HIT,Shenzhen
Differential Geometry and Topology
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Exercise Collection

zhaofeng-shu33 May 7, 2018

Exercise 1.

1.1 Show that the signed curvature k_s of any regular planar curve $\gamma(t)$ is smooth. Use this to prove that the curvature $k(t)$ is smooth if $k(t) > 0$ for any t. Give an example to show that $k(t)$ may not be smooth if $k(t) = 0$ for some t.

Solution : *Proof.* Without loss of generality, we assume $\gamma(t)$ is unit-speed, otherwise by reparametrization (which is smooth transformation) we can get a unit-speed representation of the curve. Since $\gamma''(t) = k_s N \Rightarrow k_s = \gamma''(t) \cdot N \Rightarrow k_s$ is smooth.

$k = |k_s|$, if $k(t) > 0$, then k_s can not change sign by continuity.$\Rightarrow k(t) = -k_s(t)\forall t$ or $k(t) = k_s(t)\forall t$ and k is smooth.

Counterexample: Let $\gamma(t) : (t, t^3) \Rightarrow k(t) = \frac{6|t|}{(1+9t^4)^{3/2}}$. Since *abs* function is not smooth at $t = 0$, $k(t)$ is not smooth. $\square$

1.2 Describe all curves in $\mathbb{R}^3$ which have *constant* curvature $\kappa > 0$ and *constant* torsion τ

Solution : Let $\gamma(t) = (a\cos t, b\sin t, bt)$, which is circular helix. We know that $\kappa = \frac{|a|}{a^2+b^2}$ and $\tau = \frac{b}{a^2+b^2}$, which gives $|a| = \frac{\kappa}{\kappa^2+\tau^2}, b = \frac{\tau}{\kappa^2+\tau^2}$, by the fundamental theorem of curves, all curves with constant curvature $\kappa > 0$ and constant torsion τ can be obtained by translating and rotating the helix with parameter a, b.

1.3 Let $\gamma(t)$ be a regular plane curve and let λ be a constant. The *parallel curve* γ^λ of γ is defined by

$$\gamma^\lambda(t) = \gamma(t) + \lambda \boldsymbol{n}_s(t) \tag{1}$$

Show that, if $\lambda\kappa_s(t) \neq 1$ for all values of t, then γ^λ is a regular curve and that its signed curvature is $\frac{\kappa_s}{|1-\lambda\kappa_s|}$.

Solution : *Proof.* Let T be the tangent vector of γ, $\boldsymbol{n}_s$ the vector obtained by rotating $\boldsymbol{n}_t$ anti-clockwise $90°$. Also $\tilde{T}$ be the tangent vector of γ^λ and $\tilde{\boldsymbol{n}}_s$ is obtained from $\tilde{T}$. For curve γ^λ, we choose the arc length parameter

$\tilde{s} = \int_{t_0}^t \|\gamma(v)\| dv \Rightarrow \frac{\mathrm{d}s}{\mathrm{d}t} = |1 - \lambda\kappa_s| \|\gamma'(t)\|$, and the arc length parameter for γ is denoted by s, then we have $\tilde{s}(t) = |1-\lambda\kappa_s(t)|s(t)$.

$$\begin{aligned}\frac{\mathrm{d}\gamma^\lambda(t)}{\mathrm{d}t} &= \gamma'(t) + \lambda\frac{\mathrm{d}\boldsymbol{n}_s(t)}{\mathrm{d}t} = \gamma'(t) + \lambda\frac{\mathrm{d}\boldsymbol{n}_s}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}\\ &= \gamma'(t) - \lambda\kappa_s\frac{\mathrm{d}s}{\mathrm{d}t}T = (1-\lambda\kappa_s)\frac{\mathrm{d}\gamma}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}\\ &= (1-\lambda\kappa_s)\gamma'(t) \neq 0\end{aligned}$$

Hence γ^λ is regular.

$$\tilde{T} = \frac{\mathrm{d}\gamma^\lambda}{\mathrm{d}\tilde{s}} = \frac{\frac{\mathrm{d}\gamma^\lambda(t)}{\mathrm{d}t}}{|1-\lambda\kappa_s|\frac{\mathrm{d}s}{\mathrm{d}t}} = \mathrm{sgn}\{1-\lambda\kappa_s\}\frac{\gamma'(t)}{\|\gamma'(t)\|}$$

