Tsinghua-Berkeley Shenzhen Institute
LARGE DEVIATION THEORY
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Homework 2

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2.1. (a) We divide the proof into three parts.

i. If $X_n \xrightarrow{P} X, Y_n \xrightarrow{P} Y \Rightarrow X_n + Y_n \xrightarrow{P} X + Y$.

Proof. For any $\epsilon > 0$, since the event $\{|X_n + Y_n - X - Y| > \epsilon\} \subset \{|X_n - X| > \frac{\epsilon}{2}\} \cup \{|Y_n - Y| > \frac{\epsilon}{2}\}$, we have

$$\limsup_{n\to\infty} P(|X_n + Y_n - X - Y| > \epsilon) \leq \limsup_{n\to\infty} P(|X_n - X| > \frac{\epsilon}{2}) + \limsup_{n\to\infty} P(|X_n - X| > \frac{\epsilon}{2}) = 0$$

□

ii. If $X_n \xrightarrow{P} 0$ and Y is a random variable, $X_nY \xrightarrow{P} 0$.

Proof. For any $\epsilon > 0$ and $A > 0$, since the event $\{|X_nY| > \epsilon\} \subset \{|X_n| > \frac{\epsilon}{A}\} \cup \{|Y| > A\}$, we have

$$\limsup_{n\to\infty} P(|X_nY| > \epsilon) \leq \limsup_{n\to\infty} P(|X_n| > \frac{\epsilon}{A}) + P(|Y| > A) = P(|Y| > A)$$

Let $A \to \infty$, we have $P(|Y| > A) \to 0$. Therefore $P(|X_nY| > \epsilon) \to 0$ as $n \to \infty$. □

iii. If $X_n \xrightarrow{P} 0, Y_n \xrightarrow{P} 0 \Rightarrow X_nY_n \xrightarrow{P} 0$.

Proof. For any $\epsilon > 0$ and $A > 0$, since the event $\{|X_nY_n| > \epsilon\} \subset \{|X_n| > \sqrt{\epsilon}\} \cup \{|Y_n| > \sqrt{\epsilon}\}$, we have

$$\limsup_{n\to\infty} P(|X_nY| > \epsilon) \leq \limsup_{n\to\infty} P(|X_n| > \sqrt{\epsilon}) + \limsup_{n\to\infty} P(|Y_n| > \sqrt{\epsilon}) = 0$$

□

Based on the above three conclusions, if $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, we have $(X_n - X)Y_n = (X_n - X)(Y_n - Y) + Y(X_n - X) \xrightarrow{P} 0$ and $X(Y_n - Y) \xrightarrow{P} 0$. Summing the two up we have $X_nY_n - XY \xrightarrow{P} 0$.

(b) Let $\Omega = \{0, 1\}$ with measure $P(\{0\}) = P(\{1\}) = \frac{1}{2}$. $X_n(0) = 1, X_n(1) = 0$ while $Y_n(1) = 1, Y_n(0) = 0$. Then both X_n, Y_n follow Bern($\frac{1}{2}$). Therefore, $X_n \xrightarrow{d} X, Y_n \xrightarrow{d} Y$ where we choose $X = Y = X_n$. However, $Z_n = X_nY_n = 0$ in a sense that $Z_n(0) = Z_n(1) = 0$.
Therefore, Z_n does not converge to $XY = X^2$ in distribution.

(c) By Skorokhod's representation, we can find independent uniform random variables U_n, V_n, U, V on $[0,1]$ such that $U_n, V_n(U, V)$ have the same distribution $X_n, Y_n(X, Y)$. and $U_n \xrightarrow{a.s} U$ and $V_n \xrightarrow{a.s.} V$. Then $U_nV_n \xrightarrow{a.s.} UV$. Since $X_n, Y_n(X, Y)$ are independent, so are $U_n, V_n(U, V)$. $P(U_n \le t_1, V_n \le t_2) = P(U_n \le t_1)P(V_n \le t_2) = P(X_n \le t_1)P(Y_n \le t_2) = P(X_n \le t_1, Y_n \le t_2)$. Therefore, they have the same joint distribution. Then for any bounded continuous function f, $\mathbb{E}[f(X_nY_n)] = \mathbb{E}[f(U_nV_n)]$. Using the bounded convergence theorem we have $\lim_{n\to\infty} \mathbb{E}[f(U_nV_n)] = \mathbb{E}[f(UV)] = \mathbb{E}[f(XY)]$. We get $\mathbb{E}[f(X_nY_n)] \to \mathbb{E}[f(XY)]$ for any bounded continuous function. Then $X_nY_n \xrightarrow{d} XY$.

2.2. (a) We choose $Z \sim \mathcal{N}(0, 1)$. We discretize ϵ using $\frac{1}{n}$. Then $X + \frac{Z}{n}$ converges to X in probability. $X + \frac{Z}{n} \xrightarrow{d} X$ follows.

