一、(1) 停时可以理解为在时刻 t 能否根据当前信息判断已发生某事件。

(2) 马氏过程可以理解为该过程的未来只依赖当前状态，与过去无关。

(3) $X_t = \begin{cases} W_t, & W_0 \neq 0 \\ 0, & W_0 = 0 \end{cases}$

二、(1) ① $W_0 = 0$, a.s.

② $\forall\, 0 \leq s < t$, $W_t - W_s \sim N(0, t-s)$

③ $\forall\, 0 \leq t_0 < t_1 < \cdots < t_n$, $\{W_{t_k} - W_{t_{k-1}}\}_{k=1}^{n}$ 独立.

④ $t \mapsto W_t(\omega)$ 对几乎所有 ω 连续.

⑤ $\forall t$, $W_t \in \mathcal{F}_t$.

(2) ① $W_t \sim N(0, t) \Rightarrow \mathbb{E}|W_t| < \infty$

② $\mathbb{E}[W_t | \mathcal{F}_s] = \mathbb{E}[W_s + (W_t - W_s) | \mathcal{F}_s]$

$= \mathbb{E}[W_s | \mathcal{F}_s] + \mathbb{E}[W_t - W_s | \mathcal{F}_s]$

$= W_s + 0 = W_s$.

(3) ① 布朗运动：(i) $B_0 = 0$,

(ii) $B_t - B_s = W_{t+\tau} - W_{s+\tau} \sim N(0, t-s)$

(iii) 由 W_t 强马氏性，$B_{t_k} - B_{t_{k-1}} = W_{t_k+\tau} - W_{t_{k-1}+\tau}$ 独立

(iv) $W_t(\omega)$ 连续 $\Rightarrow B_t(\omega)$ 连续

(v) $B_t = W_{t+\tau} - W_\tau \in \mathcal{F}_{t+\tau} = \mathcal{G}_t$

② $B_t = W_{t+\tau} - W_\tau$, $\mathcal{G}_0 = \mathcal{F}_\tau$.

由 W_t 强马氏性，B_t 与 $\mathcal{G}_0$ 独立.

三.(1) ① X_t 是$\mathcal{F}_t$适应的.

② $\forall s<t$, $\mathbb{E}[e^{\lambda W_t}|\mathcal{F}_s]=\mathbb{E}[e^{\lambda(W_s+(W_t-W_s))}|\mathcal{F}_s]$

$$=e^{\lambda W_s}\,\mathbb{E}[e^{\lambda(W_t-W_s)}|\mathcal{F}_s]$$

由$W_t-W_s\sim N(0,t-s)$. 上式$=e^{\lambda W_s}\cdot e^{\lambda^2(t-s)/2}\geq X_s$

(2) $\mathbb{P}(W_t>x)\leq e^{-x^2/(2t)}$,

由反射原理, $\mathbb{P}(\sup\limits_{0\le s\le t}W_s>x)=2\mathbb{P}(W_s>x)\le 2e^{-x^2/(2t)}$

由W_t的对称性, $\mathbb{P}(\sup\limits_{0\le s\le t}|W_s|>x)=\mathbb{P}(\sup\limits_{0\le s\le t}W_s>x,\ \inf\limits_{0\le s\le t}W_s<-x)$

$$=2\mathbb{P}(\sup_{0\le s\le t}W_s>x)\le 2e^{-x^2/(2t)}$$

(3) $\frac{t}{4}\le\langle X\rangle_t=\int_0^t\sigma_s^2ds\le 4t$

由鞅的极大值不等式, $\mathbb{P}(\sup\limits_{0\le s\le t}|X_s|>x)\le 2e^{-x^2/(2\langle X\rangle_t)}\le 2e^{-x^2/(8t)}$

(4) 令$\tau_2=\inf\{t>0:|W_t|>1\}$

$\tau_1\le 4\tau_2\Rightarrow\mathbb{E}e^{\mu\tau_1}\le\mathbb{E}e^{4\mu\tau_2}$

由$\lambda<\frac{\pi^2}{8}$时, $\mathbb{E}e^{\lambda\tau_2}<\infty$, 故$\mu<\frac{\pi^2}{32}$时, $\mathbb{E}e^{\mu\tau_1}<\infty$.

