Lecture 3: Compactness and Connectedness

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Abstract

1 Compactness: definitions and examples

Definition 1 *A topological space X is compact if any open cover (i.e. $X = \cup_{i \in I} U_i$ for open sets U_i) has a finite subcover $U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n}$ for some integer n.*

Example 2 *From analysis, we know that the closed interval $[a, b]$ is compact.*

Lemma 3 *Let X be a C_1 compact space. Every sequence $\{x_n\}_{n=1}^{+\infty}$ has a convergent subsequence $\{x_{n_i}\}_{i=1}^{+\infty}$ with limit in X.*

Proof. First we claim that there exists $x \in X$ such that every open neighborhood of x contains infinite terms x_n (otherwise, every point x has an open neighborhood U_x contains only finite terms. Then $X = \cup U_x$ has only finite subcover $X = \cup_{i=1}^{n} U_i$ by the compactness of X. Thus X contains only finite terms, a contradiction). Since x has a countable neighborhood basis $U_1 \supset U_2 \supset \cdots$, choose $x_i \in U_i \backslash U_{i+1}$ having the property that $x_i \to x$. ■

Now we try to prove the converse of the previous lemma for metric space (which is obviously C_1). For this, we need a concept.

Definition 4 *Let (X, d) be a bounded metric space and $X = \cup_{i \in I} U_i$ be an open cover. The Lebesgue number is*

$$L = \min_{x \in X} \sup\{d(x, U_i^c) \mid i \in I\}.$$

Let $\varphi(x) = \sup\{d(x, U_i^c) \mid i \in I\}$. Since X is bounded, i.e. $d(x, y) \le M$ for some $M > 0$ and any $x, y \in X$, the function $\varphi(x) \le M$. Since $d(x, U^c) \le d(x, y) + d(y, U^c)$, we have $\varphi(x) \le d(x, y) + \varphi(y)$. Similarly $\varphi(y) \le d(x, y) + \varphi(x)$ and $|\varphi(x) - \varphi(y)| \le d(x, y)$. This shows that $\varphi : X \to \mathbb{R}$ is continuous. Note that when $U_i \ne X$ and $x \in U_i$, we have that $\varphi(x) \ge d(x, U_i^c) > 0$.

Lemma 5 *Let X be a topological space such that every sequence has a convergent subsequence with limit in X. If $f : X \to \mathbb{R}$ is a continuous function, then f is bounded and achieve its minimal and maximal values.*

Proof. The proof is almost the same as that in analysis. We briefly repeat here. If f is not bounded, then for any $n > 0$ we have $|f(x_n)| > n$ for some point x_n. But $\{x_n\}$ has a convergent subsequence $x_{n_k} \to x_0 \in X$, implying $f(x_{n_k}) \to f(x_0)$, a contradiction. Let $m = \sup_{x \in X} f(x)$ (or $\inf_{x \in X} f(x)$), i.e. $\exists x_i$ such that $f(x_i) \to m$. Choose a convergent subsequence $x_{i_k} \to x$ to get that $f(x_{i_k}) \to f(x) = m$. ∎

Theorem 6 *Let X be a metric space. Then X is compact if and only if every sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$.*

Proof. We already know that the left-hand side implies the right-hand side. For the converse, suppose that $\{U_i, i \in I\}$ is an open cover of X. Since every sequence has a convergent subquence, X is bounded. If $U_i = X$, the claim is proved. When $U_i \neq X$ and $x \in U_i$, we have that $\varphi(x) \geq d(x, U_i^c) > 0$. Therefore, the minimum value of $\varphi(x)$ is positive. Choose $0 < \delta < \min f(x)$. Then for every $x \in X$ the ball $B(x, \delta)$ is contained in some U_i (otherwise, every U_i containing x excludes some $y_i \in B(x, \delta)$ and thus $d(x, U_i^c) \leq d(x, y_i) + d(y_i, U_i^c) \leq \delta$, a contradiction). Since $X = \cup_{x \in X} B(x, \delta)$ has a finite cover $\cup_{i=1}^n B(x_i, \delta)$ (otherwise, $\exists \{x_i\}_{i=1}^{+\infty}$ such that $d(x_i, x_j) > \delta$ for any $i \neq j$. The sequence don't have any convergent subsequence), the space X is covered by $\cup_{i=1}^n U_i$ with $B(x_i, \delta) \subset U_i$. ∎

Corollary 7 *A subset A of a Euclidean space $\mathbb{R}^n$ is compact if and only if A bounded and closed.*

2 Compactness: properties

Roughly speaking, "compact" is the infinite version of "finite" for topological spaces.

Lemma 8 *A closed subset A of a compact space X is compact.*

Proof. Note that an open set U of A is of the form $V \cap X$, where V is an open set of X and A has the subspace topology. For any open cover $A = \cup_{i \in I} U_i = \cup_{i \in I}(V_i \cap X)$, the union $A^c \cup_{i \in I} V_i$ is an open cover of X and hence has a finite subcover, which also covers A without A^c. ∎

Lemma 9 *The continuous image of a compact space is also compact.*

Proof. Let $f : X \to Y$ be continuous and X compact. For any open cover $\cup U_i \supset \operatorname{Im} f$. The preimage $f^{-1}(U_i)$ forms an open cover of X and has a finite subcover. ∎

