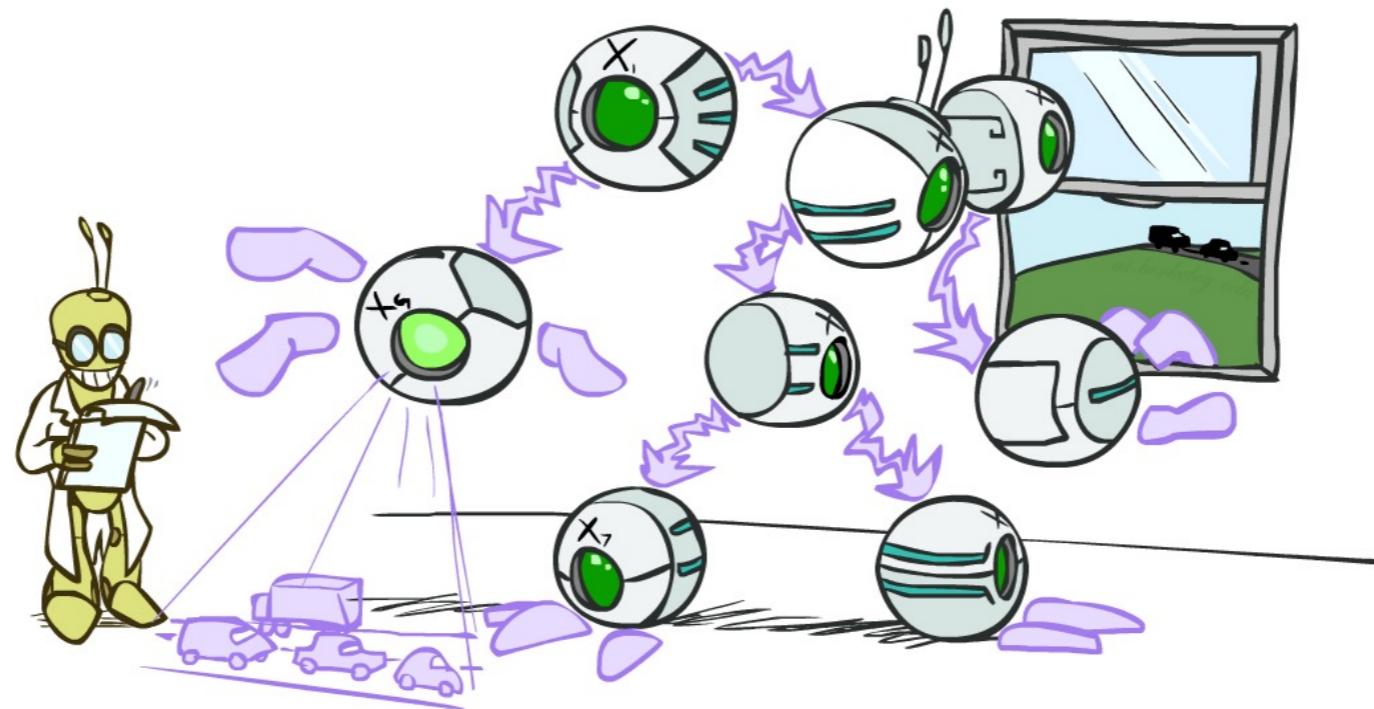


# Ve492: Introduction to Artificial Intelligence

## Bayesian Networks: Inference



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Slides adapted from <http://ai.berkeley.edu>, AIMA, UM, CMU

# Bayes' Nets

✓ Representation

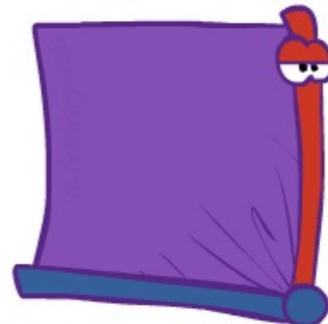
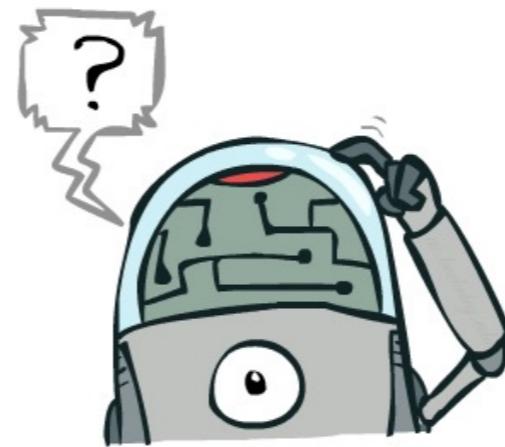
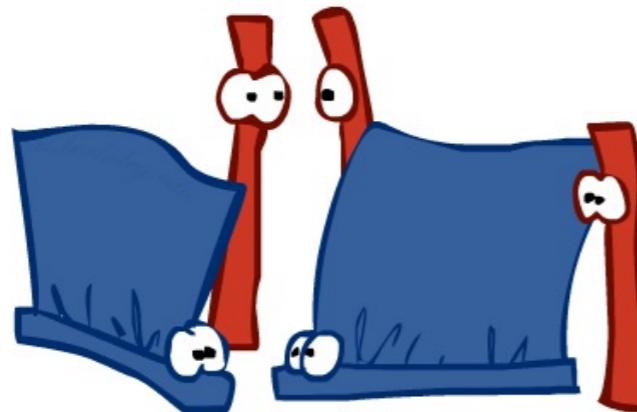
✓ Conditional Independences

## ❖ Probabilistic Inference

- ❖ Enumeration (exact, exponential complexity)
- ❖ Variable elimination (exact, worst-case exponential complexity, often better)
- ❖ Probabilistic inference is NP-complete
- ❖ Approximate inference (sampling)

# Inference

- ❖ Inference: calculating some useful quantity from a joint probability distribution
- ❖ Examples:
  - ❖ Marginal probability  $P(Q)$
  - ❖ Posterior probability  $P(Q|E = e)$
  - ❖ Most likely explanation  $\text{argmax}_q P(Q = q|E = e)$



# Inference by Enumeration

- General case:

- Evidence variables:  $E_1 \dots E_k = e_1 \dots e_k$
- Query\* variable:  $Q$
- Hidden variables:  $H_1 \dots H_r$

$X_1, X_2, \dots, X_n$   
All variables

- We have the joint and we want:

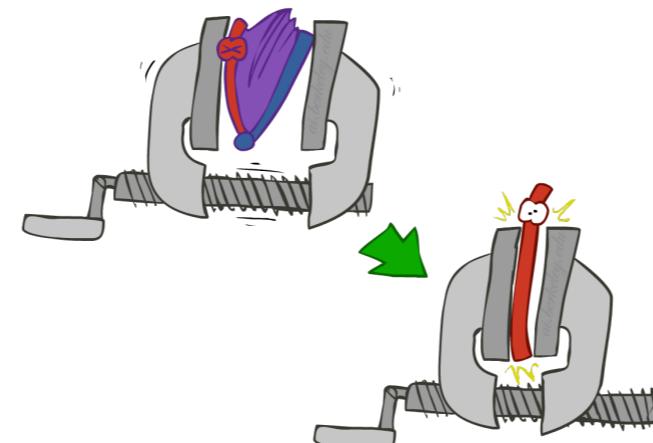
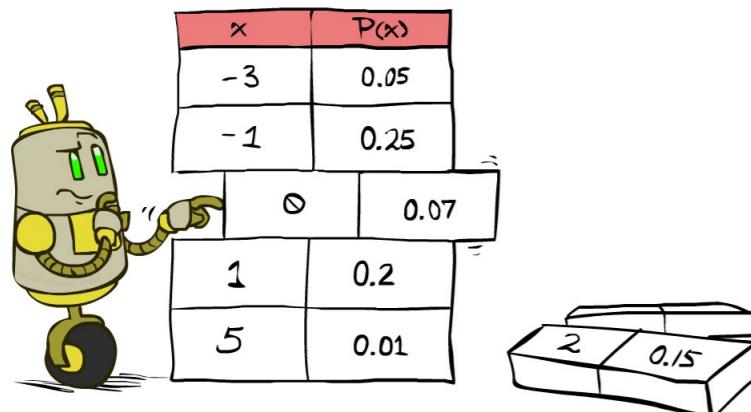
$$P(Q|e_1 \dots e_k)$$

\* Works fine with multiple query variables, too

- Step 1: Select the entries consistent with the evidence

- Step 2: Sum out  $H_1, \dots, H_r$  to get the joint of Q and evidence

- Step 3: Normalize



$$P(Q, e_1 \dots e_k) = \sum_{h_1 \dots h_r} \underbrace{P(Q, h_1 \dots h_r, e_1 \dots e_k)}_{X_1, X_2, \dots, X_n}$$

$$\times \frac{1}{Z}$$

$$Z = \sum_q P(Q, e_1 \dots e_k)$$

$$P(Q|e_1 \dots e_k) = \frac{1}{Z} P(Q, e_1 \dots e_k)$$

# Inference by Enumeration in Bayes' Net

- ❖ Given unlimited time, inference in BNs is easy
- ❖ Reminder of inference by enumeration by example:

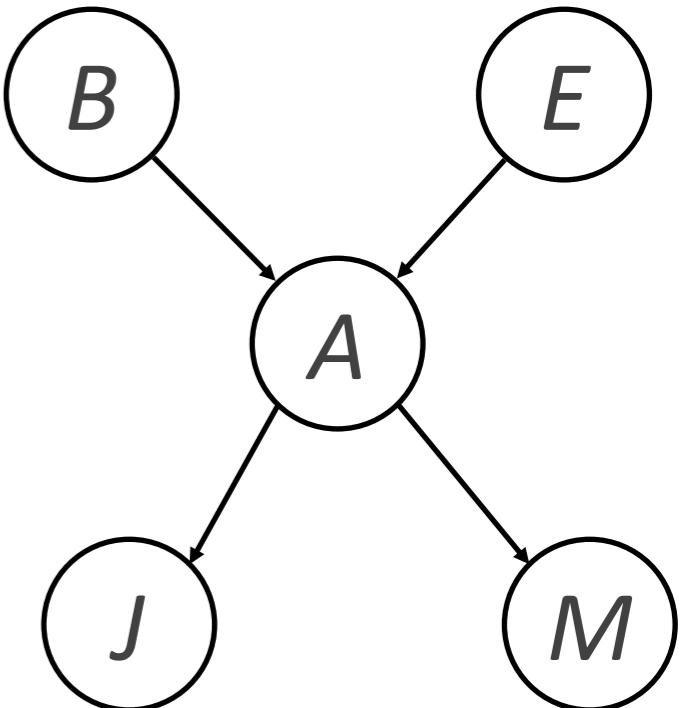
$$P(B \mid +j, +m) \propto_B P(B, +j, +m)$$

