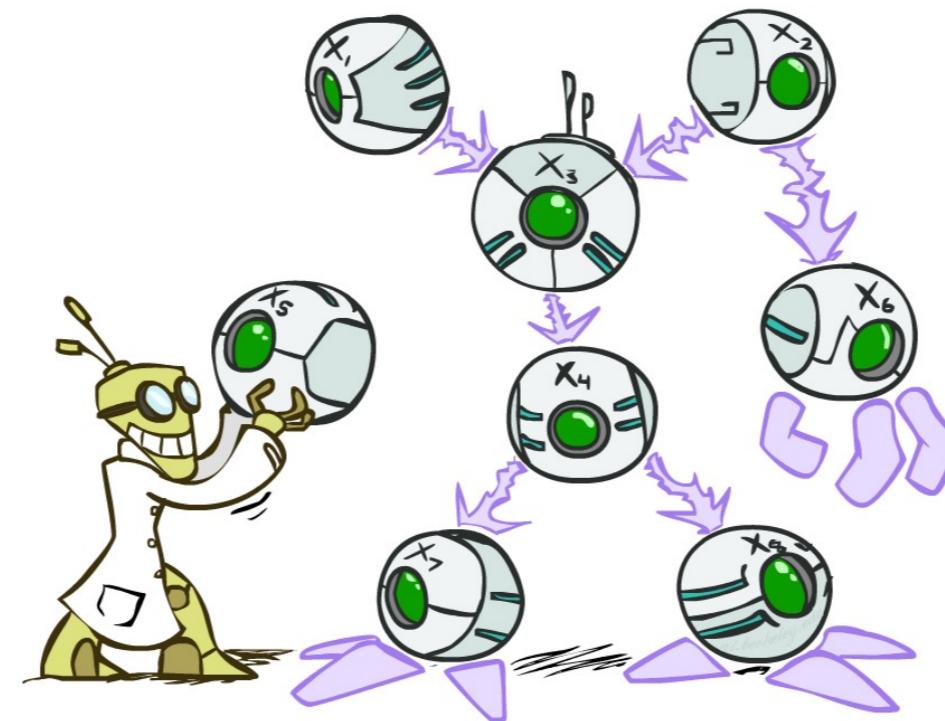


Ve492: Introduction to Artificial Intelligence

Bayesian Networks: Representation



Paul Weng

UM-SJTU Joint Institute

Slides adapted from <http://ai.berkeley.edu>, CMU, AIMA, UM

Announcements

- ❖ **Midterm**
 - ❖ Graded midterms will be available this week
- ❖ **Homework 5**
 - ❖ Due June 30 at 11:59pm
- ❖ **Project 3: RL**
 - ❖ Due July 5 at 11:59pm

