Homework 4

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4.1. We define $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$ and $Q(t) = \int_t^{+\infty} \phi(x)dx$. To show $\frac{t}{1+t^2}\phi(t) \leq Q(t) \leq \frac{\phi(t)}{t}$. We make use of integral by part and the equality $\phi'(t) = -t\phi(t)$. Then

$$\begin{aligned} Q(t) &= \int_t^{+\infty} \frac{\phi'(x)}{-x} dx \\ &= \frac{\phi(t)}{t} - \int_t^{+\infty} \frac{\phi(x)}{x^2} dx \leq \frac{\phi(t)}{t} \end{aligned}$$

which gives the upper bound. For lower bound [1], since $(\frac{\phi(u)}{u})' = \frac{u\phi'(u) - \phi(u)}{u^2} = -\frac{u^2\phi(u) + \phi(u)}{u^2} = -(1 + \frac{1}{u^2})\phi(u)$

$$\begin{aligned} \left(1 + \frac{1}{x^2}\right) Q(t) &= \int_x^{\infty} \left(1 + \frac{1}{x^2}\right) \phi(u)\, du \\ &> \int_x^{\infty} \left(1 + \frac{1}{u^2}\right) \phi(u)\, du = -\frac{\phi(u)}{u}\bigg|_x^{\infty} = \frac{\phi(x)}{x}. \end{aligned}$$

4.2. To show $\sup_{t\geq 0} f(t) = f(0) = \frac{1}{2}$ where $f(t) = Q(t)\exp(t^2/2)$, we take the derivative $f'(t) = -\phi(t)\exp(t^2/2) + tQ(t)$. From Problem 1, $Q(t) \leq \phi(t)/t$, then $f'(t) \leq -\phi(t)\exp(t^2/2) + \phi(t) \leq 0$ since $\exp(t^2/2) \geq 1$. Therefore, $f(t)$ decreases in the interval $[0, +\infty)$, and $\sup_{t\geq 0} f(t) = f(0)$.

4.3. (a) For Poisson distribution, $\mathbb{E}[e^{\lambda X}] = \sum_{k=0}^{+\infty} e^{k\lambda} \frac{\theta^k e^{-\theta}}{k!} = \exp(e^\lambda \theta - \theta)$. Therefore, the log-MGF $\psi_X(\lambda) = \theta(e^\lambda - 1)$. We solve $t = \psi_X'(\lambda)$ and get $\lambda = \log(t/\theta)$, therefore, $\psi_X^*(t) = \theta - t + t\log\frac{t}{\theta}$

(b) For Bernoulli distribution, its log-MGF $\psi_X(\lambda) = \log(pe^\lambda + 1 - p)$. We solve $t = \psi_X'(\lambda)$ and get $\lambda = \log\frac{t(1-p)}{p(1-t)}$, therefore, $\psi_X^*(t) = t\log\frac{t(1-p)}{p(1-t)} - \log\frac{1-p}{1-t} = D_{\mathrm{KL}}(\mathrm{Bern}(t)||\mathrm{Bern}(p))$

(c) For Exponential distribution,

$$\mathbb{E}[e^{\lambda X}] = \int_0^{+\infty} \theta e^{-\theta x} e^{\lambda x} dx = \begin{cases} \frac{\theta}{\theta-\lambda} & \lambda < \theta \\ +\infty & \lambda \geq \theta \end{cases}$$

Its log-MGF

$$\psi_X(\lambda) = \begin{cases} \log\frac{\theta}{\theta-\lambda} & \lambda < \theta \\ +\infty & \lambda \geq \theta \end{cases}$$

We solve $t = \psi'_X(\lambda)$ and get

$$\lambda = \begin{cases} \theta - \frac{1}{t} & t > 0 \\ -\infty & t \leq 0 \end{cases}$$

Therefore,

$$\psi^*_X(t) = \begin{cases} +\infty & t \leq 0 \\ t\theta - 1 - \log(t\theta) & t > 0 \end{cases}$$

References

[1] `https://en.wikipedia.org/wiki/Q-function#Bounds_and_approximations`