

**Homework 2**

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May 9, 2018

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- 2.1. How many topologies could be defined on the two-element set  $X = \{a, b\}$ ?

*Solution.*

- $T = \{X, \emptyset\}$
- $T = \{X, \emptyset, \{a\}\}$
- $T = \{X, \emptyset, \{b\}\}$
- $T = \{X, \emptyset, \{a\}, \{b\}\}$

- 2.2. Find the closure of  $\{(x, \sin \frac{1}{x}) | 0 < x \leq 1\}$  in the 2-dimensional Euclidean space  $\mathbb{R}^2$ .

*Proof.* We show that the closure  $\bar{A}$  of  $A \triangleq \{(x, \sin \frac{1}{x}) | 0 < x \leq 1\}$  is  $\{(0, y) | -1 \leq y \leq 1\} \cup A$ . For  $(0, y)$ ,  $|y| \leq 1$ , we can find  $(x_n, \sin \frac{1}{x_n})$ , where  $x_n = \frac{1}{2\pi n + \arcsin y}$  such that  $(x_n, \sin \frac{1}{x_n}) \rightarrow (0, y)$ .  $\square$

- 2.3. Prove that  $\mathbb{R}^2 \setminus \{(0, 0)\}$  (as a subspace of  $\mathbb{R}^2$ ) and  $\{(x, y, z) | x^2 + y^2 = 1\}$  (as a subspace of  $\mathbb{R}^3$ ) are homeomorphic.

*Proof.* We can construct a homeomorphic mapping from  $(0, \infty)$  to  $(-\infty, +\infty)$ , such as  $x \rightarrow x - \frac{1}{x}$ . Then consider the polar coordinate representation of the plane without the origin. For  $(r, \theta) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ , use mapping  $(\cos \theta, \sin \theta, r - \frac{1}{r})$  and we get a point on the cylinder  $\{(x, y, z) | x^2 + y^2 = 1\}$ . It is easy to check that the mapping is a homeomorphic.

We can also use a homeomorphic mapping from the cylinder to  $\mathbb{R} \setminus \{(0, 0)\}$  by  $(\cos \theta, \sin \theta, z) \rightarrow (e^z \cos \theta, e^z \sin \theta)$ .  $\square$

- 2.4. Let  $(X, d)$  be a metric space and  $A \subseteq X$  a closed subset. Define  $f : X \rightarrow \mathbb{R}$  by  $f(x) = \inf_{a \in A} d(x, a)$ . Prove that  $f$  is continuous and that  $f(x) = 0$  if and only if  $x \in A$ .

*Proof.* Suppose  $V$  is open in  $\mathbb{R}$ . If  $f^{-1}(V)$  is empty, it is open. We consider non-empty case in the following. For any  $x \in f^{-1}(V)$ ,  $f(x) \in V$ , we can find  $\epsilon$  such that  $(f(x) - \epsilon, f(x) + \epsilon) \subseteq V$ . We show that  $B(x, \frac{1}{2}\epsilon) \subseteq f^{-1}(V)$ . Indeed,  $\forall y \in B(x, \frac{1}{2}\epsilon)$ ,  $d(y, x) < \frac{1}{2}\epsilon$ . Then  $d(y, a) \leq d(y, x) + d(x, a) < \frac{1}{2}\epsilon + d(x, a) \Rightarrow f(y) = \inf_{a \in A} d(y, a) \leq \frac{1}{2}\epsilon + d(x, a) \Rightarrow f(y) \leq \frac{1}{2}\epsilon + \inf_{a \in A} d(x, a) < \epsilon + f(x)$ . Exchange the position of  $x$  and  $y$ :  $f(x) < \epsilon + f(y) \Rightarrow f(y) \in (f(x) - \epsilon, f(x) + \epsilon) \subseteq V$ . Therefore  $f$  is continuous.

If  $x \in A$ ,  $f(x) = 0$ ; if  $f(x) = 0$ , there exists  $\{y_n\} \subseteq A$  such that  $d(x, y_n) \rightarrow 0$ , and  $x \in \bar{A} = A$  □

- 2.5. A topological space  $X$  is called separable if there is a countable dense subset  $A$ . Prove that if two topological spaces  $X_1, X_2$  are separable, then the product  $X_1 \times X_2$  is also separable.

*Proof.* Let  $A_1, A_2$  be dense countable subset of  $X_1, X_2$  respectively.  $A_1 \times A_2$  is countable. Below we show that  $\overline{A_1 \times A_2} = X_1 \times X_2$ . We consider  $(x_1, x_2) \notin (A_1, A_2)$  and assume  $x_1 \notin A_1$  for example. For  $(x_1, x_2) \in X_1 \times X_2$  and an open set  $V \in X_1 \times X_2$  covering  $(x_1, x_2)$ .  $V = \bigcup U_i \times V_i$ , where  $U_i \in T_{X_1}, V_i \in T_{X_2}$ . Then  $(x_1, x_2) \in U_i \times V_i$  for some  $i$ . Since  $A_1$  is dense in  $X_1$  and  $x_1 \notin A_1$ ,  $u_1 \in U_i \setminus \{x_1\} \cap A_1 \neq \emptyset$ . If  $x_2 \in A_2$ , let  $v_1 = x_2$ ; if  $x_2 \notin A_2$ ,  $v_1 \in V_i \setminus \{x_2\} \cap A_2 \neq \emptyset$ .  $(u_1, v_1) \in U_i \times V_i \setminus \{(x_1, x_2)\} \cap A_1 \times A_2 \neq \emptyset \Rightarrow V \setminus \{(x_1, x_2)\} \cap A_1 \times A_2 \neq \emptyset \Rightarrow (x_1, x_2) \in (A_1 \times A_2)'$ . Therefore,  $\overline{A_1 \times A_2} = X_1 \times X_2$ . □