Theorem 10 *A compact Hausdorff X space is T_3 and T_4.*

Proof. Let x be a point and A a closed subset, which is compact by Lemma 8. When $x \notin A$, for any $y \in A$, there are open set V_y containing y but not x and open set V_x containing x but not y (since X is Hausdorff). Then $\cup V_y \supset A$. Since A is compact, there is a finite subcover $\cup_{i=1}^n V_{y_i} \supset A$. But $\cap_{i=1}^n V_{x_i}$ is an open neighborhood of x. The T_4 property is proved similarly. ■

Corollary 11 *A compact subset A of a Hausdorff space is closed.*

Proof. If A is not closed, $\exists x \in \bar{A} \backslash A$. The same proof as that of the previous theorem shows that x and A are separated by open sets, which is impossible. ■

Corollary 12 *Let X be a compact space and Y is Hausdorff space. Then a continuous bijective map $f : X \to Y$ is a homeomorphism.*

Proof. It suffices to prove that f^{-1} is continuous. For this, let $V \subset X$ be a closed set. Since V is compact by Lemma 8, $f^{-1}(V)$ is compact by Lemma 9. The previous corollary implies that $f^{-1}(V)$ is closed since Y is Hausdorff. ■

Example 13 *A (topological) manifold is a second countable Hausdorff space X such that any point $x \in X$ has an open neighborhood U homeomorphic to $\mathbb{R}^n$. Theorem 10 implies that a compact manifold X is T_4.*

3 Connectness

Definition 14 *A topological space X is connected if it is not the disjoint union of two nonempty open sets.*

Example 15 *The line $\mathbb{R}$ is connected (suppose that $\mathbb{R}$ is the disjoint union of $A \cup B$ of open sets. Choose $a \in A, b \in B$. Without loss of generality, assume $a < b$. Let $x_0 = \inf\{x \mid a < x \leq b, x \in B\}$. If $x_0 \in B$, a neighborhood of x_0 would be in B since B is open. This is impossible since x_0 is the inf. If $x_0 \in A$, then a neighborhood of x_0 would be in A. This is also impossible.)*

Lemma 16 *Continuous image of a connected space is also connected.*

Proof. Let X be connected and $f : X \to Y$ a continuous map. If $f(X) = A \cup B$, a disjoint union of open sets, then $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, impossible. ■

Example 17 *The circle $S^1 = \{x \in \mathbb{C} \mid \|x\| = 1\}$ is connected, since $S^1 = \mathrm{Im}(f(x) = e^{2\pi i x} : \mathbb{R} \to S^1)$.*

Example 18 *A connected subset of $\mathbb{R}$ is an interval.*

Lemma 19 *Let X be a topological space with a cover $\cup_{i \in I} U_i$ such that each U_i is connected. If a connected subset $A \subset X$ has $A \cap U_i \neq \emptyset$ for each $i \in I$, then X is connected.*

Proof. If $X = X_1 \cup X_2$ is a disjoint union of two non-empty sets, then $A = (A \cap X_1) \cup (A \cap X_2)$ implying $A \subset X_1$ or $A \subset X_2$. Without loss of generality, assume that $A \subset X_1$. For any connected U_i, the same argument shows that $U_i \subset X_1$ since $A \cap U_i \neq \emptyset$. This is impossible since $X = \cup U_i$. ■

Example 20 *The previous lemma shows that the Euclidean space $\mathbb{R}^n$ is connected.*

Definition 21 *A subset $A \subset X$ is a connected component if A is a maximal connected subset, i.e. A is connected, but not a proper subset of another connected subset.*

Theorem 22 *A topological space $X = \dot{\cup} X_i$ is the disjoint union of its connected components X_i. Moreover, each X_i is closed.*

Proof. It clearly that $X_i \cap X_j = \emptyset$. It's enough to prove that a connected subset A is contained in a component X_i. Choose $F = \{K \subset X \mid A \subset K,$ K is connected$\}$. Then $\cup_{K\in F} K$ is connected by Lemma 19. By the definition, $\cup_{K\in F} K$ is maximal and thus connected. Since X_i is connected, the closure $\bar{X}_i$ is also connected and thus $X_i = \bar{X}_i$. This proves the last claim. ■

Example 23 *A connected component may not be open. Let X be the subspace of rational points in $\mathbb{R}$. Each connected component is a single point, but a single point is not open. However, when the number of connected components is finite, each connected component is open.*

Now we discuss a more intuitive concept of connectedness: path connectedness.

Definition 24 *A topological space X is path-connected, if for any two points $x, y \in X$ there is a continuous map $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$.*

Example 25 *A convex subset in $\mathbb{R}^n$ is path-connected.*

Lemma 26 *A path-connected space is connected.*

Proof. Since $[0,1]$ is connected, the image $f([0,1])$ is connected. Fix a point $x \in X$. Any other point $y \in X$ is connected by a path to x. Then X is a union of connected paths with end x, thus connected by Lemma 19. ■

Example 27 *A connected space is not necessarily path-connected. Let*

$$X = \{(x,y) \mid y = \sin\frac{1}{x}, 0 < x < 1\} \cup \{(0,t) \mid t \in [-1,1]\} = B \cup A$$

with a subspace topology induced from $\mathbb{R}^2$. Then X is connected (sicne $X = \bar{B}$), but not path-connected (suppose that there is a path $f([0,1])$ connecting $(0,0)$ and $(1/2, \sin 2)$. Let $x_0 = \inf\{t \in (0,1) \mid f(t) \in B\}$. Then $f(x_0) \in A$. But when $x_n \to x_0, f(x_n)$ may not have a limit, a contradiction).

Similarly, we define the path-connected component of a topological space X as a maximal path-connected subset.