Tsinghua-Berkeley Shenzhen Institute
PROBABILITY
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Homework 7

Feng Zhao June 9, 2021

7.1. Since Y is a linear transformation of X, it's jointly Gaussian distribution. Since all X_{ij} have zero mean, so are Y_{ij}. We then compute the covariance between Y_{ij} and $Y_{i'j'}$ for $i \leq j$ and $i' \leq j'$. From $Y = U^T X U$,

$$\begin{aligned} Y_{ij} &= \sum_{k,r} u_{ik} u_{jr} X_{kr} \\ Y_{i'j'} &= \sum_{k,r} u_{i'k} u_{j'r} X_{kr} \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}[Y_{ij} Y_{i'j'}] &= 2\sum_{k=1}^{n} u_{ik} u_{jk} u_{i'k} u_{j'k} + \sum_{k<r}^{n} (u_{ik} u_{jr} + u_{ir} u_{jk})(u_{i'k} u_{j'r} + u_{i'r} u_{j'k}) \\ &= \sum_{k,r} u_{ik} u_{jr} u_{i'k} u_{j'r} + \sum_{k,r} u_{ik} u_{j'k} u_{i'r} u_{jr} \\ &= (\sum_{k=1}^{n} u_{ik} u_{i'k})(\sum_{r=1}^{n} u_{ir} u_{i'r}) + (\sum_{k=1}^{n} u_{ik} u_{j'k})(\sum_{r=1}^{n} u_{i'r} u_{jr}) \end{aligned}$$

From the orthogonality of U,

$$\mathbb{E}[Y_{ij} Y_{i'j'}] = \begin{cases} 2 & \text{if } i = i' \text{ and } j = j' \text{ and } i = j \\ 1 & \text{if } i = i' \text{ and } j = j' \text{ and } i \neq j \\ 0 & \text{if } i \neq i' \text{ or } j \neq j' \end{cases}$$

Therefore, $Y_{ij} \sim N(0, 1)$ for $i \neq j$ and $Y_{ii} \sim N(0, 2)$, and all entries on and above the diagonal of Y are independent.

7.2. We first compute the marginal distribution of X_i by recursive formula. Let Y_i be i.i.d. Bern($\frac{1}{5}$) and Y_i are independent with X_i. We have $X_i = Y_i + (1 - Y_i)\prod_{j=1}^{4}(1 - X_{i-j})$. Then $P(X_i = 1) = P(Y_i = 1) + P(Y_i = 0, X_{i-j} = 0, j = 1, 2, 3, 4) = \frac{1}{5} + P(Y_{i-j} = 0, j = 0, 1, 2, 3, 4, X_{i-5} = 1) = \frac{1}{5} + (\frac{4}{5})^5 P(X_{i-5} = 1)$. Thus we get a recursive formula, and $P(X_n = 1) \to \frac{625}{2101}$. The limit random variable equals $\frac{625}{2101}$ with probability 1.

We observe that for consecutive five numbers, there is at least 1; and if $X_n = 1$, $X_{n+1}, X_{n+2}, \ldots$ are independent with $X_1, \ldots, X_n$. Let $S_n = \frac{\sum_{i=1}^{n}}{n}$. Since $S_{5n+j} = \frac{5n}{5n+j}(S_{5n} + \frac{X_{5n+1} + \cdots + X_{5n+j}}{5n})$ for $j = 1, 2, 3, 4$ has the same limit with S_{5n}. We only need to show S_{5n} almost surely converges. Therefore, we split $S_{5n} = \frac{1}{5}\sum_{j=1}^{5} T_j$ where $T_j = \frac{1}{n}(X_j + X_{5+j} + \cdots + X_{5n-5+j})$. Since X_i is independent with X_{i+5} for any i, we can apply strong law of large number (not necessarily with the same distribution) to obtain that T_j almost surely converges.

