

Probability Theory Exercise 7 and 8

1. Let X be an $n \times n$ **symmetric** matrix, whose entries are denoted as X_{ij} , $1 \leq i, j \leq n$. Suppose that all the entries on and above the diagonal are independent, i.e., the entries X_{ij} , $1 \leq i \leq j \leq n$ are independent. Further assume that X_{ii} has $N(0, 2)$ distribution for every $1 \leq i \leq n$, and X_{ij} has $N(0, 1)$ distribution for every $1 \leq i < j \leq n$. Now let U be an $n \times n$ orthogonal matrix, and let $Y = U^T X U$. Clearly Y is symmetric. Prove that all the entries on and above the diagonal of Y are independent, and find the distributions of all these entries.

2. Let X_1, X_2, X_3, X_4 be i.i.d. Bernoulli-1/5 random variables. When $i \geq 5$, we produce X_i based on X_1, \dots, X_{i-1} as follows: If $X_{i-1} = X_{i-2} = X_{i-3} = X_{i-4} = 0$, then we set $X_i = 1$; and in all the other cases we set X_i to be a Bernoulli-1/5 random variable, independent of X_1, \dots, X_{i-1} . Prove that $\frac{1}{n} \sum_{i=1}^n X_i$ converges almost surely when $n \rightarrow \infty$ and find the distribution of the limit random variable.

3. Let Y_1, Y_2, \dots be i.i.d. random variables that only take **positive integer** values. For $i = 1, 2, \dots$, let $p_i := \mathbb{P}(Y_1 = i)$. Suppose that $\mu := \mathbb{E}[Y_1] < \infty$. Define a sequence of random variables X_n as follows:

$$X_n = \inf \{m \geq n : m = Y_1 + \dots + Y_k \text{ for some } k \geq 0\} - n.$$

For every $n = 1, 2, \dots$, further define a function

$$f(n) := \mathbb{P}(n = Y_1 + \dots + Y_k \text{ for some } k \geq 0).$$

- (i) Prove that $\{X_n\}_{n=0}^\infty$ forms a Markov chain and find the transition probabilities.
- (ii) Find the necessary and sufficient condition for the limit $\lim_{n \rightarrow \infty} f(n)$ to exist. Please state the necessary and sufficient condition in terms of p_1, p_2, \dots and prove your statement.
- (iii) When the limit $\lim_{n \rightarrow \infty} f(n)$ exists, find the limit. Express it in terms of μ and/or p_1, p_2, \dots .

4. Let $\{X_n\}_{n=0}^\infty$ be a Markov chain with state space $\{0, 1, 2, \dots\}$. The transition probabilities are

$$p_{0,1} = 1, \quad p_{i,i+1} + p_{i,i-1} = 1, \quad p_{i,i+1} = \left(\frac{i+1}{i}\right)^2 p_{i,i-1}, \quad i \geq 1$$

- (i) Suppose that $X_0 = 0$. Find the probability that the chain never returns to state 0.
- (ii) Now suppose that the transition probabilities are

$$p_{0,1} = 1, \quad p_{i,i+1} + p_{i,i-1} = 1, \quad p_{i,i+1} = \left(\frac{i+1}{i}\right)^\alpha p_{i,i-1}, \quad i \geq 1$$

for some constant α . For every $\alpha \in (-\infty, \infty)$, indicate whether the chain is positive recurrent, null recurrent, or transient. Prove your conclusion.

5. Let $\{X_n\}_{n=0}^\infty$ be a Markov chain with state space $\{0, 1, 2, \dots\}$. The transition probabilities are

$$p_{0,1} = 1, \quad p_{2n-1,2n+1} = p, \quad p_{2n-1,2n} = 1 - p, \quad p_{2n,2n+1} = p, \quad p_{2n,2n-2} = 1 - p \quad \text{for all } n \geq 1,$$

where $p \in (0, 1)$ is some constant. For every $p \in (0, 1)$, indicate whether the Markov chain is transient, null recurrent, or positive recurrent. Prove your conclusion. When the Markov chain is positive recurrent, calculate the stationary distribution.