$$= \sum_{e,a} P(B, e, a, +j, +m)$$

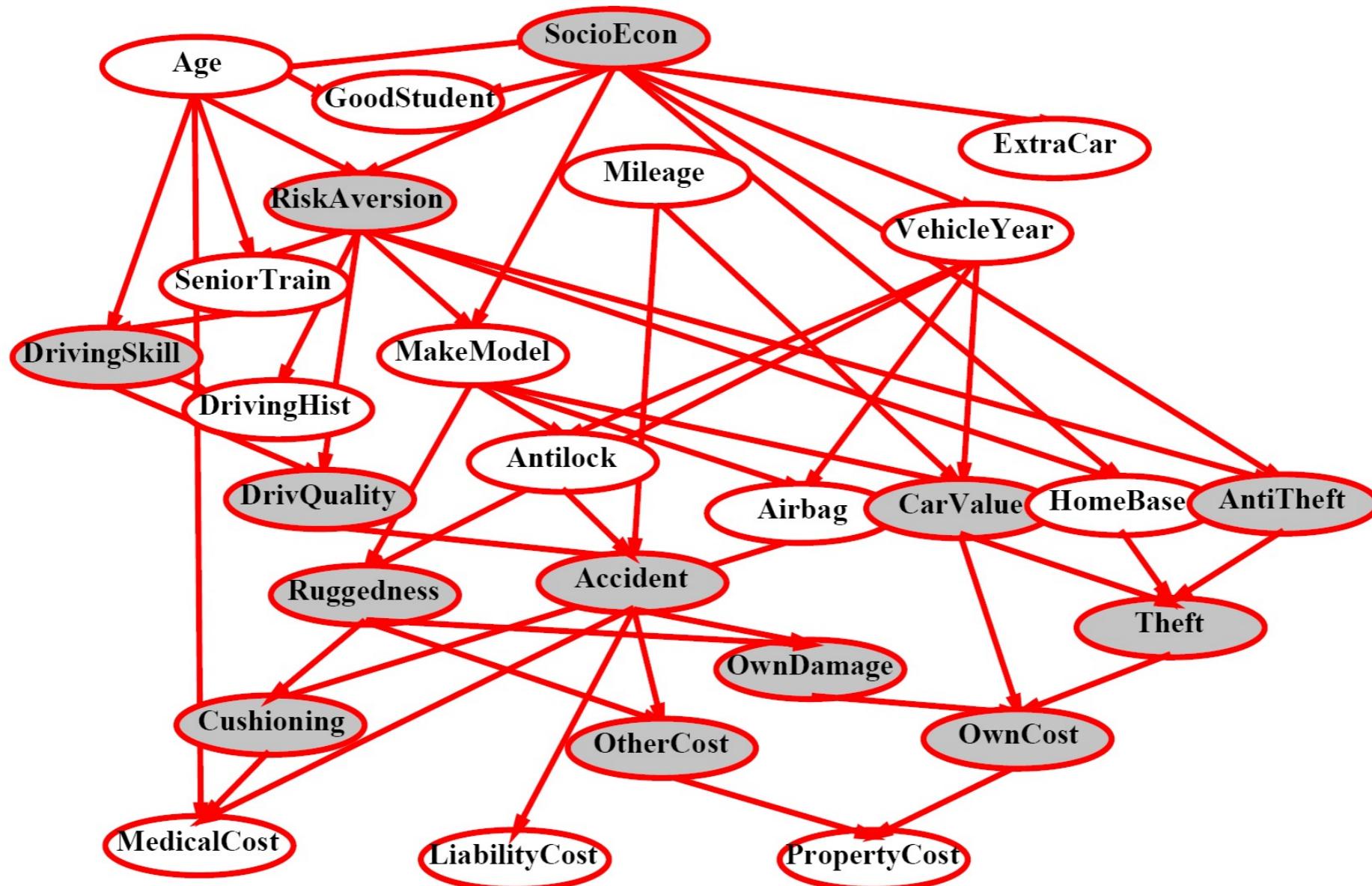
$$= \sum_{e,a} P(B)P(e)P(a|B, e)P(+j|a)P(+m|a)$$

$$= P(B)P(+e)P(+a|B, +e)P(+j|+a)P(+m|+a) + P(B)P(+e)P(-a|B, +e)P(+j|-a)P(+m|-a)$$

$$P(B)P(-e)P(+a|B, -e)P(+j|+a)P(+m|+a) + P(B)P(-e)P(-a|B, -e)P(+j|-a)P(+m|-a)$$



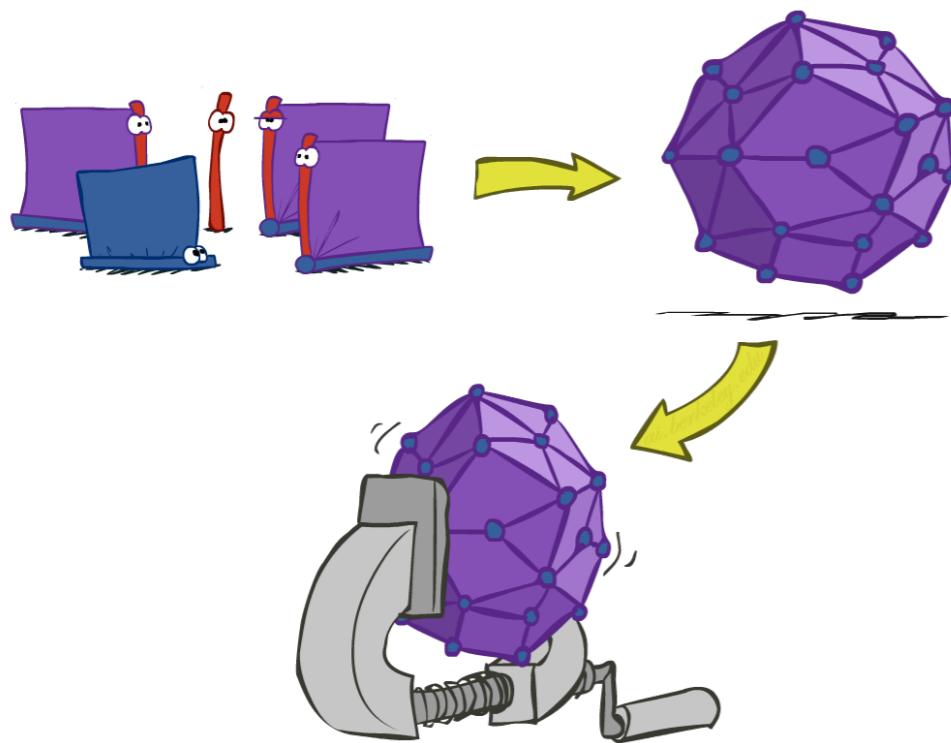
# Inference by Enumeration?



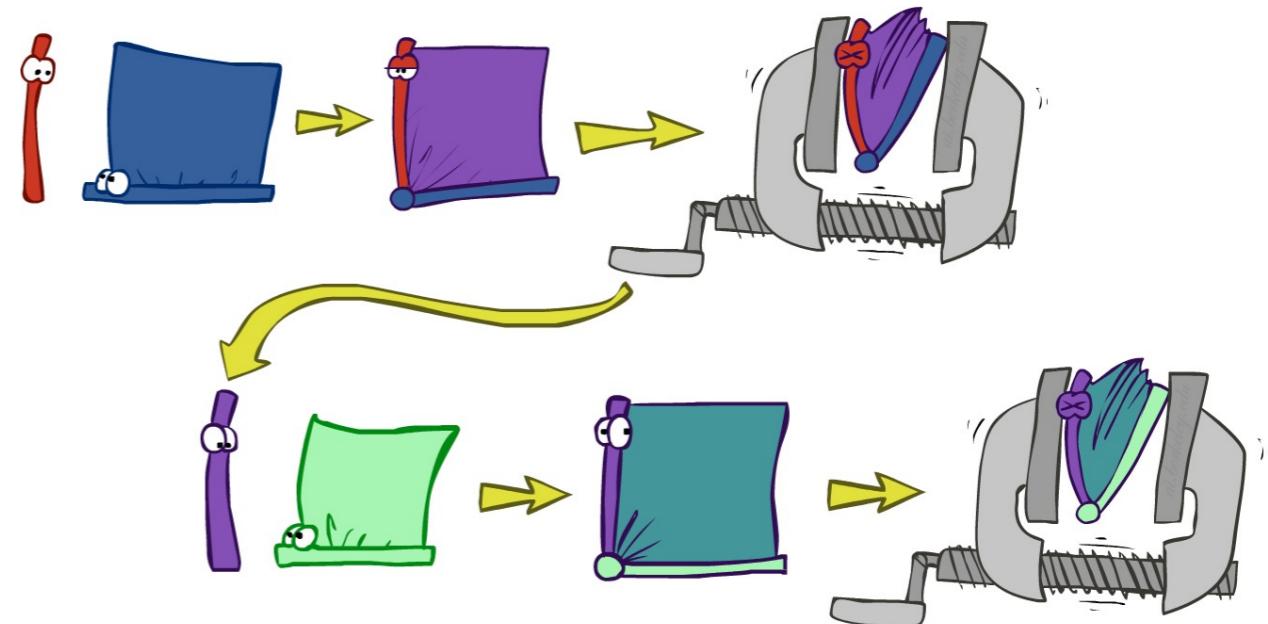
$P(\text{Antilock} | \text{observed variables}) = ?$

# Inference by Enumeration vs. Variable Elimination

- ❖ Why is inference by enumeration so slow?
  - ❖ You join up the whole joint distribution before you sum out the hidden variables

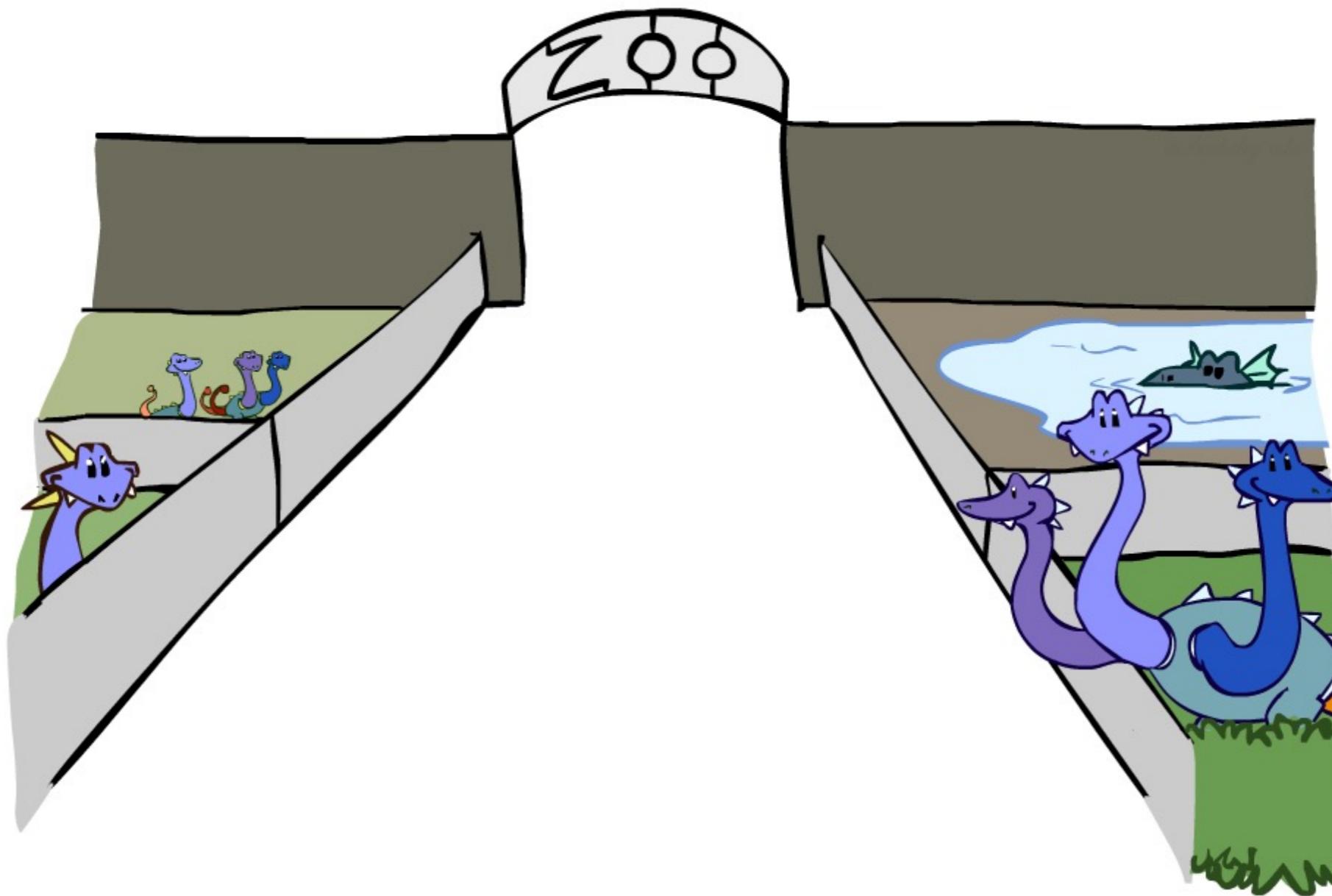


- ❖ Idea: interleave joining and marginalizing!
  - ❖ Called “Variable Elimination”
  - ❖ Still NP-hard, but usually much faster than inference by enumeration