7.3. (a) Let $X'_n = X_n + n$, which represents the first arriving position when hit $[n, +\infty)$ (starting from $X'_0 = 0$). Since $Y_1, Y_2, \dots$ are independent, it is easy to check that X'_n forms a Markov chain. Consider $P(X'_{n+1} = m' | X'_n = m)$. If $m \geq n+1$, then $P(X'_{n+1} = m | X'_n = m) = 1$. If $m = n$, then $P(X'_{n+1} = n + i | X'_n = n) = \sum_{j=0}^{+\infty} p_i p_0^j = \frac{p_i}{1-p_0}$. Therefore, the transition probabilities for X_n is given by
$P(X_{n+1} = i-1 | X_n = 0) = \frac{p_i}{1-p_0}$ for $i \geq 1$.
$P(X_{n+1} = i-1 | X_n = i) = 1$ for $i \geq 1$.

(b) $f(n) = P(X_n = 0)$. $\lim_{n\to\infty} f(n)$ exists means that the stationary distribution of this Markov chain exists. Since the state 0 has return probability 1. The chain is recurrent. We compute the expected return time as $T = 1 + \sum_{i=1}^{+\infty} i \frac{p_{i+1}}{1-p_0} = \frac{\mu}{1-p_0} < \infty$. Notice that $\mu \geq 1 - p_0$ from the definition of μ. Besides, we need the aperiodic condition to make sure the limit $\lim_{n\to\infty}$ exists. For our Markov chain, we formulate this condition as $\gcd\{i | p_i > 0\} = 1$ (gcd means the greatest common divisor). This is the sufficient and necessary condition for the existence of the limit.

(c) By the property of stationary distribution $\lim_{n\to\infty} f(n) = \frac{1}{T} = \frac{1-p_0}{\mu}$.

7.4. (a) The probability that the chain never returns to state 0 equals $\frac{6}{\pi^2}$. For this birth and death process. Let h_i be the probability that the chain hits state 0 starting from state i. Since $p_{0,1} = 1$, we only need to show that $h_1 = 1 - \frac{6}{\pi^2}$. Another boundary condition is $h_0 = 1$. The transition probability is given by

$$p_{i,i-1} = \frac{i^2}{i^2 + (i+1)^2}$$
$$p_{i,i+1} = \frac{(i+1)^2}{i^2 + (i+1)^2}$$

The recursive formula is given by

$$h_i = \frac{(i+1)^2}{i^2 + (i+1)^2} h_{i+1} + \frac{i^2}{i^2 + (i+1)^2} h_{i-1}$$

We can transform the above equation into $(i+1)^2(h_{i+1} - h_i) = i^2(h_i - h_{i-1}) \to h_{i+1} - h_i = [\prod_{j=1}^{i} \frac{j^2}{(j+1)^2}](h_1 - h_0) = \frac{1}{(i+1)^2}(h_1 - h_0)$. Therefore, the general solution is given by

$h_i - h_0 = [\sum_{j=1}^{i} \frac{1}{j^2}](h_1 - h_0) \to h_i = 1 + [\sum_{j=1}^{i} \frac{1}{j^2}](h_1 - 1)$. By the minimality and non-negativity of h_i, we have
$1 + [\sum_{j=1}^{+\infty} \frac{1}{j^2}](h_1 - 1) = 0$. Since the series sums up to $\frac{\pi^2}{6}$, we get our result that $h_1 = 1 - \frac{6}{\pi^2}$.

(b) We claim that the chain is positive recurrent for $\alpha < -1$, null recurrent for $-1 \leq \alpha \leq 1$, and transient for $\alpha > 1$. The general solution for h_i is given by $h_i = 1 + [\sum_{j=1}^{i} \frac{1}{j^\alpha}](h_1 - 1)$. If $\alpha > 1$, the series $\sum_{j=1}^{i} \frac{1}{j^\alpha}$ converges and $h_1 < 1$, therefore the chain is transient. On the other hand, if $\alpha \leq 1$, the series $\sum_{j=1}^{i} \frac{1}{j^\alpha}$ diverges and $h_i = 1$. Using $p_{0,1} = 1$ and $h_1 = 1$, we find the return probability for node 0 is 1. Then the chain is recurrent for $\alpha \leq 1$.

