\phi_{\vec{k}}F_M+\sum\nolimits_k b\cdot(\vec{k}_1\times\vec{k}_2)J_0(k_{\perp 1}\rho)\delta\phi_{\vec{k}_1}\delta F_{\vec{k}_2}-\mu_{CD}k^2\delta F_{\vec{k}}$$

$$\delta\phi_{\vec{k}}=\frac{\int d\mu J_0(k_\perp\rho)\delta F_{\vec{k}}}{\frac{T_{e0}}{T_{i0}}\int d\mu F_M[1-J_0^2(k_\perp\rho)]+(\lambda_{\vec{k}}-i\delta_{\vec{k}})}$$

Energy is defined by:

$$E(\mu)=\sum_{\vec{k}}\frac{1}{2}\left[\delta F_{\vec{k}}J_0\delta\phi_{\vec{k}}^{*}+\delta F_{\vec{k}}^{*}J_0\delta\phi_{\vec{k}}\right]$$

The time derivative of Energy is:

$$\frac{\partial}{\partial t}E(\mu)=\sum_{\vec{k}}\left[\frac{\partial\delta F_{\vec{k}}}{\partial t}J_0\delta\phi_{\vec{k}}^*+\delta F_{\vec{k}}J_0\frac{\partial\delta\phi_{\vec{k}}^*}{\partial t}+\frac{\partial\delta F_{\vec{k}}^*}{\partial t}J_0\delta\phi_{\vec{k}}+\delta F_{\vec{k}}^*J_0\frac{\partial\delta\phi_{\vec{k}}}{\partial t}\right]\frac{1}{2}$$

$$=\frac{1}{2}\sum_{\vec{k}}\left[-i\omega*J_0^2\delta\phi_{\vec{k}}^*F_M\delta\phi_{\vec{k}}+J_0\delta\phi_{\vec{k}}^*\sum_{\vec{k}=\vec{k}_1+\vec{k}_2}\vec{b}\cdot(\vec{k}_1\times\vec{k}_2)J_0(k_1\rho)\delta\phi_{\vec{k}_1}\delta F_{\vec{k}_2}-\mu_{CD}k^2\delta F_{\vec{k}}J_0\delta\phi_{\vec{k}}^*\right]$$

$$+\frac{1}{2}\sum_{\vec{k}}\left[\frac{\delta F_{\vec{k}}J_0}{\frac{T_{e0}}{T_{i0}}\int d\mu F_M[1-J_0^2(k_\perp\rho)]+\lambda_{\vec{k}}+i\delta_{\vec{k}}}\int d\mu J_0\left(i\omega*J_0F_M\delta\phi_{\vec{k}}^*+\sum_{\vec{k}=\vec{k}_1+\vec{k}_2}\vec{b}\cdot(\vec{k}_1\times\vec{k}_2)J_0(k_1\rho)\delta\phi_{\vec{k}_1}^*\delta F_{\vec{k}_2}^*-\mu_{CD}k^2\delta F_{\vec{k}}^*\right)\right]$$

$$+\frac{1}{2}\sum_{\vec{k}}\left[i\omega*J_0^2\delta\phi_{\vec{k}}F_M\delta\phi_{\vec{k}}^*+J_0\delta\phi_{\vec{k}}\sum_{\vec{k}=\vec{k}_1+\vec{k}_2}\vec{b}\cdot(\vec{k}_1\times\vec{k}_2)J_0(k_1\rho)\delta\phi_{\vec{k}_1}^*\delta F_{\vec{k}_2}^*-\mu_{CD}k^2\delta F_{\vec{k}}^*J_0\delta\phi_{\vec{k}}\right]$$

$$+\frac{1}{2}\sum_{\vec{k}}\left[\frac{\delta F_{\vec{k}}^*J_0}{\frac{T_{e0}}{T_{i0}}\int d\mu F_M[1-J_0^2(k_\perp\rho)]+\lambda_{\vec{k}}-i\delta_{\vec{k}}}\int d\mu J_0\left(-i\omega*J_0F_M\delta\phi_{\vec{k}}+\sum_{\vec{k}=\vec{k}_1+\vec{k}_2}\vec{b}\cdot(\vec{k}_1\times\vec{k}_2)J_0(k_1\rho)\delta\phi_{\vec{k}_1}\delta F_{\vec{k}_2}-\mu_{CD}k^2\delta F_{\vec{k}}\right)\right]$$

In the summation, utilize the verify conjugation formula, the first line add the third line only left:

$$-\frac{1}{2}\mu_{CD}\sum_{\vec{k}}k^2(\delta F_{\vec{k}}J_0\delta\phi_{\vec{k}}^*+\delta F_{\vec{k}}^*J_0\delta\phi_{\vec{k}})=-\mu_{CD}\sum_{\vec{k}}k^2E(\vec{k})$$

To simplify the formula, we define:

$$\Gamma(\vec{\mathrm{k}})=\frac{T_{e0}}{T_{i0}}\int d\mu F_M[1-J_0^2(k_\perp\rho)]+\lambda_{\vec{k}}$$

Thus, three terms left by the integration of the summation of second line and the fourth line:

$$\int f_{2,4}(\mu)d\mu=$$

$$=\frac{1}{2}\sum_{\vec{k}}\left[\frac{\int d\mu\delta F_{\vec{k}}J_0}{\Gamma+i\delta_{\vec{k}}}\int d\mu J_0\left(i\omega*J_0F_M\delta\phi_{\vec{k}}^*+\sum_{\vec{k}=\vec{k}_1+\vec{k}_2}\vec{b}\cdot(\vec{k}_1\times\vec{k}_2)J_0(k_1\rho)\delta\phi_{\vec{k}_1}^*\delta F_{\vec{k}_2}^*-\mu_{CD}k^2\delta F_{\vec{k}}^*\right)\right]$$

$$+\frac{1}{2}\sum_{\vec{k}}\left[\frac{\int d\mu\delta F_{\vec{k}}^*J_0}{\Gamma-i\delta_{\vec{k}}}\int d\mu J_0\left(-i\omega*J_0F_M\delta\phi_{\vec{k}}+\sum_{\vec{k}=\vec{k}_1+\vec{k}_2}\vec{b}\cdot(\vec{k}_1\times\vec{k}_2)J_0(k_1\rho)\delta\phi_{\vec{k}_1}\delta F_{\vec{k}_2}-\mu_{CD}k^2\delta F_{\vec{k}}\right)\right]$$

Use passion equation [A-2-2], the first term in the bracket gives:

$$\frac{1}{2}\sum_{\vec{k}}\frac{-4i\delta_{\vec{k}}\Gamma}{\Gamma^2+\delta_{\vec{k}}^2}\delta\phi_{\vec{k}}\delta\phi_{\vec{k}}^*i\omega*\int d\mu J_0^2F_M$$

The second term in the bracket can not be simplified:

The third term in the bracket gives:

$$-\frac{1}{2}\mu_{CD}\sum_{\vec{k}}2k^2\Gamma\delta\phi_{\vec{k}}\delta\phi_{\vec{k}}^*$$

So we get **the time derivative of total energy**:

$$\frac{\partial}{\partial t}\int d\mu E(\mu) =$$
$$-\mu_{CD}\sum_{\vec{k}}k^2\int d\mu E(\vec{k})$$
$$+2\sum_{\vec{k}}\frac{\delta_{\vec{k}}\Gamma}{\Gamma^2+\delta_{\vec{k}}^2}\delta\phi_{\vec{k}}\delta\phi_{\vec{k}}^*\omega*\int d\mu J_0^2F_M$$
$$-\mu_{CD}\sum_{\vec{k}}k^2\Gamma\delta\phi_{\vec{k}}\delta\phi_{\vec{k}}^*$$
$$+\frac{1}{2}\int d\mu\sum_{\vec{k}}\left[\delta\phi_{\vec{k}}J_0\frac{\Gamma-i\delta_{\vec{k}}}{\Gamma+i\delta_{\vec{k}}}\sum_{\vec{k}=\vec{k}_1+\vec{k}_2}\vec{b}\cdot(\vec{k}_1\times\vec{k}_2)J_0(k_1\rho)\delta\phi_{\vec{k}_1}^*\delta F_{\vec{k}_2}^*+\delta\phi_{\vec{k}}^*J_0\frac{\Gamma+i\delta_{\vec{k}}}{\Gamma-i\delta_{\vec{k}}}\sum_{\vec{k}=\vec{k}_1+\vec{k}_2}\vec{b}\cdot(\vec{k}_1\times\vec{k}_2)J_0(k_1\rho)\delta\phi_{\vec{k}_1}\delta F_{\vec{k}_2}\right]$$

The physics of these terms is clear, the first and third term are linear damping term come from the artificial dissipation μ. The second term are linear growth term comes from the $i\delta_{\vec{k}}$. The last term contribute to the non-linear growth rate, caused by the mode-mode coupling combined with $i\delta_{\vec{k}}$ driving.

