Tsinghua-Berkeley Shenzhen Institute
LARGE DEVIATION AND HIGH DIMENSIONAL STATISTICS
Spring 2021

Problem Set 4

Issued: Friday 7th May, 2021 **Due:** Friday 21st May, 2021

4.1. (*Gaussian Tail*) If $Z \sim \mathcal{N}(0,1)$, prove the following inequality

$$\frac{t}{\sqrt{2\pi}(1+t^2)} \exp\left(-\frac{t^2}{2}\right) < \mathbb{P}(Z > t) < \frac{1}{\sqrt{2\pi}t} \exp\left(-\frac{t^2}{2}\right) \quad t > 0$$

4.2. (*Improved Chernoff Bound for Gaussian Tail*) If $Z \sim \mathcal{N}(0,1)$, prove that

$$\sup_{t>0} \mathbb{P}(Z > t) \exp\left(\frac{t^2}{2}\right) = \frac{1}{2}$$

4.3. (*Rate Function*) Please compute the rate functions $\psi_X^*(t) \triangleq \sup_{\lambda\in\mathbb{R}} \lambda t - \psi_X(\lambda)$ for the following random variables:

(a) $X \sim \text{Poisson}(\theta)$ and the pmf is

$$\mathbb{P}(X = k) = \frac{\theta^k e^{-\theta}}{k!} \quad k = 0, 1, 2, \dots \quad \theta > 0$$

(b) $X \sim \text{Bern}(p) \quad p \in (0,1)$

(c) $X \sim \text{Exp}(p)$ and the pdf is

$$f_X(x) = \theta e^{-\theta x} \quad \theta > 0 \quad x \geq 0$$

(d) $X \sim \mathcal{N}(0, \sigma^2) \quad \sigma > 0$