Let k_i be the expected time of hitting 0 when starting from state i. Then k_i is the minimal non-negative solution to

$$k_0 = 0, \quad k_i = 1 + \frac{(i+1)^\alpha}{i^\alpha + (i+1)^\alpha} k_{i+1} + \frac{i^\alpha}{i^\alpha + (i+1)^\alpha} k_{i-1}$$

Then we have

$$(i+1)^\alpha (k_{i+1} - k_i) = i^\alpha (k_i - k_{i-1}) - (i^\alpha + (i+1)^\alpha)$$

Let $a_i = i^\alpha (k_i - k_{i-1})$, we get
$a_{i+1} = a_i - (i^\alpha + (i+1)^\alpha) = a_1 - \sum_{j=1}^{i} (j^\alpha + (j+1)^\alpha)$. Since $a_1 = k_1$, we get

$$k_{i+1} - k_i = \frac{k_1 - \sum_{j=1}^{i} (j^\alpha + (j-1)^\alpha)}{(i+1)^\alpha}$$

As a result, we get

$$k_i = \sum_{r=1}^{i} \frac{k_1 - \sum_{j=1}^{r} (j^\alpha + (j-1)^\alpha)}{r^\alpha}$$

To make $k_i > 0$, we have

$$k_1 > \frac{\sum_{r=1}^{i} r^{-\alpha} \sum_{j=1}^{r} j^\alpha}{\sum_{r=1}^{i} r^{-\alpha}}$$

holds for any i. To compare the order of the numerator with the denominator, we estimate the summation by integration.
$\sum_{j=1}^{r} j^\alpha \approx \frac{r^{\alpha+1} - 1}{\alpha + 1}$ for $\alpha \in (-1, 1]$. Therefore, the numerator is actually order of i^2 while the denominator is of order $i^{1-\alpha}$. Let $i \to \infty$ we have $k_1 = +\infty$ for $\alpha \in (-1, 1]$. For $\alpha = -1$, the numerator becomes $\sum_{r=1}^{i} r \log r = O(r^2 \log r)$ while the denominator is $O(r^2)$. Therefore, $\alpha = -1$ also makes $k_1 = +\infty$.

The expected return time becomes infinite for $\alpha \in [-1, 1]$.
Therefore, the chain is null-recurrent for $-1 \le \alpha \le 1$.
When $\alpha < -1$, the series $\sum_{j=1}^{r} j^{\alpha}$ converges, then we can verify that $\sum_{j=1}^{\infty}((j-1)^{\alpha} + j^{\alpha})$ is an upper bound for k_1. The expected time of returning to 0 starting from 0 is $1 + k_1$. Therefore, the state 0 is positive recurrent. So is the whole chain.

7.5. Answer: the chain is transient when $\frac{3-\sqrt{5}}{2} < p < 1$; null recurrent when $p = \frac{3-\sqrt{5}}{2}$; positive recurrent when $0 < p < \frac{3-\sqrt{5}}{2}$.

To prove our conclusion, first we consider the probability of hitting 0 starting from i. We denote it as h_i. Then the recursive formula for h_i is given by

$$\begin{aligned} h_{2i} &= (1-p)h_{2i-2} + ph_{2i+1} \\ h_{2i-1} &= ph_{2i+1} + (1-p)h_{2i} \end{aligned}$$

We can write this relationship in matrix form:

$$\begin{pmatrix} p & 1-p \\ -p & 1 \end{pmatrix} \begin{pmatrix} h_{2i+1} \\ h_{2i} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1-p \end{pmatrix} \begin{pmatrix} h_{2i-1} \\ h_{2i-2} \end{pmatrix}$$

Let $\tilde{h}_i = \binom{h_{2i+1}}{h_{2i}}$, we get the recursive relationship for $\tilde{h}_i$ as

$$\tilde{h}_{i+1} = \frac{1}{2p-p^2} \begin{pmatrix} 1 & -(1-p)^2 \\ p & p(1-p) \end{pmatrix} \tilde{h}_i$$

The transformation matrix has two eigenvalues 1 and $\frac{1-p}{2p-p^2}$. $p = \frac{3-\sqrt{5}}{2}$ makes both of the two eigenvalues equal 1. When $p > \frac{3-\sqrt{5}}{2}$, $h_{2i} = c(\frac{1-p}{2p-p^2})^{2i} < 1 \Rightarrow$ the chain is transient. When $p = \frac{3-\sqrt{5}}{2}$, we can verify that the expected return time is infinity. Therefore, the chain is null-recurrent.

