Probability Theory Exercise 8

Issued: 2020/5/26 Due: 2020/6/7

Total points: 45

Each of your first 6 homework has total points 10, and each of your last two homework has total points 45 (including this one). In total, the full score of your homework will be $6 \times 10 + 45 \times 2 = 150$**. Your final grade of this course will be 70+(your homework grade)/5.**

1. (4 points) Suppose that the marginal distributions of both X and Y are standard normal distribution $N(0,1)$, and we do not make any other assumptions on the joint distribution of X and Y. Is it possible that the distribution of $X+Y$ is not Gaussian distribution? (We view the zero random variable as a special case of Gaussian distribution, i.e., if a random variable is equal to 0 with probability 1 then we say that it has Gaussian distribution $N(0,0)$.) If your answer is yes, give such an example. If your answer is no, please explain why.

2. (3 points) Suppose that X, Y, Z are three random variables such that X and Y are independent, X and Z are independent, Y and Z are independent. Does this guarantee that X, Y, Z are mutually independent? If your answer is yes, prove it. If your answer is no, give a counterexample.

3. (7 points) Let X and Y be i.i.d. random variables with mean 0 and variance 1. Suppose that $(X+Y)/\sqrt{2}$ has the same distribution as X. Find **all** possible distributions of X that satisfy the above conditions. Prove that the distributions you find are the **only** distributions that satisfy the above conditions.

4. (4 points) Let $X \sim N(0, \sigma^2)$ be a Gaussian random variable. Prove that the limit $\lim_{x\to\infty} x e^{x^2/(2\sigma^2)} P(X \geq x)$ exists, and find the limit.

5. (i) (1 point) Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables. Find the probability $P(\min(X_1, \ldots, X_n) = X_1)$.
(ii) (3 points) Let $X_1, X_2, \ldots, X_n$ be independent random variables. Suppose that the distribution of X_i is exponential distribution with parameter $\lambda_i > 0$. Find the probability $P(\min(X_1, \ldots, X_n) = X_1)$.

6. (7 points) Let $(X_1, X_2, \ldots, X_{2n-1})$ be a random vector with density function

$$f_{X_1,\ldots,X_{2n-1}}(x_1, \ldots, x_{2n-1}) = c_n \exp\left(-\frac{1}{2}\left(x_1^2 + \sum_{i=1}^{2n-2}(x_{i+1}-x_i)^2 + x_{2n-1}^2\right)\right),$$

where c_n is the normalizing constant. (Notice that there are $2n$ **square terms, not** $2n-1$ **square terms** in the exponent of the density function.) Prove that $(X_1, X_2, \ldots, X_{2n-1})$ is a Gaussian random vector and find the value of c_n. Also find the variance $\mathrm{Var}(X_n)$. (You only need to find $\mathrm{Var}(X_n)$. You don't need to calculate the variance of every X_i.)

7. Consider a random walk on the integers. We start at $X_0 = 0$. In each step, we have $P(X_n = X_{n-1}+1) = P(X_n = X_{n-1}-1) = 1/2$. Define the random variable U as the unique positive integer such that $X_U = 0$ and $X_i \neq 0$ for all $0 < i < U$. In other words, in step U, this random walk returns to the origin for the first time. Now let $m > 0$ be a positive integer. Define another random variable N_m as the number of times this random walk visits m before step U. More precisely, we have

$$N_m := |\{i : 0 < i < U, X_i = m\}|.$$

(For a set A, $|A|$ denotes the size of A.) Note that N_m is simply the number of times this random walk visits m before returning to 0.
(i) (2 points) Find $P(N_m \geq 1)$.
(ii) (6 points) For every positive integer n, find $P(N_m = n)$.

8. (8 points) Let $X_0 = 0$ and $X_1 = 1$. For $i > 1$, let $X_i = X_{i-1} + X_{i-2}$ with probability $1/2$ and $X_i = |X_{i-1} - X_{i-2}|$ with probability $1/2$. Find the probability

$$P(\exists n \text{ such that } X_n = 3 \text{ and } X_i \neq 0 \text{ for all } 1 \leq i < n).$$

(This is the probability of the sequence $\{X_n\}$ reaching 3 before returning to the starting point 0.)