Let ϕ be a continuous mapping from a closed interval $[a, b]$ to another closed interval. If ϕ is bijective, show that ϕ is monotonic.

Proof. Without loss of generality, we assume $\phi(a) < \phi(b)$ and show that ϕ is strickly increasing. That is, $\forall x_1 < x_2 \Rightarrow \phi(x_1) < \phi(x_2)$. We proceed by contradiction. Suppose $\exists x_1 < x_2, s.t.\, \phi(x_1) \geq \phi(x_2)$. Since ϕ is injective, $\phi(x_1) > \phi(x_2)$ Discuss three cases:

1. $\phi(a) < \phi(x_2)$. Since $\phi(a) < \phi(x_2) < \phi(x_1)$, we can find $c \in (a, x_1)$ such that $\phi(c) = \phi(x_2)$.

2. $\phi(b) > \phi(x_1)$. Since $\phi(x_2) < \phi(x_1) < \phi(b)$, we can find $c \in (x_2, b)$ such that $\phi(c) = \phi(x_1)$.

3. $\phi(a) \geq \phi(x_2)$ and $\phi(b) \leq \phi(x_1)$. Since $\phi(x_2) < \phi(a) < \phi(b) < \phi(x_1)$, we can find $c \in (x_1, x_2)$ such that $\phi(c) = \phi(a)$

Contradicted with that ϕ is injective. □

Let $f : S_1 \to S_2$ be a smooth mapping between two surfaces. The tangent mapping $D_f : T_pS_1 \to t_{f(p)}S_2$ is defined by: $v \in T_p$, $v = \gamma'(t), \gamma(t) = p, D_f\gamma'(t) = (f \circ \gamma)'(t)$ Since $\gamma'(t) = \dfrac{\mathrm{d}\sigma(u(t), v(t))}{\mathrm{d}t} = \sigma_u u'(t) + \sigma_v v'(t)$ $D_f\gamma'(t) = \dfrac{\mathrm{d}(f \circ \sigma)(u(t), v(t))}{\mathrm{d}t} = (f \circ \sigma)_u u'(t) + (f \circ \sigma)_v v'(t)$ Let $D_f\sigma_u \triangleq (f \circ \sigma)_u, D_f\sigma_v \triangleq (f \circ \sigma)_v$ We can rewrite the definition as: $v \in T_p, v = a\sigma_u + b\sigma_v, D_f v = aD_f\sigma_u + bD_f\sigma_v$

Let $S_1 = S^2 = \{(x, y, z)|x^2+y^2+z^2 = r^2\}$, the sphere and $S_2 = \{(x, y, z)|x^2+y^2 = r^2\}$ the cylinder. Define

$$\begin{aligned} f : S_1 &\to S_2, \\ (x, y, z) &\longmapsto \left(\frac{rx}{\sqrt{x^2 + y^2}}, \frac{ry}{\sqrt{x^2 + y^2}}, z \right) \end{aligned}$$

Show that f is area-preserving.

Proof. Use spherical coordinate,for the sphere $x = r \cos u \cos v, y = r \cos u \sin v, z = r \sin u \Rightarrow EG - F^2 = r^2 \cos^2 u$. Under the mapping f, for the cylinder, $x' = r \cos v, y' = r \sin v, z' = r \sin u \Rightarrow E'G' - F'^2 = r^2 \cos^2 u$. Therefore, f is area-preserving. □