- ❖ First we'll need some new notation: factors

# Factor Zoo



# Factor Zoo I

$P(T, W)$

- ❖ Joint distribution:  $P(X, Y)$

- ❖ Entries  $P(x, y)$  for all  $x, y$
- ❖ Sums to 1

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

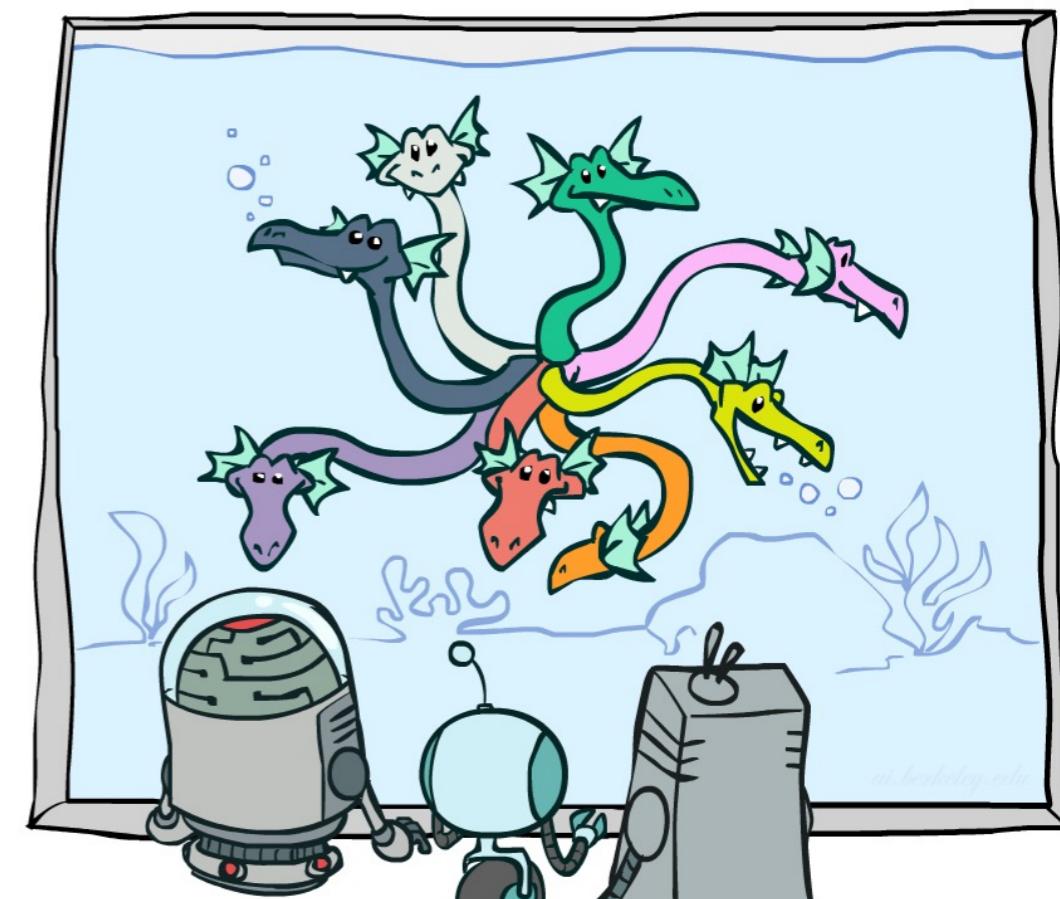
- ❖ Selected joint:  $P(x, Y)$

- ❖ A slice of the joint distribution
- ❖ Entries  $P(x, y)$  for fixed  $x$ , all  $y$
- ❖ Sums to  $P(x)$

$P(\text{cold}, W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

- ❖ Number of capitals = dimensionality of the table



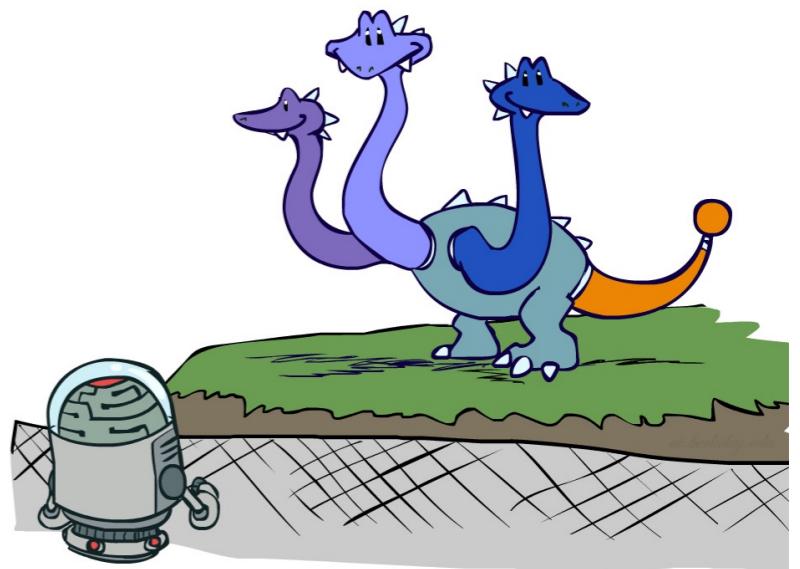
# Factor Zoo II

- ❖ Single conditional:  $P(Y | x)$

- ❖ Entries  $P(y | x)$  for fixed  $x$ , all  $y$
- ❖ Sums to 1

$P(W|cold)$

T	W	P
cold	sun	0.4
cold	rain	0.6

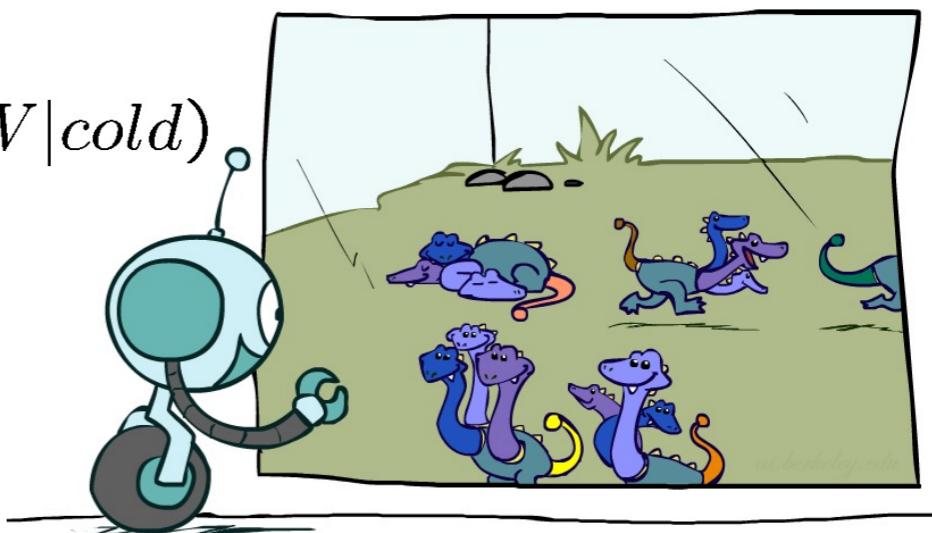


$P(W|T)$

T	W	P
hot	sun	0.8
hot	rain	0.2
cold	sun	0.4
cold	rain	0.6

$P(W|hot)$

$P(W|cold)$



# Factor Zoo III

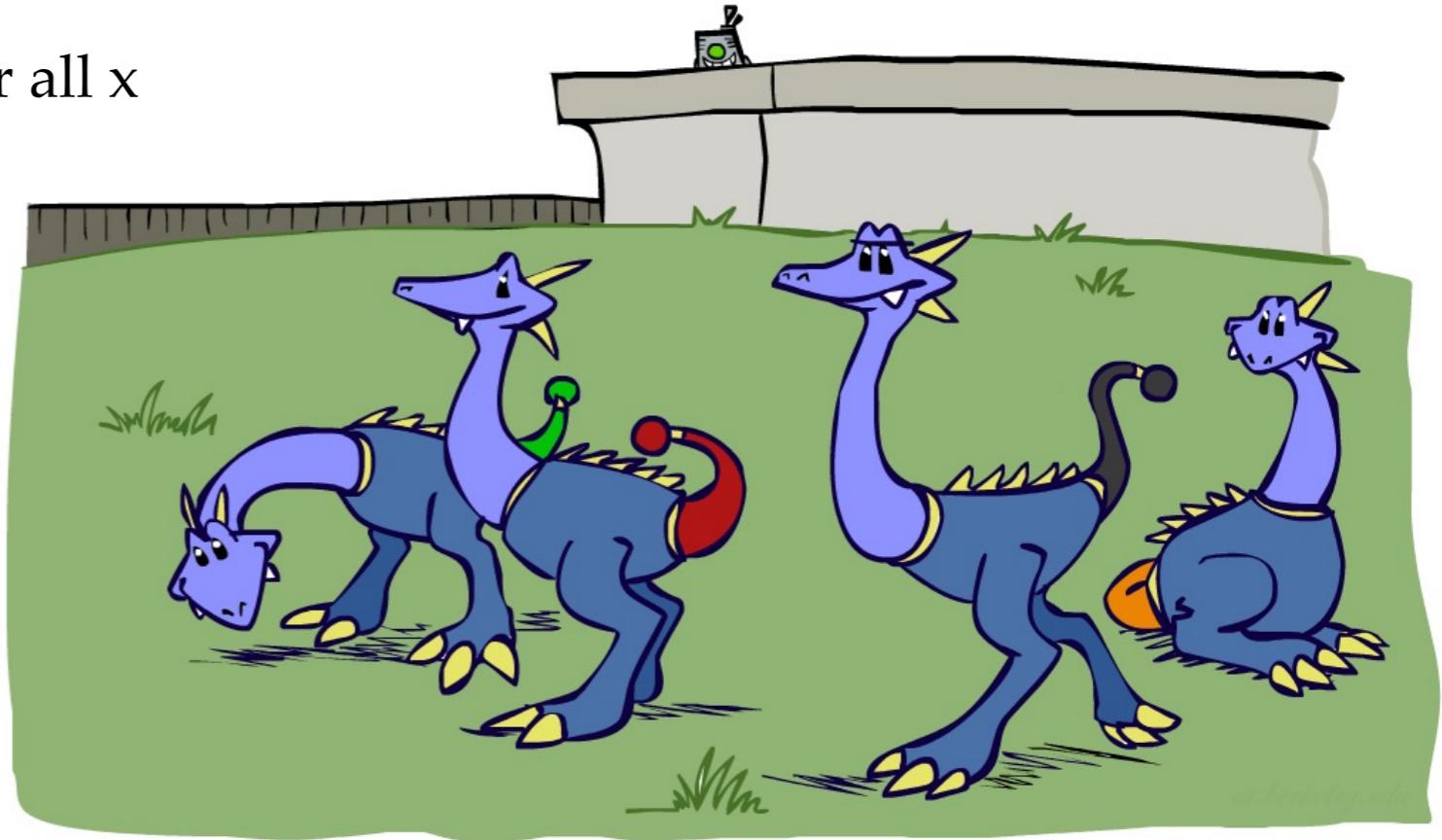
- ❖ Specified family:  $P(y | X)$

- ❖ Entries  $P(y | x)$  for fixed  $y$ , but for all  $x$
- ❖ Sums to ... who knows!

$P(rain|T)$

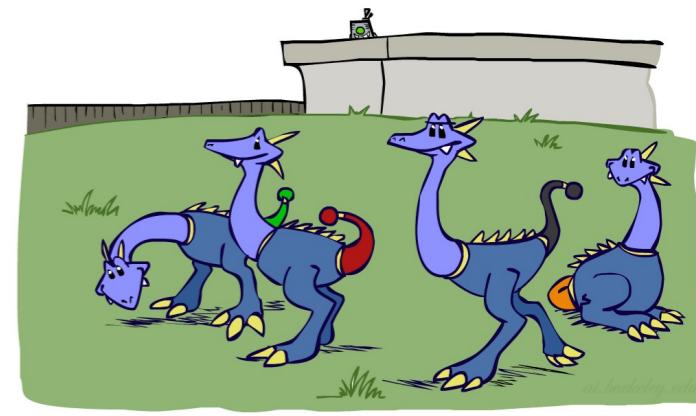
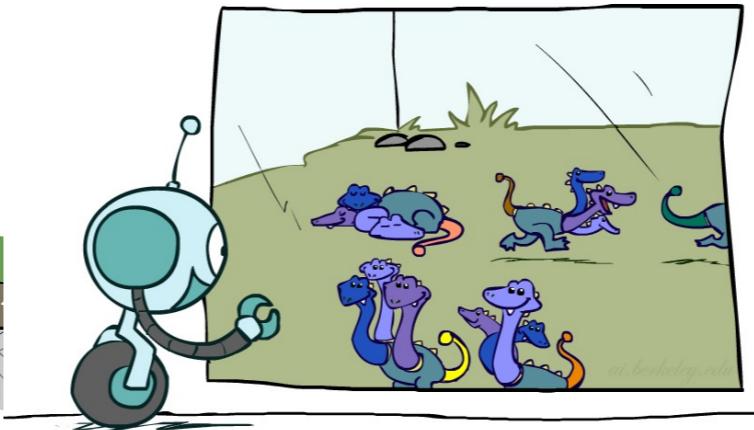
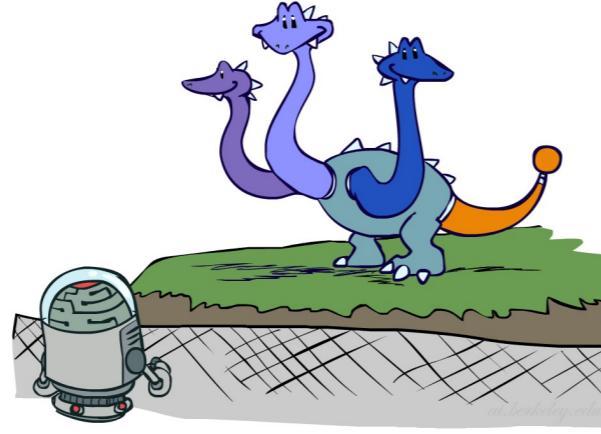
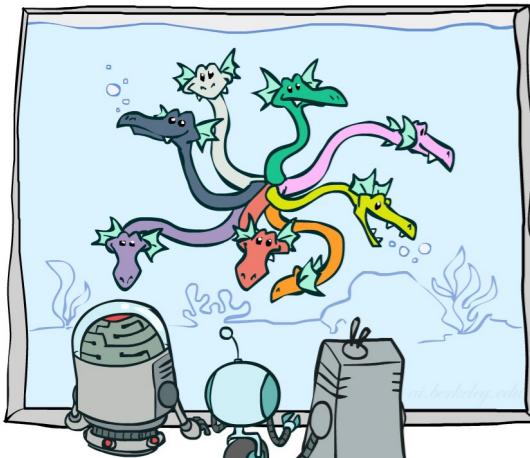
T	W	P
hot	rain	0.2
cold	rain	0.6

