Lecture 2 remark: Separation axioms and Countabilities (page 1 finished)

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April 22, 2018

1 Topological space: separation axioms

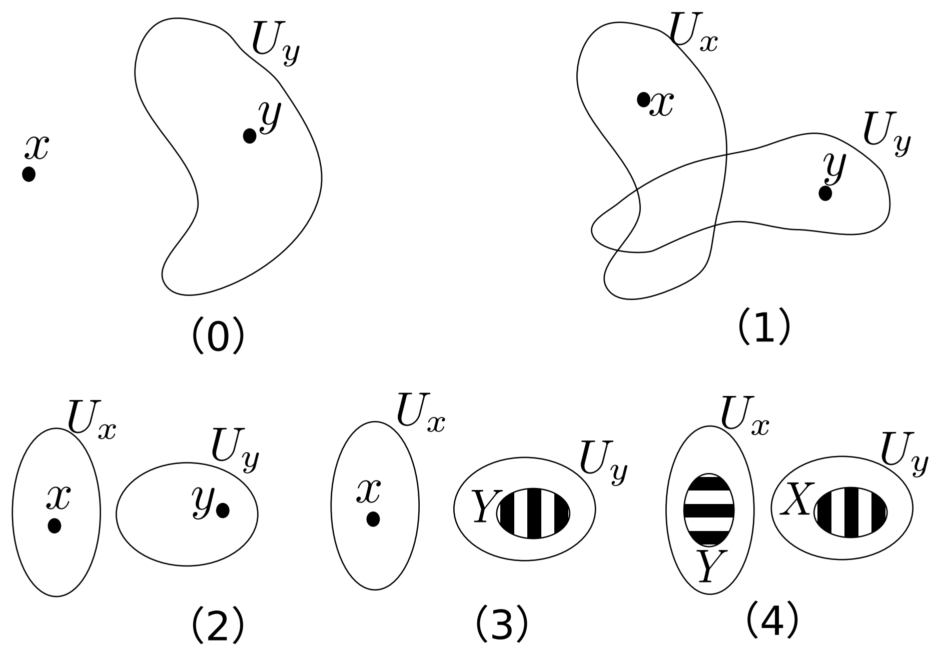

Figure 1: Illustrations on T_0 to T_4

Example 1.1 *Let $X = \{a, b\}, T = \{\varnothing, X, \{a\}\}$, Then T satisfies T_0, but does not satisfy T_1.*

Example 1.2 (Lecture 1 remark **Example 1.4** continued) *T satisfies T_1, but does not satisfy T_2 (Notice for $x \neq y, \mathbb{R}\backslash\{x\} \ni y, \not\ni x; \mathbb{R}\backslash\{y\} \ni x, \not\ni y$, thus satisfying T_1).*

Example 1.3 *If $x \notin Y$, where Y is closed, then $d(x,Y) = \inf\{d(x,y)|y \in Y\} > 0$.*

Proof. Suppose $d(x,Y) = 0$. By the definition of inferior of number set, we can find $\{y_n\}$ such that $d(x,y_n) \to 0$. For any neightborhood V of x, we can find $B(x,r) \subseteq V$. For sufficient large $n, y_n \in B(x,r) \Rightarrow V$ contains point in $Y \backslash x$. By the definition of accumulated point of Y, $x \in Y'$. Since Y is closed, if $x \in Y^c$, which is open, we can find U such that $x \in U \subseteq Y^c$, which contradicts with the fact that x is the accumulated point of Y. Therefore, $x \in Y$, which contradicts $x \notin Y$. ∎

Lemma 1.4 *A metric space X is Hausdorff (T_2), and $x_n \to x, x_n \to y$, then $x = y$.*

Proof. Suppose $x \neq y$, then there exists disjoint open set U, V s.t. $x \in U, y \in V$. Then by the definition of convergence, both U and V contain almost $\{x_n\}$, a contradiction. ∎

Lemma 1.6

(a) (X,T) is T_3 if and only if for any $x \in X$ and an open neighborhood U of x, there exists an open set V, such that $x \in V \subseteq \bar{V} \subseteq U$

(b) (X,T) is T_4 if and only if for any closed set A and an open neighborhood $U \supseteq A$, there exists an open set V, such that $A \subseteq V \subseteq \bar{V} \subseteq U$

Proof.

(a) $\Rightarrow$ U^c is closed, and $x \notin U^c$. Then by T_3 property, there exists an open set $V_1 \ni x$, another open set $V_2 \supseteq U^c$ and $V_1 \cap V_2 = \varnothing$. Then it follows that $V_1 \subseteq U$. Also $\bar{V}_1 \cap V_2 = \varnothing \Rightarrow \bar{V}_1 \subseteq U$. Therefore, choose $V = V_1$ and $x \in V \subseteq \bar{V} \subseteq U$.

$\Leftarrow$ Given x and a closed set A, $x \notin A$, then $x \in A^c$ where A^c is open. There exists an open set V such that $x \in V \subseteq \bar{V} \subseteq A^c$. Then $A \subseteq \bar{V}^c$ where $\bar{V}^c$ is open and is disjoint with open set V which contains x.

(b) $\Rightarrow$ U^c is closed, and $A \cap U^c = \varnothing$. Then by T_4 property, there exists an open set V_1, V_2 such that $A \subseteq V_1, U^c \subseteq V_2$ and $V_1 \cap V_2 = \varnothing$. Then it follows that $V_1 \subseteq U$. Also $\bar{V}_1 \cap V_2 = \varnothing \Rightarrow \bar{V}_1 \subseteq U$. Therefore, choose $V = V_1$ and $A \subseteq V \subseteq \bar{V} \subseteq U$.

$\Leftarrow$ Given two closed sets A, B, then $B \subseteq A^c$ where A^c is open. There exists an open set V such that $B \subseteq V \subseteq \bar{V} \subseteq A^c$. Then $A \subseteq \bar{V}^c$ where $\bar{V}^c$ is open and is disjoint with open set V and $B \subseteq V$.

∎

2 Countability

Example 2.0 (C_1 space) *If X is a metric space, $\forall x$, choose $N_x = \{B(x, \frac{1}{n})\}$*

Example 2.1 *$S_Q = \{(a_i, b_i) | a_i \in Q, b_i \in Q, a_i < b_i\}$ is a countable basis $\mathcal{N}$ of $\mathbb{R}$, which implies that $\mathbb{R}$ (with Euclid topology) is C_2 space.*

Definition 2.1 (Continued) *C_2 space is C_1*

Proof. Suppose $\mathcal{N}$ be a countable basis. For any $x \in X$, let $\mathcal{N}_x = \{U \in \mathcal{N} | x \in U\}$. Then $\mathcal{N}_x$ is countable. For $V \ni x, V \in T$, then $V = \bigcup U_i$. $x \in U_i$ for some i, then $U_i \in \mathcal{N}_x, U_i \subseteq V$. Therefore every neighborhood V of x contains an element of $\mathcal{N}_x$. ■

Example 2.2 (Continued)

(a) a C_2 space is separable.

(b) Discrete topology T is C_1, and if X is uncountable, then T is not C_2.

Proof.

(a) Let B be a countable topological basis of T. Define $A = \{x_V | x_V \in V, \forall V \in B\}$. A is countable. We only need to show that $\bar{A} = X$. Assume if there exists $x \in X, x \notin \bar{A}$. Then there is an open set U containing x and $U \cap A = \varnothing$. Since U is open, $U = \bigcup\limits_{V_i \in B} V_i$. $V_i \cap A = \varnothing$, but $x_{V_i} \in V_i \cap A$, a contradiction.

(b) A single point set is open in discrete topological space. Therefore, $\{x\}$ is a subset of every neighborhood of x and T is C_1. If X is uncountable, we show that X is **not separable**. Suppose A is a countable dense subset of X, then $\exists y \in X \backslash A$ and $\{y\} \supseteq y \Rightarrow y \notin \bar{A}$.

■

Example 2.3 (Continued) *We check that $\mathcal{N}$ is a basis. For any open set V in T and $x \in V$, if we can find a ball $B(y(x), \frac{1}{n(x)}) \in \mathcal{N}$ such that $x \in B(y(x), \frac{1}{n(x)}) \subseteq V$, then $V = \bigcup\limits_{x \in V} B(y(x), \dfrac{1}{n(x)})$. For $x \in Y$, it is obvious since we can find a sufficient small ball contained in V. If $x \notin Y$, since $\bar{Y} = X$, for a sufficient small ball $B(x, \frac{2}{n(x)}) \subseteq V$, $B(x, \frac{1}{n(x)})$ contains points in Y and let it be $y(x)$. Then $x \in B(y(x), \frac{1}{n(x)}) \subseteq V$.*

Example 2.4 (Lecture 1 remark **Example 1.5** continued) *T is not C_1*

Proof. $\mathcal{N}_x$ is a countable set consisting neighborhoods of x. We can find $y \in \bigcap\limits_{\mathbb{R} \backslash U \in \mathcal{N}_x} \mathbb{R} \backslash U = \mathbb{R} \backslash \bigcup\limits_{\mathbb{R} \backslash U \in \mathcal{N}_x} U$ and $y \neq x$. Then $\mathbb{R} \backslash \{y\}$ is a neighborhood of x and $y \notin U, \forall \mathbb{R} \backslash U \in \mathcal{N}_x$. Therefore $\mathbb{R} \backslash U \not\subseteq \mathbb{R} \backslash \{y\}, \forall \mathbb{R} \backslash U \in \mathcal{N}_x$. ■