Lecture 1 remark: Basic concepts of topological spaces

zhaofeng-shu33

April 24, 2018

1 Topological space: concepts and examples

Definition 1.0 *Let X be a non-empty set. The power set of X, denoted as 2^X is $\{Y|Y \subseteq X\}$*

Example 1.0 $X = \{1, 2\}, 2^X = \{\{1\}, \{2\}, \{1, 2\}, \varnothing\}$

Example 1.1 (Euclidean topology in $\mathbb{R}$) *Let $X = \mathbb{R}, T = \{U|U = \bigcup_{i\in I}(a_i, b_i)\}$*

Example 1.4 (Cocountable) $X = \mathbb{R}, T = \{\mathbb{R}\backslash Y|Y \textit{ is a countable set in } \mathbb{R}\} \cup \varnothing$

Proof. We verify that T is a topological space.

1. $X, \varnothing \in T$
2. For $\mathbb{R}\backslash Y_1 \in T, \mathbb{R}\backslash Y_2 \in T$, wher Y_1, Y_2 are countable set, their intersection $(\mathbb{R}\backslash Y_1)\cap(\mathbb{R}\backslash Y_2) = \mathbb{R}\backslash(Y_1\cup Y_2)$. Since $Y_1\cup Y_2$ is countable, $\mathbb{R}\backslash(Y_1\cup Y_2) \in T$
3. For $\mathbb{R}\backslash Y_i \in T$, since $\bigcap Y_i$ is countable, $\bigcup(\mathbb{R}\backslash Y_i \in T) = \mathbb{R}\backslash(\bigcap Y_i) \in T$

■

Example 1.5 (Cofiniteness) $X = \mathbb{R}, T = \{\mathbb{R}\backslash Y|Y \textit{ is a finite set in } \mathbb{R}\} \cup \varnothing$

Example 1.6 Proof. We show that $d(f, g) \triangleq \|f - g\|$ is a distance function. (1,2) in **Definition 1.4** are obvious. to show (3), we have

$$\begin{aligned}
|f(x) - h(x)| &\leq |f(x) - g(x)| + |g(x) - h(x)| \\
&\leq \max_{x\in[a,b]}|f(x) - g(x)| + \max_{x\in[a,b]}|g(x) - h(x)| \\
\Rightarrow \max_{x\in[a,b]}|f(x) - h(x)| &\leq \max_{x\in[a,b]}|f(x) - g(x)| + \max_{x\in[a,b]}|g(x) - h(x)|
\end{aligned}$$

That is, $d(f, h) \leq d(f, g) + d(g, h)$ ■

Example 1.7 *Let $Y \subseteq \mathbb{R}^n, d_E$ is the Euclid norm defined in* **Example 1.5**. *Then (Y, d_E) is also a metric space, called induced metric space.*

Exercise 1.10 **Proof.**

$\Rightarrow$: if $x \in A$, then $U \cap A \neq \varnothing$. Otherwise $x \in A'$, by the definition of accumulated point, $U \cap A \neq \varnothing$.

$\Leftarrow$: if $x \in A$, then $x \in \bar{A}$. Otherwise, for any neighborhood U of x. Since $U \cap A \neq \varnothing$ and $x \notin A$, U contains a point in $A \backslash \{x\}$. Therefore, $x \in A' \subseteq \bar{A}$

Below we show that $\bar{A}$ is closed.

By the above conclusion,$\forall x \notin \bar{A}$, there exists a neighborhood U of x such that $U \cap A = \varnothing$. By the definiton of $A', U \cap A' = \varnothing$. Therefore $U \cap \bar{A} = \varnothing$. Then x is an interior point of $\bar{A}^c \Rightarrow \bar{A}^c$ is open $\Rightarrow \bar{A}$ is closed. ∎

2 Continuous functions

Lemma 2.2 *(4) f is continuous at every point.*

Proof.