Bayes' Nets

- ❖ Representation
- ❖ Conditional Independences
- ❖ Probabilistic Inference

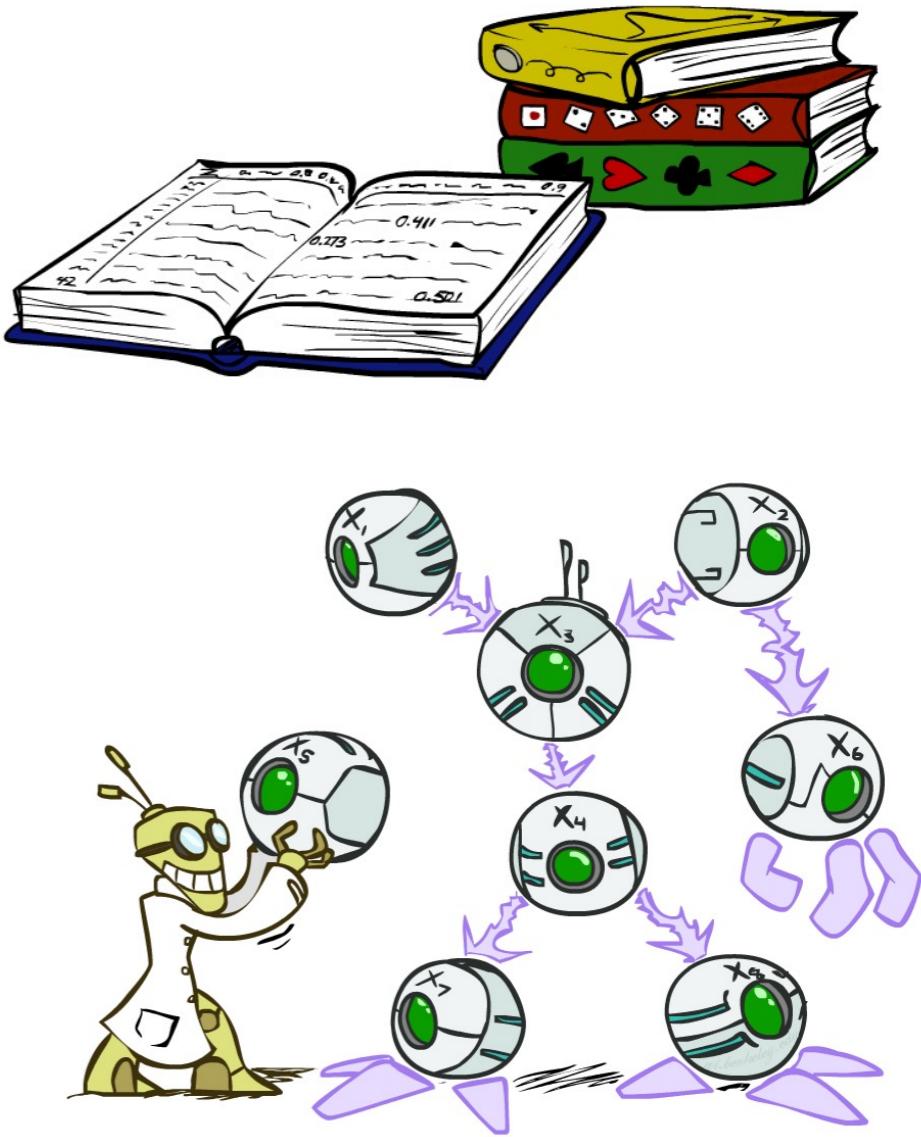
Probabilistic Models

- ❖ Models describe how (a portion of) the world works
- ❖ **Models are always simplifications**
 - ❖ May not account for every variable
 - ❖ May not account for all interactions between variables
 - ❖ “All models are wrong; but some are useful.”
 - George E. P. Box
- ❖ What do we do with probabilistic models?
 - ❖ We (or our agents) need to reason about unknown variables, given evidence
 - ❖ Example: explanation (diagnostic reasoning)
 - ❖ Example: prediction (causal reasoning)



Bayes' Nets: Big Picture

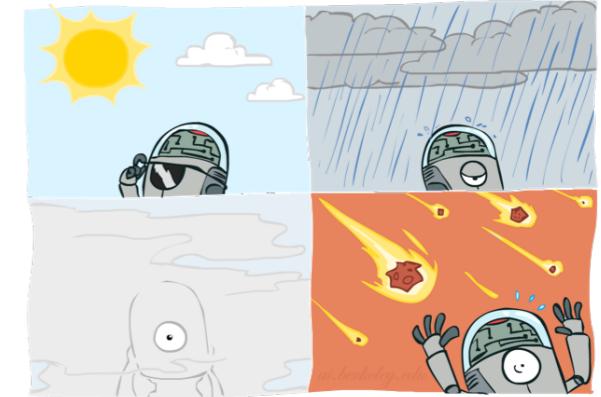
- ❖ Joint distribution can be used for inference
- ❖ Three problems with directly using full joint distribution tables as our probabilistic models:
 - ❖ Unless there are only a few variables, the joint is WAY too big to represent explicitly
 - ❖ Hard to learn (estimate) anything empirically about more than a few variables at a time
 - ❖ Computational complexity of inference
- ❖ Bayes' nets: a technique for describing complex joint distributions (models) using simple, local distributions (conditional probabilities)
 - ❖ Instance of graphical models
 - ❖ We describe how variables locally interact
 - ❖ Local interactions chain together to give global, indirect interactions