(b) The CDF of X_ϵ is $F_{X_\epsilon}(a) = P(X + \epsilon Z < a) = P(Z < \frac{a-X}{\epsilon})$. Since $X \perp\!\!\!\perp Z$, $P(Z < \frac{a-X}{\epsilon}) = \mathbb{E}[\Phi(\frac{a-X}{\epsilon})]$ where Φ is the CDF of standard Gaussian distribution. Now let $a_n \to a$, then the random variable $Y_n = \Phi(\frac{a_n-X}{\epsilon})$ converges to $\Phi(\frac{a-X}{\epsilon})$ in probability. Further Y_n is uniformly bounded by 1. By the bounded convergence theorem, we have $E[Y_n] \to E[Y]$. That is, $F_{X_\epsilon}(a)$ is continuous. Since Φ is smooth function, we can exchange the differential operation with the expectation and get the PDF of X_ϵ as $f_{X_\epsilon}(a) = \mathbb{E}[f_Z(\frac{a-X}{\epsilon})\frac{1}{\epsilon}]$.

(c) Similar to (b). Using the bounded convergence theorem we can show that $f_{X_\epsilon}(a)$ is continuous.

(d) Since derivatives of f_Z are uniformly bounded, X_ϵ has a continuous, bounded, infinitely-differentiable PDF.

2.3. Since almost surely convergence implies convergence in probability. We only need to prove the other part. The proof is divided into three lemmas.

(a) If X_i are independent, $\sum_{n=1}^{\infty} \mathbb{E}[X_n^2]$ converges and $S_n \xrightarrow{p} c$ for some constant. Then $\sum_{i=1}^{n} \mathbb{E}[X_i]$ also converges.

Proof. Since $\mathrm{Var}[S_n] \ge 0$, $S_n^2 \le \sum_{i=1}^{n} \mathbb{E}[X_i^2]$. Since $\sum_{n=1}^{\infty} \mathbb{E}[X_n^2]$ converges, $|S_n| \le M$ for all n.
$|\mathbb{E}[S_n] - c| \le 2P(|S_n - c| > \epsilon)M + (1 - P(|S_n - c| > \epsilon)\epsilon$ where $P(|S_n - c| > \epsilon)$ converges to zero. Therefore $\mathbb{E}[S_n] \to c$. □

(b) If X_i are independent random variables, $\mathbb{E}[X_n] = 0, S_n \xrightarrow{d} S$ and $|X_n| < c$ for some constant. Then $\sum_{n=1}^{\infty} \mathbb{E}[X_n^2]$ converges.

Proof. We proceed by contradiction and assume that $\sum_{n=1}^{\infty} \mathbb{E}[X_n^2] = \infty$. Since $\mathbb{E}[X_n^3] \le c\mathbb{E}[X_n^2]$, for S_n we can check the condition in Central Limit Theorem (Lyapunov):

$\frac{\sum_{i=1}^n \mathbb{E}[X_n^3]}{(\sum_{i=1}^n \mathbb{E}[X_n^2])^{3/2}} \leq \frac{c}{(\sum_{i=1}^n \mathbb{E}[X_n^2])^{1/2}} \to 0$. Therefore, $\frac{S_n}{(\sum_{i=1}^n \mathbb{E}[X_n^2])^{1/2}} \xrightarrow{d} \mathcal{N}(0,1)$. Then $P(S_n \geq t) \to 1 - \Phi(\frac{t}{\sum_{i=1}^n \mathbb{E}[X_n^2])^{1/2}}) \sim \frac{1}{2}$. On the other hand, $S_n \xrightarrow{d} S$, which implies $P(S \geq t) = \frac{1}{2}$, which is impossible for a valid random variable. Therefore, $\sum_{n=1}^\infty \mathbb{E}[X_n^2] \leq \infty$. □

(c) Let X_i be independent, and both $S_n = \sum_{i=1}^n \mathbb{E}[X_i]$ and $\sum_{i=1}^n \mathbb{E}[X_i^2]$ converge. Then S_n converges almost surely.

Proof. We first that the conclusion holds for $\mathbb{E}[X_n] = 0$. For general case, consider $X_n' = X_n - \mathbb{E}[X_n]$, which has zero mean, and $\sum_{i=1}^n \mathbb{E}[X_n'^2] = \sum_{i=1}^n \mathbb{E}[X_n^2] - \sum_{i=1}^n \mathbb{E}^2[X_n] \leq \sum_{i=1}^n \mathbb{E}[X_n^2]$. Therefore, $\sum_{i=1}^n \mathbb{E}[X_n'^2]$ converges. Using Kolmogorov's inequality for $S_k, n \leq k \leq n+m$ we have

$$P(\max_{n \leq k \leq n+m} |S_k| \leq \delta) \leq \frac{1}{\delta^2}\mathbb{E}[(\sum_{k=1}^m X_{n+k})^2] \to 0$$

since $\sum_{n=1}^\infty \mathbb{E}[X_n^2] \leq \infty$. By Cauchy Convergence criterion, S_n converges almost surely. □

For each X_n, we construct X_n' which has the same distribution but is independent with X. Consider $Y_n = X_n - X_n'$. Then $|Y_n| \leq 2$ and $\mathbb{E}[Y_n] = 0$. Since $S_n \xrightarrow{p} c$, $Y_n \xleftarrow{p} 0$, then $Y_n \xrightarrow{d} 0$. By (b) we get $\sum_{n=1}^\infty \mathbb{E}[Y_n^2] \leq \infty$. Since $\mathbb{E}[Y_n^2] = 2\mathbb{E}[X_n^2]$, we have $\sum_{n=1}^\infty \mathbb{E}[X_n^2] \leq \infty$. Then we can apply (a) and get $\sum_{i=1}^n \mathbb{E}[X_i]$ converges. Finally, we apply (c) and reach the conclusion.