

**Homework 1**

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- 1.1. Show that the signed curvature  $k_s$  of any regular planar curve  $\gamma(t)$  is smooth. Use this to prove that the curvature  $k(t)$  is smooth if  $k(t) > 0$  for any  $t$ . Give an example to show that  $k(t)$  may not be smooth if  $k(t) = 0$  for some  $t$ .

*Proof.* Without loss of generality, we assume  $\gamma(t)$  is unit-speed, otherwise by reparametrization (which is smooth transformation) we can get a unit-speed representation of the curve. Since  $\gamma''(t) = k_s N \Rightarrow k_s = \gamma''(t) \cdot N \Rightarrow k_s$  is smooth.

$k = |k_s|$ , if  $k(t) > 0$ , then  $k_s$  can not change sign by continuity.  $\Rightarrow k(t) = -k_s(t) \forall t$  or  $k(t) = k_s(t) \forall t$  and  $k$  is smooth.

Counterexample: Let  $\gamma(t) : (t, t^3) \Rightarrow k(t) = \frac{6|t|}{(1+9t^4)^{3/2}}$ . Since  $abs$  function is not smooth at  $t = 0$ ,  $k(t)$  is not smooth.  $\square$

- 1.2. Describe all curves in  $\mathbb{R}^3$  which have *constant* curvature  $\kappa > 0$  and *constant* torsion  $\tau$

*Solution.* Let  $\gamma(t) = (a \cos t, b \sin t, bt)$ , which is circular helix. We know that  $\kappa = \frac{|a|}{a^2+b^2}$  and  $\tau = \frac{b}{a^2+b^2}$ , which gives  $|a| = \frac{\kappa}{\kappa^2+\tau^2}$ ,  $b = \frac{\tau}{\kappa^2+\tau^2}$ , by the fundamental theorem of curves, all curves with constant curvature  $\kappa > 0$  and constant torsion  $\tau$  can be obtained by translating and rotating the helix with parameter  $a, b$ .

- 1.3. Let  $\gamma(t)$  be a regular plane curve and let  $\lambda$  be a constant. The *parallel curve*  $\gamma^\lambda$  of  $\gamma$  is defined by

$$\gamma^\lambda(t) = \gamma(t) + \lambda \mathbf{n}_s(t) \quad (1)$$

Show that, if  $\lambda \kappa_s(t) \neq 1$  for all values of  $t$ , then  $\gamma^\lambda$  is a regular curve and that its signed curvature is  $\frac{\kappa_s}{|1 - \lambda \kappa_s|}$ .

*Proof.* Let  $T$  be the tangent vector of  $\gamma$ ,  $\mathbf{n}_s$  the vector obtained by rotating  $\mathbf{n}_t$  anti-clockwise  $90^\circ$ . Also  $\tilde{T}$  be the tangent vector of  $\gamma^\lambda$  and  $\tilde{\mathbf{n}}_s$  is obtained from  $\tilde{T}$ . For curve  $\gamma^\lambda$ , we choose the arc length parameter  $\tilde{s} = \int_{t_0}^t \|\gamma'(v)\| dv \Rightarrow \frac{ds}{dt} = |1 - \lambda \kappa_s| \|\gamma'(t)\|$ , and the arc length parameter for  $\gamma$  is denoted by  $s$ , then we have  $\tilde{s}(t) = |1 - \lambda \kappa_s(t)| s(t)$ .

$$\begin{aligned} \frac{d\gamma^\lambda(t)}{dt} &= \gamma'(t) + \lambda \frac{d\mathbf{n}_s(t)}{dt} = \gamma'(t) + \lambda \frac{d\mathbf{n}_s}{ds} \frac{ds}{dt} \\ &= \gamma'(t) - \lambda \kappa_s \frac{ds}{dt} T = (1 - \lambda \kappa_s) \frac{d\gamma}{ds} \frac{ds}{dt} \\ &= (1 - \lambda \kappa_s) \gamma'(t) \neq 0 \end{aligned}$$

Hence  $\gamma^\lambda$  is regular.

$$\tilde{T} = \frac{d\gamma^\lambda}{d\tilde{s}} = \frac{\frac{d\gamma^\lambda(t)}{dt}}{|1 - \lambda \kappa_s| \frac{ds}{dt}} = \text{sgn}\{1 - \lambda \kappa_s\} \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

Similarly we can show that  $T = \frac{\gamma'(t)}{\|\gamma'(t)\|} \Rightarrow \tilde{\mathbf{n}}_s = \text{sgn}\{1 - \lambda \kappa_s\} \mathbf{n}_s$

$$\frac{d\tilde{T}}{d\tilde{s}} = \frac{\frac{d\tilde{T}}{dt}}{|1 - \lambda \kappa_s| \frac{ds}{dt}}$$

Since  $\kappa_s(t)$  is continuous and  $\lambda \kappa_s(t) \neq 1$ ,  $1 - \lambda \kappa_s(t)$  has constant sign. therefore  $\frac{d\tilde{T}}{d\tilde{s}} = \text{sgn}\{1 - \lambda \kappa_s\} \frac{dT}{ds} \Rightarrow \frac{d\tilde{T}}{ds} = \frac{\text{sgn}\{1 - \lambda \kappa_s\}}{|1 - \lambda \kappa_s|} \frac{dT}{ds}$  Let  $\kappa_s$  be the signed curvature of  $\gamma$  and  $\tilde{\kappa}_s$  be the signed curvature of  $\gamma^\lambda$ . Then

$$\tilde{\kappa}_s = \frac{d\tilde{T}}{d\tilde{s}} \cdot \tilde{\mathbf{n}}_s = \frac{\text{sgn}^2\{1 - \lambda \kappa_s\}}{|1 - \lambda \kappa_s|} \frac{dT}{ds} \cdot \mathbf{n}_s = \frac{1}{|1 - \lambda \kappa_s|} \kappa_s$$

□

- 1.4. Another approach to the curvature of a unit-speed plane curve  $\gamma$  at a point  $\gamma(s_0)$  is to look for the 'best approximating circle' at this point.