6. Let $\{(X_n, Y_n)\}_{n=0}^\infty$ be a 2-dimensional symmetric random walk. Namely, this is a Markov chain where (X_{n+1}, Y_{n+1}) takes one of the following 4 values with equal probability: $(X_n + 1, Y_n)$, $(X_n - 1, Y_n)$, $(X_n, Y_n + 1)$, $(X_n, Y_n - 1)$. Suppose that $X_0 = Y_0 = 0$.

- (i) Define $T := \inf\{n \geq 0 : \max(|X_n|, |Y_n|) = 3\}$. Find the value of $\mathbb{E}[T]$ and $\mathbb{P}(X_T = 3, Y_T = 0)$.
- (ii) Now define $T := \inf\{n \geq 0 : |X_n| + |Y_n| = 3\}$. Find the value of $\mathbb{E}[T]$ and $\mathbb{P}(X_T = 3, Y_T = 0)$.
- (iii) Now define $T := \inf\{n \geq 0 : \max(-X_n, |Y_n|) = 2\}$. Find the value of $\mathbb{E}[T]$ and $\mathbb{P}(X_T = -2, Y_T = 0)$.
- (iv) Now define $T := \inf\{n \geq 0 : \max(X_n, Y_n) = 2\}$. Find the value of $\mathbb{E}[T]$.

7. Let a_1, a_2, a_3, \dots be a sequence of real numbers. Let X_1, X_2, X_3, \dots be i.i.d. random variables with distribution $P(X_i = 1) = P(X_i = -1) = 1/2$ for all i .

- (i) Suppose that $\sum_{i=1}^{\infty} a_i^2 < \infty$. Find the probability $\mathbb{P}\left(\left|\sum_{i=1}^{\infty} a_i X_i\right| < \infty\right)$.
- (ii) Suppose that $\sum_{i=1}^{\infty} a_i^2 = \infty$. Find the probability $\mathbb{P}\left(\left|\sum_{i=1}^{\infty} a_i X_i\right| < \infty\right)$.

8. Produce a sequence of random variables $\{X_n\}_{n \geq 0}$ as follows: Let X_0 and X_1 be some fixed constants. For $i > 1$, let $X_i = X_{i-1} + X_{i-2}$ with probability $1/2$ and $X_i = |X_{i-1} - X_{i-2}|$ with probability $1/2$.

- (i) Suppose that $X_0 = 0$ and $X_1 = 1$. Find the probability

$$\mathbb{P}(\exists n \text{ such that } X_n = 3 \text{ and } X_i \neq 0 \text{ for all } 1 \leq i < n).$$

(This is the probability of the sequence $\{X_n\}$ reaching 3 before returning to the starting point 0.)

- (ii) Now suppose that $X_0 = 1$ and $X_1 = 2$. Find the probability

$$\mathbb{P}(\exists n \text{ such that } X_n = X_{n+1} = 1).$$

9. Produce a sequence of random variables $\{X_n\}_{n \geq 0}$ as follows: Let $X_0 = q$ with probability 1, where $q \in (0, 1)$ is some constant. For $n \geq 1$, let $X_n = X_{n-1}^2$ with probability $1/2$ and $X_n = 2X_{n-1} - X_{n-1}^2$ with probability $1/2$. Prove that $\{X_n\}_{n \geq 0}$ converges almost surely, and find the distribution of the limit random variable.

10. Given an integer $n \geq 1$, define $(X_1^{(n)}, X_2^{(n)}, \dots, X_n^{(n)})$ as a random vector uniformly distributed in the ball

$$(X_1^{(n)})^2 + (X_2^{(n)})^2 + \dots + (X_n^{(n)})^2 \leq n.$$

Find the limit joint distribution of the random vector $(X_1^{(n)}, X_2^{(n)}, X_3^{(n)})$ as $n \rightarrow \infty$.