Bayes' Net Notation

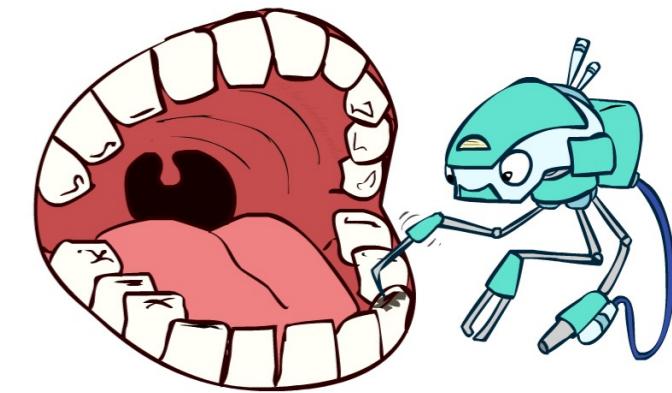
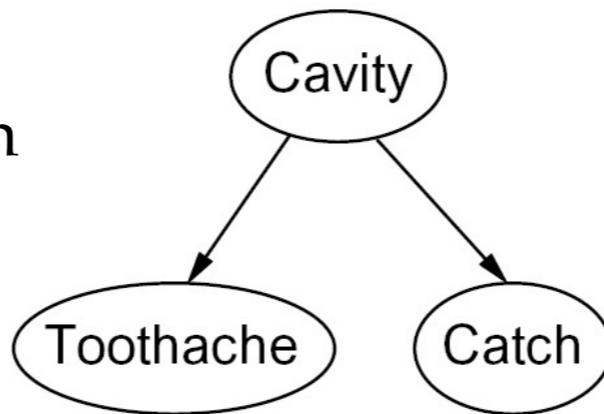
- ❖ Nodes: variables (with domains)

- ❖ Can be assigned (observed) or unassigned (unobserved)



- ❖ Arcs: interactions

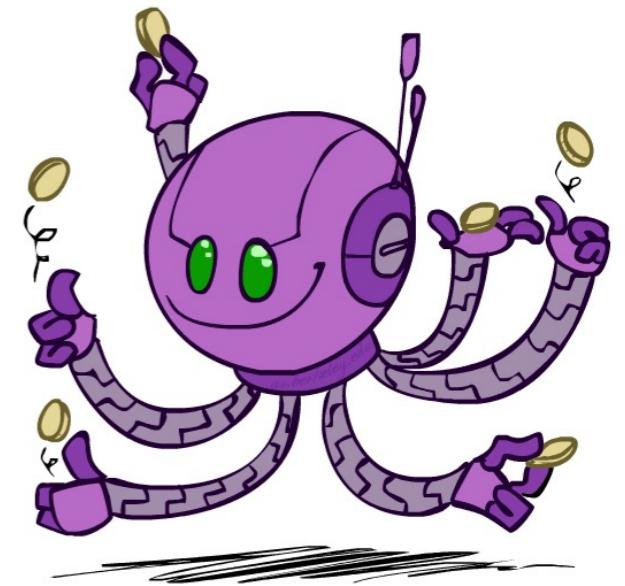
- ❖ Similar to CSP constraints
 - ❖ Indicate “direct influence” between variables
 - ❖ Formally: encode conditional independence (more later)



- ❖ For now: imagine that arrows mean direct causation (in general, they don't!)