We can then *define* the curvature of  $\gamma$  to be the reciprocal of the radius of this circle.

Carry out this programme by showing that the center of the circle which passes through three nearby points  $\gamma(s_0)$  and  $\gamma(s_0 \pm \delta_s)$  on  $\gamma$  approaches the point

$$\epsilon(s_0) = \gamma(s_0) + \frac{1}{\kappa_s(s_0)} \mathbf{n}_s(s_0) \quad (2)$$

as  $\delta_s$  tends to zero. The circle  $\mathcal{C}$  with center  $\epsilon$  passing through  $\gamma(s_0)$  is called the *osculating circle* to  $\gamma$  at the point  $\gamma(s_0)$ , and  $\epsilon(s_0)$  is called the *centre of curvature* of  $\gamma$  at  $\gamma(s_0)$ . The radius of  $\mathcal{C}$  is  $1/|\kappa_s(s_0)| = 1/\kappa(s_0)$ , where  $\kappa$  is the curvature of  $\gamma$  - this is called the *radius of curvature* of  $\gamma$  at  $\gamma(s_0)$ .

*Solution.* The line segment bisector of  $\gamma(s_0), \gamma(s_0 + \delta_s)$  has the parametrized form ( $t_1$  is the parameter):

$$\ell_1(t_1) : \frac{1}{2}(\gamma(s_0) + \gamma(s_0 + \delta_s)) + t_1(\gamma(s_0 + \delta_s) - \gamma(s_0)) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Similarly, the line segment bisector of  $\gamma(s_0), \gamma(s_0 - \delta_s)$  has the parametrized form ( $t_2$  is the parameter):

$$\ell_2(t_2) : \frac{1}{2}(\gamma(s_0) + \gamma(s_0 - \delta_s)) + t_2(\gamma(s_0 - \delta_s) - \gamma(s_0)) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The intersection of  $\ell_1(t_1)$  and  $\ell_2(t_2)$  is the center of the approximating circle

To simplify the notation, let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, a = \gamma(s_0 + \delta_s) - \gamma(s_0), b = \gamma(s_0 - \delta_s) - \gamma(s_0). \text{ The}$$

intersection point satisfies  $\ell_1(t_1) = \ell_2(t_2) \Rightarrow \frac{1}{2}(a - b) = t_2 b J - t_1 a J$ .

Since  $aJ$  is perpendicular with  $a$  ( $J$  is counterclockwise  $90^\circ$  rotation matrix), dot product both sides by  $a$ . we can solve  $t_2$  as:  $t_2 = \frac{(a-b) \cdot a}{2bJ \cdot a}$ . Then the center of circle can be expressed by  $a, b, J$  as:

$$\epsilon(s_0, \delta_s) = \gamma(s_0) + \frac{b}{2} + \frac{(a-b) \cdot a}{2bJ \cdot a} bJ$$

Since  $\delta_s$  is small, we can expand  $a, b$  as:

$$a = \gamma'(s_0)\delta_s + \gamma''(s_0)\frac{\delta_s^2}{2} + o(\delta_s^2) \quad (3a)$$

$$b = -\gamma'(s_0)\delta_s + \gamma''(s_0)\frac{\delta_s^2}{2} + o(\delta_s^2) \quad (3b)$$

By the definition of  $\mathbf{n}_s, k_s$ , we have  $\mathbf{n}_s(s_0) = \gamma'(s_0)J, \gamma''(s_0) = \kappa_s \mathbf{n}_s(s_0)$ , from (3b) we have

$$bJ = -\frac{k_s \delta_s^2}{2} \gamma'(s_0) - \frac{\delta_s}{k_s} \gamma''(s_0) + o(\delta_s^2) \quad (4)$$

Since  $\gamma'(s_0)$  is perpendicular with  $\gamma''(s_0)$ ,  $2bJ \cdot a = -2\delta_s^3 \kappa_s + o(\delta_s^3)$  and  $\frac{1}{2bJ \cdot a} = \frac{1}{-2\delta_s^3 \kappa_s} (1 + o(1))$ , also from  $||\delta_s|| = 1$  we can compute  $(a - b) \cdot a = 2\delta_s^2 + o(\delta_s^3) \Rightarrow \frac{(a-b) \cdot a}{2bJ \cdot a} bJ = \frac{\gamma''(s_0)}{\kappa_s^2} + o(1) \Rightarrow \epsilon(s_0, \delta_s) = \gamma(s_0) + \frac{1}{\kappa_s(s_0)} \mathbf{n}_s(s_0)$ . It follows that  $\epsilon(s_0) = \epsilon(s_0, \delta_s)$  as  $\delta_s \rightarrow 0$ .