$$\left. \begin{array}{l} P(rain|hot) \\ P(rain|cold) \end{array} \right\}$$

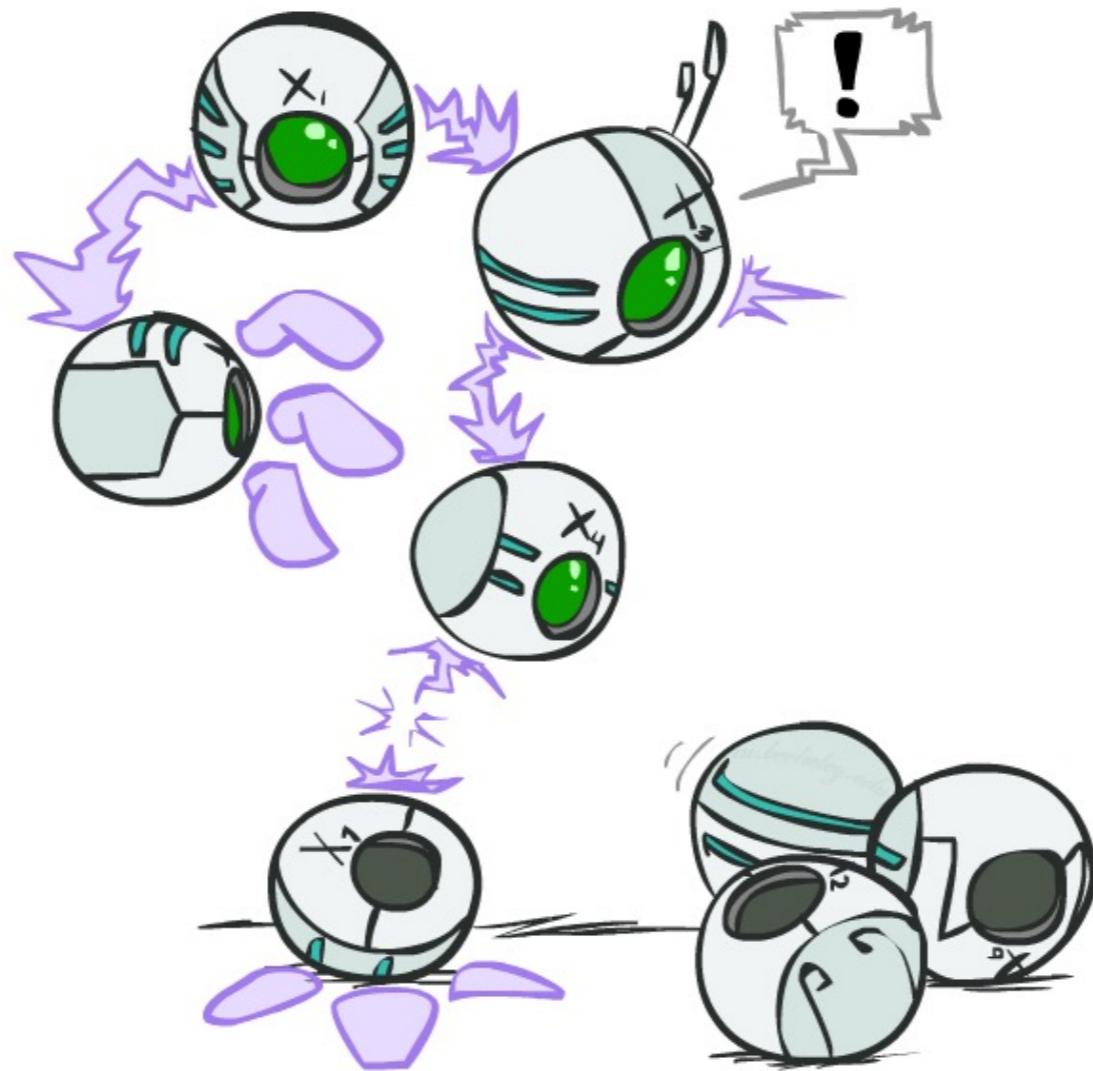


# Factor Zoo Summary

- ❖ In general, when we write  $P(Y_1 \dots Y_N | X_1 \dots X_M)$ 
  - ❖ It is a “factor,” a multi-dimensional array
  - ❖ Its values are  $P(y_1 \dots y_N | x_1 \dots x_M)$
  - ❖ Any assigned (=lower-case) X or Y is a dimension missing (selected) from the array



# Variable Elimination (VE)



# Example: Traffic Domain

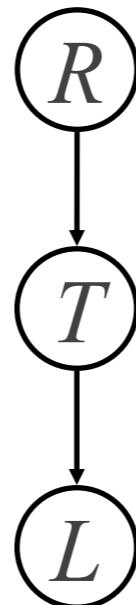
- ❖ Random Variables

- ❖ R: Raining
- ❖ T: Traffic
- ❖ L: Late for class!

$$P(L) = ?$$

$$= \sum_{r,t} P(r, t, L)$$

$$= \sum_{r,t} P(r)P(t|r)P(L|t)$$



$$P(R)$$

+r	0.1
-r	0.9

$$P(T|R)$$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$$P(L|T)$$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

# Inference by Enumeration: Procedural Outline

- ❖ Track objects called **factors** 
- ❖ Initial factors are local CPTs (one per node)

$$P(R)$$

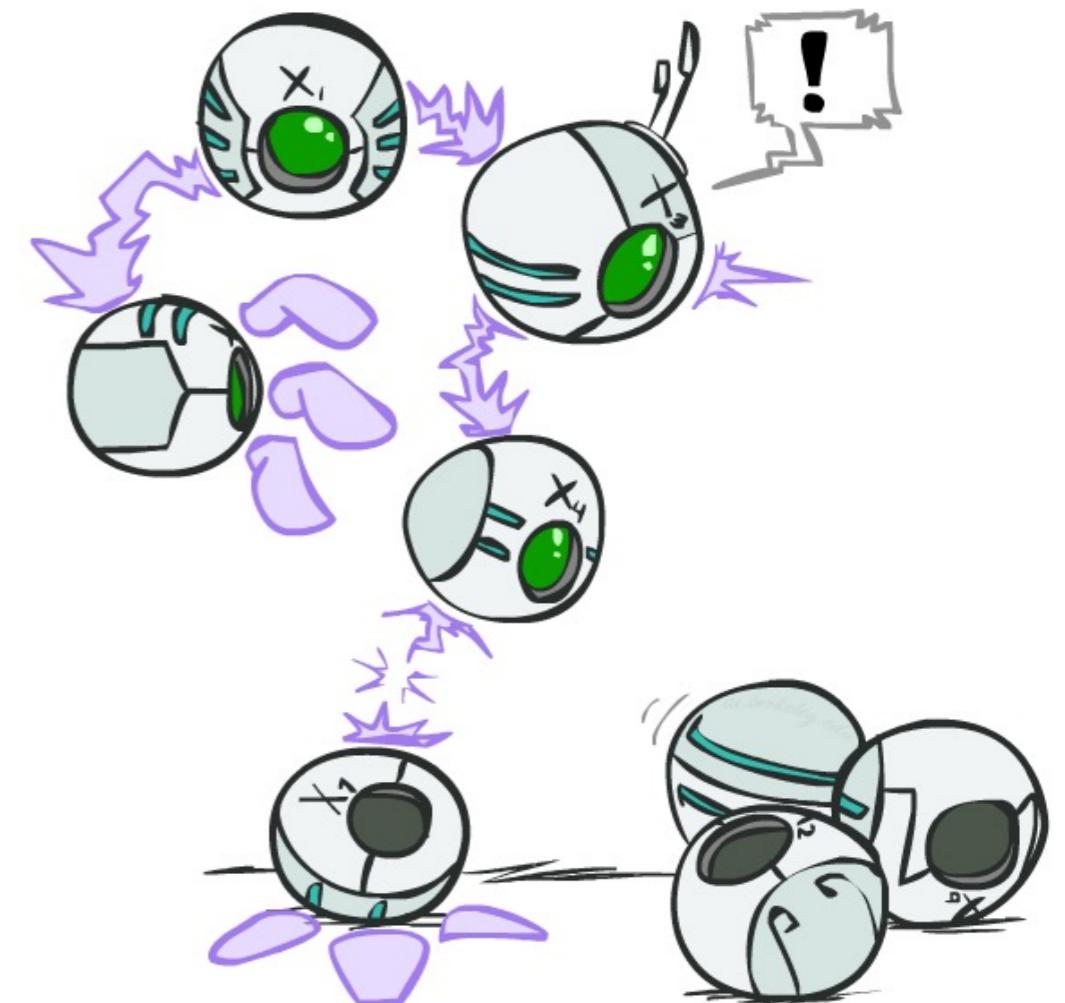
+r	0.1
-r	0.9

$$P(T|R)$$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$$P(L|T)$$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9



- ❖ Any known values are selected
  - ❖ E.g. if we know  $L = +\ell$ , the initial factors are

$$P(R)$$

+r	0.1
-r	0.9

$$P(T|R)$$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

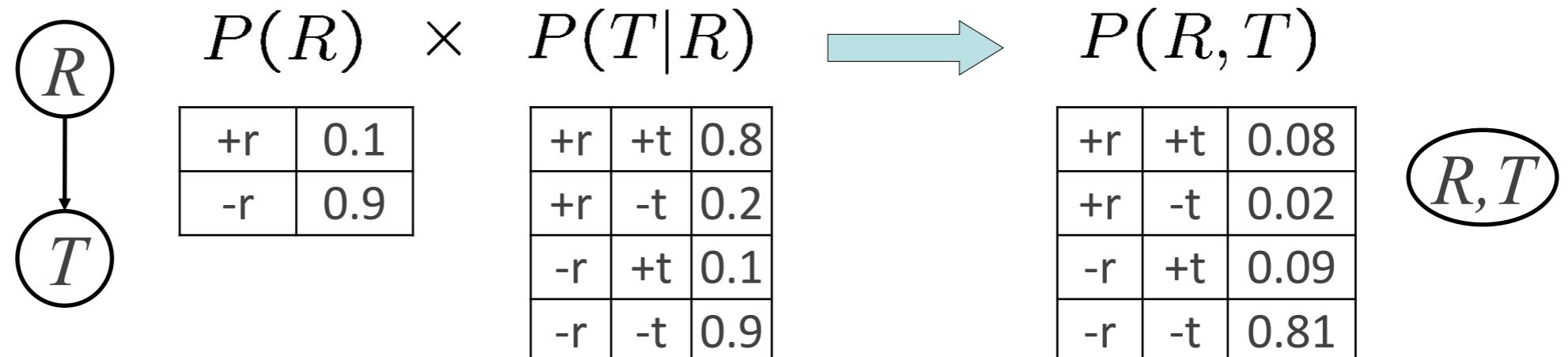
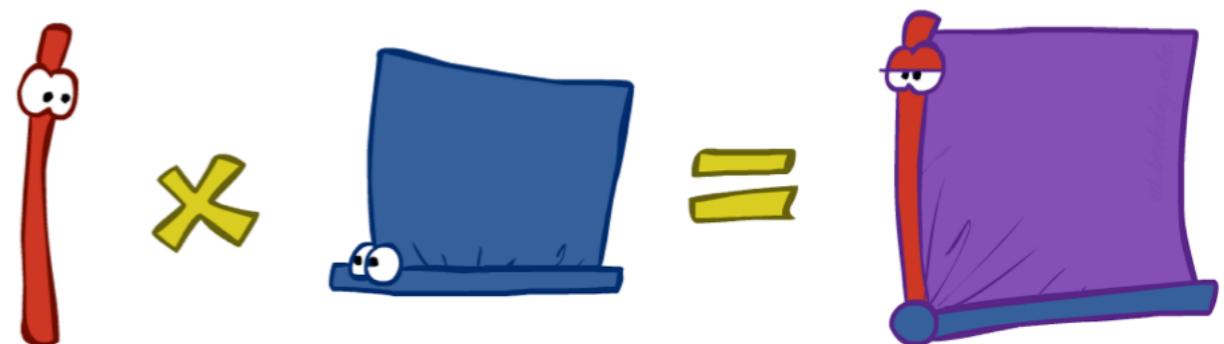
$$P(+\ell|T)$$

