Probability Theory Exercise 7

Issued: 2020/5/22 Due: 2020/6/7
Total points: 45

Each of your first 6 homework has total points 10, and each of your last two homework has total points 45 (including this one). In total, the full score of your homework will be $6\times 10+45\times 2=150$. Your final grade of this course will be 70+(your homework grade)/5.

2. Let $X_1, X_2, \dots$ be i.i.d. Cauchy random variables with PDF $f(x)=\frac{1}{\pi(1+x^2)}$. Let $S_n=X_1+X_2+\cdots+X_n$.
(i) (2 points) Does $\{S_n/n\}_{n=1}^{\infty}$ converge in distribution? Please explain why. If the answer is yes, what is the limit distribution?
(ii) (1 point) Does $\{S_n/n^2\}_{n=1}^{\infty}$ converge in distribution? Please explain why. If the answer is yes, what is the limit distribution?
(iii) (1 point) Does $\{S_n/\sqrt{n}\}_{n=1}^{\infty}$ converge in distribution? Please explain why. If the answer is yes, what is the limit distribution?

1. (5 points) Consider a branching process whose offspring distribution has expectation μ and variance σ^2 (see the definition of branching process and offspring distribution in the lecture slides). For $n=0,1,2,\dots$, let X_n be the number of individuals in the nth generation. Assume that $X_0=1$. What is $\mathrm{Var}(X_n)$? Please express it in terms of n, μ and σ^2.

3. Let $X_1, X_2, X_3, \dots$ be independent random variables with distribution $P(X_i=i)=P(X_i=-i)=1/2$ for all i. (Note that the distribution of each X_i is **different**!!) Define $S_n=X_1+X_2+\cdots+X_n$ for every positive integer n.
(i) (3 points) Does $\{S_n/n^2\}_{n=1}^{\infty}$ converge in distribution? Please explain why. If the answer is yes, what is the limit distribution?
(ii) (4 points) Does $\{S_n/n^{3/2}\}_{n=1}^{\infty}$ converge in distribution? Please explain why. If the answer is yes, what is the limit distribution?
(iii) (2 points) For every real number $x\in\mathbb{R}$, find the limit $\lim_{n\to\infty} P\Big(\frac{S_n}{n}\le x\Big)$.

4. (9 points) Let $X_1, X_2, \dots, X_{2n}$ be i.i.d. random variables with the distribution $P(X_i=1)=P(X_i=-1)=1/2$ for all $1\le i\le 2n$. For every $1\le i\le 2n$, define $S_i=X_1+X_2+\cdots+X_i$. What is the conditional probability $P(S_i\ge 0 \text{ for all } 1\le i\le 2n \mid S_{2n}=0)$?

5. (9 points) Let $X_1, X_2, X_3, \dots$ be i.i.d. Bernoulli-p random variables, i.e., $P(X_i=1)=p\in(0,1)$ and $P(X_i=0)=1-p$ for all i. Define another sequence $\{L_n\}_{n=1}^{\infty}$ of integer-valued random variables as follows: If $X_n=0$, then we define $L_n=0$. If $X_n=1$, then we define L_n as the unique positive integer such that $X_n=X_{n+1}=X_{n+2}=\cdots=X_{n+L_n-1}=1, X_{n+L_n}=0$. Prove that there is a function of p, which we denote as $f(p)$, such that

$$\limsup_{n\to\infty}\frac{L_n}{\log(n)}=f(p) \quad \text{with probability } 1.$$

Also solve the exact expression of $f(p)$. (We use natural logarithm in this problem.)

6. Let $X_1, X_2, X_3, \dots$ be i.i.d. random variables with distribution $P(X_i=1)=P(X_i=1/2)=1/2$ for all i. For all positive integer n, define $Y_n=\prod_{i=1}^{n} X_i$ and $S_n=\sum_{i=1}^{n} Y_i$.

(i) (6 points) Does $\{S_n\}_{n=1}^{\infty}$ converge almost surely to some limit random variable? Please explain why. If the answer is yes, find the mean and variance of the limit random variable.
(ii) (3 points) Now suppose that the distribution of X_i is $P(X_i = 2) = P(X_i = 1/4) = 1/2$ for all i, and the definition of Y_n and S_n remain the same. In this case, does $\{S_n\}_{n=1}^{\infty}$ converge almost surely to some limit random variable? Please explain why.