Example: Coin Flips

- ❖ N independent coin flips



- ❖ No interactions between variables: **absolute independence**

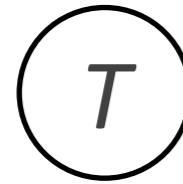
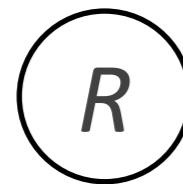
Example: Traffic

- ❖ **Variables:**

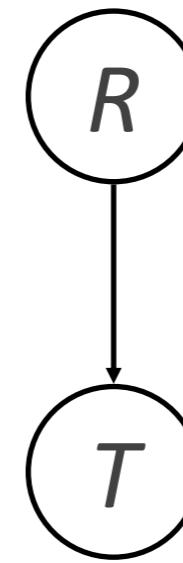
- ❖ R: It rains
- ❖ T: There is traffic



- ❖ Model 1: independence



- ❖ Model 2: rain causes traffic



- ❖ Why is an agent using model 2 better?

Example: Traffic II

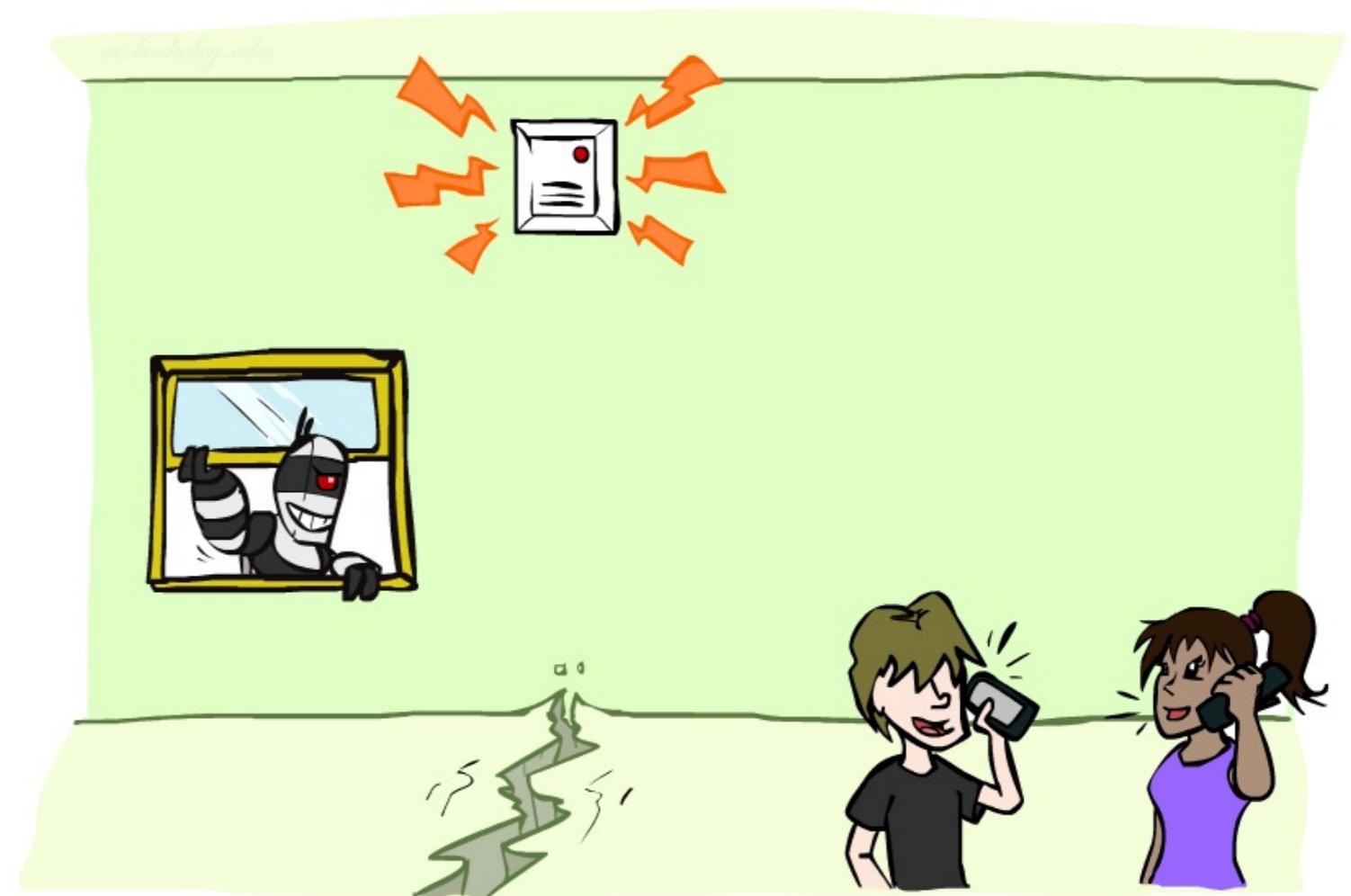
- ❖ Let's build a causal graphical model!
- ❖ **Variables**
 - ❖ T: Traffic
 - ❖ R: It rains
 - ❖ L: Low pressure
 - ❖ D: Roof drips
 - ❖ B: Ballgame
 - ❖ C: Cavity