+t	+l	0.3
-t	+l	0.1

- ❖ Procedure: Join all factors, then eliminate all hidden variables

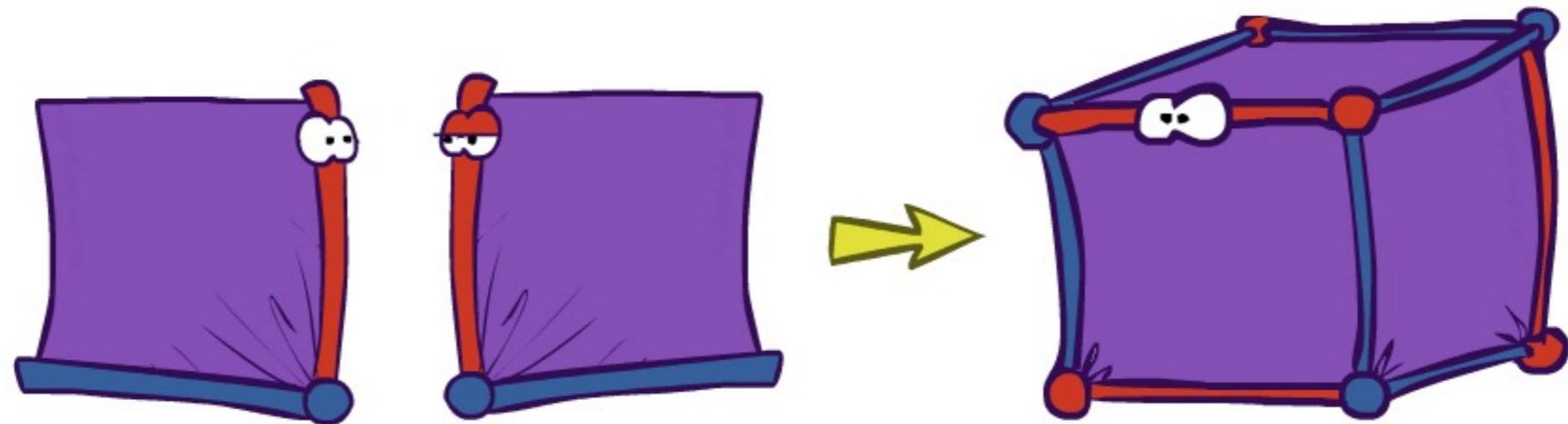
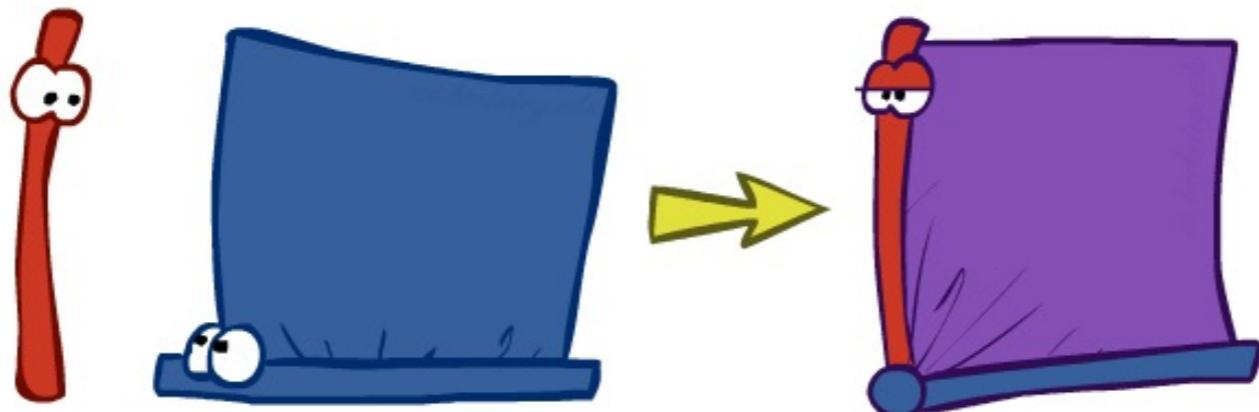
# Operation 1: Join Factors

- ❖ First basic operation: joining factors
- ❖ Combining factors:
  - ❖ Just like a database join
  - ❖ Get all factors over the joining variable
  - ❖ Build a new factor over the union of the variables involved
- ❖ Example: Join on R



- ❖ Computation for each entry: pointwise products  $\forall r, t : P(r, t) = P(r) \cdot P(t|r)$

# Example: Multiple Joins



# Example: Multiple Joins

$R$

 $P(R)$ 

+r	0.1
-r	0.9

$T$

 $P(T|R)$ 

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$L$

 $P(L|T)$ 

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Join R

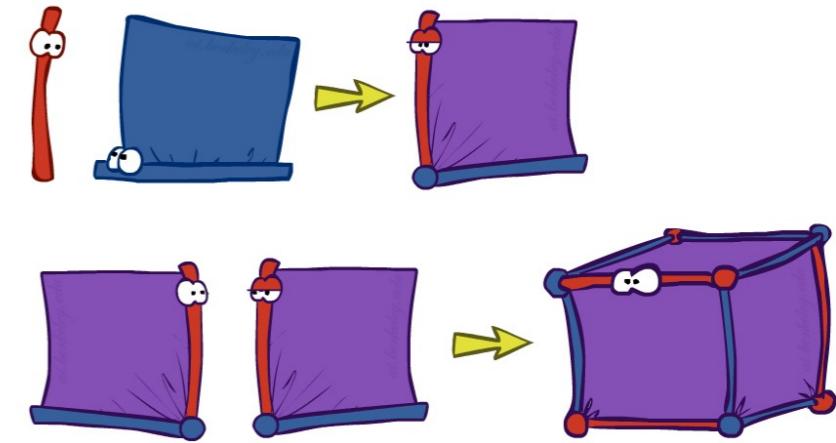
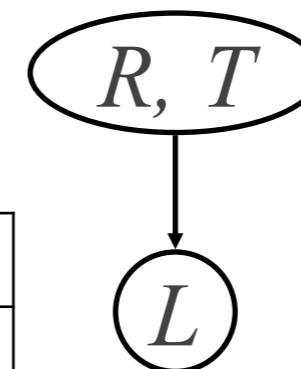
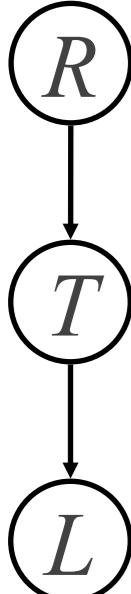
$P(R, T)$

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81

Join T

$P(R, T, L)$

+r	+t	+l	0.024
+r	+t	-l	0.056
+r	-t	+l	0.002
+r	-t	-l	0.018
-r	+t	+l	0.027
-r	+t	-l	0.063
-r	-t	+l	0.081
-r	-t	-l	0.729



$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

# Operation 2: Eliminate

- ❖ Second basic operation: **marginalization**
- ❖ Take a factor and sum out a variable
  - ❖ Shrinks a factor to a smaller one
  - ❖ A **projection** operation
- ❖ Example:

$$P(R, T)$$

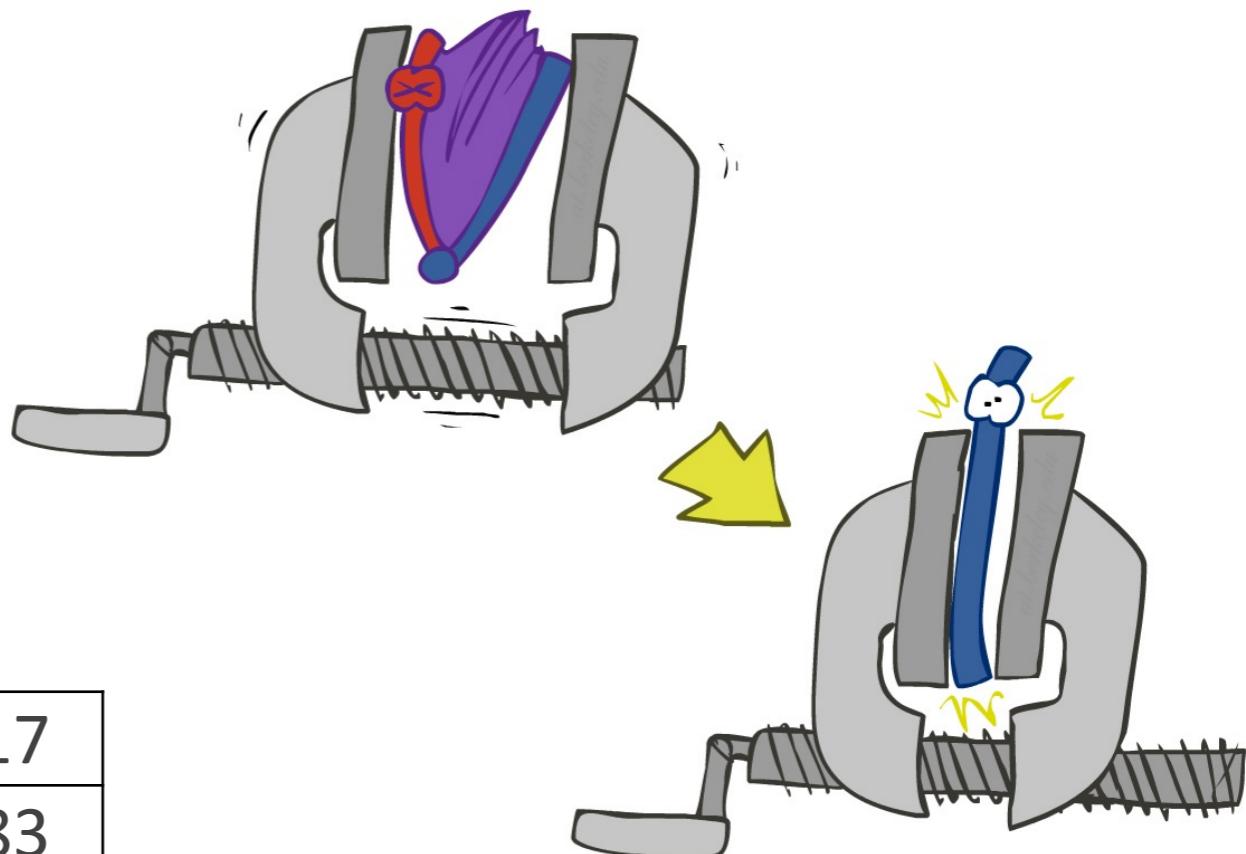
+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81

sum  $R$



$$P(T)$$

+t	0.17
-t	0.83



# Multiple Elimination

$P(R, T, L)$

$+r$	$+t$	$+l$	$P(R, T, L)$
$+r$	$+t$	$+l$	0.024
$+r$	$+t$	$-l$	0.056
$+r$	$-t$	$+l$	0.002
$+r$	$-t$	$-l$	0.018
$-r$	$+t$	$+l$	0.027
$-r$	$+t$	$-l$	0.063
$-r$	$-t$	$+l$	0.081
$-r$	$-t$	$-l$	0.729