(2)$\Rightarrow$ (3): if C_Y is closed in Y, C_Y^c is open in Y. $[f^{-1}(C_Y)]^c = f^{-1}(C_Y^c)$ is open by (2). Therefore $f^{-1}(C_Y)$ is closed.

(4)$\Rightarrow$ (1): Let U_Y be an open set in Y, $y \in f^{-1}(U_Y)$. Since f is continuous at y and U_Y is an open neighborhood of $f(y)$, by pointwise continuous definition $f^{-1}(U_Y)$ is an open set. ∎

Example 2.3 $X = (0, 2\pi) \subseteq (\mathbb{R}, d), Y = \{z \in \mathbb{C} | \|z\| = 1\} \subseteq \mathbb{C} = (\mathbb{R}^2, d)$. *Let $f : X \to Y$ such that $f(t) = e^{it}$. By geometric intuition f is continuous (The interval $(2\pi - \epsilon, 2\pi]$ is open in X). However, the inverse function $f^{-1} : Y \to X$ is not continuous. The preimage of interval $(2\pi - \epsilon, 2\pi]$ is mapped to a circular arc with the point $(1, 0)$, which is not open in Y.*

Lemma 2.4 **Proof.** For an open set U_Z in Z, $g^{-1}(U_Z)$ is open in Y since g is continuous. $f^{-1} \circ g^{-1}(U_Z)$ is open in X since f is continuous. $(g \circ f)^{-1}(U_Z) = f^{-1} \circ g^{-1}(U_Z)$ for any U_Z, hence $g \circ f$ is continuous. ∎

Example 2.5 *Let $X = (\mathbb{R}, d), x_i \to x$ iff $\forall \epsilon > 0, \exists N, s.t. \forall n > N, |x_n - x| < \epsilon$*

Proof.

$\Rightarrow \forall \epsilon > 0$, choose $V = (x - \epsilon, x + \epsilon)$, since $x_i \to x, \exists N, s.t. \forall n > N, x_n \in V$. That is, $|x_n - x| < \epsilon$.

$\Leftarrow \forall V \ni x$, since V is open, $V = \bigcup_{i \in I}(x_i - \epsilon, x_i + \epsilon)$. Then $\exists i_0 \in I, s.t. x \in (x_{i_0} - \epsilon_{i_0}, x_{i_0} + \epsilon_{i_0})$. Choose $\epsilon = \epsilon_{i_0} - |x - x_{i_0}|$, then $(x - \epsilon, x + \epsilon) \subseteq (x_{i_0} - \epsilon_{i_0}, x_{i_0} + \epsilon_{i_0})$. For the chosen $\epsilon, \exists N, .s.t. \forall n > N, x_n \in (x - \epsilon, x + \epsilon) \subseteq V$. Therefore $x_i \to x$. ∎

Example 2.6 (**Example 2.3** continued) *We can use* **Corollary 2.6** *to show that f^{-1} is not continuous. Construct $x_i \to x = (1, 0)$, such that* $\mathrm{Re}[x_{2n}] < 0, \mathrm{Re}[x_{2n+1}] > 0$. *Then $f^{-1}(x_{2n}) \to 2\pi, f^{-1}(x_{2n+1}) \to 0 \Rightarrow f^{-1}(x_n)$ is not convergent.*

Example 2.7 *We show that if $x_n \to x$ in the topology T, then $x = x_n$ for sufficiently large n.*

Proof. We proceed by contradiction. Suppose there exists $\{x_{n_k}\}, s.t. \, n_k \to \infty$ but $x_{n_k} \neq x$. Then we construct $V = X \backslash \{x_{n_k}\}_{k=1}^{\infty} \in T$. Obviously, $x \in V$, but by the definition of convergence, $x_n \nrightarrow x$ ■

3 Homeomorphism

Example 3.2 *$f : x \to \tan(\frac{\pi}{2} \frac{2x-a-b}{b-a})$ maps (a, b) to $\mathbb{R}$.*

We can extend this result to map the unit disk in $\mathbb{R}^2$ to the whose plane by $(r, \theta) \to (\tan(\frac{\pi}{2} r), \theta)$ (in polar coordinate).

4 Construct new topologies: subs and products

Example 4.0 *Let $X = (\mathbb{R}, d_E), Y = [0, 1]$, then $T_Y = \{\bigcup_{i \in I} (a_i, b_i) \cap Y\}$.e.g. $[0, \frac{1}{2}] \in T_Y$*

Example 4.1 *Let (X, d) be a metric space, $Y \subseteq X$. Then (Y, d) is a topological space with $T_Y^d = \{\bigcup B_Y(y_i, r_i)\}$, where $B_X(y_i, r_i) = \{x \in X | d(x, y_i) < r_i\}$ and $B_Y(y_i, r_i) = B_X(y_i, r_i) \cap Y = \{x \in Y | d(x, y_i) < r_i\}$.*

Example 4.2 We show that A is homeomorphic to $\mathbb{R}^2$ by the spherical representation of complex numbers. see The Extended Plane and Its Spherical Representation for detail.

Remark 4.3 (on **Definition 4.3**) *The topology basis S of X satisfies*

1. $\forall V \in T, V = \bigcup S_i, S_i \in S$
2. $\forall S_1, S_2 \in S, S_1 \cap S_2 = \bigcup_{i \in I} S_i$

Lemma 4.4 (**Proof.** continued) We also need to show that if T is a topology containing S, then the right hand side set $\subseteq T$. Since T is a topology space, $U_i \in S \subseteq X, \bigcup U_i \in T$. $\{U | U = \bigcup U_i, U_i \in S\} \subseteq T$,and the proof is complete.

Exercise 4.6 Proof. $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \times V_2)$. Since $U_1 \cap U_2 \in T_X, V_1 \cap V_2 \in T_Y$, $(U_1 \cap U_2) \times (V_1 \times V_2) \in \{U \times V | U \in T_X, V \in T_Y\}$. ■

Remark 4.6 *From* Exercise **4.6**, $\{U \times V | U \in T_X, V \in T_Y\}$ is a topology basis of the product topology on $U \times V$.

Example 4.6 *Let $X = (\mathbb{R}, d_E)$, $X \times X = (\mathbb{R}^2, T_p)$, where T_p is the product topology on $\mathbb{R}^2$. Let T_E be the metric topology on $\mathbb{R}^2$. Then $T_P = T_E$.*

Proof. T_E is the set of union of open ball in $\mathbb{R}^2$ and T_p is the set of union of rectangle.

$T_E \subseteq T_p$: For $x \in B(x_i, r_i)$, we can find a rectangle such that $x \in (a(x), b(x)) \times (c(x), d(x)) \subseteq B(x_i, r_i)$. Therefore $B(x_i, r_i) = \bigcup_{x \in B(x_i, r_i)} (a(x), b(x)) \times (c(x), d(x))$.

$T_p \subseteq T_E$: $(a_i, b_i) \times (c_i, d_i) = \bigcup B(y_i, r_i)$ ■

Lemma 4.7 Proof.

$\Rightarrow$: $f_1 = P_1 \circ f$, where $P_1 : X_1 \times X_2 \to Y, P_1(x_1, x_2) = x_1$. For any open set $V \subseteq X, P_1^{-1}(V) = V \times X_2$ is open in the product topology of $X_1 \times X_2$. Therefore P_1 is continuous and the composite function f_1 is continuous. Similarly, f_2 is continuous.

$\Leftarrow$: For any open set $V \in \langle T_1 \times T_2 \rangle, V = \bigcup_{i \in I} V_i \times U_i, f^{-1}(V) = \bigcup_{i \in I} f^{-1}(V_i \times U_i) = \bigcup_{i \in I} f_1^{-1}(V_i) \times f_2^{-1}(U_i)$. Since f_1, f_2 are continuous and V_i, U_i are open, $f_1^{-1}(V_i), f_2^{-1}(U_i)$ are open in Y. Then $f^{-1}(V)$ is open and f is continuous. $\blacksquare$