$$\lambda = \frac{1}{\sqrt{\varepsilon}}$$

$$c_1 e^{-\frac{ix}{\sqrt{\varepsilon}}} + c_2 e^{+\frac{ix}{\sqrt{\varepsilon}}}$$

$y(0)=0,\quad y(1)=1$

$$y(x,\varepsilon) = \frac{\sin\frac{x}{\sqrt{\varepsilon}}}{\sin\frac{1}{\sqrt{\varepsilon}}} \qquad \varepsilon \neq (n\pi)^{-2};$$

RK: 1) 若 $f(x,\lambda) \sim \lambda\phi_1 + \phi_2 + \frac{\phi_3}{\lambda} + \cdots$ $($~~ϕ_0~~

$y'' + f\, y = 0 \qquad y = e^S$

$S = \sum_{n=0}^{\infty} g_n(\lambda)\psi_n(x)$; 只考虑第一项

$g_0^2 = O(\lambda) \Rightarrow g_0 = \lambda^{\frac{1}{2}} \qquad \left(g_0\psi_0'' + g_0^2\psi_0'^2 = O(\lambda); \right)$

2) $f(x,\lambda) = F(x)G(\lambda)$

$\mu^2 = G(\lambda) \Rightarrow y'' + \mu^2 F(x)\, y = 0$

转化为前面的形式

RK 奇异摄动转正则摄动

$Q(x) \sim ax + bx^2$ as $x \to 0$

$$\Rightarrow \int_0^x \sqrt{|Q|} \sim \int_0^x \sqrt{at+bt^2}\, dt$$

Taylor 展开

$$\sim \int_0^x \sqrt{at}\left(1+\frac{bt}{2a}\right) dt$$

$$= \frac{2}{3} a^{\frac{1}{2}} x^{\frac{3}{2}} + \frac{b}{5a^{1/2}} x^{\frac{5}{2}}$$

$|x|$ needs

$$\frac{1}{\varepsilon} \frac{b}{5a^{\frac{1}{2}}} x^{\frac{5}{2}} \ll 1$$

$\Rightarrow$ $x \ll \varepsilon^{\frac{2}{5}}$ as $\varepsilon \to 0$

precise region $\varepsilon^{\frac{2}{3}} \ll x \ll \varepsilon^{\frac{2}{5}}$

In II if $t \gg 1$, $Ai(t)$

$$\sim \frac{1}{2\sqrt{\pi}} t^{-\frac{1}{4}} e^{-\frac{2}{3} t^{\frac{3}{2}}}$$

$t = \varepsilon^{-\frac{2}{3}} a^{\frac{1}{3}} x$; $x \gg \varepsilon^{\frac{2}{3}}$

$$Bi(t) \sim \frac{1}{\sqrt{\pi}}\, t^{-\frac{1}{4}} e^{\frac{2}{3} t^{\frac{3}{2}}}$$

In I∩II $\quad y \sim y_2(x) \sim \frac{1}{\sqrt{\pi}}\, a^{-\frac{1}{12}} \varepsilon^{-\frac{1}{6}} x^{-\frac{1}{4}}$

$$\left(\frac{b_1}{2} e^{-\frac{2a^{\frac12}}{3\varepsilon} x^{\frac32}} + b_2\, e^{\frac{2a^{\frac12}}{3\varepsilon} x^{\frac32}} \right)$$

$$a^{-\frac14} a_1 = \frac{1}{\sqrt{\pi}}\, \varepsilon^{\frac16} a^{-\frac{1}{12}} b_2$$

$$a^{-\frac14} a_2 = \frac{1}{\sqrt{\pi}}\, \varepsilon^{\frac16} a^{\frac{1}{12}} \frac{b_1}{2}$$

~~文~~ 自己课下完成 II∩III 的匹配

① Matching ~~for I∩III~~

region $[\varepsilon^{\frac23}, \varepsilon^{\frac23}]$

是越来越小的,

Matching Variable is $t = \varepsilon^{-\frac23} a^{\frac13} x$

Matching region

$$1 \ll t \ll \varepsilon^{-\frac{4}{15}}$$

RK [1] WKB ~~me~~ analysis is not sensitive to the order of the diff eq.

