HIT,Shenzhen
Differential Geometry and Topology
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Homework 3

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3.1. Show that applying an isometry of $\mathbb{R}^3$ does not change the first fundamental form. What is the effect of a dilation (i.e. a map $\mathbb{R}^3 \to \mathbb{R}^3$ given by $x \to ax$ for some constant a $\neq 0$)?

Proof. An isometry in $\mathbb{R}^3$ has the form $f : x \to xP + a$.
$(f \circ \sigma)_u = \sigma_u P, (f \circ \sigma)_v = \sigma_v P \Rightarrow E(f \circ \sigma) = \sigma_u P P^T \sigma_u^T = \sigma_u \sigma_u^T = E(\sigma)$.
Similarly, F, G are also unchanged under the isometry and the first fundamental form remains the same.

For dilation, $g : x \to ax$.
$(g \circ \sigma)_u = a\sigma_u, (g \circ \sigma)_v = a\sigma_v \Rightarrow E(g \circ \sigma) = a^2 \sigma_u \cdot \sigma_u = a^2 E(\sigma)$.
Similarly, $F(g \circ \sigma) = a^2 F(\sigma), G(g \circ \sigma) = a^2 G(\sigma)$. If $a = \pm 1$, the first fundamental form is unchanged; otherwise, it changes. □

3.2. Let $\gamma : (a, b) \to \mathbb{R}^3$ be a unit speed curve. The surface of tangent developable is given by $\sigma(u, v) = \gamma(u) + v\gamma'(u)$

(1) Compute the first fundamental form of σ; Show that the first fundamental form is independent of the torsion of γ;
(2) Show that the tangent developables of two curves γ_1, γ_2 are locally isometric if their curvature functions are the same;
(3) Show that the tangent developable σ is locally isometric to a plane.

Proof.

(1) $\sigma_u = \gamma'(u) + v\gamma''(u), \sigma_v = \gamma'(u)$. Since $\gamma'_u \circ \gamma''_u = 0$, the first fundamental form is $(1 + v^2\kappa^2)du^2 + 2dudv + dv^2$, where κ is the curvature of the curve. From this expression, we see that the first fundamental form is indepedent with the torsion τ of γ.
(2) We further suppose both γ_1, γ_2 are regular. For $\sigma_1(u, v)$ on the tangent developable of γ_1, we map it to $\sigma_2(u, v)$ on the tangent developable of γ_2. Since $\gamma_1' \neq 0$, we can find a neighborhood $N_1 = N(\sigma_1(u, v), \epsilon_1) \cap \sigma_1$, such that $\sigma_1^{-1}(N_1)$ and N_1 is one-to-one. Similarly we can find $\sigma_2(u, v) \in N_2 \subseteq \sigma_2$ such that $\sigma_2^{-1}(N_2)$ and N_2 is one-to-one. Let $K = \sigma_1^{-1}(N_1) \cap \sigma_2^{-1}(N_2)$, then $\sigma_1(K)$ and $\sigma_2(K)$ are one-to-one. Therefore, we construct a locally smooth mapping f from $\sigma_1(K)$ to $\sigma_2(K)$ as $\sigma_2 \circ \sigma_1^{-1}$. The first fundamental form of σ_1 and $f \circ \sigma_1$ are the same from (1). It follows that f is isometric.
(3) By fundamental theorem of curves, it is possible to construct a planar curve with $\kappa(u)$ as curvature. From (2) σ is locally isometric to a plane.

□

3.3. Show that Enneper's surface

$$\sigma(u,v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2) \tag{1}$$

is conformally parametrized.

Proof. $\sigma_u = (1 - u^2 + v^2, 2uv, 2u), \sigma_v = (2uv, 1 - v^2 + u^2, -2v)$. The first fundamental form is $(1 + u^2 + v^2)^2(du^2 + dv^2)$, which is proportional to the first fundamental form of plane. Therefore, the surface is conformally parametrized. □