Tsinghua-Berkeley Shenzhen Institute
INFERENCE AND INFORMATION
Fall 2017

Problem Set 5

Issued: Tuesday 28th November, 2017 **Due:** Monday 4th December, 2017

5.1. Assume $p_{\mathsf{y}}(y;x)$ is a continuously differentiable function of x. Show that

$$\lim_{\delta \to 0} \frac{D\left(p_{\mathsf{y}}(\cdot;x) \,\|\, p_{\mathsf{y}}(\cdot;x+\delta)\right)}{\delta^2} = \frac{1}{2} J_{\mathsf{y}}(x),$$

so that for small δ we can make the approximation

$$D\left(p_{\mathsf{y}}(\cdot;x) \,\|\, p_{\mathsf{y}}(\cdot;x+\delta)\right) \approx \frac{\delta^2}{2} J_{\mathsf{y}}(x),$$

where $J_{\mathsf{y}}(x)$ is the Fisher information in y about x:

$$J_{\mathsf{y}}(x) = -\mathbb{E}\left[\frac{\partial^2}{\partial x^2} \log p_{\mathsf{y}}(\mathsf{y};x)\right].$$

5.2. Suppose, for $i = 1, 2$

$$\mathsf{y}_i = x + \mathsf{w}_i$$

where x is an unknown but non-zero constant, w_1 and w_2 are statistically independent, zero-mean Gaussian random variables with

$$\begin{aligned} \mathrm{Var}(\mathsf{w}_1) &= 1 \\ \mathrm{Var}(\mathsf{w}_2) &= \begin{cases} 1 & x > 0 \\ 2 & x < 0 \end{cases}. \end{aligned}$$

(a) Calculate the Cramér-Rao bound for unbiased estimators of x based on observation of

$$\underline{\mathsf{y}} = \begin{bmatrix} \mathsf{y}_1 \\ \mathsf{y}_2 \end{bmatrix}.$$

(b) Show that a minimum variance unbiased estimator $\hat{x}_{\mathrm{MVU}}(\underline{\mathsf{y}})$ does not exist.

Hint: Consider the estimators

$$\begin{aligned} \hat{x}_1(\underline{\mathsf{y}}) &= \frac{1}{2}\mathsf{y}_1 + \frac{1}{2}\mathsf{y}_2, \\ \hat{x}_2(\underline{\mathsf{y}}) &= \frac{2}{3}\mathsf{y}_1 + \frac{1}{3}\mathsf{y}_2. \end{aligned}$$

5.3. A random variable y is observed and used to decide between two hypotheses, H_0 and H_1. Under each of these two hypotheses, y is as follows:

$$H_0 : \mathsf{y} = s_0 + \mathsf{w}$$
$$H_1 : \mathsf{y} = s_1 + \mathsf{w}$$

where s_0 and s_1 are known scalars with $s_0 < s_1$, and w is a zero mean normal random variable with known variance σ^2. Assume symmetric costs (i.e., $C_{00} = C_{11} = 0, C_{10} = C_{01} = 1$).

The *a priori* probabilities $\mathbb{P}(\mathsf{H} = H_0) = 1 - p$ and $\mathbb{P}(\mathsf{H} = H_1) = p$ are unknown; hence we wish to conduct a minimax test.

(a) What is the least favorable prior and its corresponding decision rule $\hat{H}_{\mathrm{M}}(y)$?

(b) Assuming now that the priors $1 - p$ and p are known, what is the expected cost of the minimax decision rule $\hat{H}_{\mathrm{M}}(y)$ calculated in part (a), and how does it compare to the expected cost (Bayesian risk) of the optimum Bayes' decision rule $\hat{H}_{\mathrm{B}}(y)$? Determine an expression for the expected costs in terms of the unspecified parameters, and then evaluate both costs when $p = (\mathrm{e}^2 + 1)^{-1}, s_0 = 0, s_1 = 2$, and $\sigma^2 = 1$.

Hint: Use Q-function (`https://en.wikipedia.org/wiki/Q-function`) to simplify your expressions.

5.4. (a) Let $p_{\mathsf{y}}(y; x)$ be a member of the exponential family, i.e.,

$$p_{\mathsf{y}}(y; x) = \exp[\lambda(x)t(y) - \alpha(x) + \beta(y)]$$

for some functions $\lambda(\cdot), t(\cdot), \alpha(\cdot)$ and $\beta(\cdot)$. Put $\mathsf{z} = \mathsf{y} + a$, where a is a known constant. Is $p_{\mathsf{z}}(z; x)$ in the exponential family, as well?

(b) Let y_1 and y_2 be two independent identically distributed continuous random variables with a distribution $p_{\mathsf{y}}(y; x)$ that is in the canonical (one-parameter) exponential family. That is, $p_{\mathsf{y}}(y; x) = \exp(xy - \alpha(x) + \beta(y))$ for some functions α and β. Put $\mathsf{z} = \mathsf{y}_1 + \mathsf{y}_2$.

Show that $p_{\mathsf{z}}(z; x)$ is also in the canonical exponential family. That is, show that $p_{\mathsf{z}}(z; x) = \exp(xz - \alpha'(x) + \beta'(z))$ for some functions α' and β'. Express α' in terms of α, and β' in terms of β.

This result means that the canonical exponential family possesses a kind of stability characteristic.