Solution to Midterm Exam

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1. omit

2. omit

3. $\mathbb{E}[X|X^2 = k^2] = kP[X = k|X^2 = k^2] - kP[X = -k|X^2 = k^2] = k\frac{P[X=k]-P[X=-k]}{P[X=k]+P[X=-k]}$. Therefore, we get $\mathbb{E}[X|X^2 = 0] = 0, \mathbb{E}[X|X^2 = 1] = \frac{1}{2}, \mathbb{E}[X|X^2 = 4] = 2, \mathbb{E}[X|X^2 = 9] = 0$. Combining the same value, we have $P(Y = 0) = \frac{1}{3}$, $P(Y = \frac{1}{2}) = \frac{4}{9}, P(Y = 2) = \frac{2}{9}$.

4. In this problem, we use $Y_i = X_i - \mathbb{E}[X_i]$ to simplify the notation.

   (a) Yes.

$$
\begin{aligned}
\mathbb{E}[(S-\mathbb{E}[S])^3] &= \mathbb{E}[\sum_{i=1}^{10}(X_i - \mathbb{E}[X_i])^3] \\
&= \sum_{i=1}^{10}\mathbb{E}[(X_i - \mathbb{E}[X_i])^3] + 3\sum_{1\le i<j\le 10}\mathbb{E}[(X_i-\mathbb{E}[X_i])^2(X_j-\mathbb{E}[X_j])] \\
&+ 6\sum_{1\le i<j<k\le 10}\mathbb{E}[(X_i-\mathbb{E}[X_i])(X_j-\mathbb{E}[X_j])(X_k-\mathbb{E}[X_k])]
\end{aligned}
$$

   Since $X_i \perp\!\!\!\perp X_j$ for $i \neq j$, $\mathbb{E}[(X_i-\mathbb{E}[X_i])^2(X_j-\mathbb{E}[X_j])] = \mathbb{E}[(X_i - \mathbb{E}[X_i])^2]\,\mathbb{E}[(X_j-\mathbb{E}[X_j])] = 0$ and $\mathbb{E}[(X_i-\mathbb{E}[X_i])(X_j-\mathbb{E}[X_j])(X_k - \mathbb{E}[X_k])] = 0$. Therefore, $\mathbb{E}[(S-\mathbb{E}[S])^3] = \sum_{i=1}^{10}\mathbb{E}[(X_i-\mathbb{E}[X_i])^3]$ holds.

   Another proof, based on induction. Let $S_n = \sum_{i=1}^n Y_i$. For $i = 2$, we have $\mathbb{E}[S_2^3] = \mathbb{E}[(Y_1+Y_2)^3] = \mathbb{E}[Y_1^3] + \mathbb{E}[Y_2^3]$. Suppose $\mathbb{E}[S_n^3] = \mathbb{E}[\sum_{i=1}^n Y_i^3]$, then

$$
\begin{aligned}
\mathbb{E}[S_{n+1}^3] &= \mathbb{E}[(S_n + Y_{n+1})^3] \\
&\overset{(a)}{=} \mathbb{E}[S_n^3] + \mathbb{E}[Y_{n+1}^3] \\
&\overset{(b)}{=} \mathbb{E}[\sum_{i=1}^{n+1} Y_i^3]
\end{aligned}
$$

In the above equations, (a) comes from the independence of Y_{n+1} and S_n while (b) comes from the induction assumption. Therefore, $\mathbb{E}[S_n^3] = \mathbb{E}[\sum_{i=1}^n Y_i^3]$ holds for any n. Take $n = 10$, and we have $\mathbb{E}[(S - \mathbb{E}[S])^3] = \sum_{i=1}^{10} \mathbb{E}[(X_i - \mathbb{E}[X_i])^3]$.

(b) It is not true. Consider $X_i \sim N(0, 1)$. Then $\mathbb{E}[X_i] = 0$, $\mathbb{E}[X_i^4] = 3$. Then the right hand side evaluates to 30 while the left hand side $\mathbb{E}[S^4] = 300$.

