笔记整理

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1 草稿

涡量方程: (**??**)式给出了不可压流的 N-S 方程，对于一般情形，在(**??**)式中代入(**??**)式（取 $\mu' = 0$）于是有

$$\nabla \cdot \boldsymbol{T} = -\nabla p + \mu \nabla^2 \overrightarrow{v} + \mu \frac{\partial^2 v_i}{\partial x_i \partial x_j} \overrightarrow{e}_j + \frac{\partial}{\partial x_i}(-\frac{2}{3}\mu \nabla \cdot \overrightarrow{v} \delta_{ij}) \overrightarrow{e}_j, \text{eq}(\textbf{??}) \text{ invalid} \tag{1}$$

$$= -\nabla p + \mu \nabla^2 \overrightarrow{v} + \frac{\mu}{3} \nabla (\nabla \cdot \overrightarrow{v}) \tag{2}$$

因此我们得到一般形式的 N-S 方程:

$$\frac{\partial \overrightarrow{v}}{\partial t} + \overrightarrow{v} \cdot \nabla \overrightarrow{v} = \overrightarrow{f} - \frac{\nabla p}{\rho} + \frac{\mu}{\rho}(\frac{1}{3}\nabla(\nabla \cdot \overrightarrow{v}) + \nabla^2 \overrightarrow{v}) \tag{3}$$

通过对上式两边取旋度可以得到涡量方程，为此，首先推导：

$$2\overrightarrow{v} \cdot \nabla \overrightarrow{v} = \nabla(\overrightarrow{v} \cdot \overrightarrow{v}) + 2(\nabla \times \overrightarrow{v}) \times \overrightarrow{v} \tag{4}$$

$$\begin{aligned}
\text{RHS} =& \frac{\partial v_i^2}{\partial x_j} \overrightarrow{e_j} + 2(\epsilon_{ijk} \frac{\partial v_k}{\partial x_j} \overrightarrow{e_i}) \times \overrightarrow{v} \\
=& \frac{\partial v_i^2}{\partial x_j} \overrightarrow{e_j} + 2(\epsilon_{min} \epsilon_{ijk} \frac{\partial v_k}{\partial x_j} v_n) \overrightarrow{e_m} \\
=& 2v_i \frac{\partial v_i}{\partial x_j} \overrightarrow{e_j} - 2((\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}) \frac{\partial v_k}{\partial x_j} v_n) \overrightarrow{e_m} \\
=& 2v_i \frac{\partial v_i}{\partial x_j} \overrightarrow{e_j} - 2(\frac{\partial v_k}{\partial x_j} v_k) \overrightarrow{e_j} + 2(\frac{\partial v_k}{\partial x_j} v_j) \overrightarrow{e_k} \\
=& 2(\frac{\partial v_k}{\partial x_j} v_j) \overrightarrow{e_k} \\
=& \text{ LHS}
\end{aligned}$$

再推导

$$\nabla \times (\overrightarrow{A} \times \overrightarrow{B}) = (\nabla \cdot \overrightarrow{B} + \overrightarrow{B} \cdot \nabla) \overrightarrow{A} - (\nabla \cdot \overrightarrow{A} + \overrightarrow{A} \cdot \nabla) \overrightarrow{B} \tag{5}$$

$$
\begin{aligned}
\text{LHS} =& \nabla \times (\epsilon_{ijk} A_j B_k \overrightarrow{e_i}) \\
=& \epsilon_{mni}\epsilon_{ijk} \frac{\partial (A_j B_k)}{\partial x_n} \overrightarrow{e_m} \\
=& (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})(B_k \frac{\partial A_j}{\partial x_n} + A_j \frac{\partial B_k}{\partial x_n}) \overrightarrow{e_m} \\
=& (B_k \frac{\partial A_j}{\partial x_k} + A_j \frac{\partial B_k}{\partial x_k}) \overrightarrow{e_j} - (B_k \frac{\partial A_j}{\partial x_j} + A_j \frac{\partial B_k}{\partial x_j}) \overrightarrow{e_k} \\
=& \text{RHS}
\end{aligned}
$$

因此，由(3)式我们有:

$$
\begin{aligned}
\nabla \times \text{LHS} =& \nabla \times (\frac{\partial \overrightarrow{v}}{\partial t} + \overrightarrow{v} \cdot \nabla \overrightarrow{v}) \\
=& \frac{\partial}{\partial t}(\nabla \times \overrightarrow{v}) + \nabla \times (\overrightarrow{v} \cdot \nabla \overrightarrow{v}) \\
=& \frac{\partial \overrightarrow{w}}{\partial t} + \nabla \times (\nabla \frac{|\overrightarrow{v}|^2}{2} + \overrightarrow{w} \times \overrightarrow{v}) \\
=& \frac{\partial \overrightarrow{w}}{\partial t} + (\nabla \cdot \overrightarrow{v})\overrightarrow{w} - (\nabla \cdot \overrightarrow{w})\overrightarrow{v} + (\overrightarrow{v} \cdot \nabla)\overrightarrow{w} - (\overrightarrow{w} \cdot \nabla)\overrightarrow{v}, \nabla \cdot \overrightarrow{w} = 0 \\
=& \frac{D\overrightarrow{w}}{Dt} + (\nabla \cdot \overrightarrow{v})\overrightarrow{w} - (\overrightarrow{w} \cdot \nabla)\overrightarrow{v}
\end{aligned}
$$