Entropy is defined by:

$$S(\mu)=\sum_{\vec{k}}\delta F_{\vec{k}}\delta F_{\vec{k}}^*/F_M$$

The time derivative of Entropy is:

$$\frac{\partial}{\partial t}S(\mu)=\sum_{\vec{k}}\frac{1}{F_M}\left[\frac{\partial\delta F_{\vec{k}}}{\partial t}\delta F_{\vec{k}}^*+\delta F_{\vec{k}}\frac{\partial\delta F_{\vec{k}}^*}{\partial t}\right]$$
$$=\sum_{\vec{k}}\left[-i(\omega*)J_0\delta\phi_{\vec{k}}\delta F_{\vec{k}}^*+\frac{\delta F_{\vec{k}}^*}{F_M}\sum_{\vec{k}=\vec{k}_1+\vec{k}_2}\vec{b}\cdot(\vec{k}_1\times\vec{k}_2)J_0(k_1\rho)\delta\phi_{\vec{k}_1}\delta F_{\vec{k}_2}-\frac{1}{F_M}\mu_{CD}k^2\delta F_{\vec{k}}\delta F_{\vec{k}}^*\right]$$
$$+\sum_{\vec{k}}\left[i(\omega*)J_0\delta\phi_{\vec{k}}^*\delta F_{\vec{k}}+\frac{\delta F_{\vec{k}}}{F_M}\sum_{\vec{k}=\vec{k}_1+\vec{k}_2}\vec{b}\cdot(\vec{k}_1\times\vec{k}_2)J_0(k_1\rho)\delta\phi_{\vec{k}_1}^*\delta F_{\vec{k}_2}^*-\frac{1}{F_M}\mu_{CD}k^2\delta F_{\vec{k}}^*\delta F_{\vec{k}}\right]$$

Then, we integral the equation by μ. Finally we get **the time derivative of Entropy**:

$$\frac{\partial}{\partial t}\int S(\mu)d\mu=2(\frac{a}{L_n})D_{i\delta_{\vec{k}}}-2\mu_{CD}\sum_{\vec{k}}k^2\int S(\mu)d\mu$$

The first term in the two brackets gives the $D_{i\delta_{\vec{k}}}$, the second term vanishes while apply the verify conjugation formula, the third term gives an damping term of entropy.

Drift kinetics

The $[1-J_0^2(Gk_\perp\rho)]$ term should be divided by G^2, when G go to 0.

As: $J_0(x)=1-\dfrac{x^2}{4}+\dfrac{x^4}{64}+O(high)$

This equation tells us how the Bessel function is effected by G.

Usually, we keep the second order of Bessel function:

$$\lim_{G\to 0} J_0(G\hat{k}_\perp\hat{\rho})=1-\frac{(G\hat{k}_\perp\hat{\rho})^2}{4}$$

The Poisson equation will reduce to 2D H-M form, while The $[1-J_0^2(Gk_\perp\rho)]$ term is divided by G^2.

$$\lim_{G\to 0}\delta\phi_{\vec{k}}=\lim_{G\to 0}\frac{\int d\mu J_0(Gk_\perp\rho)\delta F_{\vec{k}}}{\frac{T_{e0}}{T_{i0}}\int d\mu F_M[1-J_0^2(Gk_\perp\rho)]\frac{1}{G^2}+(\lambda_{\vec{k}}-i\delta_{\vec{k}})}$$

$$=\lim_{G\to 0}\frac{\int d\mu\delta F_{\vec{k}}}{\frac{T_{e0}}{T_{i0}}\int d\mu\,\mathrm{e}^{-\mu}\left[1-\left(1-\frac{G^2k^2\rho^2}{4}\right)^2\right]\frac{1}{G^2}+(\lambda_{\vec{k}}-i\delta_{\vec{k}})}$$

$$=\lim_{G\to 0}\frac{N_k}{\frac{T_{e0}}{T_{i0}}\int d\mu\,\mathrm{e}^{-\mu}\frac{G^2k^2 2\frac{T_{i0}}{T_{e0}}\mu}{2}\frac{1}{G^2}+(\lambda_{\vec{k}}-i\delta_{\vec{k}})}$$

$$=\frac{N_k}{k^2+(\lambda_{\vec{k}}-i\delta_{\vec{k}})}$$

Nonlinear conservation Verify conjugation

We prove that the following two formulas equal to zero. These two formulas could verify conjugation of potential function and distribution function in the code.

$$\sum_{\vec{k}} \delta F_{\vec{k}}^{\ *} \sum_{\vec{k}=\vec{k}_1+\vec{k}_2} \hat{z}\cdot\vec{k}_1\times\vec{k}_2 J_0(k_1\rho)\phi_{\vec{k}_1}\delta F_{\vec{k}_2} = \sum_{\vec{k}}\sum_{\vec{k}_1}\sum_{\vec{k}_2}\hat{z}\cdot\vec{k}_1\times\vec{k}_2 J_0(k_1\rho)\phi_{\vec{k}_1}\delta F_{\vec{k}_2}\delta F_{-\vec{k}}\delta(\vec{k}_1+\vec{k}_2-\vec{k})$$

$$=\sum_{\vec{k}}\sum_{\vec{k}_1}\sum_{\vec{k}_2}\hat{z}\cdot\vec{k}_1\times\vec{k}_2 J_0(k_1\rho)\phi_{\vec{k}_1}\delta F_{\vec{k}_2}\delta F_{\vec{k}}\delta(\vec{k}_1+\vec{k}_2+\vec{k})$$

