HIT,Shenzhen
DIFFERENTIAL GEOMETRY AND TOPOLOGY
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Homework 4

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4.1. Prove that $S = \{(-\infty, a) | a \text{ is rational}\}$ is a topology basis of the real line $\mathbb{R}$ (with some appropriately defined topology).

Proof.

(a) $\mathbb{R} = \bigcup_{a \in Q} (-\infty, a)$

(b) $(-\infty, a) \cap (-\infty, b) = (-\infty, \min\{a, b\})$

The topology generated by S is $T = \{(-\infty, b) | b \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$ □

4.2. Let $X = \mathbb{R}$ be the real line and S the set of all irrational numbers. Define $T = \{U \backslash A | U \text{ is open in } \mathbb{R} \text{ and } A \subseteq S\}$.

(a) Show that T is a topology.

(b) Show that (X, T) has T_2, but not T_3 property.

(c) Show that (X, T) is first countable.

(d) Prove that S is a discrete subspace of (X, T). Therefore, S is not separable.

(e) Prove that (X, T) is not C_2.

Proof.

(a) (i) $X, \varnothing \in T$

(ii) $\bigcup_{i \in I} (U_i \backslash A_i) = (\bigcup_{i \in I} U_i) \backslash B, B \subseteq \bigcup_{i \in I} A_i \subseteq S$. Therefore, $\bigcup_{i \in I} (U_i \backslash A_i) \in T$

(iii) $(U_1 \backslash A_1) \cap (U_2 \backslash A_2) = (U_1 \cap U_2) \backslash (A_1 \cup A_2) \in T$

(b) $\forall x, y \in \mathbb{R}, x \neq y$. Let $d = \frac{|x-y|}{2}$, then $x \in (x - d, x + d) \in T, y \in (y - d, y + d) \in T$ and $(x - d, x + d) \cap (y - d, y + d) = \varnothing$. Therefore, the topology satisfies T_2.

Consider a rational point p and the closed set $S (= Q^c)$. Suppose $U \backslash A$ is an open neighborhood containing p, then all rational points within U are in this set. Take an rational number q from U, $q \neq p$. For any open neighborhood $V \backslash B$ containing S, $q \in V$, where V is $\mathbb{R}$. $\Rightarrow (U \backslash A) \cap (V \backslash B) \neq \varnothing$ and the topology defined in this problem is not T_3.

(c) Let $\mathcal{N} = \{\{x\} \cup (B(x, \frac{1}{n})\backslash S), n = 1, 2, \dots\}$. For a neighborhood $U\backslash A$ of x, we can find sufficiently large n such that $B(x, \frac{1}{n}) \subseteq U \Rightarrow \{x\} \cup (B(x, \frac{1}{n})\backslash Q^c) \subseteq U\backslash A$. Therefore, (X, T) is C_1.

(d) $\forall S_1 \subset S, S_1 = S \cap (\mathbb{R}\backslash(S\backslash S_1))$ is open in S. Thus S has discrete topology. Let $A \subseteq S$, since each single point set is open, the accumulated point set of A is empty. If A is dense in S, $\bar{A} = S \Rightarrow A = S$ and A is uncountable. Hence S cannot be separable.

(e) Assume (X, T) is C_2, then (S, T) is C_2. C_2 space is separable, a contradiction.

□

4.3. Show that a compact metric space is separable and thus is C_2.

Proof. $\forall n \in \mathbb{N}, X = \bigcup_{x\in X} B(x, \dfrac{1}{n}) = \bigcup_{i=1}^{m(n)} B(x_{n_i}, \dfrac{1}{n})$. We choose $A = \{x_{n_i} | n \in \mathbb{N}, i = 1, 2, \dots, m(n)\}$. Then A is countable, and we verify $\bar{A} = X$. $\forall x \in X\backslash A$ and a neighborhood U of x, we can find sufficiently large n such that $B(x, \frac{1}{n}) \subseteq U$. For this n, there exists n_i such that $x \in B(x_{n_i}, \frac{1}{n}) \Rightarrow x_{n_i} \in B(x, \frac{1}{n}) \Rightarrow U \cap A \neq \varnothing \Rightarrow x \in A'$. Therefore, A is dense in X. And by known conclusion, seperable metric space is C_2. □

4.4. Let $\sigma : U \to \Sigma$ be a parameterization of a regular surface Σ with an open subset $U \subseteq \mathbb{R}^2$.

(1) Prove that a surface Σ is part of a plane if and only if its second fundamental form is always zero.

(2) Prove that a surface Σ is part of a sphere if and only if its second fundamental form Π_p is a non-zero-constant multiple of its first fundamental form I_p, i.e. $\Pi_p = cI_p$ for some real number c and each $p \in \Sigma$.

Proof.

(1) $\Rightarrow$: the second partial derivative of σ with respect to u, v is zero, therefore the second fundamental form is zero.
$\Leftarrow$: $\overrightarrow{n} = \sigma_u \times \sigma_v$, We know that $\sigma_u \cdot \overrightarrow{n} = 0 \Rightarrow \sigma_{uu} \cdot \overrightarrow{n} + \sigma_u \cdot \overrightarrow{n}_u = 0$. Since $L = \sigma_{uu} \cdot \overrightarrow{n} = 0$, we have $\sigma_u \cdot \overrightarrow{n}_u = 0$. Also $\sigma_{uv} \cdot \overrightarrow{n} + \sigma_u \cdot \overrightarrow{n}_v = 0 \Rightarrow \sigma_v \cdot \overrightarrow{n}_u = 0$. From $\overrightarrow{n} \cdot \overrightarrow{n} = 1 \Rightarrow \overrightarrow{n} \cdot \overrightarrow{n}_u = 0$. While $\overrightarrow{n}, \sigma_u, \sigma_v$ is an orthogonal basis of $\mathbb{R}^3$, $\overrightarrow{n}_u = 0$. By the same deduction, $\overrightarrow{n}_v = 0$. Then $\overrightarrow{n}$ is a constant unit vector and integrate $\sigma_u \cdot \overrightarrow{n} = 0$ gives $\sigma \cdot \overrightarrow{n} = c$, which is the equation of a plane in $\mathbb{R}^3$.

(2) $\Rightarrow$: See the next problem solution for detail.
$\Leftarrow$: We know that $\binom{\overrightarrow{n}_u}{\overrightarrow{n}_v} = -BA^{-1}\binom{\sigma_u}{\sigma_v}$. Since $B = cA$, $\overrightarrow{n}_u = -c\sigma_u, \overrightarrow{n}_v = -c\sigma_v \Rightarrow \overrightarrow{n} + c\sigma = a \Rightarrow \|\sigma - \frac{a}{c}\| = |\frac{1}{c}|$. That is, Σ is part of a sphere.

□

4.5. Compute the Gauss curvature of the sphere S^2 of radius r.

Proof. We provide two methods:

(a) Use the parameterization $(u, v) \longmapsto (r\cos u\cos v, r\cos u\sin v, r\sin u)$ The first fundamental form matrix $A = r^2\begin{pmatrix}1 & 0\\ 0 & \cos^2 u\end{pmatrix}$. The second fundamental form matrix $B = rA$. Therefore the Gaussian curvature $K = \det(BA^{-1}) = \frac{1}{r^2}$

(b) The intersection curve γ of a normal section with the sphere is always a great circle $\Rightarrow k_1 = k_2 = \pm\frac{1}{r} \Rightarrow K = \frac{1}{r^2}$.

□