Probability Theory Exercise 3

1. Suppose that $\{X_n\}_{n=1}^{\infty}$ is a sequence of non-decreasing random variables, i.e., $X_{n+1}(\omega) \geq X_n(\omega)$ for all $n \geq 1$ and all $\omega \in \Omega$. Define another (possibly extended-valued) random variable X as the pointwise limit of $\{X_n\}_{n=1}^{\infty}$, i.e., let $X(\omega) = \lim_{n\to\infty} X_n(\omega)$ for all $\omega \in \Omega$. Do these conditions guarantee that $E[X] = \lim_{n\to\infty} E[X_n]$? If your answer is yes, prove this statement. If your answer is no, give a counterexample.

2. Let X be a random variable with probability density function:

$$f_X(x) = \begin{cases} c\sqrt{4-x^2}, & \text{for } -2 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

for some constant c.
(a) Find the constant c.
(b) Find $E[X^k]$ for all $k = 1, 2, \ldots$

3. Let $X_1, \ldots, X_n$ be n independent Bernoulli random variables. (We don't assume that $X_1, \ldots, X_n$ have the same distribution!) Let $Y_1, \ldots, Y_n$ be another n independent Bernoulli random variables. (We don't assume that $Y_1, \ldots, Y_n$ have the same distribution, either!) Let $X = X_1 + \cdots + X_n$ and $Y = Y_1 + \cdots + Y_n$. Suppose that $P(X_i = 1) \geq P(Y_i = 1)$ for all $i = 1, 2, \ldots, n$. Does this guarantee that $P(X \geq k) \geq P(Y \geq k)$ for **all** $k = 1, 2, \ldots, n$? If your answer is yes, prove this statement. If your answer is no, give a counterexample.

4. Let U and V be independent random variables, such that U is uniformly distributed over the interval $[0, 1]$, and V is a exponential random variable with parameter 1
(a) Calculate $E\left[\frac{V^2}{1+U}\right]$
(b) Calculate $P\{U \leq V\}$.
(c) Find the joint probability density function of Y and Z, where $Y = U^2$ and $Z = UV$. Be sure to indicate where the joint pdf is zero.

5. Let X_1, X_2, X_3 be three i.i.d. exponential random variables with the same parameter $\lambda > 0$. Find the value of $P(X_1 > X_2 + X_3)$.

6. Let X_1, X_2, X_3 be three independent Gaussian random variables. Suppose that both X_1 and X_2 have mean 0 and variance 2, and suppose that X_3 is a standard Gaussian random variable (mean 0 and variance 1). Let Y_1 and Y_2 be the two eigenvalues of the random matrix

$$\begin{bmatrix} X_1 & X_3 \\ X_3 & X_2 \end{bmatrix}.$$

Find the joint probability density function of Y_1 and Y_2. (You don't need to calculate the normalizing constant.)