1. A simple Version of Courrège Theorem:

Assume that $\mathcal{L}$ is a local operator on $C_c^\infty(\mathbb{R}^d)$ satisfying the positive maximal principle, then

$$\mathcal{L}\varphi(x) = \frac{1}{2}a_{ij}(x)\,\partial_{ij}\varphi(x) + b_i(x)\,\partial_i\varphi(x) - c(x)\,\varphi(x).$$

Here $a : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $a(x)$ is a nonnegative definite symmetric matrix;

$b : \mathbb{R}^d \to \mathbb{R}^d$;

$c : \mathbb{R}^d \to [0,\infty)$.

Pf: • We first show that if $\varphi(0) = \nabla\varphi(0) = \nabla^2\varphi(0) = 0$, then $L\varphi(0) = 0$. Let $\eta(x) = \eta(|x|)$. $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is smooth and decreasing, and $\eta(r) = 1$ if $r \in [0,1]$, $\eta(r) = 0$ if $r > 2$.

Define $\varphi_R(x) := \varphi(x)\cdot\eta_R(x)$. Then for each $R \in (0,1)$,

$$|\varphi_R(x)| \le C|x|^3\,|\eta_R(x)| \le CR|x|^2\eta_R(x) \le Rf(x)$$

where $f(x) = C|x|^2\eta(x)$.

Thus $$RLf(0) \le L\varphi_R(0) = L\varphi(0).$$

$$L\varphi(0) \le L\varphi_R(0) \le -RLf(0).$$

Let $R \to 0$, one sees $L\varphi(0) = 0$.

• Now assume $\varphi \in C_c^\infty$. Let $f_0(x) = \eta(x)$, $f_i(x) = \partial_i\varphi(0)\,x_i\,\eta(x)$. $f_{ij}(x) = \partial_{ij}\varphi(0)\,x_i x_j\cdot\eta(x)$. Let $a_{ij}^0 = Lf_{ij}(0)$

$b_i^0 = Lf_i(0)$, $c^0 = -Lf_0(0) = -L\eta(0) \geq 0$.

Noting that $\phi = \varphi - \frac{1}{2}\sum_{i,j} f_{ij} - \sum_i f_i - f_0 \in C_c^\infty$,

$\phi(0) = \nabla\phi(0) = \nabla^2\phi(0) = 0$, by step 1, we get

$L\phi(0) = 0$, i.e.

$$L\varphi(0) = \frac{1}{2} a_{ij}^0 \partial_{ij}\varphi(0) + b_i^0 \partial_i \varphi(0) - c^0\varphi(0).$$

□.

2. We give a proof for C-Z lemma, phrased in martingale language.

Pf: We can assume that f is supported on $Q = [0,1]^d$ and $\lambda > \int_Q f$. Consider the dyadic martingale $f_n = \mathbb{E}(f \mid \mathcal{F}_n)$, where $\mathcal{F}_n$ is the partition of Q into 2^{dn} equal cubes. Let $T = \inf\{n : f_n > \lambda\}$.

One sees $\{x : T(x) = n\}$ is the union of some cubes in $\mathcal{F}_n$,

Say $\{T = n\} = \bigcup_{i \le K_n} Q_{i,n}$, $(K_n \le 2^{dn})$.

Let $Q'_{i,n}$ be the element of $\mathcal{F}_{n-1}$ containing $Q_{i,n}$, then

$$\lambda < \frac{1}{|Q_{i,n}|}\int_{Q_{i,n}} f \le 2^d \cdot \frac{1}{|Q'_{i,n}|}\int_{Q'_{i,n}} f \le 2^d \lambda.$$

Thus, $\{T < \infty\} = \bigcup Q_i$ and

$$\lambda < \fint_{Q_i} f \le 2^d \lambda.$$

For any $x \in \{T = \infty\}$, by definition, $\sup_n f_n(x) \le \lambda$.

martingale convergence Thm implies

$$f(x) = \lim_n f_n(x) \le \lambda, \quad \text{a.e. } x \in \{T = \infty\}$$

□.