I: $x \gg \varepsilon^{\frac{2}{3}}$ $x > 0$

III $x \ll -\varepsilon^{\frac{2}{3}}$ $x < 0$

II $|x| \ll 1$ $x = \frac{1}{6}\varepsilon$

In I, WKB analysis

$$y(x,\varepsilon) = a_1 Q(x)^{-\frac{1}{4}} e^{\frac{1}{\varepsilon}\int_0^x \sqrt{Q(t)}\,dt}$$
$$+ a_2 Q(x)^{-\frac{1}{4}} e^{-\frac{1}{\varepsilon}\int_0^x \sqrt{Q(t)}\,dt}$$

In III $Q < 0$

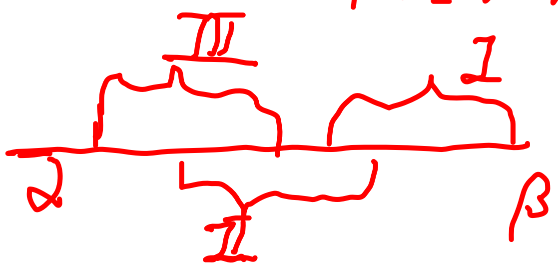
$$y(x,\varepsilon) = (-Q(x))^{-\frac{1}{4}} \cdot$$
$$\left\{ c_1 \exp\left(\frac{i}{\varepsilon}\int_0^x \sqrt{-Q(t)}\,dt\right) + c_2 \exp\left(-\frac{i}{\varepsilon}\int_0^x \sqrt{-Q(t)}\,dt\right) \right\}$$

In II, $\varepsilon^2 y'' \approx axy$

In III $t = \varepsilon^{-\frac{2}{3}} a^{\frac{1}{3}} x$ $\varepsilon^2 y'' = axy$

$\Rightarrow y'' = ty$

$$y(x,\varepsilon) \sim y_2(x) = b_1 Ai(\varepsilon^{-\frac{2}{3}} a^{\frac{1}{3}} x) + b_2 Bi(\varepsilon^{-\frac{2}{3}} a^{\frac{1}{3}} x);$$



Moreover, for $|x|$ sufficiently small, $Q(x) \sim ax$, hence

$$\int_0^x \sqrt{|Q(t)|}\,dt = \sqrt{a}\int_0^x \sqrt{t}\,dt = \frac{2}{3}\sqrt{a}\, x^{\frac{3}{2}}.$$

考虑 I∩II: $y \sim y_1(x) \sim$ sign(x) $a^{-\frac{1}{4}} x^{-\frac{1}{4}}$

$$\left(a_1 e^{\frac{2a^{\frac{1}{2}}x^{\frac{3}{2}}}{2\varepsilon}} + a_2 e^{-\frac{2a^{\frac{1}{2}}x^{\frac{3}{2}}}{2\varepsilon}}\right); \quad (*)$$

Rk: Precise region of validity of (*) depends on $Q(x)$

$\xi = \lambda x$,

$$y'' + \lambda^{-2} f\left(\frac{\xi}{\lambda}, \lambda\right) = 0.$$

但 $\xi \in (\lambda\alpha, \lambda\beta)$ 没有一致性.

不可以

Turning pts.

$$\varepsilon^2 y'' = Q(x)\, y \qquad x \in (\alpha, \beta)$$

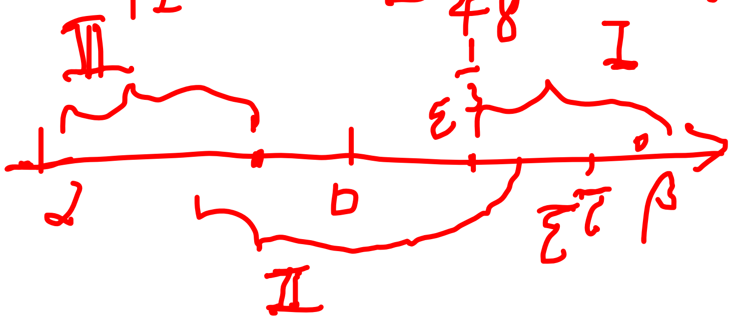
WLOG assume $Q(x) \sim ax$ as $x \to 0$ $(a>0)$ moreover, assume

$$\begin{cases} Q(x) > 0, & 0 < x < \beta \\ Q(x) < 0, & \alpha < x < 0 \end{cases}$$

WKB $\Longrightarrow$ $\psi_0(x) \sim \pm \frac{2}{3} a^{\frac{1}{2}} x^{\frac{3}{2}}$

$$\psi_1(x) = -\frac{1}{4} \ln x$$

$$\psi_2(x) \sim \pm \frac{5}{48} a^{-\frac{1}{2}} x^{-\frac{3}{2}}$$



II 段用 Airy Equation 方法.

RK: For $\varepsilon \neq 0$, fixed, $\frac{\alpha(x)}{\varepsilon^2}$ analytic at $x=0$,

渐近匹配 Asymptotic matching

Validity of physical optics:

$$\left.\begin{array}{l}\frac{1}{\varepsilon}\psi_0 \gg \psi_1 \gg \varepsilon\psi_2 \\ \varepsilon\psi_2 \ll 1\end{array}\right\}$$ 得出 x 的范围

as $\varepsilon \to 0^+$

$$x \gg \varepsilon^{\frac{2}{3}}$$

2) Can not apply to nonlinear case, can apply to inhomogeneous linear diff. eq.

"bounded layer" 边界层方法

多重尺度法.