Similarly we can show that $T = \frac{\gamma'(t)}{\|\gamma'(t)\|} \Rightarrow \tilde{\boldsymbol{n}}_s = \mathrm{sgn}\{1-\lambda\kappa_s\}\boldsymbol{n}_s$

$$\frac{\mathrm{d}\tilde{T}}{\mathrm{d}\tilde{s}} = \frac{\frac{\mathrm{d}\tilde{T}}{\mathrm{d}t}}{|1-\lambda\kappa_s|\frac{\mathrm{d}s}{\mathrm{d}t}}$$

Since $\kappa_s(t)$ is continuous and $\lambda\kappa_s(t) \neq 1$, $1-\lambda\kappa_s(t)$ has constant sign. therefore $\frac{\mathrm{d}\tilde{T}}{\mathrm{d}t} = \mathrm{sgn}\{1-\lambda\kappa_s\}\frac{\mathrm{d}T}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}\tilde{T}}{\mathrm{d}s} = \frac{\mathrm{sgn}\{1-\lambda\kappa_s\}}{|1-\lambda\kappa_s|}\frac{\mathrm{d}T}{\mathrm{d}s}$ Let κ_s be the signed curvature of γ and $\tilde{\kappa}_s$ be the signed curvature of γ^λ. Then

$$\tilde{\kappa}_s = \frac{\mathrm{d}\tilde{T}}{\mathrm{d}s}\cdot\tilde{\boldsymbol{n}}_s = \frac{\mathrm{sgn}^2\{1-\lambda\kappa_s\}}{|1-\lambda\kappa_s|}\frac{\mathrm{d}T}{\mathrm{d}s}\cdot\boldsymbol{n}_s = \frac{1}{|1-\lambda\kappa_s|}\kappa_s$$

□

1.4 Another approach to the curvature of a unit-speed plane curve γ at a point $\gamma(s_0)$ is to look for the 'best approximating circle' at this point. We can then *define* the curvature of γ to be the reciprocal of the radius of this circle.

Carry out this programme by showing that the center of the circle which passes through three nearby points $\gamma(s_0)$ and $\gamma(s_0 \pm \delta_s)$ on γ approaches the point

$$\boldsymbol{\epsilon}(s_0) = \gamma(s_0) + \frac{1}{\kappa_s(s_0)}\boldsymbol{n}_s(s_0) \tag{2}$$

as δ_s tends to zero. The circle $\mathcal{C}$ with center $\boldsymbol{\epsilon}$ passing through $\gamma(s_0)$ is called the *osculating circle* to γ at the point $\gamma(s_0)$, and $\epsilon(s_0)$ is called the *centre of curvature* of γ at $\gamma(s_0)$. The radius of $\mathcal{C}$ is $1/|\kappa_s(s_0)| = 1/\kappa(s_0)$, where κ is the curvature of γ - this is called the *radius of curvature* of γ at $\gamma(s_0)$.

Solution : The line segment bisector of $\gamma(s_0), \gamma(s_0+\delta_s)$ has the parametrized form (t_1 is the parameter):

$$\ell_1(t_1) : \frac{1}{2}(\gamma(s_0)+\gamma(s_0+\delta_s)) + t_1(\gamma(s_0+\delta_s)-\gamma(s_0))\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Similary, the line segment bisector of $\gamma(s_0), \gamma(s_0-\delta_s)$ has the parametrized form (t_2 is the parameter):

$$\ell_2(t_2) : \frac{1}{2}(\gamma(s_0)+\gamma(s_0-\delta_s)) + t_2(\gamma(s_0-\delta_s)-\gamma(s_0))\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The intersection of $\ell_1(t_1)$ and $\ell_2(t_2)$ is the center of the approximating circle

