Lecture 1: Basic concepts of topological spaces

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1 Topological space: concepts and examples

A topological space is a set of points, along with a set of neighborhoods for each point, satisfying a set of axioms.

Definition 1.1 *Let X be a non-empty set. A collection T of subsets of X is called a topology if it satisfies the following.*
(1) $X, \emptyset \in T$;
(2) (finite intersection) If $U, V \in T$, then $U \cap V \in T$;
(3) (infinite union) If each $U_i \in T$ for an index set I, then $\cup_{i \in I} U_i \in T$.

Example 1.2 *(Euclidean topology) Let $X = \mathbb{R}^n$, the n-dimensional Euclidean space and T the collection of open sets.*

Example 1.3 *(discrete topology) Let X be a set and T the power set of X, i.e. T is collection of all subsets of X.*

A metric space is a set X on which we could talk about "distance".

Definition 1.4 *A metric space X is a set together with a distance function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that*
(1) $d(x, y) \geq 0$ and $d(x, x) = 0$;
(2) $d(x, y) = d(y, x)$ for any $x, y \in X$;
(3) $d(x, y) \leq d(x, z) + d(z, y)$ for any $x, y, z \in X$.

Example 1.5 *The set $X = \mathbb{R}^n$ is a metric space under the distace function $d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.*

Example 1.6 *Let $C[a, b]$ be the set of continuous functions on the interval $[a, b]$. For each $f \in C[a, b]$, define its norm $\|f\| = \max_{x \in [a,b]} |f(x)|$. The distance between two functions f, g is defined as $d(f, g) = \|f - g\|$.*

Let $B(x_0, r) = \{x \in X \mid d(x, x_0) < r\}$ be the ball of radius r with center x_0.

Lemma 1.7 *Let X be a metric space. The set $\tau_d = \{U \mid U = \cup B(x_i, r_i)\}$ is a topology of X, called metric topology.*

Proof. It's enough to check the "finite intersection", while the other two are obvious. For any $x \in U \cap V$, we have $x \in B(x_i, r_i) \cap B(x'_i, r'_i)$. Choose $r = \min(r_i - d(x, x_i), r'_i - d(x, x'_i))$. Then $B(x, r) \subset U \cap V$. ∎

Definition 1.8 *Let T be a topology on X. An element $U \in T$ is called an open set and a subset $V \subset X$ is closed if the complement V^c is open.*

Definition 1.9 *(1) Let $x \in A \subset X$. The point x is an interior point of A if there exists open set $U \subset A$ containing x. We call A a neighborhood of x. The set of interior points in A is denoted by A°.*

(2) Let $x \in X, A \subset X$. The point x is an accumulated point of A if every neighborhood of x contains a point in $A \backslash \{x\}$. The set of all accumulated points of A is denoted by A'. The closure $\bar{A}$ of A is $A \cup A'$.

Exercise 1.10 *Prove that $x \in \bar{A}$ if and only if every neighborhood U of x has $U \cap A \neq \emptyset$. Prove that $\bar{A}$ is closed.*

Definition 1.11 *A subset Y of a topological space X is dense if $\bar{Y} = X$. If Y is countable, we call X a separable topological space.*

Example 1.12 *The set of points with rational coordinates (called rational points) is dense in $\mathbb{R}^n$. The set of polynomial functions is dense in $C[0, 1]$.*

2 Continuous functions

Definition 2.1 *Let $(X, T_X), (Y, T_Y)$ be two topological spaces. A function $f : X \to Y$ is continuous at a point $x \in X$ if every open set U_Y containing $f(x)$ we have $f^{-1}(U_Y)$ is open. If f is continuous at every point of X, we simply call that f is continuous.*

Lemma 2.2 *Let $f : X \to Y$ be a function. The following are equivalent.*
(1) f is continuous;
(2) the preimage $f^{-1}(U_Y)$ is open if U_Y is open in Y;
(3) the preimage $f^{-1}(C_Y)$ is closed if C_Y is closed in Y;

Proof. It's obvious that (2) and (3) are equivalent by the definition of closeness. ∎

Example 2.3 *Let (X, T) be any topological space and (X, dis) the discrete topological space. Then the identity map $f : (X, dis) \to (X, T)$ is continuous, but not the other way.*

It's not hard to prove the following.

