Probability Theory Exercise 4

1. (i) Let $X \sim \text{Ca}(t)$ be a Cauchy random variable with parameter $t > 0$. It has PDF

$$f_X(x) = \frac{1}{\pi} \frac{t}{t^2 + x^2}, \quad \forall x \in \mathbb{R}.$$

Calculate the moment generating function $M_X(s)$ for all $s \in \mathbb{R}$.

(ii) Does there exist a random variable X such that $\left|\mathbb{E}[X^k]\right| < \infty$ for all integers $k \geq 1$ but the moment generating function $M_X(s) = \infty$ for all $s \neq 0$? If your answer is yes, give such an example. If your answer is no, prove your conclusion.

2. Suppose that X is a nonnegative random variable and that $M_X(s) < \infty$ for all $s \in (-\infty, a]$, where $a > 0$ is a positive number.

(a) Show that $\mathbb{E}\left[X^k\right] < \infty$, for every positive integer k

(b) Show that $\mathbb{E}\left[X^k e^{sX}\right] < \infty$, for every positive integer k and every $s < a$

(c) Show that $\left(e^{hX} - 1\right)/h \leq X e^{hX}$ for $h > 0$.

(d) Use the Dominated Convergence Theorem to show that

$$\mathbb{E}[X] = \mathbb{E}\left[\lim_{h \downarrow 0} \frac{e^{hX} - 1}{h}\right] = \lim_{h \downarrow 0} \frac{\mathbb{E}\left[e^{hX}\right] - 1}{h}$$

3. Let $X \sim N(0, \sigma^2)$ be a Gaussian random variable. Prove that the limit $\lim_{x \to \infty} x e^{x^2/(2\sigma^2)} P(X \geq x)$ exists, and find the limit.

4. (i) Let $X_1, X_2, \ldots, X_n$ be i.i.d. **continuous** random variables. Find the probability $P(\min(X_1, \ldots, X_n) = X_1)$.

(ii) Let $X_1, X_2, \ldots, X_n$ be i.i.d. Bernoulli(p) random variables. Find the probability $P(\min(X_1, \ldots, X_n) = X_1)$.

(iii) Let $X_1, X_2, \ldots, X_n$ be independent random variables. Suppose that the distribution of X_i is exponential distribution with parameter $\lambda_i > 0$. Find the probability $P(\min(X_1, \ldots, X_n) = X_1)$.

5. Let $X_1, \ldots, X_n$ be n i.i.d. standard Gaussian random variables. Let $X = \max(X_1, \ldots, X_n)$. Prove that there exists a constant β_0 such that $\lim_{n \to \infty} P(X \geq \beta\sqrt{\log(n)}) = 0$ if $\beta > \beta_0$; and $\lim_{n \to \infty} P(X \geq \beta\sqrt{\log(n)}) = 1$ if $\beta < \beta_0$. Also find the value of β_0. What is $\lim_{n \to \infty} P(X \geq \beta\sqrt{\log(n)})$ when $\beta = \beta_0$? (Hint: Use the conclusion of Problem 3.)

6. Consider a branching process whose offspring distribution has expectation μ and variance σ^2 (see the definition of branching process and offspring distribution in the lecture slides). For $n = 0, 1, 2, \ldots$, let X_n be the number of individuals in the nth generation. Assume that $X_0 = 1$. What is $\text{Var}(X_n)$? Please express it in terms of n, μ and σ^2.