Sum out R

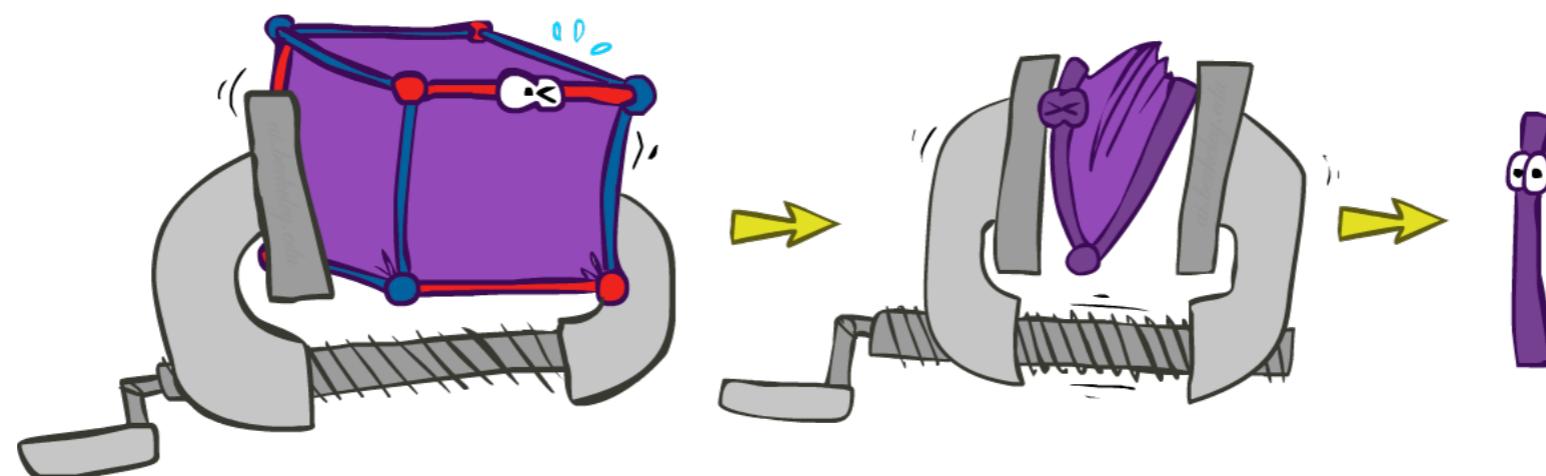
$P(T, L)$

$+t$	$+l$	$P(T, L)$
$+t$	$+l$	0.051
$+t$	$-l$	0.119
$-t$	$+l$	0.083
$-t$	$-l$	0.747

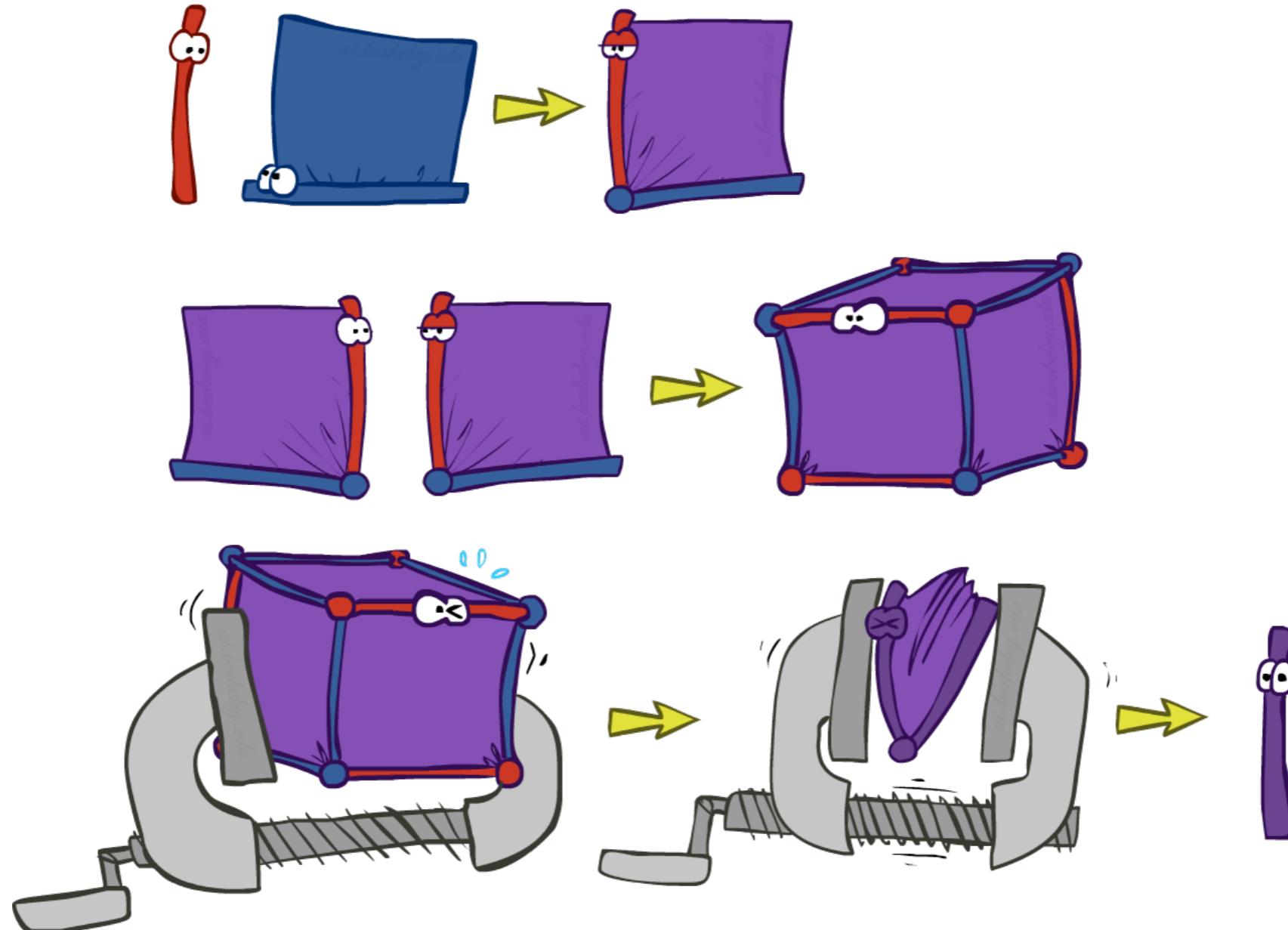
Sum out T

$P(L)$

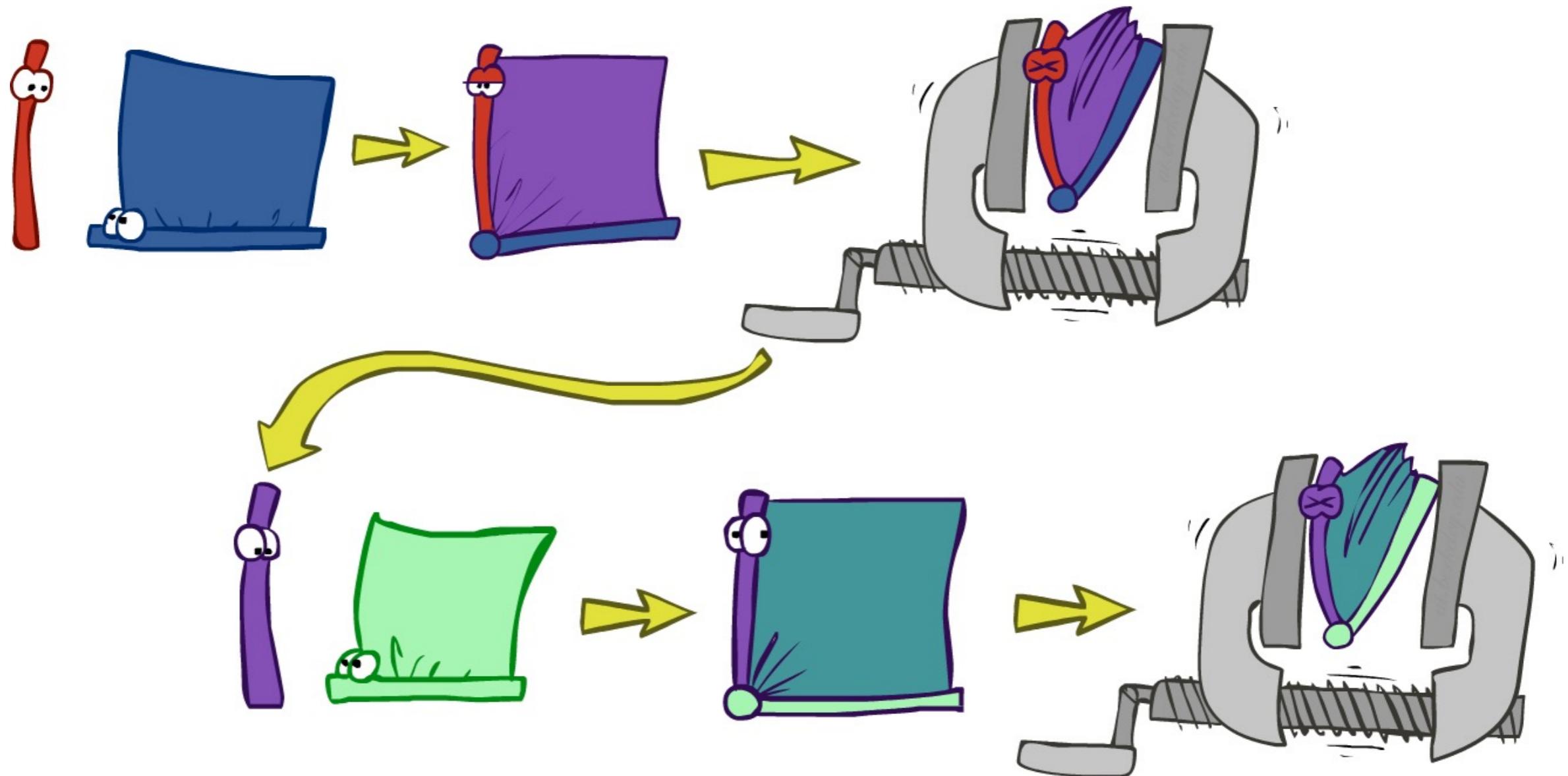
$+l$	$P(L)$
$+l$	0.134
$-l$	0.886



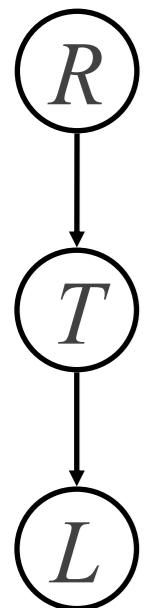
# Thus Far: Multiple Join, Multiple Eliminate (= Inference by Enumeration)



# Marginalizing Early (= Variable Elimination)



# Traffic Domain



$$P(L) = ?$$

❖ Inference by Enumeration

$$= \sum_t \sum_r P(L|t) P(r) \underbrace{P(t|r)}_{\text{Join on } r}$$

$\underbrace{\phantom{P(t|r)}}_{\text{Join on } t}$

$\underbrace{\phantom{P(t|r)}}_{\text{Eliminate } r}$

$\underbrace{\phantom{P(t|r)}}_{\text{Eliminate } t}$

❖ Variable Elimination

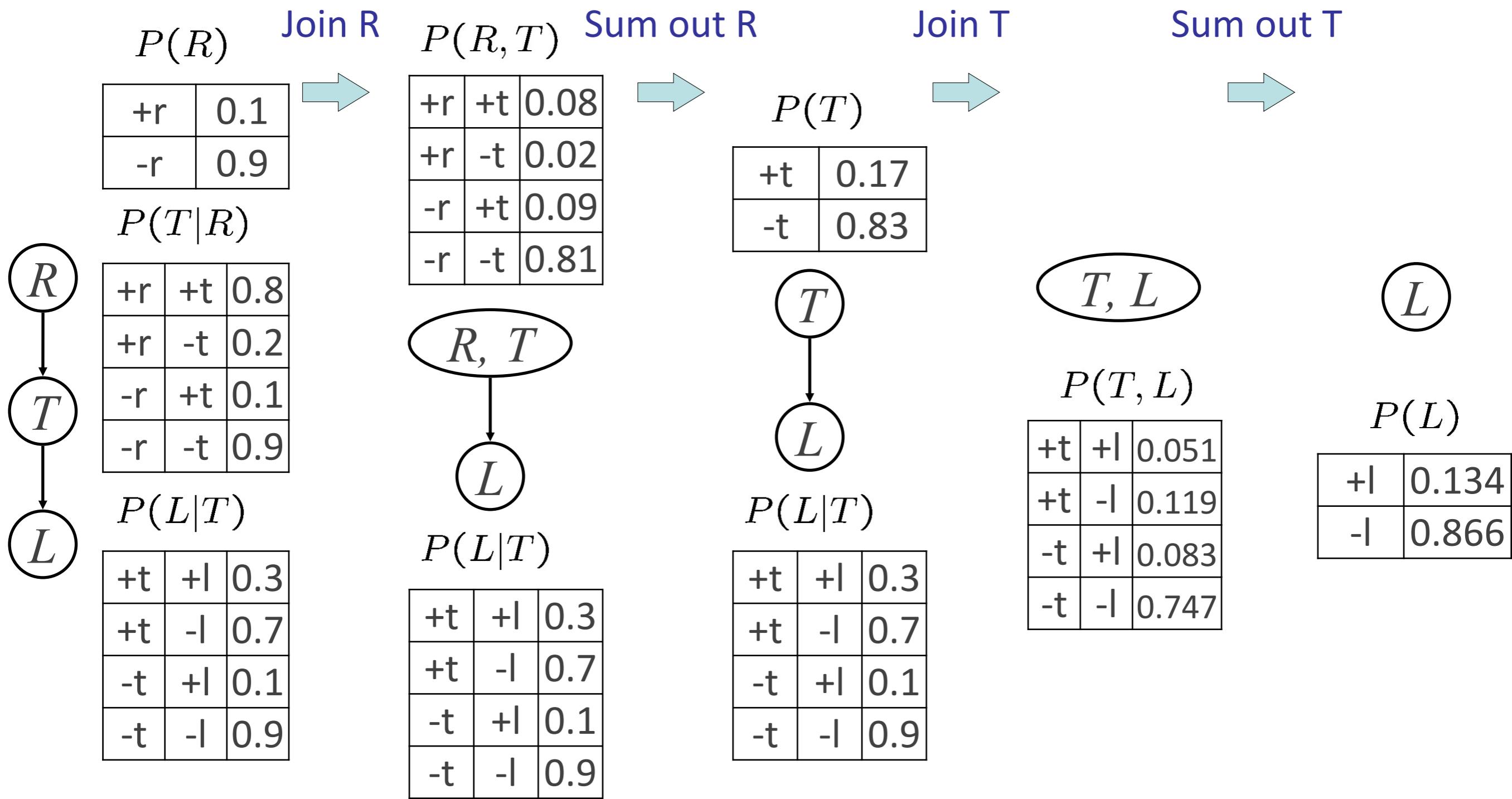
$$= \sum_t P(L|t) \sum_r \underbrace{P(r) P(t|r)}_{\text{Join on } r}$$