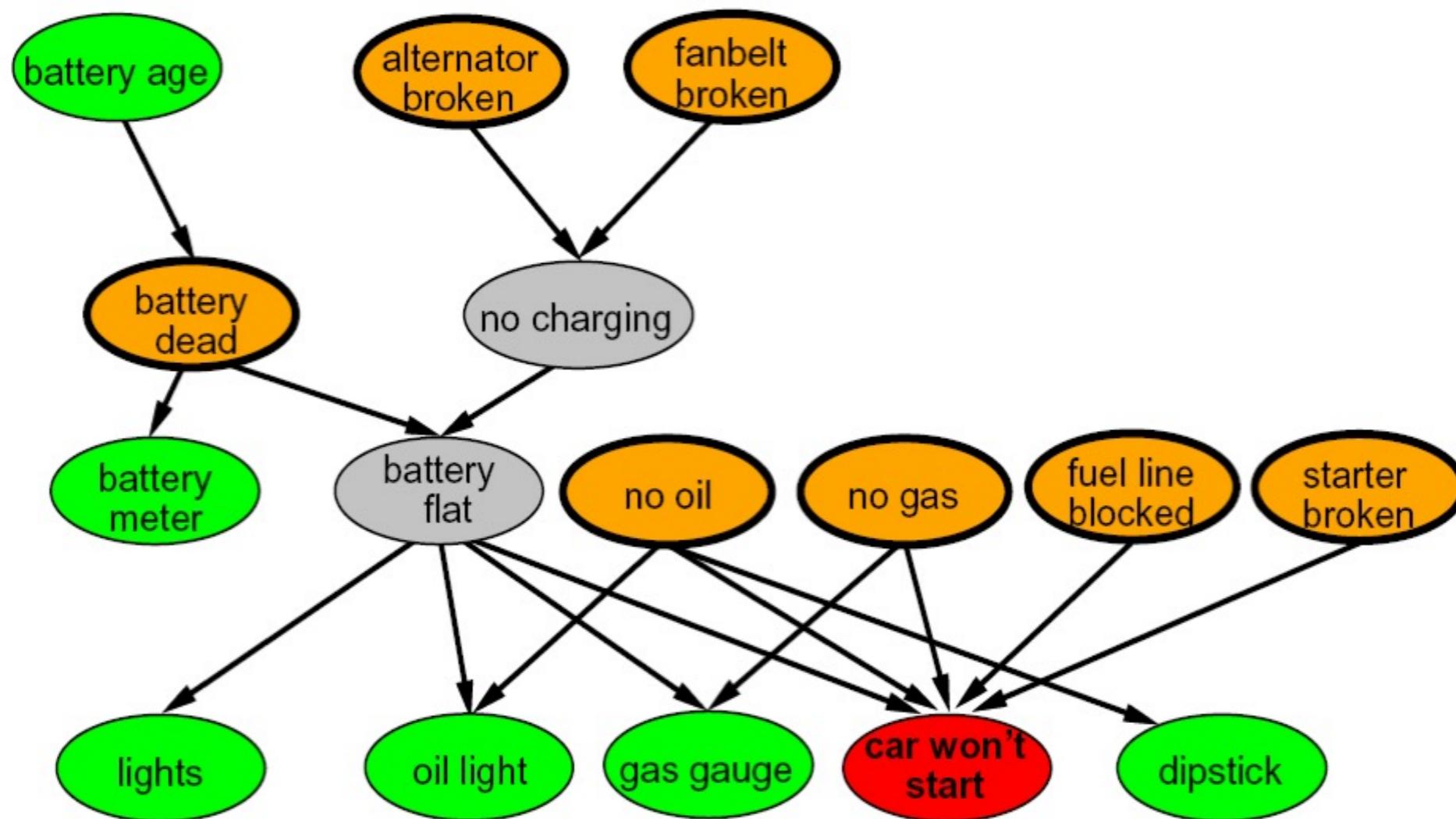
Example: Alarm Network

- ❖ **Variables**

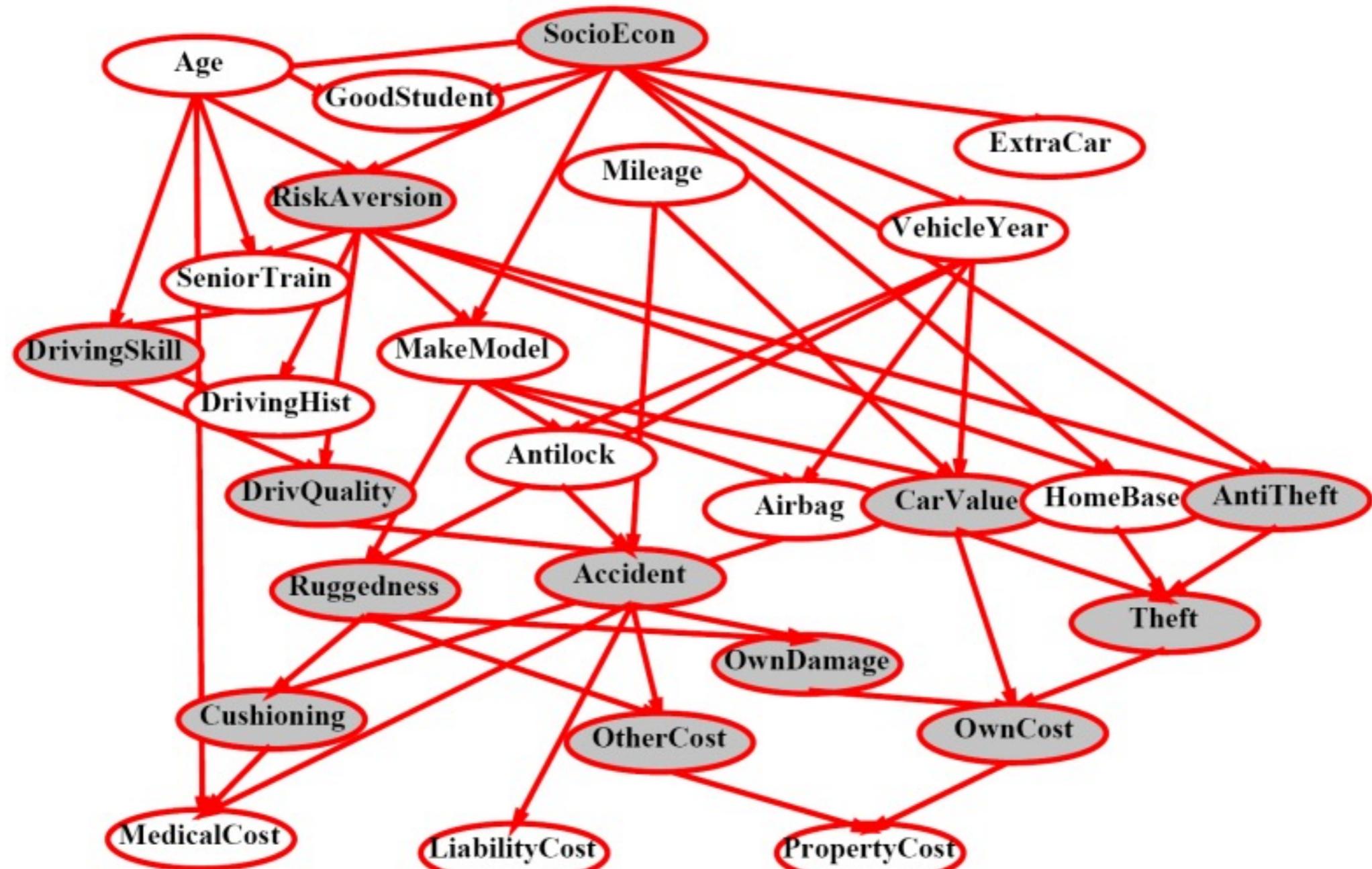
- ❖ B: Burglary
- ❖ A: Alarm goes off
- ❖ M: Mary calls
- ❖ J: John calls
- ❖ E: Earthquake!



