HIT,Shenzhen
DIFFERENTIAL GEOMETRY AND TOPOLOGY
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Homework 1

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1.1. Show that the signed curvature k_s of any regular planar curve $\gamma(t)$ is smooth. Use this to prove that the curvature $k(t)$ is smooth if $k(t) > 0$ for any t. Give an example to show that $k(t)$ may not be smooth if $k(t) = 0$ for some t.

Proof. Without loss of generality, we assume $\gamma(t)$ is unit-speed. Otherwise by reparametrization (which is smooth transformation) we can get a unit-speed representation of the curve.

$$\gamma''(t) = k_s N \Rightarrow k_s = \gamma''(t) \cdot N \Rightarrow k_s \text{ is smooth.}$$

We know $k = |k_s|$. If $k(t) > 0$, k_s can not change sign by continuity.$\Rightarrow k(t) = -k_s(t)\,\forall t$ or $k(t) = k_s(t)\,\forall t$ and k is smooth.

Counterexample: Let $\gamma(t) : (t, t^3) \Rightarrow k(t) = \frac{6|t|}{(1+9t^4)^{3/2}}$. Since *abs* function is not smooth at $t = 0$, $k(t)$ is not smooth. □

1.2. Describe all curves in $\mathbb{R}^3$ which have *constant* curvature $\kappa > 0$ and *constant* torsion τ

Solution. Let $\gamma(t) = (a\cos t, b\sin t, bt)$, which is circular helix. We know that $\kappa = \frac{|a|}{a^2+b^2}$ and $\tau = \frac{b}{a^2+b^2}$, which gives $|a| = \frac{\kappa}{\kappa^2+\tau^2}, b = \frac{\tau}{\kappa^2+\tau^2}$, by the fundamental theorem of curves, all curves with constant curvature $\kappa > 0$ and constant torsion τ can be obtained by translating and rotating the helix with parameter a, b.

1.3. Let $\gamma(t)$ be a regular plane curve and let λ be a constant. The *parallel curve* γ^λ of γ is defined by

$$\gamma^\lambda(t) = \gamma(t) + \lambda \boldsymbol{n}_s(t) \tag{1}$$

Show that, if $\lambda\kappa_s(t) \neq 1$ for all values of t, then γ^λ is a regular curve and that its signed curvature is $\frac{\kappa_s}{|1-\lambda\kappa_s|}$.

Proof. Let T be the tangent vector of γ, $\boldsymbol{n}_s$ the vector obtained by rotating T anti-clockwise 90°. Also $\widetilde{T}$ is the tangent vector of γ^λ and $\tilde{\boldsymbol{n}}_s$ is obtained from $\widetilde{T}$.

$$\begin{aligned}
\frac{\mathrm{d}\gamma^\lambda(t)}{\mathrm{d}t} &= \gamma'(t) + \lambda\frac{\mathrm{d}\boldsymbol{n}_s(t)}{\mathrm{d}t} = \gamma'(t) + \lambda\frac{\mathrm{d}\boldsymbol{n}_s}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} \\
&= \gamma'(t) - \lambda\kappa_s\frac{\mathrm{d}s}{\mathrm{d}t}T = \gamma'(t) - \lambda\kappa_s\frac{\mathrm{d}\gamma}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} \\
&= (1-\lambda\kappa_s)\gamma'(t) \neq 0
\end{aligned}$$

We choose the arc length parameter s^λ for curve γ^λ, and the arc length parameter for γ is denoted by s.

$$s^\lambda = \int_{t_0}^{t} \|\frac{\mathrm{d}\gamma^\lambda}{\mathrm{d}v}\| dv \Rightarrow \frac{\mathrm{d}s^\lambda}{\mathrm{d}t} = \|\frac{\mathrm{d}\gamma^\lambda}{\mathrm{d}t}\| = |1-\lambda\kappa_s|\|\gamma'(t)\|$$

$$\begin{aligned}\widetilde{T} &= \frac{\mathrm{d}\gamma^\lambda}{\mathrm{d}s^\lambda} = \frac{\frac{\mathrm{d}\gamma^\lambda}{\mathrm{d}t}}{\frac{\mathrm{d}s^\lambda}{\mathrm{d}t}} = \frac{(1-\lambda\kappa_s)\gamma'(t)}{|1-\lambda\kappa_s|\|\gamma'(t)\|} \\ &= \mathrm{sgn}\{1-\lambda\kappa_s\}\frac{\gamma'(t)}{\|\gamma'(t)\|} = \mathrm{sgn}\{1-\lambda\kappa_s\}T \\ &\Rightarrow \tilde{\boldsymbol{n}}_s = \mathrm{sgn}\{1-\lambda\kappa_s\}\boldsymbol{n}_s\end{aligned}$$

Since $\kappa_s(t)$ is continuous and $\lambda\kappa_s(t) \neq 1$, $1-\lambda\kappa_s(t)$ has constant sign. Therefore $\gamma^\lambda \neq 0$ and γ^λ is regular.

$$\frac{\mathrm{d}\widetilde{T}}{\mathrm{d}s^\lambda} = \frac{\frac{\mathrm{d}\widetilde{T}}{\mathrm{d}t}}{\frac{\mathrm{d}s^\lambda}{\mathrm{d}t}} = \frac{\mathrm{sgn}\{1-\lambda\kappa_s\}\frac{\mathrm{d}T}{\mathrm{d}t}}{|1-\lambda\kappa_s|\|\gamma'(t)\|} = \frac{\mathrm{sgn}\{1-\lambda\kappa_s\}}{|1-\lambda\kappa_s|}\frac{\mathrm{d}T}{\mathrm{d}s}$$