If the Markov chain is positive recurrent. Using $x = xP$ we can get $x_0 = c$ and

$$\begin{aligned} x_{2n} &= \frac{p^{n-1}c(2-p)^{n-1}}{(1-p)^n}(n \ge 1) \\ x_{2n+1} &= \frac{p^n c(2-p)^n}{(1-p)^n}(n \ge 0) \end{aligned}$$

This result can be proved using induction ($x = xP$) on:

$$\begin{aligned} (1-p)(x_{2n-1} + x_{2n+2}) &= x_{2n} \\ p(x_{2n-1} + x_{2n}) &= x_{2n+1} \end{aligned}$$

for $n \geq 1$ while the initial condition is $x_0 = x_1 = c, x_2(1-p) = x_0$. Although the transition matrix P is infinite, we can still use the steady state equation.

c is the normalization constant such that $\sum_{n=0}^{+\infty} x_n = 1$. We obtain $c = \frac{p^2-3p+1}{(1-p)(3-p)}$. The series converges if $\frac{p(2-p)}{1-p} < 1$, from which we can get the necessary condition on p for the Markov chain to be positive-recurrent: $p < \frac{3-\sqrt{5}}{2}$. We further guess that the chain is null-recurrent when $p = \frac{3-\sqrt{5}}{2}$ and transient when $p > \frac{3-\sqrt{5}}{2}$. When the chain is positive recurrent, the stationary distribution is given by

$$\begin{aligned}
x_0 &= \frac{p^2-3p+1}{(1-p)(3-p)} \\
x_{2n} &= \frac{p^2-3p+1}{(1-p)(3-p)} \frac{p^{n-1}(2-p)^{n-1}}{(1-p)^n} (n \geq 1) \\
x_{2n+1} &= \frac{p^2-3p+1}{(1-p)(3-p)} \frac{p^n(2-p)^n}{(1-p)^n} (n \geq 0)
\end{aligned}$$

7.6. (i) Let $x_1 = \mathbb{E}[T|X_0 = Y_0 = 0], x_2 = \mathbb{E}[T|X_0 = 1, Y_0 = 0], x_3 = \mathbb{E}[T|X_0 = 2, Y_0 = 0], x_4 = \mathbb{E}[T|X_0 = 1, Y_0 = 1], x_5 = \mathbb{E}[T|X_0 = 1, Y_0 = 2], x_6 = \mathbb{E}[T|X_0 = 2, Y_0 = 2]$. By the symmetric property, we can establish a linear equation system about $x_1, \ldots, x_6$ as

$$\begin{aligned}
x_1 &= \frac{1}{4}(4x_2) + 1 \\
x_2 &= \frac{1}{4}(x_3 + 2x_4 + x_1) + 1 \\
x_3 &= \frac{1}{4}(x_2 + 2x5) + 1 \\
x_4 &= \frac{1}{4}(2x_2 + 2x_5) + 1 \\
x_5 &= \frac{1}{4}(x_3 + x_4 + x_6) + 1 \\
x_6 &= \frac{1}{4}(2x_5 + 1)
\end{aligned}$$

Using symbolic computation software (like `sympy` we can get $x_1 = \frac{135}{13}$, which is $\mathbb{E}[T]$ as required. $P(X_T = 3, Y_T = 0)$ is the probability that the particle reaches $(3, 0)$ before it reaches all other particles on T. By solving a linear equation system with 15 unknown variables,