To simplify the notation, let
$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $a = \gamma(s_0+\delta_s)-\gamma(s_0), b = \gamma(s_0-\delta_s)-\gamma(s_0)$. The intersection point satisfies $\ell_1(t_1) = \ell_2(t_2) \Rightarrow \frac{1}{2}(a-b) = t_2 bJ - t_1 aJ$. Since aJ is perpendicular with a (J is counterclockwise $90°$ rotation matrix), dot product both sides by a. we can solve t_2 as: $t_2 = \frac{(a-b)\cdot a}{2bJ\cdot a}$. Then the center of circle can be expressed by a, b, J as:

$$\boldsymbol{\epsilon}(s_0, \delta_s) = \gamma(s_0) + \frac{b}{2} + \frac{(a-b)\cdot a}{2bJ\cdot a} bJ$$

Since δ_s is small, we can expand a, b as:

$$a = \gamma'(s_0)\delta_s + \gamma''(s_0)\frac{\delta_s^2}{2} + o(\delta_s^2) \tag{3a}$$

$$b = -\gamma'(s_0)\delta_s + \gamma''(s_0)\frac{\delta_s^2}{2} + o(\delta_s^2) \tag{3b}$$

By the definition of $\boldsymbol{n}_s, k_s$, we have $\boldsymbol{n}_s(s_0) = \gamma'(s_0)J, \gamma''(s_0) = \kappa_s \boldsymbol{n}_s(s_0)$, from (3b) we have

$$bJ = -\frac{k_s\delta_s^2}{2}\gamma'(s_0) - \frac{\delta_s}{k_s}\gamma''(s_0) + o(\delta_s^2) \tag{4}$$

Since $\gamma'(s_0)$ is perpendicular with $\gamma''(s_0), 2bJ\cdot a = -2\delta_s^3\kappa_s + o(\delta_s^3)$ and $\frac{1}{2bJ\cdot a} = \frac{1}{-2\delta_s^3\kappa_s}(1+o(1))$, also from $||\delta_s|| = 1$ we can compute $(a-b)\cdot a = 2\delta_s^2 + o(\delta_s^3) \Rightarrow \frac{(a-b)\cdot a}{2bJ\cdot a} bJ = \frac{\gamma''(s_0)}{\kappa_s^2} + o(1) \Rightarrow \boldsymbol{\epsilon}(s_0, \delta_s) = \gamma(s_0) + \frac{1}{\kappa_s(s_0)}\boldsymbol{n}_s(s_0)$. It follows that $\boldsymbol{\epsilon}(s_0) = \boldsymbol{\epsilon}(s_0, \delta_s)$ as $\delta_s \to 0$.

Exercise 2.

2.1 How many topologies could be defined on the two-element set $X = \{a, b\}$?

Solution :

- $T = \{X, \varnothing\}$
- $T = \{X, \varnothing, \{a\}\}$
- $T = \{X, \varnothing, \{b\}\}$
- $T = \{X, \varnothing, \{a\}, \{b\}\}$

2.2 Find the closure of $\{(x, \sin\frac{1}{x}|0 < x \leq 1)\}$ in the 2-dimensional Euclidean space $\mathbb{R}^2$

Solution : *Proof.* We show that the closure $\bar{A}$ of $A = \{(x, \sin\frac{1}{x}|0 < x \leq 1)\}$ is $\{(0, y)| -1 \leq y \leq 1\} \cup A$. For $(0, y), |y| \leq 1$, we can find $(x_n, \sin\frac{1}{x_n})$, where $x_n = \frac{1}{2\pi n + \arcsin y}$ such that $(x_n, \sin\frac{1}{x_n}) \to (0, y)$. $\square$

2.3 Prove that $\mathbb{R}^2\backslash\{(0,0)\}$ (as a subspace of $\mathbb{R}^2$) and $\{(x, y, z)|x^2 + y^2 = 1\}$(as a subspace of $\mathbb{R}^2$) are homeomorphic.