Lemma 2.4 *If $f : X \to Y, g : Y \to Z$ are continuous, then the composite $g \circ f : X \to Z$ is continuous as well.*

In Analysis, we have the concept of convergent sequence. In topology, we could have a similar thing:

Definition 2.5 *Let $x_1, x_2, \cdots, x_n, \cdots \in X$. We say $\lim_{n\to\infty} x_n = x_0$ if every neighborhood U of x_0 contains almost all terms x_i (i.e. $\exists N > 0$ we have $x_n \in U$ when $n > N$).*

Corollary 2.6 *If $f : X \to Y$ is a continuous map between topological spaces and $x_n \to x$ in X, then $f(x_n) \to f(x)$.*

Proof. For every neighborhood U of $f(x)$, we find an open set U' containing $f(x)$. Since f is continuous, $f^{-1}(U')$ is open in X containing x. Therefore, all most all terms x_i lie in $f^{-1}(U')$ and then $f(x_i)$ lie in U'. ■

But the converse of the previous corollary is not true.

Example 2.7 *Let $X = \mathbb{R}$, the set of real numbers, and T the set of subsets $X\backslash Q$, the complements of countable subsets Q in X. If $x_n \to x$ in the topology T, then $x = x_n$ for sufficiently large n. This implies that the identity map $f : (X, T) \to (X, dis)$ maps every convergent sequence to a convergent sequence. But f is not continuous, eg. $f^{-1}([0, 1])$ is not open.*

3 Homeomorphism

Definition 3.1 *A map $f : X \to Y$ between topological spaces is a homeomorphism if*

(1) f is bijective; (2) f and its inverse f^{-1} are both continuous.

Two topological spaces X, Y are homeomorphic if there exists a homeomorphism between them.

In topology, we are mainly interested in the invariants under homeomorphisms, i.e. the numbers/functions unchanged by homeomorphisms.

Example 3.2 *An open interval (a, b) and $\mathbb{R}$ are homeomorphic.*

Example 3.3 *The exponential map $e : [0, 1) \to S^1 = \{x \in \mathbb{C} \mid \|x\| = 1\}$ given by $t \longmapsto e^{2\pi i t}$ is bijective and continuous. But the inverse f^{-1} is not continuous (a sequence $x_i \to e^0$ may have its image $f^{-1}(x_i) \to 0$ or 1), hence not a homeomorphism.*

4 Construct new topologies: subs and products

Definition 4.1 *Let (X, T) be a topological space and A a subset of X. The subspace (or induced) topology T_A of A is $T_A = \{U \cap A \mid U \in T\}$.*

Example 4.2 *Let A be a subset of a Euclidean space $\mathbb{R}^n$. The subspace topology of A is the metric topology on A induced from $\mathbb{R}^n$. For example, $A = S^2\backslash N$, a sphere in $\mathbb{R}^3$ without the north pole, is homeomorphic to the plane $\mathbb{R}^2$.*

Definition 4.3 *Let X be a set and S a set of subsets of X. Suppose that (1) $\cup_{B \in S} B = X$; (2) for any $B_1, B_2 \in S$ we have $B_1 \cap B_2 = \cup_{i \in I} B_i$ for some index set $I, B_i \in S$. The topology $\langle S \rangle$ generated by S is the minimal topology on X containing S. The set S is called a topology basis of $\langle S \rangle$.*

Lemma 4.4 $\langle S \rangle = \{U \mid U = \cup U_i, U_i \in S\}$.

Proof. Since $\langle S \rangle$ is a topology, it's clear that the union $\cup U_i \in \langle S \rangle$. It's enough to check that $\langle S \rangle$ is a topology. Note that both $\emptyset$ and $X \in \langle S \rangle$. The intersection $(\cup U_i) \cap (\cup U_j) = \cup (U_i \cap U_j) \in \langle S \rangle$, since each $U_i \cap U_j \in \langle S \rangle$ by the requirement of S. ■

Definition 4.5 *For two topological spaces $(X, T_X), (Y, T_Y)$, the product topology on $X \times Y$ is the smallest topology containing $\{U \times V \mid U \in T_X, V \in T_Y\}$.*

Exercise 4.6 *Check that the intersection of two elements in $\{U \times V \mid U \in T_X, V \in T_Y\}$ still lies in $\{U \times V \mid U \in T_X, V \in T_Y\}$.*

Lemma 4.7 *Let $f : Y \to X_1 \times X_2$ be a map given by $y \longmapsto (f_1(y), f_2(y))$. Then f is continuous if and only if f_1, f_2 are continuous.*