Probability Theory Spring 2021 Exam

1. **State** Monotone Convergence Theorem and Dominated Convergence Theorem. **(No need to prove.)**

2. **State** Central Limit Theorem. **(No need to prove.)**

3. Let X be a discrete random variable with distribution $\mathbb{P}(X=-3)=1/9,\ \mathbb{P}(X=-1)=1/9,\ \mathbb{P}(X=0)=1/9,\ \mathbb{P}(X=1)=3/9,\ \mathbb{P}(X=2)=2/9,\ \mathbb{P}(X=3)=1/9$. Define another random variable $Y=\mathbb{E}[X|X^2]$. Find the distribution of Y.

4. Let $X_1, X_2, \ldots, X_{10}$ be 10 independent random variables. Define $S = X_1 + X_2 + \cdots + X_{10}$.
(i) Do the above conditions guarantee the following equality? If your answer is yes, prove the equality. If your answer is no, give a counterexample.

$$\mathbb{E}[(S-\mathbb{E}[S])^3] = \sum_{i=1}^{10} \mathbb{E}[(X_i - \mathbb{E}[X_i])^3]$$

(ii) Do the above conditions guarantee the following equality? If your answer is yes, prove the equality. If your answer is no, give a counterexample.

$$\mathbb{E}[(S-\mathbb{E}[S])^4] = \sum_{i=1}^{10} \mathbb{E}[(X_i - \mathbb{E}[X_i])^4]$$

5. Let $X_1, X_2, X_3, \ldots$ be i.i.d. Bernoulli-$1/2$ random variables, i.e., $\mathbb{P}(X_i=0)=\mathbb{P}(X_i=1)=1/2$ for all $i \geq 1$. For every positive integer $n \geq 1$, define

$$Y_n = \sum_{i=1}^{n} \frac{X_i}{2^i}.$$

Prove that the sequence of random variables $\{Y_n\}_{n=1}^{\infty}$ converges almost surely and find the distribution of the limit random variable.

6. Let (X_1, X_2, X_3) be a Gaussian random vector with $\mathbb{E}[X_1]=1, \mathbb{E}[X_2]=2, \mathbb{E}[X_3]=3$ and covariance matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(i) Calculate $\mathbb{E}[(X_1 - \mathbb{E}[X_1|X_2, X_3])^2]$.
(ii) Calculate $\mathbb{E}[(X_1 - \mathbb{E}[X_1|X_2 + X_3])^2]$.
(iii) Calculate $\mathbb{E}[(X_1 - \mathbb{E}[X_1|X_2 - X_3])^2]$.

7. Define a sequence of **independent** random variables $X_1, X_2, X_3, \ldots$ as follows. Let $X_1 = 0$ with probability 1. For every integer $n \geq 2$, X_n only takes three values $-n, 0, n$ with the following probabilities:

$$\mathbb{P}(X_n = n) = \mathbb{P}(X_n = -n) = \frac{1}{2n \ln n}, \qquad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n \ln n}.$$

For every $n \geq 1$, define $Y_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$.

(i) Calculate $\lim_{n\to\infty} \mathbb{E}[Y_n^2]$.
(ii) Does $\{Y_n\}_{n=1}^{\infty}$ converge in probability? If your answer is yes, find the distribution of the limit random variable. If your answer is no, please explain why.
(iii) Define an event $A = \{\omega : |X_n(\omega)| = n \text{ for infinitely many } n\}$. Find the probability $\mathbb{P}(A)$.
(iv) Does $\{Y_n\}_{n=1}^{\infty}$ converge almost surely? If your answer is yes, find the distribution of the limit random variable. If your answer is no, please explain why.