Solution : *Proof.* We can construct a homeomorphic mapping from $(0, \infty)$ to $(-\infty, +\infty)$, such as $x \to x - \frac{1}{x}$. Then consider the polar coordinate representation of the plane without the origin. For $(r, \theta) \in \mathbb{R}^2\backslash\{(0,0)\}$, use mapping $(\cos\theta, \sin\theta, r - \frac{1}{r})$ and we get a point on the cylinder $\{(x, y, z)|x^2 + y^2 = 1\}$. It is easy to check that the mapping is a homeomorphic. $\square$

2.4 Let (X, d) be a metric space and $A \subseteq X$ a closed subset. Define $f : X \to \mathbb{R}$ by $f(x) = \inf_{a\in A} d(x, a)$. Prove that f is continuous and that $f(x) = 0$ if and only if $x \in A$.

Solution : *Proof.* Suppose V is open, then for any $x \in f^{-1}(V), f(x) \in V$, we can find ϵ such that $(f(x) - \epsilon, f(x) + \epsilon) \subseteq V$. We show that $B(x, \frac{1}{2}\epsilon) \subseteq f^{-1}(V)$. Indeed, $\forall y \in B(x, \frac{1}{2}\epsilon), d(y, x) < \frac{1}{2}\epsilon$. Then $d(y, a) \leq d(y, x) + d(x, a) < \frac{1}{2}\epsilon + d(x, a) \Rightarrow$ $f(y) = \inf_{a\in A} d(y, a) \leq \frac{1}{2}\epsilon + d(x, a) \Rightarrow f(y) \leq \frac{1}{2}\epsilon + \inf_{a\in A} d(x, a) < \epsilon + f(x)$.
Exchange the position of x and y:
$f(x) < \epsilon + f(y) \Rightarrow f(y) \in (f(x) - \epsilon, f(x) + \epsilon) \subseteq V$. Therefore f is continuous.

If $x \in A$, $f(x) = 0$; if $f(x) = 0$, there exists $\{y_n\}$ such that $d(x, y_n) \to 0$, and $x \in \bar{A} = A$ $\square$

2.5 A topological space X is called separable if there is a countable dense subset A. Prove that if two topological spaces X_1, X_2 are separable, then the product $X_1 \times X_2$ is also separable.

Solution : *Proof.* Let A_1, A_2 be dense countable subset of X_1, X_2 respectively. $A_1 \times A_2$ is countable. Below we show that $\overline{A_1 \times A_2} = X_1 \times X_2$. We consider $(x_1, x_2) \notin (A_1, A_2)$ and assume $x_1 \notin A_1$ for example. For $(x_1, x_2) \in X_1 \times X_2$ and an open set $V \in X_1 \times X_2$ covering (x_1, x_2). $V = \bigcup U_i \times V_i$, where $U_i \in T_{X_1}, V_i \in T_{X_2}$.

Then $(x_1, x_2) \in U_i \times V_i$ for some i. Since A_1 is dense in X_1 and $x_1 \notin A_1$, $U_i \backslash \{x_1\} \cap A_1 \neq \varnothing$. $U_i \times V_i \backslash \{(x_1, x_2)\} \cap A_1 \times A_2 \neq \varnothing \Rightarrow V \backslash \{(x_1, x_2)\} \cap A_1 \times A_2 \neq \varnothing \Rightarrow (x_1, x_2) \in (A_1 \times A_2)'$. Therefore, $\overline{A_1 \times A_2} = X_1 \times X_2$. □

Exercise 3.

3.1 Show that applying an isometry of $\mathbb{R}^3$ does not change the first fundamental form. What is the effect of a dilation (i.e. a map $\mathbb{R}^3 \to \mathbb{R}^3$ given by $x \to ax$ for some constant a $\neq 0$)?

Solution : *Proof.* An isometry in $\mathbb{R}^3$ has the form $f : x \to xP + a$.
$(f \circ \sigma)_u = \sigma_u P, (f \circ \sigma)_v = \sigma_v P \Rightarrow E(f \circ \sigma) = \sigma_u P P^T \sigma_v^T = \sigma_u \sigma_v^T = E(\sigma)$. Similarly, F, G are also unchanged under the isometry and the first fundamental form remains the same. □

3.2 Let $\gamma : (a, b) \to \mathbb{R}^3$ be a unit speed curve. The surface of tangent developable is given by $\sigma(u, v) = \gamma(u) + v\gamma'(u)$

(1) Compute the first fundamental form of σ; Show that the first fundamental form is independent of the torsion of γ;

(2) Show that the tangent developables of two curves γ_1, γ_2 are locally isometric if their curvature functions are the same;

(3) Show that the tangent developable σ is locally isometric to a plane.