(5). 可以推广到d维.

因为W上定理在d维仍成立. 可以对每个分量单独处理.

只在计算时系数会相差一个常数.

四. 由Girsanov定理, $\frac{dQ}{dP}=\exp(\int_0^\infty h_s dW_s-\frac{1}{2}\int_0^\infty h_s^2 ds)$

$$\Rightarrow \log\frac{dQ}{dP}=\int_0^\infty h_s dW_s-\frac{1}{2}\int_0^\infty h_s^2 ds$$

令$\tilde{W}_t=W_s-h_s$. 则$\tilde{W}_s$

$$\log\frac{dQ}{dP}=\int_0^\infty h_s(d\tilde{W}_s+h_s ds)-\frac{1}{2}\int_0^\infty h_s^2 ds$$

$$=\int_0^\infty h_s d\tilde{W}_s+\frac{1}{2}\int_0^\infty h_s^2 ds$$

$\therefore H(\mathbb{Q}\|\mathbb{P}) = \mathbb{E}[\int_0^{t_0} h_s d\tilde{W}_s] + \mathbb{E}[\frac{1}{2}\int_0^{t_0} h_s^2 ds]$

$= \frac{1}{2}\int_0^{t_0} h_s^2 ds$

五、设 W_t 为 d 维布朗运动. 令 $\tau = \inf\{t>0: W_t \in \partial D\}$.

$u(x) = \mathbb{E}_x u(W_\tau) - \mathbb{E}_x[\int_0^\tau \frac{1}{2}\Delta u(W_s) ds] = \mathbb{E}_x u(W_\tau)$

则 $u(0) = \mathbb{E}_0 u(W_\tau)$

由 $\mathbb{P}(W_\tau = 0) = 0$, $\mathbb{P}(|W_\tau| = 1) = 1$, 故 $u(0) = 0$. 与题设 $u(0)=1$ 矛盾.

六、由 a 对称正定, 故 $\exists\, \sigma_t$ s.t. $a_t = \frac{1}{2}\sigma_t\sigma_t^T$

设 $X_t = \int_0^t \sigma_s dW_s + \int_0^t b_s ds + x_0$.

由 Itô 公式,

$u(X_{t\wedge\tau_D}) = u(x_0) + \int_0^{\tau_D\wedge t} L u(X_t) dt + \int_0^{\tau_D\wedge t} \sigma \partial_i u dW_s$

两边取期望有. $\mathbb{E}u(X_{t\wedge\tau_D}) - u(x_0) = \mathbb{E}[\int_0^{t\wedge\tau_D} L u(X_t) dt] + 0 \geq 0$

由 $Lu(x) \geq 0$, $x \in D$.

又 u 在 $x_0 \in D$ 有最大值. 故 $u(X_{t\wedge\tau_D}) \leq u(X_0)$, $\forall t$.

$\Rightarrow \mathbb{E}u(X_{t\wedge\tau_D}) = u(X_0)$. 且 $\mathbb{P}(u(X_{t\wedge\tau_D}) < u(X_0)) = 0$.

若 u 在 D 上不恒为常数. 且 x_0 在 D 内部.

则 $\exists\, r_x > 0$. s.t. $B(x_0, r_x) \subset D$, 且 $\exists\, y \in D$ 内部. $u(y) < u(X_0)$.

则 $\exists\, r_y > 0$. s.t. $B(y, r_y) \subset D$, 且 $u(y') < u(X_0)$. $y' \in B(y, r_y)$.

令 $\varepsilon = \frac{1}{2}\min\{r_x, r_y\}$. φ 为连接 x_0 和 y 的线段

则 φ 与 ∂D 距离大于 ε.

由 support th. x_t 有正概率在 y 的 ε 范围内.

$\tau_{B(y,\frac{r_0}{2})} = \inf\{t>0: X_t \in B(y, \frac{r_0}{2})\}$.

由 support th. $\exists C_1>0$. $\mathbb{P}^{x_0}(\tau_{B(y,\frac{r_0}{2})} < \tau_D) \geq C_1$.

$u(X_{\tau_{B(y,\frac{r_0}{2})}}) < u(x_0)$.