$$=\sum_{\vec{k}}\sum_{\vec{k}_1}\sum_{\vec{k}_2}\hat{z}\cdot(-\vec{k}_2-\vec{k})\times\vec{k}_2 J_0(k_1\rho)\phi_{\vec{k}_1}\delta F_{\vec{k}_2}\delta F_{\vec{k}}\delta(\vec{k}_1+\vec{k}_2+\vec{k})$$

$$=\sum_{\vec{k}}\sum_{\vec{k}_1}\sum_{\vec{k}_2}\hat{z}\cdot(-\vec{k})\times\vec{k}_2 J_0(k_1\rho)\phi_{\vec{k}_1}\delta F_{\vec{k}_2}\delta F_{\vec{k}}\delta(\vec{k}_1+\vec{k}_2+\vec{k})$$

We could exchange the subscripts when their range is the same.

(we exchange the $\vec{k}$ and $\vec{k}_2$) $=\sum_{\vec{k}}\sum_{\vec{k}_1}\sum_{\vec{k}_2}\hat{z}\cdot(-\vec{k}_2)\times\vec{k} J_0(\vec{k}_1\rho)\phi_{\vec{k}_1}\delta F_{\vec{k}_2}\delta F_{\vec{k}}\delta(\vec{k}_1+\vec{k}_2+\vec{k})$

so $\sum_{\vec{k}} \delta F_{\vec{k}}^{\ *} \sum_{\vec{k}=\vec{k}_1+\vec{k}_2} \hat{z}\cdot\vec{k}_1\times\vec{k}_2 J_0(\vec{k}_1\rho)\phi_{\vec{k}_1}\delta F_{\vec{k}_2} = 0$

$$\sum_{\vec{k}} J_0(k\rho)\delta\phi_{\vec{k}}^{\ *} \sum_{\vec{k}=\vec{k}_1+\vec{k}_2} \hat{z}\cdot\vec{k}_1\times\vec{k}_2 J_0(k_1\rho)\delta\phi_{\vec{k}_1}\delta F_{\vec{k}_2}$$

$$=\sum_{\vec{k}}\sum_{\vec{k}_1}\sum_{\vec{k}_2}\hat{z}\cdot\vec{k}_1\times\vec{k}_2 J_0(k_1\rho)\delta\phi_{\vec{k}_1}\delta F_{\vec{k}_2} J_0(k\rho)\delta\phi_{-\vec{k}}\delta(\vec{k}_1+\vec{k}_2-\vec{k})$$

$$=\sum_{\vec{k}}\sum_{\vec{k}_1}\sum_{\vec{k}_2}\hat{z}\cdot\vec{k}_1\times\vec{k}_2 J_0(k_1\rho)\delta\phi_{\vec{k}_1}\delta F_{\vec{k}_2} J_0(k\rho)\delta\phi_{\vec{k}}\delta(\vec{k}_1+\vec{k}_2+\vec{k})$$

$$=\sum_{\vec{k}}\sum_{\vec{k}_1}\sum_{\vec{k}_2}\hat{z}\cdot\vec{k}_1\times\left(-\vec{k}_1-\vec{k}\right) J_0(k_1\rho)\delta\phi_{\vec{k}_1}\delta F_{\vec{k}_2} J_0(k\rho)\delta\phi_{\vec{k}}\delta(\vec{k}_1+\vec{k}_2+\vec{k})$$

$$=\sum_{\vec{k}}\sum_{\vec{k}_1}\sum_{\vec{k}_2}\hat{z}\cdot\vec{k}_1\times(-\vec{k}) J_0(k_1\rho)\delta\phi_{\vec{k}_1}\delta F_{\vec{k}_2} J_0(k\rho)\delta\phi_{\vec{k}}\delta(\vec{k}_1+\vec{k}_2+\vec{k})$$

We exchange the k and k1, then we again get:

$$\sum_{\vec{k}}\sum_{\vec{k}_1}\sum_{\vec{k}_2}\hat{z}\cdot\vec{k}\times(-\vec{k}_1) J_0(k_1\rho)\delta\phi_{\vec{k}_1}\delta F_{\vec{k}_2} J_0(k\rho)\delta\phi_{\vec{k}}\delta(\vec{k}_1+\vec{k}_2+\vec{k})$$

Thus $\sum_{\vec{k}} J_0(k\rho)\delta\phi_{\vec{k}}^{\ *} \sum_{\vec{k}=\vec{k}_1+\vec{k}_2} \hat{z}\cdot\vec{k}_1\times\vec{k}_2 J_0(k_1\rho)\delta\phi_{\vec{k}_1}\delta F_{\vec{k}_2} = 0$

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