$$\begin{aligned} 4x_1 - x_2 - x_6 &= 0 & 4x_2 - x_1 - x_3 - x_7 &= 0 & 4x_3 - x_2 - x_4 - x_8 &= 0 \\ 4x_4 - x_3 - x_5 - x_9 &= 0 & 4x_5 - x_4 - x_{10} &= 0 & 4x_6 - x_1 - x_7 - x_{11} &= 0 \\ 4x_7 - x_2 - x_6 - x_8 - x_{12} &= 0 & 4x_8 - x_3 - x_7 - x_9 - x_{13} &= 0 & 4x_9 - x_4 - x_8 - x_10 - x_{14} &= 0 \\ 4x_{10} - x_5 - x_9 - x15 &= 0 & 4x_{11} - 2x_6 - x_{12} &= 0 & 4x_{12} - 2x_7 - x_{11} - x_{13} &= 0 \\ 4x_{13} - 2x_8 - x_{12} - x_{14} &= 0 & 4x_{14} - 2x_9 - x_{13} - x_{15} &= 0 & 4x_{15} - 2x_{10} - x_{14} - 1 &= 0 \end{aligned}$$

we get $P(X_T = 3, Y_T = 0) = x_{13} = \frac{1}{13}$.

(ii) Using same method as above, we have $\mathbb{E}[T] = \frac{39}{7}$ and $P(X_T = 3, Y_T = 0) = \frac{1}{28}$

(iii) The basic idea is first consider $T_N := \int\{n \geq 0 : |Y_n| \leq 2, -2 \leq X_n \leq N\}$. and let $N \to \infty$. Let $x_n = \mathbb{E}[T|X_0 = n-2, Y_n = 1], y_n = \mathbb{E}[T|X_0 = n-2, Y_n = 0]$. By symmetric property, $x_n = \mathbb{E}[T|X_0 = n-2, Y_n = -1]$. The recursive formulas for x_n, y_n are

$$x_n = 1 + \frac{x_{n-1} + x_{n+1} + y_n}{4}$$
$$y_n = 1 + \frac{2x_n + y_{n+1} + y_{n-1}}{4}$$

Let $A_n = (x_n, y_n, x_{n-1}, y_{n-1})^T, q = (-4, -4, 0, 0)^T$ and the matrix P is defined as

$$P = \begin{pmatrix} 4 & -1 & -1 & 0 \\ -2 & 4 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Then we have $A_{n+1} = PA_n + Q$ Let v satisfies $(I - P)v = q$ which allows $v = (6, 8, 6, 8)^T$, and $\widetilde{A}_n = A_n - v$, then we have $\widetilde{A}_{n+1} = P\widetilde{A}_n$, which is a homogeneous recurrence equation system. The characteristic function is $\lambda^4 - 8\lambda^2 + 16\lambda^2 - 8\lambda + 1 = 0$, which can be transformed as $\mu^2 - 8\mu + 14 = 0$ where $\mu = \lambda + \frac{1}{\lambda}$. Suppose $\widetilde{A}_{n+1} = (\tilde{x}_{n+1}, \tilde{y}_{n+1}, \tilde{x}_n, \tilde{y}_n)$. The general formula for $\tilde{x}_n$ is $\tilde{x}_n = a\lambda_1^n + b\lambda_2^n + c\lambda_3^n + d\lambda_4^n$. Using $\tilde{x}_n = \frac{\tilde{x}_{n-1}+\tilde{x}_{n+1}+\tilde{y}_n}{4}$ we get $\tilde{y}_n = a(1-\lambda_1-\frac{1}{\lambda_1})\lambda_1^n + b(1-\lambda_2-\frac{1}{\lambda_2})\lambda_2^n + c(1-\lambda_3-\frac{1}{\lambda_3})\lambda_3^n + d(1-\lambda_4-\frac{1}{\lambda_4})\lambda_4^n$. The boundary condition is that $\tilde{x}_0 = -6, \tilde{y}_0 = -8, \tilde{x}_N = -6, \tilde{y}_N = -8$. After analysis of the four roots $\lambda_1 \ldots \lambda_4$, we find there are exactly two roots which are larger than 1. Assume they are $\lambda_2 > 1, \lambda_4 > 1$. Then from the two equations $\tilde{x}_N = -6, \tilde{y}_N = -8$ we get $b, d \to 0$ as $N \to \infty$. Therefore, we only need to solve a, c in the limit case, that is

$$a + c = -6$$
$$a(1 - \lambda_1 - \frac{1}{\lambda_1}) + c(1 - \lambda_3 - \frac{1}{\lambda_3}) = -8$$

