Probability Theory Exercise 1

1. Suppose that X, Y, Z are three random variables such that X and Y are independent, X and Z are independent, Y and Z are independent. Does this guarantee that X, Y, Z are mutually independent? If your answer is yes, prove it. If your answer is no, give a counterexample.

2. Suppose that we have a vector consisting of 8 random binary digits that are mutually independent. Each digit is either 0 or 1 with equal probability. Define the following four events:
$A_1 :=$ "No two consecutive digits are the same".
$A_2 :=$ "Some cyclic shift of the vector is equal to 01100110"
$A_3 :=$ "The vector contains exactly four ones and four zeros"
$A_4 :=$ "There are at least six consecutive ones in the vector"
Find the probabilities $P(A_1), P(A_2), P(A_3), P(A_4), P(A_1|A_3)$ and $P(A_2|A_3)$.

3. Suppose each corner of a cube is colored red with probability $p \in (0, 1)$, independently of other corners. Define A as the event that there is at least one face of the cube whose all four corners are colored red.
(1) Find the conditional probability of A given that exactly 5 corners of the cube are colored red.
(2) Find the conditional probability of A given that at least 5 corners of the cube are colored red.
(3) Find the unconditional probability $P(A)$.

4. (1) Let $\Omega = \{H, T\}^2$ (two coin tosses). Let $\mathcal{C} = \{\{HH, HT\}, \{HH, TH\}\}$. What is the size of $\sigma(\mathcal{C})$?
(2) Let $\Omega = \{H, T\}^3$ (three coin tosses). Let

$$\mathcal{C} = \{\{HHH, HHT, TTH\}, \{HHH, TTH, THT\}, \{HHH, HHT, THT\}\}.$$

What is the size of $\sigma(\mathcal{C})$?

5. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of independent events defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}(A_n) < 1$ for all n and $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = 1$. Find the value of $\mathbb{P}(\cap_{i=1}^{\infty} \cup_{n=i}^{\infty} A_n)$ and prove your conclusion.

6. (1) Let X_1 and X_2 be two random variables defined on a measurable space $(\Omega, \mathcal{F})$. Prove that $X_1 + X_2$ is a random variable.
(2) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables defined on a measurable space $(\Omega, \mathcal{F})$. Suppose that $\{X_n\}_{n=1}^{\infty}$ converges pointwise to X, i.e., $X(\omega) = \lim_{n\to\infty} X_n(\omega)$ for all $\omega \in \Omega$. Prove that X is also a random variable.