Lecture 2: Separation axioms and Countabilities

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1 Topological space: separation axioms

We already know that a topological space is a set of points, together with a set of neighborhoods for each point. Now we study more special (practically useful) neighborhoods in terms of separation axioms.

Definition 1.1 *Let (X, T) be a topological space. The topology satisfies the*

(0) T_0 axiom, if every two points $x \neq y$ there is an open neighborhood U containing one but not the other point.

(1) T_1 axiom, if any two points $x \neq y$, there are open neighborhoods $U_x \ni x, U_y \ni y$ but $y \notin U_x, x \notin U_y$;

(2) T_2 axiom (i.e. Hausdorff space), if any two distinct points x, y, there are open neighborhoods $U_x \ni x, U_y \ni y$ but $U_x \cap U_y = \emptyset$;

(3) T_3 axiom, if any point x and closed set Y with $x \notin Y$, there are open neighborhoods $U_x \ni x, U_y \supset Y$ but $U_x \cap U_y = \emptyset$;

(4) T_4 axiom, if any two disjoint closed sets x, y, there are open neighborhoods $U_x \supset x, U_y \supset y$ but $U_x \cap U_y = \emptyset$;

Example 1.2 *A metric space (X, d) satisfies T_i, $i = 0, 1, 2, 3, 4$.*

Proof. For any distinct $x, y \in X$, the distance $d(x, y) > 0$. Choose $U_x = B(x, \frac{d(x,y)}{2})$ and $U_y = B(y, \frac{d(x,y)}{2})$ to chect the axioms T_1, T_2. For any point x and a closed set Y with empty intersection, the distance $d(x, Y) = \inf\{d(x, y) \mid y \in Y\} > 0$ (otherwise, $d(x, y_n) \to 0$ and $x \in \bar{Y} = Y$ for some y_n). Choose $U_x = B(x, \frac{d(x,Y)}{2})$ and $U_y = \cup_{y \in Y} B(y, \frac{d(x,y)}{2})$ to chect the axioms T_3.

For disjoint closed sets A, B, choose $U_A = \cup_{a \in A} B(a, \frac{d(a,B)}{2})$ (by the previous paragraph, $d(a, B) > 0$) and $U_B = \cup_{b \in B} B(b, \frac{d(b,A)}{2})$. We claim that $U_A \cap U_B = \emptyset$. Otherwise, there would be a point $x \in B(a, \frac{d(a,B)}{2}) \cap B(b, \frac{d(b,A)}{2})$ for some $a \in A, b \in B$. But $d(a, b) \leq d(a, x) + d(x, b) < \frac{d(a,B)}{2} + \frac{d(b,A)}{2} \leq \frac{d(a,b)}{2} + \frac{d(b,a)}{2} = d(a, b)$, a contradiction. ∎

Lemma 1.3 $T_2 \Longrightarrow T_1 \Longrightarrow T_0$.

Proof. It's clear that T_2 implies T_1 and T_1 implies T_0. ∎

Example 1.4 *[Zariski topology] Let $X = \mathbb{C}$, the set of complex numbers. Define $T = \{\mathbb{C}\backslash\text{zero}(f) \mid f$ is a polynomial$\}$, where $\text{zero}(f)$ is the roots of f. Check that T is a topology and satisfies T_1 but not T_2.*

Proof. Since $\text{zero}(0) = \mathbb{C}$ and $\text{zero}(1) = \emptyset$, we see that $X, \emptyset \in T$. Moreover, $\text{zero}(f) \cup \text{zero}(g) = \text{zero}(fg)$ and $\cap_{i\in I}\text{zero}(f_i) = \text{zero}(f)$, where f is a generator of the ideal $\langle f_i, i \in I\rangle$ (a principal ideal). This proves that T is a topology. For any two distinct points $c, c' \in \mathbb{C}$, we see that $c \in \mathbb{C}\backslash\text{zero}(x - c')$ and $c' \in \mathbb{C}\backslash\text{zero}(x - c)$ proving T_1. But the intersection of any two nonempty open sets $\mathbb{C}\backslash\text{zero}(f), \mathbb{C}\backslash\text{zero}(g)$ are non-empty, implying T is not T_2. ■

Lemma 1.5 *A topological space X is T_1 space if and only if a single point is closed.*

Proof. If a point x is not closed, then $x \neq \bar{x}$. By the definition of closure, every neighborhood of x containing another point $y \in \bar{x}$. This implies that x, y are not separated.

If every point is closed, for any two distinct points x, y, the complements $X\backslash x, X\backslash y$ are open and separate x, y. ■

Corollary 1.6 *If X is a T_1 space, then $T_4 \Longrightarrow T_3 \Longrightarrow T_2$.*

Exercise 1.7 *Give examples to show that T_4 does not imply T_3 and T_3 does not imply T_2.*

2 Countability

Definition 2.1 *A topological space (X, T) is*

(1) first countable (i.e. C_1 space) if every point x has a countable set $\mathcal{N}_x$ consisting of neighborhoods of x such that every neighborhood N of x contains an element of $\mathcal{N}_x$. We call $\mathcal{N}_x$ a neighborhood basis of x.

(2) second countable (i.e. C_2 space) if T has a countable basis $\mathcal{N}$, i.e. every open set $U \in T$ is a union of elements in $\mathcal{N}$ and the intersection of any two open sets in $\mathcal{N}$ is still in $\mathcal{N}$.

It's clear that a C_2 space is C_1 by taking $\mathcal{N}_x = \{U \in \mathcal{N} \mid x \in U\}$.

Example 2.2 *A metric space is not necessary C_2, eg. $X = \mathbb{R}$, with metric defined by $d(x, y) = 1$ if $x \neq y$. The metric topology on X is discrete. A basis must contain each point, hence uncountable. On the other hand, a C_2 space is separable, since the subspace consisting of one point from each open set in a basis is dense.*

Example 2.3 *A separable metric space X is second countable, since it has a countable dense subspace Y a countable basis $\mathcal{N} = \{B(y, \frac{1}{n}) \mid y \in Y, n \in \mathbb{N}\}$.*

Lemma 2.4 *If X has a countable neighborhood basis $\mathcal{N}_x$ at x, then there exists countable neighborhood basis $\{U_i\}_{i=1}^{+\infty}$ such that $U_{i+1} \subset U_i$.*

Proof. Suppose that $\mathcal{N}_x = \{V_i\}_{i=1}^{+\infty}$. Choose $U_i = \cap_{1\leq j\leq i} V_j$. ■

Theorem 2.5 *Let X be a C_1 topological space and $f : X \to Y$ map between topological spaces. Then f is continuous at a point x_0 if and only if $f(x_n) \to f(x_0)$ when $x_n \to x_0$.*

Proof. We already know the "only if" part. If f is not continuous at x_0, for some neighborhood U containing $f(x_0)$ the preimage $f^{-1}(U)$ is not a neighborhood of x_0. This implies that every open neighborhood of x_0 contains a point $y \notin f^{-1}(U)$. Since x_0 has a countable neighborhood basis, the previous lemma implies that x_0 has basis $\{U_i\}_{i=1}^{+\infty}$ such that $U_{i+1} \subset U_i$. Therefore, there exists $y_n \in U_n \backslash f^{-1}(U)$ such that $y_n \to x_0$. But $f(y_n) \to f(x_0)$ implies that almost all $f(y_n) \in U$. This is a contradiction. ■