笔记整理

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1 草稿

理想流体运动的基本方程理想流体应力张量 $T_{ij} = -p\delta_{ij}$，因此动量方程(**??**)式化为:

$$\frac{\partial \overrightarrow{v}}{\partial t} + \overrightarrow{v} \cdot \nabla \overrightarrow{v} = \overrightarrow{f} - \frac{1}{\rho} \nabla \cdot p \tag{1}$$

(1)式即欧拉方程，相应的(**??**)化为

$$\frac{D(e + \frac{1}{2}|\overrightarrow{v}|^2)}{Dt} = \overrightarrow{f} \cdot \overrightarrow{v} - \frac{1}{\rho} \nabla \cdot (p \overrightarrow{v}) + (\dot{q} + q_R) \tag{2}$$

综合连续性方程(**??**)式,(1),(**??**) 共 5 个方程，但未知数有 $\overrightarrow{v}, e, p, \rho$ 6 个，因此需补充热力学方程才能使方程组封闭。

实际上，我们可以通过动量方程(1)式得到动能的变化率:

$$\frac{D}{Dt}(\frac{1}{2}|\overrightarrow{v}|^2) = \overrightarrow{f} \cdot \overrightarrow{v} - \frac{1}{\rho} \overrightarrow{v} \cdot \nabla p \tag{3}$$

将(2)式与(3)作差得:

$$\begin{aligned} \frac{De}{Dt} =& -\frac{p}{\rho} \nabla \cdot \overrightarrow{v} + \dot{q} + q_R, \text{by } (\textbf{??}) \\ =& \frac{p}{\rho^2} \frac{D\rho}{Dt} + \dot{q} + q_R \\ =& -p \frac{D}{Dt} \left(\frac{1}{\rho} \right) + \dot{q} + q_R \end{aligned} \tag{4}$$

定义 $i = e + \frac{p}{\rho}$ 为气体的焓，则可以得到

$$\frac{Di}{Dt} = \dot{q} + q_R + \frac{1}{\rho} \frac{Dp}{Dt} \tag{5}$$

对于绝热状态下的理想常比热完全气体，我们有

$$\begin{aligned}\frac{De}{Dt} =&C_V\frac{DT}{Dt}, e = C_V T\\ =&\frac{C_V}{R}\frac{D}{Dt}\left(\frac{p}{\rho}\right), p = \rho RT\\ =&\frac{1}{\gamma-1}\frac{D}{Dt}\left(\frac{p}{\rho}\right), C_P - C_V = R, \frac{C_P}{C_V} = \gamma\end{aligned} \tag{6}$$

将(4)式去掉产热项，与(6)式结合可以得到

$$\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \tag{7}$$

(7)式即为对于气体补充的 ρ 和 p 的关系的热力学方程。

对于匀质不可压缩的液体，补充 $\nabla\cdot\overrightarrow{v} = 0$ 的方程，此时(**??**)式恒成立，动力学方程与热力学方程解耦，因此我们可以联立求解:

$$\begin{aligned}\nabla\cdot\overrightarrow{v} =&0\\ \frac{\partial\overrightarrow{v}}{\partial t} + \overrightarrow{v}\cdot\nabla\overrightarrow{v} =&\overrightarrow{f} - \frac{1}{\rho}\nabla\cdot p\end{aligned} \tag{8}$$

得到 $\overrightarrow{v}, p$ 再代入能量方程求其他参量。

$$\tag{9}$$

对于球坐标，基矢量随坐标变量的变化规律为 [8]:

$$\frac{\partial\overrightarrow{e_r}}{\partial\theta} =\overrightarrow{e_\theta},\ \frac{\partial\overrightarrow{e_r}}{\partial\varphi} = \overrightarrow{e_\varphi}\sin\theta \tag{10}$$

$$\frac{\partial\overrightarrow{e_\theta}}{\partial\theta} = -\overrightarrow{e_r},\ \frac{\partial\overrightarrow{e_\theta}}{\partial\varphi} = \overrightarrow{e_\varphi}\cos\theta \tag{11}$$

$$\frac{\partial\overrightarrow{e_\varphi}}{\partial\varphi} = -\left(\overrightarrow{e_r}\sin\theta + \overrightarrow{e_\theta}\cos\theta\right) \tag{12}$$

球坐标系下的梯度算子表示为:

$$\nabla = \frac{\partial}{\partial r}\overrightarrow{e_r} + \frac{1}{r}\frac{\partial}{\partial\theta}\overrightarrow{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\overrightarrow{e_\varphi} \tag{13}$$

因此球坐标系下对于矢量 $\overrightarrow{v} = v_r\overrightarrow{e_r} + v_\theta\overrightarrow{e_\theta} + v_\varphi\overrightarrow{e_\varphi}$ 的散度为:

$$\begin{aligned}&(\frac{\partial}{\partial r}\overrightarrow{e_r} + \frac{1}{r}\frac{\partial}{\partial\theta}\overrightarrow{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\overrightarrow{e_\varphi})\cdot(v_r\overrightarrow{e_r} + v_\theta\overrightarrow{e_\theta} + v_\varphi\overrightarrow{e_\varphi}) =\\ &\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_r}{r}\frac{\partial\overrightarrow{e_r}}{\partial\theta}\cdot\overrightarrow{e_\theta} + (v_r\frac{\partial\overrightarrow{e_r}}{\partial\varphi} + v_\theta\frac{\partial\overrightarrow{e_\theta}}{\partial\varphi} + v_\varphi\frac{\partial\overrightarrow{e_\varphi}}{\partial\varphi})\cdot\frac{\overrightarrow{e_\varphi}}{r\sin\theta}\\ =&\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_r}{r} + (v_r\sin\theta + v_\theta\cos\theta)\frac{1}{r\sin\theta}\\ =&\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{2v_r}{r} + \frac{v_\theta\cos\theta}{r\sin\theta}\end{aligned}$$

所以
$$\nabla \cdot \overrightarrow{v} = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(v_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} \tag{14}$$
$$\tag{15}$$

References

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[5] https://en.wikipedia.org/wiki/Curl_(mathematics)

[6] https://en.wikipedia.org/wiki/Fundamental_solution

[7] https://en.wikipedia.org/wiki/Green%27s_function#Green.27s_functions_for_the_Laplacian

[8] https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates