Tsinghua-Berkeley Shenzhen Institute
INFERENCE AND INFORMATION
Fall 2017

Problem Set 4

Issued: Tuesday 21st November, 2017 **Due:** Monday 27th November, 2017

4.1. Assume we have n i.i.d. random variables $\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n$ with mean μ and variance σ^2. Prove that the mean estimator m_n and variance estimator v_n are unbiased, i.e., $\mathbb{E}[\mathsf{m}_n] = \mu, \mathbb{E}[\mathsf{v}_n] = \sigma^2$, where m_n and v_n are defined as

$$\mathsf{m}_n \triangleq \frac{1}{n}\sum_{i=1}^{n} \mathsf{x}_i,$$
$$\mathsf{v}_n \triangleq \frac{1}{n-1}\sum_{i=1}^{n}(\mathsf{x}_i - \mathsf{m}_n)^2.$$

4.2. Let x be a scalar random variable with a known distribution that is symmetrical about zero. Define $g_k \triangleq \mathbb{E}[\mathsf{x}^k]$. Let $\mathsf{y} = \mathsf{x}^3$. Express your answers in terms of the g_k's.

(a) Find $\hat{\mathsf{x}}_{\mathrm{BLS}}(\mathsf{y})$ the Bayes Least-Squares estimator of x given y.

(b) Find $\hat{\mathsf{x}}_{\mathrm{LLS}}(\mathsf{y})$ the Linear Least-Squares estimator of x given y.

(c) Suppose we are given two more observations $\mathsf{z}_1 = \mathsf{x} + \mathsf{w}$ and $\mathsf{z}_2 = \mathsf{x} - 2\mathsf{w}$, where $\mathsf{w} \sim \mathcal{N}(0, 1)$, independent of x and y. Find $\hat{\mathsf{x}}_{\mathrm{LLS}}(\mathsf{y}, \mathsf{z}_1, \mathsf{z}_2)$.

(d) Suppose now that $\mathsf{v} = \mathsf{x}^2$. Find $\hat{\mathsf{x}}_{\mathrm{BLS}}(\mathsf{v})$.

4.3. (a) Let

$$p_{\mathsf{y}}(y; x) = \begin{cases} x & \text{if } 0 \le y \le 1/x \\ 0 & \text{otherwise} \end{cases}$$

for $x > 0$. Show that there exist no unbiased estimators $\hat{x}(\mathsf{y})$ for x. (Note that because only $x > 0$ are possible values, an unbiased estimator need only be unbiased for $x > 0$ rather than all x.)

(b) Suppose instead that

$$p_{\mathsf{y}}(y; x) = \begin{cases} 1/x & \text{if } 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

for $x > 0$. Does a minimum-variance unbiased estimator for x based on y exist? If your answer is yes, determine $\hat{x}_{\mathrm{MVU}}(\mathsf{y})$. If your answer is no, explain.

4.4. The data $\mathsf{x}[n] = ar^n + \mathsf{w}[n]$ for $n = 0, \cdots, N-1$ are observed. The random variables $\mathsf{w}[0], \cdots, \mathsf{w}[N-1]$ are i.i.d. Gaussian random variables with zero mean and variance σ^2. r is a non-zero constant. Find the Cramér-Rao bound for a. Does an efficient estimator exist? If so, what is it and what is its variance?