Tsinghua-Berkeley Shenzhen Institute
LARGE DEVIATION AND HIGH DIMENSIONAL STATISTICS
Spring 2021

Problem Set 1

Issued: Monday 1st March, 2021 **Due:** Monday 15th March, 2021

1.1. (a) Show that the union of countably many countable sets is countable.

(b) A real number x is rational if $x = m/n$, where m is an integer and n is a nonzero integer. Show that the set of rational numbers $\mathbb{Q}$ is countable

1.2. Let $\Omega = [0, 1)$ and let $\mathcal{F}_0$ be the collection of finite unions $\cup_{i=1}^{n}[a_i, b_i)$ for $0 \leq a_1 < b_1 \leq \cdots \leq a_n < b_n \leq 1$ (consisting of the empty set). For any $A \in \mathcal{F}_0$, let $\mathbb{P}[A] = 1$ if one of the $b_i = 1$, and $\mathbb{P}[A] = 0$ otherwise.

(a) Prove that $\mathcal{F}_0$ is an algebra (field) but not a σ-algebra (σ-field).

(b) Show that $\mathbb{P}[\cdot]$ is a non-negative finitely additive set-function on $\mathcal{F}_0$.

(c) Show that $\mathbb{P}[\cdot]$ is not countably additive on $\mathcal{F}_0$.

1.3. Let $\{X_n\}$ be a sequence of independent non-negative random variables. Show that sequence X_n is almost surely bounded if and only if $\sum_{n=1}^{\infty} \mathbb{P}[X_n > c] < \infty$ for some c. (Hint: X_n a.s. bounded simply means $\mathbb{P}[\sup_n X_n = \infty] = 0$.)

1.4. Suppose that $X_1, X_2, \cdots, X_n, \cdots$ are random variables defined on the same probability space. Show that $\max\{X_1, X_2\}$, $\sup_n X_n$, and $\limsup_{n\to\infty} X_n$ are random variables.