Example Bayes' Net: Car



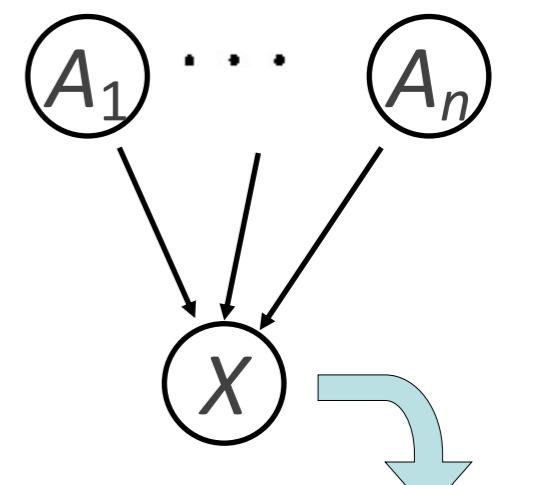
Example Bayes' Net: Insurance



Bayes' Net Definition and Semantics

- ❖ A set of nodes, one per variable
- ❖ A directed, **acyclic** graph
- ❖ A conditional distribution for each node
 - ❖ A collection of distributions over X , one for each combination of parents' values

$$P(X|a_1 \dots a_n)$$



$$P(X|A_1 \dots A_n)$$

- ❖ CPT: conditional probability table
- ❖ Description of a noisy “causal” process

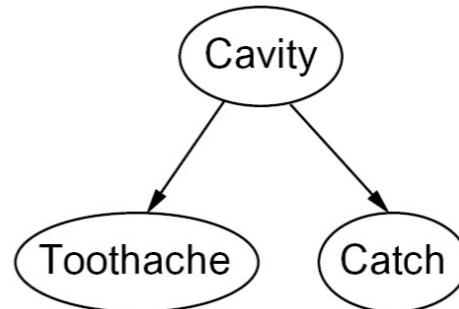
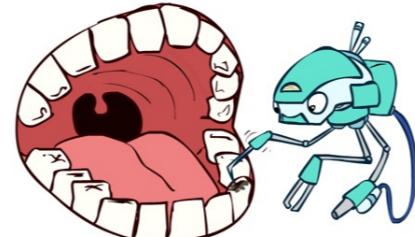
A Bayes net = Topology (graph) + Local Conditional Probabilities

Probabilities in BNs

- ❖ Bayes' nets **implicitly** encode joint distributions
 - ❖ As a product of local conditional distributions
 - ❖ To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$

- ❖ Example:



$$P(+\text{cavity}, +\text{catch}, -\text{toothache})$$

Probabilities in BNs

- ❖ Why are we guaranteed that setting

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$

results in a proper joint distribution?