Assume $\lambda_1 > \lambda_3$, we can get $1 - \lambda_1 - \frac{1}{\lambda_1} = \sqrt{2}$ and $1 - \lambda_3 - \frac{1}{\lambda_3} = -\sqrt{2}$ by solving $\mu^2 - 8\mu = 14$. Therefore, we get $a = -3 - 2\sqrt{2}$ and $c = -3 + 2\sqrt{2}$. The required expectation $\mathbb{E}[T] = a(1 - \lambda_1 - \frac{1}{\lambda_1})\lambda_1^2 + c(1 - \lambda_3 - \frac{1}{\lambda_3})\lambda_3^2 + 6$, in which we need the value of λ_1 and λ_3, given by

$$\lambda_1 = \frac{k - \sqrt{k^2 - 4}}{2} \text{ where } k = 4 - \sqrt{2}$$
$$\lambda_3 = \frac{k - \sqrt{k^2 - 4}}{2} \text{ where } k = 4 + \sqrt{2}$$

After simplification, we get $\mathbb{E}[T] = -8 + (8 + 5\sqrt{2})\sqrt{7 - 4\sqrt{2}} + (-8 + 5\sqrt{2})\sqrt{7 + 4\sqrt{2}} \approx 6.16$. Using similar techniques in (i) we can compute $P(X_T = -2, Y_T = 0) = 8 - \frac{(2\sqrt{2}-1)\sqrt{7-4\sqrt{2}}+(2\sqrt{2}+1)\sqrt{7+4\sqrt{2}}}{2} \approx 0.13$

(iv) $\mathbb{E}[T] = +\infty$. To prove this, we first reduce the 2d random walk to 1d. Define $S_{2n} = \sum_{i=1}^{n}(X_i + Y_i)$ and $S_{2n+1} = S_{2n} + X_{n+1}$. Then $\mathbb{E}[T] = \sum_{n=1}^{+\infty} 2nP(S_2 \neq 2, \ldots, S_{2n-2} \neq 2, S_{2n} = 2)$. Notice that S_{2n+1} is an even number and $S_{2n+1} \neq 2$ always holds. Therefore, $\mathbb{E}[T] = \sum_{n=1}^{+\infty} 2nP(S_1 \neq 2, \ldots, S_{2n-1} \neq 2, S_{2n} = 2) = \sum_{n=1}^{+\infty} nP(S_1 \neq 2, \ldots, S_{n-1} \neq 2, S_n = 2)$. That's to say, $\mathbb{E}[T]$ is the expected time for a particle hits 2 in a 1d random walk (starting from $S_0 = 0$. We can also transform the problem to the expected time for a particle hits 0 starting from 2. First assume the walk is bounded with $[0, N]$, then the expected time is $2(N - 2)$. Let $N \to \infty$, we get $\mathbb{E}[T] = +\infty$.

7.7. (a) Let $\sum_{i=1}^{\infty} a_i^2 = D$. By Chebyshev's inequality, $P(|\sum_{i=1}^{n} a_i X_i| > k) \leq \frac{\sum_{i=1}^{n} a_i^2}{4k^2} \leq \frac{D}{4k^2}$ holds for any n. Define the event A_k as $\{\exists n, s.t. |\sum_{i=1}^{n} a_i X_i| > k\}$. Then $A_k \supset A_{k+1}$, and $P(A_k) \leq \frac{D}{4k^2}$. Therefore, $\sum_{k=1}^{\infty} P(A_k) < +\infty$. By Borel-Cantelli Lemma, $P(\lim_{k\to\infty} A_k) = 0$. That is, $P(\sum_{i=1}^{n} a_i X_i \text{ diverges}) = 0 \implies P(|\sum_{i=1}^{\infty} a_i X_i| < \infty) = 1$.