Let κ_s be the signed curvature of γ and $\tilde{\kappa}_s$ be the signed curvature of γ^λ. Then

$$\tilde{\kappa}_s = \frac{\mathrm{d}\widetilde{T}}{\mathrm{d}s}\cdot\tilde{\boldsymbol{n}}_s = \frac{\mathrm{sgn}^2\{1-\lambda\kappa_s\}}{|1-\lambda\kappa_s|}\frac{\mathrm{d}T}{\mathrm{d}s}\cdot\boldsymbol{n}_s = \frac{1}{|1-\lambda\kappa_s|}\kappa_s$$

□

1.4. Another approach to the curvature of a unit-speed plane curve γ at a point $\gamma(s_0)$ is to look for the 'best approximating circle' at this point. We can then *define* the curvature of γ to be the reciprocal of the radius of this circle.

Carry out this programme by showing that the center of the circle which passes through three nearby points $\gamma(s_0)$ and $\gamma(s_0 \pm \delta_s)$ on γ approaches the point

$$\boldsymbol{\epsilon}(s_0) = \gamma(s_0) + \frac{1}{\kappa_s(s_0)}\boldsymbol{n}_s(s_0) \tag{2}$$

as δ_s tends to zero. The circle $\mathcal{C}$ with center $\boldsymbol{\epsilon}$ passing through $\gamma(s_0)$ is called the *osculating circle* to γ at the point $\gamma(s_0)$, and $\epsilon(s_0)$ is called the *centre of curvature* of γ at $\gamma(s_0)$. The radius of $\mathcal{C}$ is $1/|\kappa_s(s_0)| = 1/\kappa(s_0)$, where κ is the curvature of γ - this is called the *radius of curvature* of γ at $\gamma(s_0)$.

Solution. The line segment bisector of $\gamma(s_0), \gamma(s_0 + \delta_s)$ has the parametrized form (t_1 is the parameter):

$$\ell_1(t_1) : \frac{1}{2}(\gamma(s_0) + \gamma(s_0 + \delta_s)) + t_1(\gamma(s_0 + \delta_s) - \gamma(s_0)) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Similary, the line segment bisector of $\gamma(s_0), \gamma(s_0 - \delta_s)$ has the parametrized form (t_2 is the parameter):

$$\ell_2(t_2) : \frac{1}{2}(\gamma(s_0) + \gamma(s_0 - \delta_s)) + t_2(\gamma(s_0 - \delta_s) - \gamma(s_0)) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The intersection of $\ell_1(t_1)$ and $\ell_2(t_2)$ is the center of the approximating circle.

To simplify the notation, let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, a = \gamma(s_0 + \delta_s) - \gamma(s_0), b = \gamma(s_0 - \delta_s) - \gamma(s_0)$. The intersection point satisfies $\ell_1(t_1) = \ell_2(t_2) \Rightarrow \frac{1}{2}(a - b) = t_2 bJ - t_1 aJ$. Since aJ is perpendicular with a (J is counterclockwise $90°$ rotation matrix), dot product both sides by a. we can solve t_2 as: $t_2 = \frac{(a-b)\cdot a}{2bJ\cdot a}$. Then the center of circle can be expressed by a, b, J as:

$$\boldsymbol{\epsilon}(s_0, \delta_s) = \gamma(s_0) + \frac{b}{2} + \frac{(a-b)\cdot a}{2bJ\cdot a} bJ$$

Since δ_s is small, we can expand a, b as:

$$a = \gamma'(s_0)\delta_s + \gamma''(s_0)\frac{\delta_s^2}{2} + o(\delta_s^2) \tag{3a}$$

$$b = -\gamma'(s_0)\delta_s + \gamma''(s_0)\frac{\delta_s^2}{2} + o(\delta_s^2) \tag{3b}$$

By the definition of $\boldsymbol{n}_s, \kappa_s$, we have $\boldsymbol{n}_s(s_0) = \gamma'(s_0)J, \gamma''(s_0) = \kappa_s \boldsymbol{n}_s(s_0)$, from (**??**) we have

$$bJ = -\frac{\kappa_s \delta_s^2}{2}\gamma'(s_0) - \frac{\delta_s}{\kappa_s}\gamma''(s_0) + o(\delta_s^2) \tag{4}$$

Since $\gamma'(s_0)$ is perpendicular with $\gamma''(s_0)$, $\|\gamma'\| = 1, \|\gamma''\| = \kappa_s$,

$$2bJ\cdot a = -2\delta_s^3\kappa_s + o(\delta_s^3) \Rightarrow \frac{1}{2bJ\cdot a} = \frac{1}{-2\delta_s^3\kappa_s}(1 + o(1))$$

$$(a-b)\cdot a = 2\delta_s^2 + o(\delta_s^3)$$

$$\frac{(a-b)\cdot a}{2bJ\cdot a} bJ = \frac{\gamma''(s_0)}{\kappa_s^2} + o(1) \Rightarrow \boldsymbol{\epsilon}(s_0, \delta_s) = \gamma(s_0) + \frac{1}{\kappa_s(s_0)}\boldsymbol{n}_s(s_0) + o(1)$$

It follows that $\boldsymbol{\epsilon}(s_0) = \boldsymbol{\epsilon}(s_0, \delta_s)$ as $\delta_s \to 0$.