故 $\mathbb{P}(u(X_{t\wedge\tau_D}) < u(x_0)) \geq \mathbb{P}^{x_0}(\tau_{B(y,\frac{r_0}{2})} < \tau_D) \geq C_1 > 0$. 矛盾.

故 $x_0 \in \partial D$.

七. 设 A 有一个元素 / α_t 有 Markov 性质.

由 "Stochastic Differential Equation", Th 11.2.1.

$J(t,x;\alpha) = \mathbb{E}_{t,x}\, g(X_T^\alpha) = \mathbb{E}_{t,x}[\mathbb{E}_{t,x}[g(X_T^\alpha) | \mathcal{F}_{t+\Delta t}]]$

$= \mathbb{E}_{t,x}[\mathbb{E}_{t+\Delta t, x(t+\Delta t)}\, g X_T^\alpha]$

$= \mathbb{E}_{t,x}[J(t+\Delta t, x(t+\Delta t); \alpha)]$.

设 α^*, s.t. $u(t+\Delta t, x(t+\Delta t)) = J(t+\Delta t, x(t+\Delta t); \alpha^*)$

则令 $\alpha_s = \begin{cases} \alpha', & t \leq s < t+\Delta t \\ \alpha^*, & s \geq t+\Delta t \end{cases}$ 其中 $\alpha' \in \mathcal{A}$.

则 $u(t,x) \leq J(t,x;\alpha) = \mathbb{E}_{t,x}[J(t+\Delta t, x(t+\Delta t); \alpha^*)]$

$= \mathbb{E}_{t,x}[u(t+\Delta t, x(t+\Delta t))]$. (*)

由 Itô 公式. $du = \partial_t u\, dt + \sum_{i=1}^d \partial_i u\, dX_i + \frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d \partial_{ij}^2 u\, dX_i dX_j$

$= \partial_t u\, dt + \sum_{i=1}^d \partial_i u\, \sigma\, dW_t + \frac{1}{2}\sum_{k,i,j} \sigma_{ik}\sigma_{jk}\partial_{ij}^2 u\, dt$.

$\Rightarrow u(t+\Delta t, X(t+\Delta t)) - u(t,x) = \int_t^{t+\Delta t} (\partial_t u + \frac{1}{2}\sum_{k,i,j}^d \sigma_{ik}\sigma_{jk}\partial_{ij}^2 u)\, dt$

代入 (*) 得 $u(t,x) \leq u(t,x) + \mathbb{E}_{t,x}[\int_t^{t+\Delta t} (\partial_t u + \frac{1}{2}\sum_{k,i,j}^d \sigma_{ik}\sigma_{jk}\partial_{ij}^2 u)\, dt]$

$\Rightarrow \mathbb{E}_{t,x}[\int_t^{t+\Delta t} (\partial_t u + \frac{1}{2}\sum \sigma_{ik}\sigma_{jk}\partial_{ij}^2 u)\, dt] \geq 0$.

令$\Delta t\to 0$. 得 $\partial_t u+\frac{1}{2}\sum \sigma_{ik}\sigma_{jk}\partial_{ij}^2 u\geq 0$.

设α^*最优. 即 $u(t,x)=J(t,x;\alpha^*)=\mathbb{E}_{t,x}\,g(X_T^{\alpha^*})$

$u(t,x)$满足某时间和Dirichlet问题. 则 $\partial_t u+\frac{1}{2}\sum\sigma_{ik}\sigma_{jk}(\alpha^*)\partial_{ij}^2 u=0$.

两边取$\inf\limits_{\alpha\in A}$得 $\partial_t u+\frac{1}{2}\inf\limits_{\alpha\in A}\sigma_{ik}\sigma_{jk}\partial_{ij}^2 u\leq 0$.

故 $\partial_t u+\frac{1}{2}\inf\limits_{\alpha\in A}\sigma_{ik}\sigma_{jk}\partial_{ij}^2 u=0$.

$u(T)=\inf\limits_{\alpha\in A}J(T,x;\alpha)=\mathbb{E}_{T,x}\,g(X_T^{\alpha})=g$

在一定条件下 th 1.2.3 可推广至一般适应过程.