$\underbrace{\phantom{P(r) P(t|r)}}_{\text{Eliminate } r}$

$\underbrace{\phantom{P(r) P(t|r)}}_{\text{Join on } t}$

$\underbrace{\phantom{P(r) P(t|r)}}_{\text{Eliminate } t}$

# Marginalizing Early! (aka VE)



# Evidence

- ❖ If evidence, start with factors that select that evidence
  - ❖ No evidence uses these initial factors:

$$P(R)$$

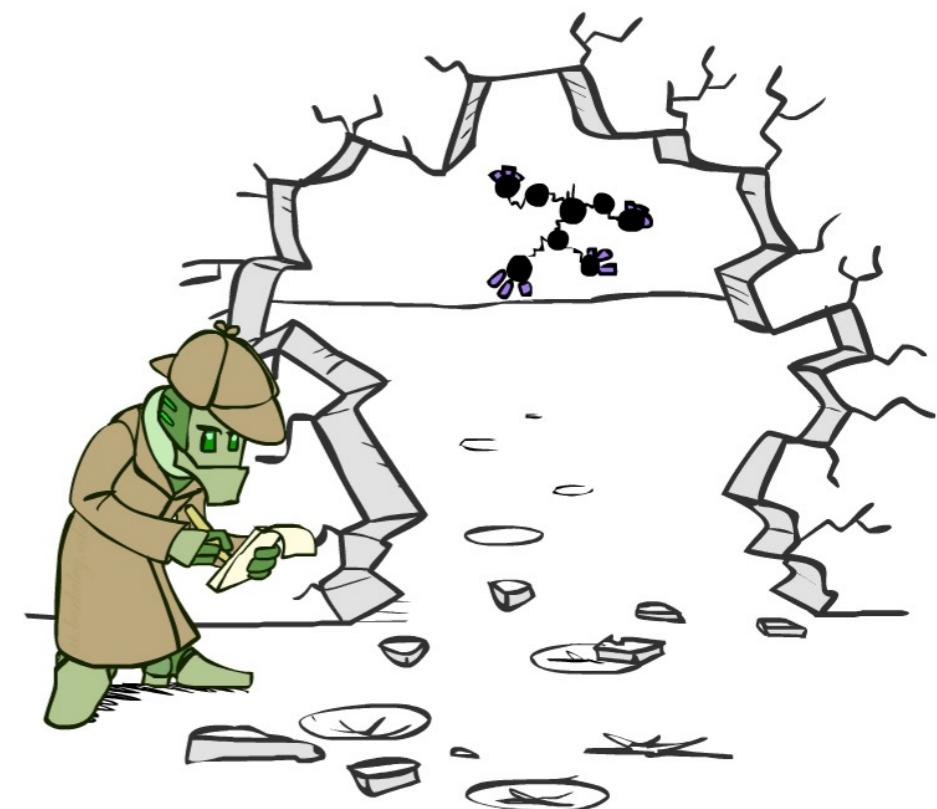
+r	0.1
-r	0.9

$$P(T|R)$$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

$$P(L|T)$$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9



- ❖ Computing  $P(L|+r)$ , the initial factors become:

$$P(+r)$$

+r	0.1
----	-----

$$P(T|+r)$$

+r	+t	0.8
+r	-t	0.2

$$P(L|T)$$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

- ❖ We eliminate all vars other than query + evidence

# Evidence ctd.

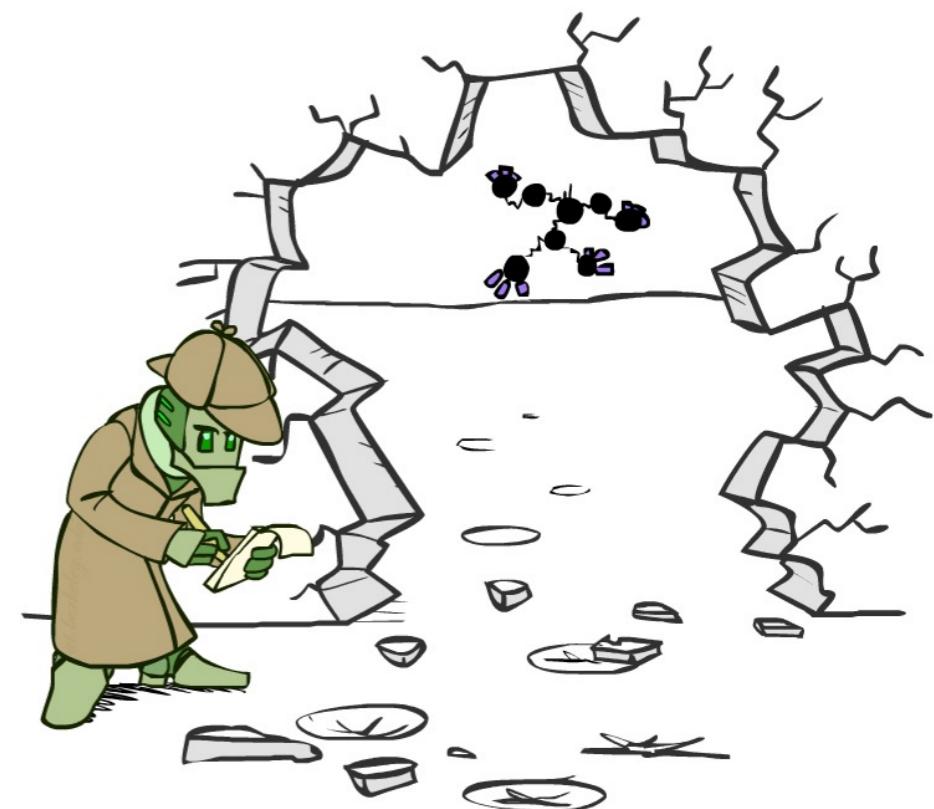
- ❖ Result will be a selected joint of query and evidence
  - ❖ E.g. for  $P(L | +r)$ , we would end up with:

$P(+r, L)$       Normalize       $P(L | +r)$



+r	+l	0.026
+r	-l	0.074

+l	0.26
-l	0.74



- ❖ How to get the answer?
  - ❖ Just normalize this!

# General Variable Elimination

- ❖ Query:  $P(Q|E_1 = e_1, \dots, E_k = e_k)$

- ❖ Start with initial factors:

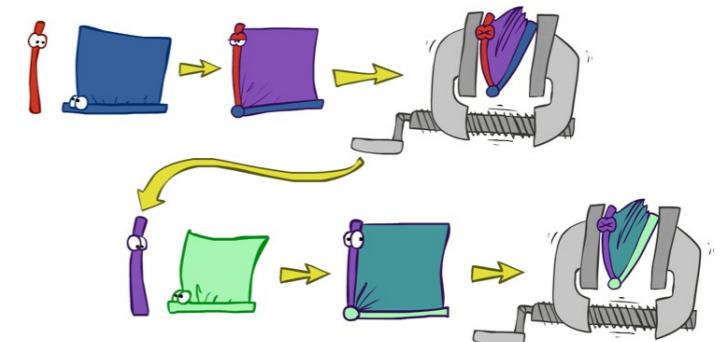
- ❖ Local CPTs (but instantiated by evidence)

A cartoon character with a yellow body and a grey head is holding a small table titled "P(x)". The table has columns for "x" and "P(x)". The rows contain values: -3 (0.05), -1 (0.25), 0 (0.07), 1 (0.2), 5 (0.01), and a row for 2 with a value of 0.15.

x	P(x)
-3	0.05
-1	0.25
0	0.07
1	0.2
5	0.01
2	0.15

- ❖ While there are still hidden variables (not Q or evidence):

- ❖ Pick a hidden variable H
  - ❖ Join all factors depending on H
  - ❖ Eliminate (sum out) H



- ❖ Join all remaining factors and normalize

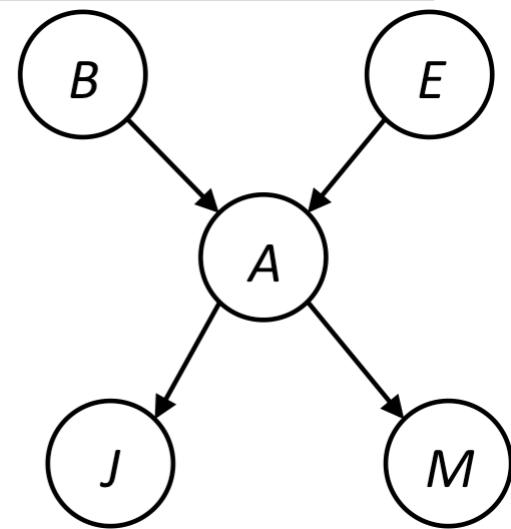
$$f \times \text{[blue card]} = \text{[purple card]} \times \frac{1}{Z}$$

# Example

$$P(B|j, m) \propto P(B, j, m)$$



$P(B)$	$P(E)$	$P(A B, E)$	$P(j A)$	$P(m A)$
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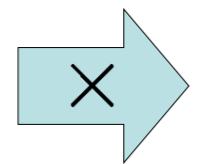


Choose A

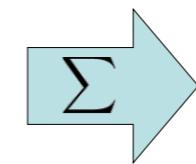
$$P(A|B, E)$$

$$P(j|A)$$

$$P(m|A)$$



$$P(j, m, A|B, E)$$



$$P(j, m|B, E)$$

$P(B)$	$P(E)$	$P(j, m B, E)$
--------	--------	----------------

# Example ctd.

$P(B)$	$P(E)$	$P(j, m B, E)$
--------	--------	----------------

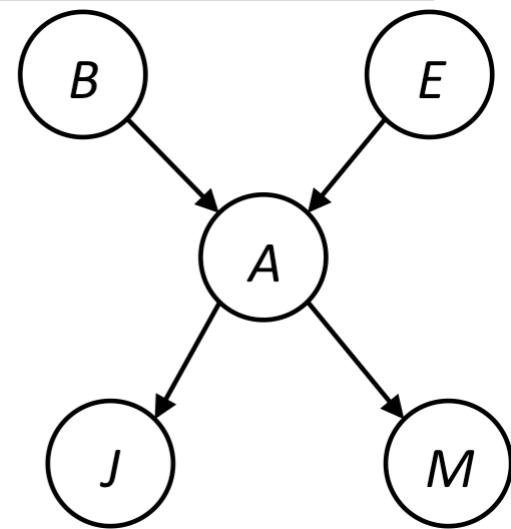
Choose E

$$\begin{array}{ccc} P(E) & \xrightarrow{\times} & P(j, m, E|B) \\ P(j, m|B, E) & & \xrightarrow{\sum} P(j, m|B) \end{array}$$

$P(B)$	$P(j, m B)$
--------	-------------

Finish with B

$$\begin{array}{ccccc} P(B) & \xrightarrow{\times} & P(j, m, B) & \xrightarrow{\text{Normalize}} & P(B|j, m) \\ P(j, m|B) & & & & \end{array}$$

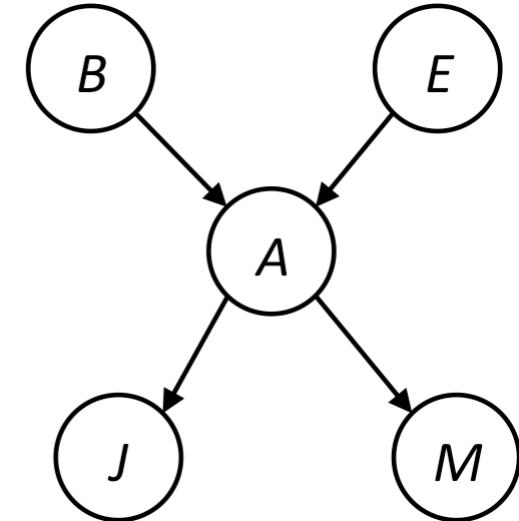


# Same Example in Equations

$$P(B|j, m) \propto P(B, j, m)$$

$$P(B) \quad P(E) \quad P(A|B, E) \quad P(j|A) \quad P(m|A)$$

$$\begin{aligned} P(B|j, m) &\propto P(B, j, m) \\ &= \sum_{e,a} P(B, j, m, e, a) \\ &= \sum_{e,a} P(B)P(e)P(a|B, e)P(j|a)P(m|a) \\ &= \sum_e P(B)P(e) \sum_a P(a|B, e)P(j|a)P(m|a) \\ &= \sum_e P(B)P(e)f_1(B, e, j, m) \\ &= P(B) \sum_e P(e)f_1(B, e, j, m) \\ &= P(B)f_2(B, j, m) \end{aligned}$$



- ❖ marginal can be obtained from joint by summing out
- ❖ use Bayes' net joint distribution expression
- ❖ use  $x^*(y+z) = xy + xz$
- ❖ joining on a, and then summing out gives  $f_1$
- ❖ use  $x^*(y+z) = xy + xz$
- ❖ joining on e, and then summing out gives  $f_2$

All we are doing is exploiting  $uw\bar{y} + uw\bar{z} + u\bar{x}\bar{y} + u\bar{x}\bar{z} + v\bar{w}\bar{y} + v\bar{w}\bar{z} + v\bar{x}\bar{y} + v\bar{x}\bar{z} = (u+v)(w+x)(y+z)$  to improve computational efficiency!