Solution : *Proof.*

(1) $\sigma_u = \gamma'(u) + v\gamma''(u), \sigma_v = \gamma'(u)$. Since $\gamma'_u \circ \gamma''_u = 0$, the first fundamental form is $(1 + v^2\kappa^2)du^2 + 2dudv + dv^2$, where κ is the curvature of the curve. From this expression, we see that the first fundamental form is indepedent with the torsion τ of γ.

(2)

(3) We construct a planar curve with $\kappa(u)$ as curvature. By fundamental theorem of curves, it is possible. Then

□

3.3 Show that Enneper's surface

$$\sigma(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2) \tag{5}$$

is conformally parametrized.

Solution : *Proof.* $\sigma_u = (1-u^2+v^2, 2uv, 2u), \sigma_v = (2uv, 1-v^2+u^2, -2v)$. The first fundamental form is $(1+u^2+v^2)^2(du^2+dv^2)$, which is proportional to the first fundamental form of plane. Therefore, the surface is conformally parametrized. □

Exercise 4.

4.1 Prove that $S = \{(-\infty, a)|a \text{ is rational}\}$ is a topology basis of the real line $\mathbb{R}$ (with some appropriately defined topology).

Solution : *Proof.*

.1. $\mathbb{R} = \bigcup_{a\in Q}(-\infty, a)$

.2. $(-\infty, a) \cap (-\infty, b) = (-\infty, \min\{a, b\})$

The topology generated by S is $T = \{(-\infty, b)|b \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$ □

4.2 Let $X = \mathbb{R}$ be the real line and S the set of all irrational numbers. Define $T = \{U\backslash A|U \text{ is open in } \mathbb{R} \text{ and } A \subseteq S\}$.

(a) Show that T is a topology.

(b) Show that (X, T) has T_2, but not T_3 property.

(c) Show that (X, T) is first countable.

(d) Prove that S is a discrete subspace of (X, T). Therefore, S is not separable.

(e) Prove that (X, T) is not C_2.

Solution : *Proof.*

(a) (i) $X, \varnothing \in T$

(ii) $\bigcup_{i\in I}(U_i\backslash A_i) = (\bigcup_{i\in I} U_i)\backslash B, B \subseteq \bigcup_{i\in I} A_i \subseteq S$. Therefore, $\bigcup_{i\in I}(U_i\backslash A_i) \in T$

(iii) $(U_1\backslash A_1) \cap (U_2\backslash A_2) = (U_1 \cap U_2)\backslash(A_1 \cup A_2) \in T$

(b) $\forall x, y \in \mathbb{R}, x \neq y$. Let $d = \frac{|x-y|}{2}$, then $x \in (x-d, x+d) \in T, y \in (y-d, y+d) \in T$ and $(x-d, x+d) \cap (y-d, y+d) = \varnothing$. Therefore, the topology satisfies T_2.
Consider a rational point p and a closed set S. Suppose $U\backslash A$ is an open neighborhood containing p, then all rational points within U are in this set. Take an irrational number q from U. For any open neighborhood $Y\backslash B$ containing S, $q \in Y$, where Y is open in $\mathbb{R}$. We can find a rational number

within Y, sufficiently close to q such that this rational number is also in U. Therefore $(U\backslash A)\cap(V\backslash B)\neq\varnothing$ and the topology defined in this problem is not T_3.

(c) Let $\mathcal{N}=\{\{x\}\cup B(x,\frac{1}{n})\backslash S, n=1,2,\dots\}$. For a neighborhood $U\backslash A$ of x, we can find sufficiently large n such that $B(x,\frac{1}{n})\subseteq U\Rightarrow\{x\}\cup B(x,\frac{1}{n})\backslash Q^c\subseteq U\backslash A$. Therefore, (X,T) is C_1.

(d) $\forall S_1\subset S, S_1=S\cap(\mathbb{R}\backslash(S\backslash S_1))$ is open in S. Thus S has discrete topology. Each single point set is open. Therefore the accumulated point set of A is empty. $\bar{A}=S\Rightarrow A=S$ and A is uncountable. Hence S cannot be separable.

