Tsinghua-Berkeley Shenzhen Institute
LARGE DEVIATION AND HIGH DIMENSIONAL STATISTICS
Spring 2021

Problem Set 2

Issued: Monday 22nd March, 2021 **Due:** Monday 5th April, 2021

2.1. Show that

(a)
$$X_n \xrightarrow{\text{p}} X,\ Y_n \xrightarrow{\text{p}} Y \quad \Rightarrow \quad X_n Y_n \xrightarrow{\text{p}} XY;$$

(b) however,
$$X_n \xrightarrow{\text{d}} X,\ Y_n \xrightarrow{\text{d}} Y \quad \not\Rightarrow \quad X_n Y_n \xrightarrow{\text{d}} XY;$$

(c) when $X_n \perp\!\!\!\perp Y_n$ and $X \perp\!\!\!\perp Y$,
$$X_n \xrightarrow{\text{d}} X,\ Y_n \xrightarrow{\text{d}} Y \quad \Rightarrow \quad X_n Y_n \xrightarrow{\text{d}} XY.$$

2.2. Show that for every X on $(\mathbb{R}, \mathcal{B})$, there exist a sequence $X_n \xrightarrow{\text{d}} X$ such that every X_n has a continuous, bounded, infinitely-differentiable PDF. Steps:

(a) Show that $X_\epsilon = X + \epsilon Z \xrightarrow{\text{d}} X$ as $\epsilon \to 0$.

(b) Let $X \perp\!\!\!\perp Z$ and $Z \sim \mathcal{N}(0, 1)$. Show that CDF of X_ϵ is continuous (Hint: BCT) and differentiable with derivative
$$f_{X_\epsilon}(a) = \mathbb{E}\left[f_Z\left(\frac{a - X}{\epsilon}\right)\frac{1}{\epsilon}\right].$$

(c) Show that $a \mapsto f_{X_\epsilon}(a)$ is continuous.

(d) Conclude the proof (Hint: derivatives of f_Z are uniformly bounded on $\mathbb{R}$).

2.3. Let $S_n = \sum_{j=1}^{n} X_j$ be a sum of independent random variables X_j with $|X_j| \le 1$ almost surely. Show that S_n converges in probability if and only if it converges almost surely (to a finite value).