# Another Variable Elimination Example

Query:  $P(X_3|Y_1 = y_1, Y_2 = y_2, Y_3 = y_3)$

Start by inserting evidence, which gives the following initial factors:

$$p(Z)p(X_1|Z)p(X_2|Z)p(X_3|Z)p(y_1|X_1)p(y_2|X_2)p(y_3|X_3)$$

Eliminate  $X_1$ , this introduces the factor  $f_1(Z, y_1) = \sum_{x_1} p(x_1|Z)p(y_1|x_1)$ , and we are left with:

$$p(Z)f_1(Z, y_1)p(X_2|Z)p(X_3|Z)p(y_2|X_2)p(y_3|X_3)$$

Eliminate  $X_2$ , this introduces the factor  $f_2(Z, y_2) = \sum_{x_2} p(x_2|Z)p(y_2|x_2)$ , and we are left with:

$$p(Z)f_1(Z, y_1)f_2(Z, y_2)p(X_3|Z)p(y_3|X_3)$$

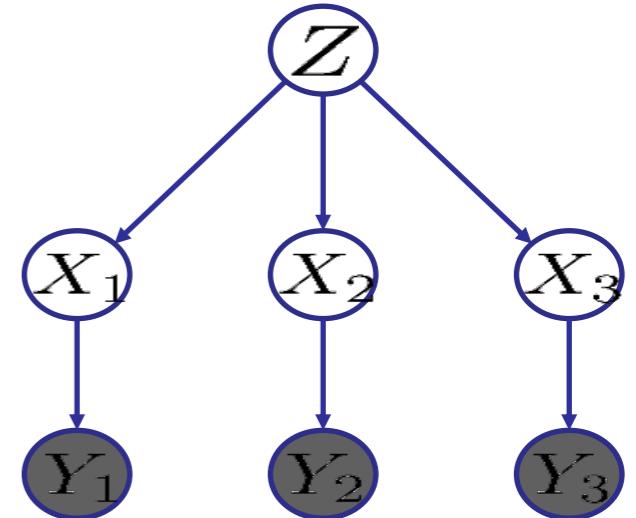
Eliminate  $Z$ , this introduces the factor  $f_3(y_1, y_2, X_3) = \sum_z p(z)f_1(z, y_1)f_2(z, y_2)p(X_3|z)$ , and we are left:

$$p(y_3|X_3), f_3(y_1, y_2, X_3)$$

No hidden variables left. Join the remaining factors to get:

$$f_4(y_1, y_2, y_3, X_3) = P(y_3|X_3)f_3(y_1, y_2, X_3).$$

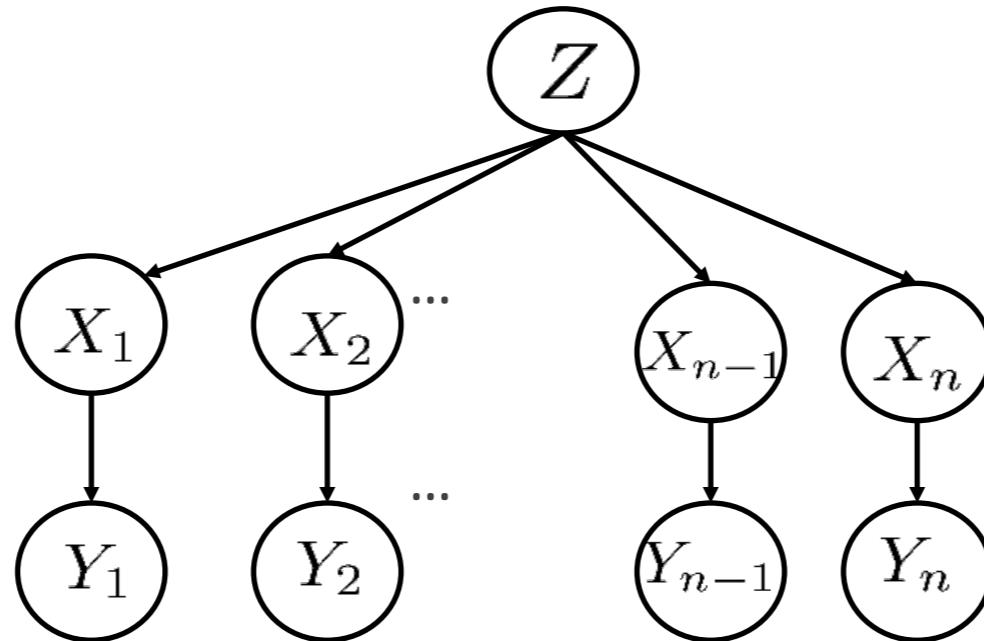
Normalizing over  $X_3$  gives  $P(X_3|y_1, y_2, y_3)$ .



Computational complexity critically depends on the largest factor being generated in this process. Size of factor = number of entries in table. In example above (assuming binary) all factors generated are of size 2 -- as they all only have one variable (Z, Z, and X3 resp.).

# Quiz: Variable Elimination Ordering

- ❖ Assume all variables are binary.
- ❖ For the query  $P(X_n | y_1, \dots, y_n)$  work through the following two different orderings as done in previous slide:  $Z, X_1, \dots, X_{n-1}$  and  $X_1, \dots, X_{n-1}, Z$ .



- ❖ What is the size of the maximum factor generated for each of the orderings?
- ❖ In general, the ordering can greatly affect efficiency.

# VE: Computational and Space Complexity

- ❖ The computational and space complexity of variable elimination is determined by the largest factor
- ❖ The elimination ordering can greatly affect the size of the largest factor.
  - ❖ E.g., previous slide's example  $O(2^n)$  vs.  $O(1)$
- ❖ Does there always exist an ordering that only results in small factors?
  - ❖ No!

# Worst Case Complexity?

## ❖ CSP:

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3 \vee \neg x_4) \wedge (x_2 \vee \neg x_2 \vee x_4) \wedge (\neg x_3 \vee \neg x_4 \vee \neg x_5) \wedge (x_2 \vee x_5 \vee x_7) \wedge (x_4 \vee x_5 \vee x_6) \wedge (\neg x_5 \vee x_6 \vee \neg x_7) \wedge (\neg x_5 \vee \neg x_6 \vee x_7)$$

$$P(X_i = 0) = P(X_i = 1) = 0.5$$

$$Y_1 = X_1 \vee X_2 \vee \neg X_3$$

...

$$Y_8 = \neg X_5 \vee X_6 \vee X_7$$

$$Y_{1,2} = Y_1 \wedge Y_2$$

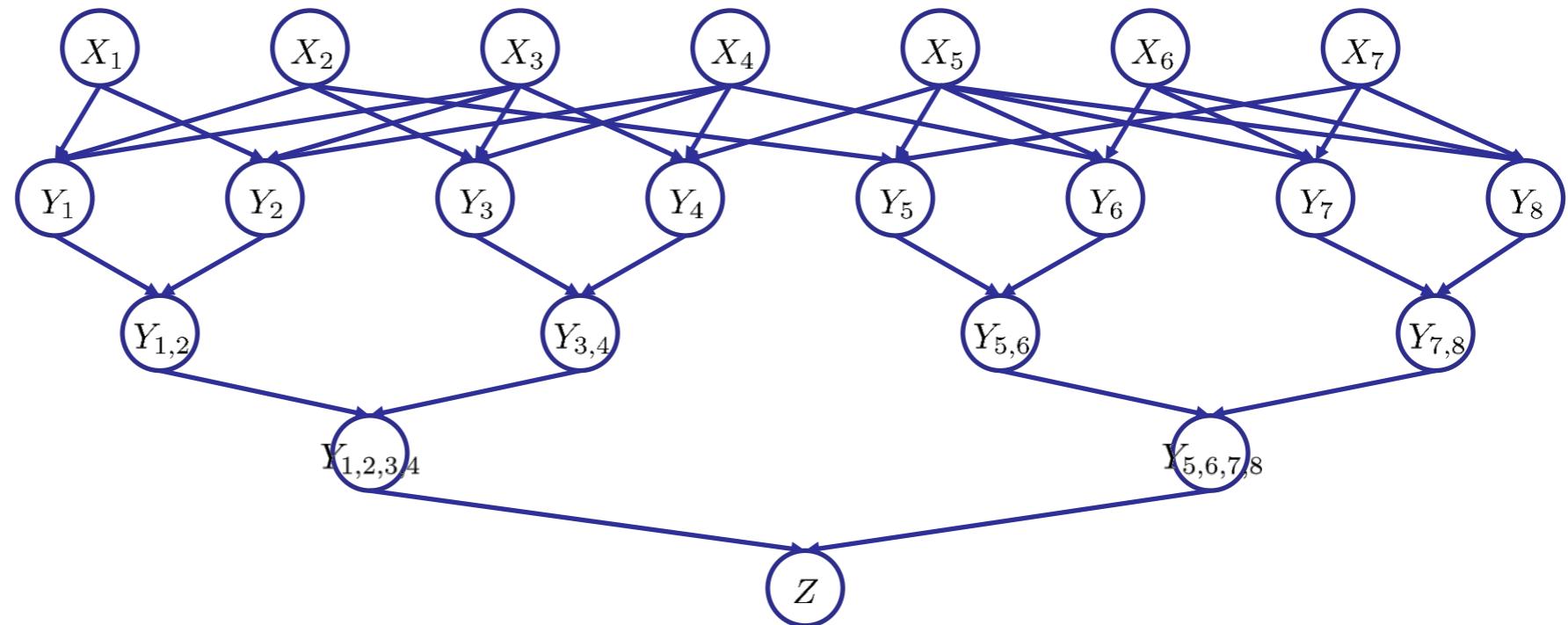
...

$$Y_{7,8} = Y_7 \wedge Y_8$$

$$Y_{1,2,3,4} = Y_{1,2} \wedge Y_{3,4}$$

$$Y_{5,6,7,8} = Y_{5,6} \wedge Y_{7,8}$$

$$Z = Y_{1,2,3,4} \wedge Y_{5,6,7,8}$$



- ❖ If we can answer  $P(z)$  equal to zero or not, we answered whether the 3-SAT problem has a solution.
- ❖ Hence inference in Bayes' nets is NP-hard. No known efficient probabilistic inference in general.

# Polytrees

- ❖ A polytree is a directed graph with no undirected cycles
- ❖ For polytrees you can always find an ordering that is efficient
  - ❖ Try it!!
- ❖ Cut-set conditioning for Bayes' net inference
  - ❖ Choose set of variables such that if removed only a polytree remains
  - ❖ Exercise: Think about how the specifics would work out!

# Bayes' Nets

✓ Representation

✓ Conditional Independences

- ❖ Probabilistic Inference

- ✓ Enumeration (exact, exponential complexity)
- ✓ Variable elimination (exact, worst-case exponential complexity, often better)
- ✓ Probabilistic inference is NP-complete
  - ❖ Approximate inference (sampling)