3. To prove support thm, we need some lemmas. We only prove the one-dimensional case.

Lemma. Let W be a B.M. Then for any $\lambda, t > 0$,

$$\mathbb{P}\Big(\sup_{s\in[0,t]} |W_s| \le \lambda \Big) \geq 1 - 2e^{-\frac{\lambda^2}{2t}}$$

Proof. Consider $X_t = e^{a|W_t|}$. By Doob's inequality

$$\mathbb{P}\Big(\sup_{s\in[0,t]} |W_s| > \lambda \Big) = \mathbb{P}\Big(\sup_{s\in[0,t]} X_s > e^{a\lambda} \Big)$$

$$\le e^{-a\lambda}\, \mathbb{E} e^{a|W_t|} \le 2e^{-a\lambda} e^{a^2 t/2}$$

By taking $a = \lambda/t$, we get

$$\mathbb{P}\Big(\sup_{s\in[0,t]} |W_s| > \lambda \Big) \le 2e^{-\frac{\lambda^2}{2t}}. \qquad \square$$

Lemma.

$$\mathbb{P}_0\Big(\sup_{t\in[0,1]} |W_t| \le \varepsilon \Big) \geq C(\varepsilon) > 0.$$

Proof. By the scaling property of B.M. ($\frac{1}{\varepsilon} W_t \overset{d}{=} W_{t/\varepsilon^2}$) we need to show

$$\mathbb{P}_0\Big(\sup_{t\in[0,\varepsilon^{-2}]} |W_t| \le 1 \Big) \geq C(\varepsilon).$$

Claim: a). $\inf_{|x|\le 1/3} \mathbb{P}_x\big(|W_\delta| \le \frac{1}{3} \big) \geq \frac{1}{3}$, for any $\delta \ll 1$.

b). $\inf_{|x|\le 1/3} \mathbb{P}_x\Big(\sup_{t\in[0,\delta]} |W_t| \le 1 \Big) \geq \frac{5}{6}$

Thus, $\inf_{|x|\le 1/3} \mathbb{P}_x\Big(\sup_{t\in[0,\delta]} |W_t| \le 1,\ |W_\delta| \le \frac{1}{3} \Big) \geq \frac{1}{2}$

By the Markov property of w,

$$\inf_{|x|\le 1/3} \mathbb{P}_x\Big(\sup_{t\in[0,k\delta]} |W_t| \le 1 \Big) \ge \Big(\frac{1}{2}\Big)^k .$$

This implies

$$\mathbb{P}_0\Big(\sup_{t\in[0,\varepsilon^{-2}]} |W_t| \le 1 \Big) \ge \Big(\frac{1}{2}\Big)^{\varepsilon^{-2}\delta^{-1}+1}$$

$\square$.

Proof of Thm:

Step 1. Assume $x=\varphi=b=0$. Let $\tau(t) = \inf\{u : \int_0^u \sigma^2(X_s)ds > t\}$.

Then $Y_t = X_{\tau(t)}$ is a B.M. Noting that $K^{-1} \le \tau(t)/t \le K$, one sees

$$P\Big(\sup_{t\in[0,1]} |X_t| \le \varepsilon \Big) \ge P\Big(\sup_{t\in[0,K]} |Y_t| \le \varepsilon\Big) > 0 .$$

Step 2. Assume $x=0$, $\varphi\in C^1$.

Define p.m. Q by

$$\frac{dQ}{dP} = \exp\Big(\int_0^1 (-b(X_s)+\varphi'(s))\,\sigma^{-1}(X_s)\,dW_s - \frac{1}{2}\int_0^1 [(-b(X_s)+\varphi'(s))\,\sigma^{-1}(X_s)]^2 ds \Big).$$

Let $M_t = \int_0^t \sigma(X_s)dW_s$. By Girsanov's Thm, under Q

$$X_t - \varphi(t) = M_t - \Big(-\int_0^t b(X_s)ds + \varphi(t)\Big) \quad \text{is}$$

a martingale with quadratic variation $\frac{1}{2}\int_0^t \sigma^2(X_s)ds$.

By step 1.
$$Q\Big(\sup_{t\in[0,1]} |X_t - \varphi(t)| \le \varepsilon \Big) > 0 .$$

Since $Q << P$, we ~~got~~

$$P\left(\sup_{t\in[0,1]} |X_t - \varphi(t)| \le \varepsilon \right) > 0 \quad . \qquad (*)$$

Step 3. For $\varphi \in C^0$. We can find a function $\phi \in C^1$ and $\sup_{t\in[0,1]} |\phi(t) - \varphi(t)| \le \frac{\varepsilon}{2}$. Thus, $(*)$ holds for any $\varphi \in C^0$. $\square$.