Tsinghua-Berkeley Shenzhen Institute
LARGE DEVIATION THEORY
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Homework 1

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1.1. (a) Suppose $A_i = \bigcup_{j\in N}\{a_{ij}\}$, to show $\bigcup_{i\in N} A_i$ is countable. We construct a mapping $(i,j) \to \frac{(i+j)(i+j+1)}{2} + j$, which is a 1-1 mapping from N^2 to N. Therefore, $\bigcup_{i\in N} A_i$ is countable.

(b) Q can be regarded as the subset of N^2. By the property that "Any subset of a countable set is countable", we conclude that Q is countable.

1.2. (a) We can verify $\emptyset \in \mathcal{F}_0$. For $A = \cup_{i=1}^n [a_i, b_i)$, $A^c = [0, a_1) \bigcup \cup_{i=1}^{n-1}[b_i, a_{i+1}) \bigcup [b_n, 1) \in \Omega$. Finally, for $A, B \in \mathcal{F}_0$ we can decompose it into $A = A_1 \cup C, B = B_1 \cup C$ where $C = A \cap B, A_1 = A \backslash C, B_1 = B \backslash C, \in \mathcal{F}_0$. We can verify that both A_1, B_1, C are finite unions of intervals (left-close, right-open), and A_1, B_1, C are disjoint. Then $A \cup B = A_1 \cup B_1 \cup C$ is the collection of interval representations, also finite unions of intervals. $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}_0$. Therefore, $\mathcal{F}_0$ is an algebra. Consider $A_i = [\frac{1}{n}, 1)$, then $\cup_{i=2}^{+\infty} A_i = (0,1) \notin \mathcal{F}_0$. Therefore, $\mathcal{F}_0$ is not a σ-algebra.

(b) Let $A = \cup_{i=1}^n [a_i, b_i)$, $B = \cup_{i=1}^m [c_i, d_i)$, $b_n = d_m = 1$ is impossible since $A \cap B = \emptyset$. If one of $b_n, d_m = 1$, $P(A \cup B) = 1$ and $P(A) + P(B) = 1$. If $b_n, d_m < 1$, $P(A \cup B) = 0$ and $P(A) + P(B) = 0$. In either case, $P(A \cup B) = P(A) + P(B)$. The finite additivity is satisfied.

(c) Consider $A_n = [1 - \frac{1}{n}, 1 - \frac{1}{n+1})$ for $n \geq 2$, $A_n, n = 2, 3, \ldots$ are disjoint, whose union is $[\frac{1}{2}, 1)$. But $P([\frac{1}{2}, 1)) = 1$ while $P(A_n) = 0$. The property of countable additivity is not satisfied.

1.3. Suppose $\sum_{n=1}^\infty P[X_n > c]$ converges, then for any ϵ, there exists N such that $\sum_{n=N}^\infty P[X_n > c] < \epsilon$. By union bound, $P[\sup_n X_n = \infty] \leq P[\exists n > N, X_n > c] \leq \sum_{n=N}^\infty P[X_n > c] < \epsilon$. Since ϵ can be arbitrarily small, $P[\sup_n X_n = \infty] = 0$.

For the other side, we will show that $\forall c > 0, \sum_{n=1}^\infty P[X_n > c] = \infty \Rightarrow P[\sup_n X_n = \infty] = 1$. Let $A_n = \{w | X_n(w) > c\}$, then A_n are independent, by Borel-Contelli Lemma, we have $P[\cap_{n=1}^\infty \cup_{i=n}^\infty \{w | X_n(w) > c\}] = 1$. Therefore, $P[\cup_{i=n}^\infty \{w | X_n(w) > c\}] = 1$ holds for any c, n. Then $P[\cap_{i=n}^\infty \{w | X_n(w) \leq c\}] = 0 \Rightarrow P[\{w | \sup_n X_n(w) \leq c\}] = 0$. By union bound, $P[\cup_{c=1}^\infty \{w | \sup_n X_n(w) \leq c\}] = 0$, then taking the complement we have $P[\{\sup_n X_n = \infty\}] = 1$.

1.4. For any $c \in \mathbb{R}$,
$\{w | \max\{X_1(w), X_2(w)\} \leq c\} = \{w | X_1(w) \leq c\} \cap \{w | X_1(w) \leq c\} \in \mathcal{F}$,
$\{w | \sup_n X_n(w) \leq c\} = \cap_n \{w | X_n(w) \leq c\} \in \mathcal{F}$,
$\{w | \limsup_{n\to\infty} X_n(w) \geq c\} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{w | X_n(w) \geq c\} \in \mathcal{F}$
($\limsup_{n\to\infty} X_n(w) = \lim_{n\to\infty}(\sup_{i\geq n} X_i(w))$ is the limit of a decreasing series). Therefore, $\max\{X_1, X_2\}, \sup_n X_n, \limsup_{n\to\infty} X_n$ are random variables.