而上式右端：

$$
\begin{aligned}
\nabla \times \text{LHS} =& \nabla \times \left(\overrightarrow{f} - \frac{\nabla p}{\rho} + \frac{\mu}{\rho}(\frac{1}{3}\nabla(\nabla \cdot \overrightarrow{v}) + \nabla^2 \overrightarrow{v}) \right) \\
=& \nabla \times \overrightarrow{f} - \nabla \frac{1}{\rho} \times \nabla p + \frac{\mu}{\rho} \nabla \times \nabla^2 \overrightarrow{v} + \nabla \left(\frac{\mu}{\rho} \right) \times (\frac{1}{3}\nabla(\nabla \cdot \overrightarrow{v}) + \nabla^2 \overrightarrow{v})
\end{aligned}
$$

因此我们由不可压的 N-S 方程得到了涡量方程的一般形式：

$$
\frac{D\overrightarrow{w}}{Dt} + (\nabla \cdot \overrightarrow{v})\overrightarrow{w} - (\overrightarrow{w} \cdot \nabla)\overrightarrow{v} = \nabla \times \overrightarrow{f} - \nabla \frac{1}{\rho} \times \nabla p + \frac{\mu}{\rho} \nabla \times \nabla^2 \overrightarrow{v} + \nabla \left(\frac{\mu}{\rho} \right) \times (\frac{1}{3}\nabla(\nabla \cdot \overrightarrow{v}) + \nabla^2 \overrightarrow{v}) \tag{6}
$$

从上面的涡量方程可以看出，对于理想正压流体，若质量力有势，方程右端项为零。由连续性方程(**??**)式左端可化简为

$$
\frac{1}{\rho}\frac{D\overrightarrow{w}}{Dt} + \frac{D}{Dt}\left(\frac{1}{\rho}\right)\overrightarrow{w} - \left(\frac{\overrightarrow{w}}{\rho}\right) \cdot \nabla \overrightarrow{v} = 0 \tag{7}
$$

即整理为

$$
\frac{D}{Dt}\left(\frac{\overrightarrow{w}}{\rho}\right) = \left(\frac{\overrightarrow{w}}{\rho}\right) \cdot \nabla \overrightarrow{v} \tag{8}
$$

Lamb 型方程：考虑对理想气体，由欧拉方程(??)式和(4)式得到

$$\frac{\partial \vec{v}}{\partial t} + \nabla(\frac{1}{2}|\vec{v}|^2) - \vec{v} \times \vec{\Omega} = \vec{f} - \frac{1}{\rho}\nabla \cdot p \tag{9}$$

其中 $\vec{v} \times \Omega$ 被称为 Lamb 矢量，若考虑质量力有势的正压定常流体，上式沿流线或涡线积分即可得到 Bernoulli 守恒方程:

$$\int_l \vec{s} \cdot (\nabla(\frac{1}{2}|\vec{v}|^2) - \vec{v} \times \vec{\Omega})ds = \int_l \vec{s} \cdot (-\nabla\Pi - \nabla\mathbb{P})ds \tag{10}$$

注意到 $\vec{s}$ 与 $\vec{v}$ 或 $\vec{\Omega}$ 平行，因此 $\vec{s} \cdot (\vec{v} \times \vec{\Omega}) = 0$, 于是得到

$$\frac{1}{2}v^2 + \mathbb{P} + \Pi = C \tag{11}$$

针对(9)式，如考虑用常比热完全气体的等熵过程 ($\frac{p}{\rho^\gamma} = c$) 代替正压的条件，则压力势项可改写为

$$\begin{aligned}
\mathbb{P} &= \int \frac{dp}{\rho} \\
&= \int c\frac{d\rho^\gamma}{\rho} \\
&= \int c\gamma d\rho^{\gamma-2} d\rho \\
&= \int c\frac{\gamma}{\gamma-1}\rho^{\gamma-1} \\
&= \frac{\gamma}{\gamma-1}\frac{p}{\rho}
\end{aligned}$$

因此对常比热完全气体的等熵过程，Bernoulli 方程为

$$\frac{1}{2}v^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho} + \Pi = C \tag{12}$$