(e) Assume (X,T) is C_2, then (S,T) is C_2. C_2 space is separable, a contradiction.

□

4.3 Show that a compact metric space is separable and thus is C_2.

Solution : *Proof.* $\forall n\in\mathbb{N}, X=\bigcup_{x\in X}B(x,\dfrac{1}{n})=\bigcup_{i=1}^{m(n)}B(x_{n_i},\dfrac{1}{n})$. We choose $A=\{x_{n_i}|n\in\mathbb{N}, i=1,2,\dots,m(n)\}$. Then A is countable, and we verify $\bar{A}=X$. $\forall x\in X\backslash A$ and a neighborhood U of x, we can find sufficiently large n such that $B(x,\frac{1}{n})\subseteq U$. For this n, there exists n_i such that $x\in B(x_{n_i},\frac{1}{n})\Rightarrow x_{n_i}\in B(x,\frac{1}{n})\Rightarrow U\cap A\neq\varnothing$. Therefore, A is dense in X. And by known conclusion, seperable metric space is C_2. □

4.4 Let $\sigma: U\to\Sigma$ be a parameterization of a regular surface Σ with an open subset $U\subseteq\mathbb{R}^2$.

(1) Prove that a surface Σ is part of a plane if and only if its second fundamental form is always zero.

(2) Prove that a surface Σ is part of a sphere if and only if its second fundamental form Π_p is a non-zero-constant multiple of its first fundamental form I_p, i.e. $\Pi_p=cI_p$ for some real number c and each $p\in\Sigma$.

Solution : *Proof.*

(1) $\Rightarrow$: the second partial derivative of σ with respect to u,v is zero, therefore the second fundamental form is zero.
$\Leftarrow$: We know that $\sigma_u\cdot\vec{n}=0\Rightarrow\sigma_{uu}\cdot\vec{n}+\sigma_u\cdot\vec{n}_u=0$. Since $L=\sigma_{uu}\cdot\vec{n}=0$, we have $\sigma_u\cdot\vec{n}_u=0$. Also $\sigma_{uv}\cdot\vec{n}+\sigma_u\cdot\vec{n}_v=0\Rightarrow\sigma_v\cdot\vec{n}_u=0$. From $\vec{n}\cdot\vec{n}=1\Rightarrow\vec{n}\cdot\vec{n}_u=0$. While $\vec{n},\sigma_u,\sigma_v$ is an orthogonal basis of $\mathbb{R}^3$, $\vec{n}_u=0$. By the same deduction, $\vec{n}_v=0$. Then $\vec{n}$ is a constant unit vector and integrate $\sigma_u\cdot\vec{n}=0$ gives $\sigma\cdot\vec{n}=c$, which is the equation of a plane in $\mathbb{R}^3$.

(2) $\Rightarrow$: See the next problem solution for detail.

$\Leftarrow$: We know that $\binom{\vec{n}_u}{\vec{n}_v} = -BA^{-1}\binom{\sigma_u}{\sigma_v}$. Since $B = cA$,
$\vec{n}_u = -c\sigma_u, \vec{n}_v = -c\sigma_v \Rightarrow \vec{n} + c\sigma = a \Rightarrow \|\sigma - \frac{a}{c}\| = |\frac{1}{c}|$. That is, Σ is part of a sphere.

$\square$

4.5 Compute the Gauss curvature of the sphere S^2 of radius r.

Solution : *Proof.* We provide two methods:

.1. Use the parameterization $(u, v) \longmapsto (r\cos u\cos v, r\cos u\sin v, r\sin u)$ The first fundamental form matrix $A = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\cos^2 u \end{pmatrix}$. The second fundamental form matrix $B = A$. Therefore the Gaussian curvature $K = \det(BA^{-1}) = \frac{1}{r^2}$

.2. The intersection curve γ of a normal section with the sphere is always a great circle $\Rightarrow k_1 = k_2 = \pm\frac{1}{r} \Rightarrow K = \frac{1}{r^2}$.

$\square$