Differential Geometry and Topology Tutorial 1
2018/04/13

The Lecturer Name________________________________

Instructions. Please write down your answers in details.

1. Show that the signed curvature k_s of any regular planar curve $\gamma(t)$ is smooth. Use this to prove that the curvature $k(t)$ is smooth if $k(t) > 0$ for any t. Given an example to show that $k(t)$ may not be smooth if $k(t) = 0$ for some t.

2.

2.3.2 Describe all curves in $\mathbb{R}^3$ which have *constant* curvature $\kappa > 0$ and *constant* torsion τ.

3.

2.2.5 Let $\boldsymbol{\gamma}(t)$ be a regular plane curve and let λ be a constant. The *parallel curve* $\boldsymbol{\gamma}^\lambda$ of $\boldsymbol{\gamma}$ is defined by

$$\boldsymbol{\gamma}^\lambda(t) = \boldsymbol{\gamma}(t) + \lambda \mathbf{n}_s(t).$$

Show that, if $\lambda\kappa_s(t) \neq 1$ for all values of t, then $\boldsymbol{\gamma}^\lambda$ is a regular curve and that its signed curvature is $\kappa_s/|1 - \lambda\kappa_s|$.

4.

2.2.6 Another approach to the curvature of a unit-speed plane curve $\boldsymbol{\gamma}$ at a point $\boldsymbol{\gamma}(s_0)$ is to look for the 'best approximating circle' at this point. We can then *define* the curvature of $\boldsymbol{\gamma}$ to be the reciprocal of the radius of this circle.

Carry out this programme by showing that the centre of the circle which passes through three nearby points $\boldsymbol{\gamma}(s_0)$ and $\boldsymbol{\gamma}(s_0 \pm \delta s)$ on $\boldsymbol{\gamma}$ approaches the point

$$\boldsymbol{\epsilon}(s_0) = \boldsymbol{\gamma}(s_0) + \frac{1}{\kappa_s(s_0)}\mathbf{n}_s(s_0)$$

as δs tends to zero. The circle $\mathcal{C}$ with centre $\boldsymbol{\epsilon}(s_0)$ passing through $\boldsymbol{\gamma}(s_0)$ is called the *osculating circle* to $\boldsymbol{\gamma}$ at the point $\boldsymbol{\gamma}(s_0)$, and $\boldsymbol{\epsilon}(s_0)$ is called the *centre of curvature* of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s_0)$. The radius of $\mathcal{C}$ is $1/|\kappa_s(s_0)| = 1/\kappa(s_0)$, where κ is the curvature of $\boldsymbol{\gamma}$ – this is called the *radius of curvature* of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s_0)$.