(b) By the Kolmogorov's zero and one law (Chapter 6 of [1]), the probability can only be 0 or 1. Let $S_n = \sum_{i=1}^{n} a_i X_i$ $\mathbb{E}[S_n] = 0, \mathrm{Var}[S_n] = \sum_{i=1}^{n} a_i^2$, by the Kolmogorov's three series theorem, if S_n converges almost surely, then $\mathrm{Var}[S_n]$ converges. Now we have $\mathrm{Var}[S_n] \to \infty$, therefore $P(|\sum_{i=1}^{\infty} a_i X_i| < \infty) = 0$.

7.8. (a) The probability is $\frac{2}{7}$. Let $h_{i,j}$ represents the probability of the sequence $\{X_n\}$ reaching 3 before returning to 0 with the initial condition $X_0 = i, X_1 = j$. We are required to compute $h_{0,1}$. The boundary condition includes $h_{1,0} = 0, h_{2,3} = 1, h_{1,3} = 1$ and so on. The relationship between different $h_{i,j}$ is given by $h_{i,j} = \frac{1}{2}(h_{j,i+j} + h_{j,|i-j|})$. From $h_{0,1} = h_{1,1}, h_{1,1} = \frac{1}{2}h_{1,2}, h_{1,2} = \frac{1}{2}h_{2,1} + \frac{1}{2}, h_{2,1} = \frac{1}{2}h_{1,1} + \frac{1}{2} \Rightarrow h_{0,1} = \frac{3}{7}$.

(b) Let $Y_n = (X_n, X_{n+1})$, then $(Y_0, Y_1, \dots,)$ is a Markov chain. First we show that $P(\exists n \text{ such that } Y_n = (j,i) | Y_0 = (i, i+j)) = p$ for any $i \geq 1, j \geq 1$. Let $+$ represents the choice of $X_i = X_{i-1} + X_{i-2}$ and $-$ represents the choice of $X_i = |X_{i-1} - X_{i-2}|$. The consecutive occurrence of $+--$ makes the state return to itself, otherwise, X_i becomes too large to make the pattern $X_n = j, X_{n+1} = i$ occurs. In summary, if the number of $+$ is r, the number of $-$ should be $2r+2$. This property is irrelevant with i, j. Similarly we can show that $P(\exists n \text{ such that } Y_n = (j,i) | Y_0 = (i+j, i)) = q$. The relationship of p, q can be solved by a linear equation system. We consider $p_{i,j} = P(\exists n \text{ such that } Y_n = (1,1) | Y_0 = (i,j))$. Then $p_{1,2} = \frac{1}{2}p_{2,3} + \frac{1}{2}p_{2,1}, p_{2,1} = \frac{1}{2} + \frac{1}{2}p_{1,3}$. Notice that $p_{1,2} = p, p_{2,1} = q$. Besides, starting from $Y_0 = (2,3)$, the sequence needs to pass $Y_m = (1,2)$ before it can reach $Y_n = (1,1)$. Therefore, $p_{2,3} = p_{1,2}P(\exists n \text{ such that } Y_n = (1,2) | Y_0 = (2,3)) = p^2$. Similar analysis gives $p_{1,3} = pq$. The linear equation system for p, q is

$$
\begin{aligned}
p &= \frac{1}{2}p^2 + \frac{1}{2}q \\
q &= \frac{1}{2} + \frac{1}{2}pq
\end{aligned}
$$

Eliminating q, we have $(p-1)(p^2 - 3p + 1) = 0$. Notice that $p_{1,2} = \frac{1}{2}P(\exists n, Y_n = (2,1) | Y_0 = (1,2)) \leq \frac{1}{2}$. Therefore, we should choose the root $p = \frac{3-\sqrt{5}}{2}$. That is, $P(\exists n \text{ such that } X_n = X_{n+1} = 1) = \frac{3-\sqrt{5}}{2}$.