同理可推出对完全气体的等温过程

$$\frac{1}{2}v^2 + RT\ln p + \Pi = C \tag{13}$$

下面考虑流体相对等转速坐标系 $O'x'y'$ 下的 Bernoulli 方程，转动坐标系相对静止坐标系的关系如下图所示：这里，我们去掉正压流体的假设，而附加绝热条件，于是(??)式焓的随体导数可简化为:

$$\begin{aligned}
\frac{Di}{Dt} &= \frac{1}{\rho}\frac{Dp}{Dt} \\
\Rightarrow \vec{v} \cdot \nabla i &= \frac{1}{\rho}\vec{v} \cdot \nabla p
\end{aligned}$$

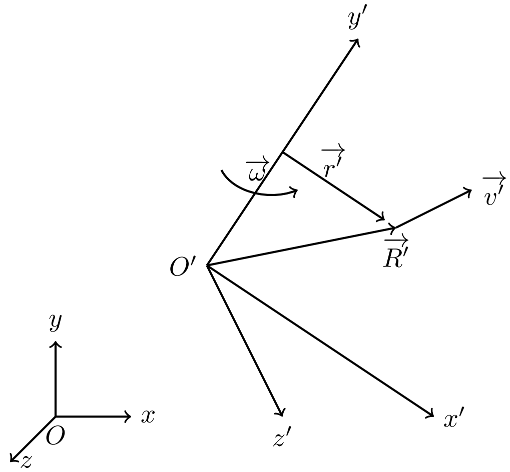

Figure 1: 等转速坐标系下的 Bernoulli 方程推导示意

最后一式用到了定常流的条件，由于 $\overrightarrow{v}$ 的任意性，所以 $\nabla i = \frac{\nabla p}{\rho}$, 即 $\frac{\nabla p}{\rho}$ 有势函数 i。

从(9)式出发，由于是在转动坐标系中，我们对 $\overrightarrow{f}$ 有加速加项的修正，即以 $\overrightarrow{f} - \overrightarrow{a}$ 代替(9)式中的 $\overrightarrow{f}$。$\overrightarrow{a}$ 的表达式由(14)式给出

$$\overrightarrow{a} = \overrightarrow{\omega} \times (\overrightarrow{\omega} \times \overrightarrow{R'}) + 2\overrightarrow{\omega} \times \overrightarrow{v'} \tag{14}$$

注意到 $\overrightarrow{a}$ 的第二项 $\overrightarrow{v'}$ 含 $\overrightarrow{v'}$，如沿相对流线积分，同样由 $\overrightarrow{s'}$ 与 $\overrightarrow{v'}$ 平行的性质得其积分为零，因此，只需考虑 $\overrightarrow{a}$ 的第一项

$$\overrightarrow{\omega} \times (\overrightarrow{\omega} \times \overrightarrow{R'}) = (\overrightarrow{\omega} \cdot \overrightarrow{R'})\overrightarrow{\omega} - (\overrightarrow{\omega} \cdot \overrightarrow{\omega})\overrightarrow{R'} \tag{15}$$

我们这里设 $\overrightarrow{\omega}$ 沿 y' 轴，即 $\overrightarrow{\omega} = \omega\overrightarrow{e_{y'}}$, 考虑 $(\overrightarrow{\omega} \cdot \overrightarrow{\omega})\overrightarrow{R'}$ 在 $\overrightarrow{e_{y'}}$ 方向的投影为 $\omega^2(\overrightarrow{R'} \cdot \overrightarrow{e_{y'}})$, 而 $(\overrightarrow{\omega} \cdot \overrightarrow{R'})\overrightarrow{\omega} = \omega^2(\overrightarrow{R'} \cdot \overrightarrow{e_{y'}})\overrightarrow{e_{y'}}$, 因此(15)式可化简为 $(\overrightarrow{\omega} \cdot \overrightarrow{\omega})\overrightarrow{R'}$ 在 $O'x'z'$ 平面上的投影长度的相反数：

$$\overrightarrow{\omega} \times (\overrightarrow{\omega} \times \overrightarrow{R'}) = -\omega^2\overrightarrow{r'} \tag{16}$$

其中 $\overrightarrow{r'}$ 为 $\overrightarrow{R'}$ 在 $Ox'z'$ 平面上的投影向量, 并假设转动角速度 ω 为常数，则

$$\overrightarrow{\omega} \times (\overrightarrow{\omega} \times \overrightarrow{R'}) = -\nabla'(\frac{1}{2}\omega^2 r'^2) \tag{17}$$

这里 ∇' 表示相对转运坐标系的梯度算子，所以 $\overrightarrow{a}$ 项在沿相对流线积分得到 $\frac{1}{2}\omega^2 r'^2$。综合上面的结果，我们得到等转速坐标系下的 Bernoulli 方程为：

$$\frac{1}{2}v'^2 + \Pi + i - \frac{1}{2}\omega^2 r'^2 = c \tag{18}$$

$$\tag{19}$$

$$\tag{20}$$

References

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[9] https://en.wikipedia.org/wiki/Cauchy%E2%80%93Euler_equation