HIT,Shenzhen
Differential Geometry and Topology
Spring 2018

Homework 2

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2.1. How many topologies could be defined on the two-element set $X = \{a, b\}$?

Solution.

- $T = \{X, \varnothing\}$
- $T = \{X, \varnothing, \{a\}\}$
- $T = \{X, \varnothing, \{b\}\}$
- $T = \{X, \varnothing, \{a\}, \{b\}\}$

2.2. Find the closure of $\{(x, \sin\frac{1}{x}|0 < x \leq 1)\}$ in the 2-dimensional Euclidean space $\mathbb{R}^2$

Proof. We show that the closure $\bar{A}$ of $A = \{(x, \sin\frac{1}{x}|0 < x \leq 1)\}$ is $\{(0, y)| -1 \leq y \leq 1\} \cup A$. For $(0, y), |y| \leq 1$, we can find $(x_n, \sin\frac{1}{x_n})$, where $x_n = \frac{1}{2\pi n + \arcsin y}$ such that $(x_n, \sin\frac{1}{x_n}) \to (0, y)$. □

2.3. Prove that $\mathbb{R}^2 \backslash \{(0, 0)\}$ (as a subspace of $\mathbb{R}^2$) and $\{(x, y, z)|x^2 + y^2 = 1\}$(as a subspace of $\mathbb{R}^2$) are homeomorphic.

Proof. We can construct a homeomorphic mapping from $(0, \infty)$ to $(-\infty, +\infty)$, such as $x \to x - \frac{1}{x}$. Then consider the polar coordinate representation of the plane without the origin. For $(r, \theta) \in \mathbb{R}^2 \backslash \{(0, 0)\}$, use mapping $(\cos\theta, \sin\theta, r - \frac{1}{r})$ and we get a point on the cylinder $\{(x, y, z)|x^2 + y^2 = 1\}$. It is easy to check that the mapping is a homeomorphic. □

2.4. Let (X, d) be a metric space and $A \subseteq X$ a closed subset. Define $f : X \to \mathbb{R}$ by $f(x) = \inf_{a \in A} d(x, a)$. Prove that f is continuous and that $f(x) = 0$ if and only if $x \in A$.

Proof. Suppose V is open, then for any $x \in f^{-1}(V), f(x) \in V$, we can find ϵ such that $(f(x) - \epsilon, f(x) + \epsilon) \subseteq V$. We show that $B(x, \frac{1}{2}\epsilon) \subseteq f^{-1}(V)$. Indeed, $\forall y \in B(x, \frac{1}{2}\epsilon), d(y, x) < \frac{1}{2}\epsilon$. Then $d(y, a) \leq d(y, x) + d(x, a) < \frac{1}{2}\epsilon + d(x, a) \Rightarrow f(y) = \inf_{a \in A} d(y, a) \leq \frac{1}{2}\epsilon + d(x, a) \Rightarrow f(y) \leq \frac{1}{2}\epsilon + \inf_{a \in A} d(x, a) < \epsilon + f(x)$. Exchange the position of x and y: $f(x) < \epsilon + f(y) \Rightarrow f(y) \in (f(x) - \epsilon, f(x) + \epsilon) \subseteq V$. Therefore f is continuous.

If $x \in A$, $f(x) = 0$; if $f(x) = 0$, there exists $\{y_n\}$ such that $d(x, y_n) \to 0$, and $x \in \bar{A} = A$ □

2.5. A topological space X is called separable if there is a countable dense subset A. Prove that if two topological spaces X_1, X_2 are separable, then the product $X_1 \times X_2$ is also separable.

Proof. Let A_1, A_2 be dense countable subset of X_1, X_2 respectively. $A_1 \times A_2$ is countable. Below we show that $\overline{A_1 \times A_2} = X_1 \times X_2$. We consider $(x_1, x_2) \notin (A_1, A_2)$ and assume $x_1 \notin A_1$ for example. For $(x_1, x_2) \in X_1 \times X_2$ and an open set $V \in X_1 \times X_2$ covering (x_1, x_2). $V = \bigcup U_i \times V_i$, where $U_i \in T_{X_1}, V_i \in T_{X_2}$. Then $(x_1, x_2) \in U_i \times V_i$ for some i. Since A_1 is dense in X_1 and $x_1 \notin A_1$, $U_i \backslash \{x_1\} \cap A_1 \neq \varnothing$. $U_i \times V_i \backslash \{(x_1, x_2)\} \cap A_1 \times A_2 \neq \varnothing \Rightarrow V \backslash \{(x_1, x_2)\} \cap A_1 \times A_2 \neq \varnothing \Rightarrow (x_1, x_2) \in (A_1 \times A_2)'$. Therefore, $\overline{A_1 \times A_2} = X_1 \times X_2$. □