Lecture 3 remark: Compactness and Connectedness

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1 Compactness: definitions and examples

Theorem 1.6 **Proof.**
$\Rightarrow$: Since a metric space is C_1. ■

2 Compactness: properties

Theorem 2.10 (Continued) *We show that a compact Hausdorff space X is T_4.*

Proof. Let A, B be two closed set of X, $\forall x \in A$, since X is T_3, we can find $V_x \ni x, V_y \supseteq B$ and $V_x \cap V_y = \varnothing$. $\bigcup\limits_{x \in A} V_x$ is an open cover of A, since X is compact, we can find a finite subcover $\bigcup_{i=1}^n V_{x_i}$ covering A, which is disjoint with open set $\bigcap_{i=1}^n V_{y_i}$ covering B. Therefore X is T_4. ■

3 Connectness

Lemma 3.19 *If B is a connected subset of X, then $\bar{B}$ is also connected.*

Proof. Assume $\bar{B} = B_1 \dot{\bigcup} B_2, B_1$ and B_2 are open. $B = (B \cap B_1) \dot{\bigcup} (B \cap B_2)$, since B is connected, either $B \cap B_1$ or $B \cap B_2$ is empty. Suppose $B \cap B_1$ is empty, since $B \subseteq B_1 \cup B_2$, $B \subseteq B_2$. $\exists x \in B_1 \subseteq \bar{B}$. x is an accumulated point of B. But B_1 is an open set containing x and is disjoint with B, a contradiction. ■

Lemma 3.22 *Connected component of X is closed.*

Proof. Since A is a connected component of X, by **Lemma 3.19** $\bar{A}$ is connected. $\bar{A} \subseteq A$, and by the definition of connected component, $\bar{A} = A$. Therefore A is closed. ■

Example 3.23 (Continued) *If X has only finitely many connecte components, then each component A is both closed and open.*

Proof. Let $X = \bigcup_{i=1}^{n} A_i$, then $A_i = (X \backslash A_i)^c = \bigcap_{\substack{j=1 \\ j \neq i}}^{n} A_j^c$ ■

Lemma 3.25 *If $f : X \to Y$ is continuous and X is path-connected, then $f(X)$ is also path-connected.*

Proof. For any two points $x, y \in f(X)$, $f^{-1}(x), f^{-1}(y) \in X$(choose one if having multiple preimage). Then there exists a continuous curve $g : [0,1] \to X, g(0) = f^{-1}(x), g(1) = f^{-1}(y)$. $f \circ g$ is a continuous mapping from $[0,1]$ to Y, with $(f \circ g)(0) = x, (f \circ g)(1) = y$. Therefore $f(X)$ is path-connected. ■

Lemma 3.26 (Another Proof) **Proof.** Assume $X = U_1 \dot{\bigcup} U_2, U_1$ and U_2 are open. Since U_1, U_2 are not empty, we can find $x \in U_1, y \in U_2$. Since X is path-connected, we can find a continuous mapping from $[0,1]$ to X such that $f(0) = x, f(1) = y$. Then $[0,1] = f^{-1}(U_1 \cap f([0,1])) \dot{\bigcup} f^{-1}(U_2 \cap f([0,1]))$, a contradiction. ■