7.9. First we have $\mathbb{E}[X_n | X_{n-1}] = \frac{1}{2}X_{n-1}^2 + \frac{1}{2}(2X_{n-1} - X_{n-1}^2) = X_{n-1}$. $2X_{n-1} - X_{n-1}^2 = X_{n-1}(2 - X_{n-1}) \leq (\frac{X_{n-1} + 2 - X_{n-1}}{2})^2 = 1 \Rightarrow 0 \leq X_n \leq 1$. Therefore, $(X_n, n = 0, 1, \dots)$ is a martingale. By the Martingale Convergence Theorem, $X_n \xrightarrow{a.s.} X$. By the law of total expectation, $\mathbb{E}[X_n] = \mathbb{E}[\mathbb{E}[X_n | X_{n-1}]] = \mathbb{E}[X_{n-1}] = \mathbb{E}[X_0] = q$. By the dominated convergence theorem, $\mathbb{E}[X] = \lim_{n\to\infty} \mathbb{E}[X_n] = q$. Since $0 \leq X \leq 1$, $\mathbb{E}[X^k] \leq \mathbb{E}[X] = q$. On the other hand,

$$
\begin{aligned}
\mathbb{E}[X_n^k] &= \mathbb{E}[\mathbb{E}[X_n^k | X_{n-1}]] = \mathbb{E}[\frac{1}{2}(X_{n-1}^2)^k + \frac{1}{2}(2X_{n-1} - X_{n-1}^2)^k] && (1) \\
&= \mathbb{E}[X_{n-1}^k \left(\frac{X_{n-1}^k + (2 - X_{n-1})^k}{2}\right)] \geq \mathbb{E}[X_{n-1}^k] && (2)
\end{aligned}
$$

Therefore, $\mathbb{E}[X_n^k]$ is an increasing sequence about n for fixed k. Since $\mathbb{E}[X_n^k] \leq q$, $\lim_{n\to\infty} \mathbb{E}[X_n^k]$ exists. By the dominated convergence theorem, $\mathbb{E}[X^k] = \lim_{n\to\infty} \mathbb{E}[X_n^k]$. In equation (1), let $k = 2$, we have $\mathbb{E}[X_n^2] = \mathbb{E}[X_{n-1}^4] + 2\,\mathbb{E}[X_{n-1}^2] - 2\,\mathbb{E}[X_n^3]$. Taking the limit $n \to \infty$ on both sides of the equality, we have
$\mathbb{E}[X^4] + \mathbb{E}[X^2] - 2\,\mathbb{E}[X^3] = 0 \Rightarrow \mathbb{E}[X^2(X-1)^2] = 0$. Therefore, X only takes value 0 or 1. It is a Bernoulli random variable. By $\mathbb{E}[X] = q$, the parameter of this Bernoulli distribution is q.

7.10. We claim that $(X_1^{(n)}, X_2^{(n)}, X_3^{(n)})$ follows the standard normal distribution, and they are independent. Let $V_n(R)$ be the volume of the n dimensional ball with radius R. The formula for $V_n(R)$ is given by $V_n(R) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} R^n$. Then the joint distribution for $(X_1^{(n)}, X_2^{(n)}, X_3^{(n)})$ is given by

$$\begin{aligned}
p_n(x_1, x_2, x_3) &= \frac{1}{V_n(n)} \int_{x_4^2 + \cdots + x_n^2 \leq n - S} dx_4 \ldots dx_n \\
&= \frac{1}{V_n(n)} V_{n-3}(\sqrt{n-S}) \text{ where } S = x_1^2 + x_2^2 + x_3^2 \\
&= \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}}} n^{-n/2} \frac{\pi^{\frac{n-3}{2}}}{\Gamma(\frac{n-3}{2}+1)} (n-S)^{\frac{n-3}{2}}
\end{aligned}$$

Now we take the limit $n \to \infty$ in the above equation for fixed S, using the Stirling's formula for Gamma function: $\Gamma(x+1) \sim \sqrt{2\pi x}(\frac{x}{e})^x$, we have

$$\begin{aligned}
p_n(x_1, x_2, x_3) \sim &\frac{e^{-s/2}}{\pi^{3/2}} \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2}-\frac{1}{2})} \cdot (n-S)^{\frac{3}{2}} \\
\sim &\frac{\exp(-\frac{S}{2})}{(2\pi)^{\frac{3}{2}}}
\end{aligned}$$

Therefore, the limit distribution for $(X_1^{(n)}, X_2^{(n)}, X_3^{(n)})$ is the standard normal distribution.

References

[1] 测度论与概率论基础，程士宏编著，北京大学出版社，2004 年