Another counter-example: Choose $X_i = 0$ for $i = 3, \dots, 10$. And X_1, X_2 are random variables with non-zero variance. Then

$$\begin{aligned}
\mathbb{E}[(S - \mathbb{E}[S])^4] &= \mathbb{E}[(Y_1 + Y_2)^4] \\
&= \mathbb{E}[Y_1^4] + \mathbb{E}[Y_2^4] + 6\,\mathbb{E}[Y_1^2]\,\mathbb{E}[Y_2^2] \\
&> \mathbb{E}[Y_1^4] + \mathbb{E}[Y_2^4] \\
&= \sum_{i=1}^{10} \mathbb{E}[(X_i - \mathbb{E}[X_i])^3]
\end{aligned}$$

5. $Y_n(\omega)$ is monotonically increasing and bounded by 1. Therefore, $Y_n \xrightarrow{a.s.} Y$. Heuristically we guess Y is uniform distribution over the interval $[0, 1]$.

6. (a) $1 - (1, 1)\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1}\binom{1}{1} = \frac{1}{3}$

(b) We first compute the covariance matrix of $(X_1, X_2 + X_3)$ as $\begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix}$ Then $1 - \frac{4}{6} = \frac{1}{3}$

(c) Since $\mathrm{Cov}[X_1, X_2 - X_3] = 0$, $X_1 \perp\!\!\!\perp X_2 - X_3$ and $\mathbb{E}[X_1|X_2 - X_3] = \mathbb{E}[X_1]$ therefore, the value is $\mathrm{Var}[X_1] = 1$.

7. (i) Since $\mathbb{E}[X_i^2] = i^2 \frac{1}{i \ln i} = \frac{i}{\ln i}$, $\mathbb{E}[Y_n] = \frac{1}{n^2} \sum_{i=2}^n \mathbb{E}[X_i^2] = \frac{1}{n^2} \sum_{i=2}^n \frac{i}{\ln i}$. Let $f(x) = \frac{x}{\ln x}$. Since $f'(x) = \frac{\ln x - 1}{\ln^2 x} > 0$ for $x > e$. Then $f(i)$ is increasing for $i >= 3$. $\mathbb{E}[X_i^2] \leq \frac{1}{n^2}(\frac{2}{\ln 2} + (n-3)\frac{n}{\ln n}) \to 0$. Therefore, $\lim_{n\to\infty} \mathbb{E}[Y_n^2] = 0$.

(ii) We first have $\mathbb{E}[Y_n] = 0$. By Chebyshev's inequality, $P(|Y_n| > \epsilon) \leq \frac{\mathbb{E}[Y_n^2]}{\epsilon^2} \to 0$ as $n \to \infty$ from (i).

(iii) Let $A_n = \{\omega : |X_n(\omega)| = n\}$. Then $A = \limsup_{n\to\infty} A_n$. $P(A_n) = \frac{1}{n \log n}$, and $\sum_{n=1}^{+\infty} P(A_n) = \sum_{n=2}^{\infty} \frac{1}{n \log n} = +\infty$ by approximating it with $\int_2^{+\infty} \frac{1}{x \log x} dx$. Also A_n are independent events. By Borel-Cantelli lemma, we have $P(A) = 1$.

(iv) If $|X_n(\omega) = n|$, then

$$\begin{aligned}
|Y_n(\omega) - Y_{n-1}(\omega)| &= \Big|\frac{X_n(\omega)}{n} - \frac{\sum_{i=1}^{n-1} X_i(\omega)}{(n-1)n}\Big| \\
&\geq \Big|\frac{X_n(\omega)}{n}\Big| - \Big|\frac{\sum_{i=1}^{n-1} X_i(\omega)}{(n-1)n}\Big| \\
&= 1 - \Big|\frac{\sum_{i=1}^{n-1} X_i(\omega)}{(n-1)n}\Big| \\
&\overset{(a)}{\geq} \frac{1}{2}
\end{aligned}$$

(a) holds since

$$\begin{aligned}
|\sum_{i=1}^{n-1} X_i(\omega)| &\leq \sum_{i=1}^{n-1} |X_i(\omega)| \\
&\leq \sum_{i=1}^{n-1} i = \frac{1}{2}n(n-1)
\end{aligned}$$

Therefore, for the event $B_n := \{w|Y_n(w) - Y_{n-1}(w)| > \frac{1}{2}\}$, we have $P(A_n) \leq P(B_n)$. From the analysis in (iii), we have $\sum_{n=1}^{+\infty} P(B_n) = +\infty$. Therefore, the probability of $\{w\big| |Y_n(w) - Y_{n-1}(w)| > \frac{1}{2}$ for infinitely many $n\}$ is 1. By Cauchy's convergence test, the probability of $\{w|Y_n(\omega)$ does not converge$\}$ is 1.