# Probability Theory Exercise 2

1. We throw 3 (six-sided) dices one by one. What is the probability that we obtain 3 numbers in strictly increasing order? What about the probability that we obtain 3 numbers in strictly decreasing order?

2. For a discrete random variable $X$ that takes $n$ values $\{x_1, x_2, \dots, x_n\}$, its entropy is defined as

$$H(X) := -\sum_{k=1}^{n} p_k \log(p_k), \text{ where } p_k := P(X = x_k) \text{ for } k = 1, 2, \dots, n.$$

Prove the following two statements:
(i) Among all the discrete random variables whose range is $\{x_1, x_2, \dots, x_n\}$ and expectation is $\mu$, the maximum entropy distribution has the following shape:

$$p_k = Cr^{x_k} \text{ for } k = 1, 2, \dots, n.$$

Here $C$ and $r$ are constants that can be determined by the requirements $\sum_{k=1}^{n} p_k = 1$ and $\sum_{k=1}^{n} p_k x_k = \mu$. (Hint: Use "Log sum inequality".)
(ii) Among all the discrete random variables whose range is the countably infinite set $\{x_1, x_2, \dots\}$ and expectation is $\mu$, the maximum entropy distribution has the following shape:

$$p_k = Cr^{x_k} \text{ for } k = 1, 2, \dots$$

For the case of $x_k = k$ for all $k$, conclude that this is a geometric distribution, and find the values of $C$ and $r$.

3. **(Conditionally convergent series)** (i) Find the value of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

(ii) Find the value of the following infinite sum

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots$$

Note that this new series is just a reordering of the original infinite series in (i). More precisely, in the original series, every positive term is followed by one negative term while in the new series, every positive term is followed by two negative terms.
(iii) Find the value of

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

This series is obtained by taking absolute values of each term in the original series in (i).
(In this problem, we have seen a concrete example of a conditionally convergent series, and we have shown that by rearranging the order of a conditionally convergent series, we can make it sum up to a different value.)

4. (i) Find $\mathbb{E}[X^3]$ and $\mathbb{E}[X^4]$ for $X \sim$Geometric$(p)$.
(ii) Find $\mathbb{E}[X^3]$ and $\mathbb{E}[X^4]$ for $X \sim$Poisson$(\lambda)$.

5. Suppose that there are $k$ people. Each of them independently picks a uniformly random number from the set $\{1, 2, \dots, n\}$. We say that a collision happens if there exist two people picking the same number.

Let $k = \lceil n^\beta \rceil$, where $\beta$ is some constant that does not change with $n$. Prove that there is a constant $\beta_0 \in (0, 1)$ such that if $\beta > \beta_0$, then a collision happens with probability 1 when $n \to \infty$; and if $\beta < \beta_0$, then a collision happens with probability 0 when $n \to \infty$. Also find the value of $\beta_0$.

6. Roll a die many times. Let $X_i$ be the number obtained in the $i$th roll. Then $X_i$ has uniform distribution over the set $\{1, 2, 3, 4, 5, 6\}$, and the random variables $X_1, X_2, \dots, X_i, \dots$ are independent. Now define $S_n = \sum_{i=1}^n X_i$, i.e., $S_n$ is the sum of the first $n$ rolls.
(1) What is the probability that the number 2020 appears in the sequence $\{S_n\}_{n=1}^\infty$? Or in other words, calculate the probability $\mathbb{P}(\exists n \text{ s.t. } S_n = 2020)$. (Hint: Let $f(m) = \mathbb{P}(\exists n \text{ s.t. } S_n = m)$. Find a recursive formula for $f(m)$, and then write a computer program to calculate $f(2020)$)
(2) Let $f(m) = \mathbb{P}(\exists n \text{ s.t. } S_n = m)$. Now assume that the sequence $\{f(m)\}_{m=1}^\infty$ converges. What is the limit $\lim_{m\to\infty} f(m)$? (Hint: Try to decompose the probability of the complement of the event $\exists n$ s.t. $S_n = m$)
(3) Prove that the sequence $\{f(m)\}_{m=1}^\infty$ does converge.