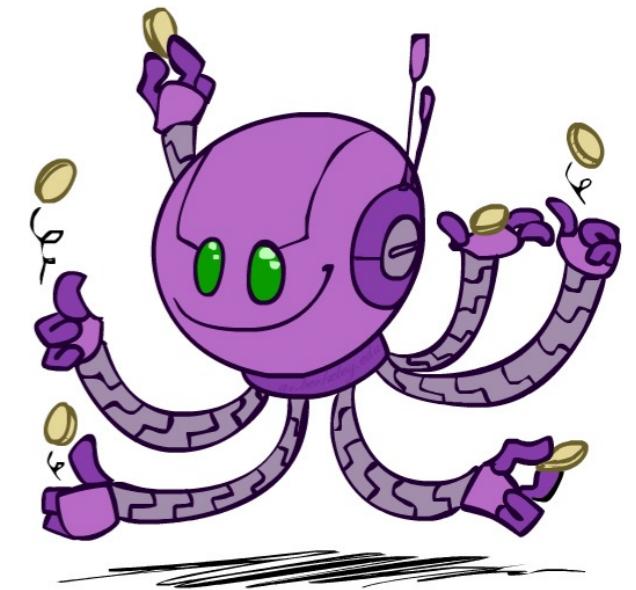
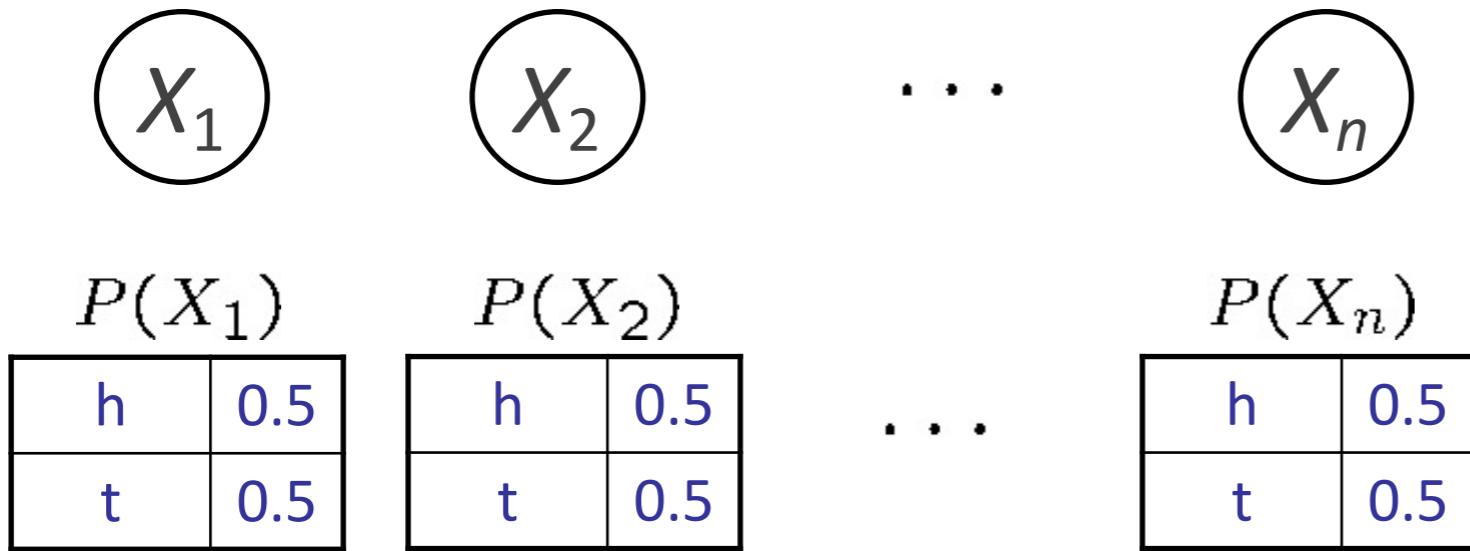
- ❖ Chain rule (valid for all distributions): $P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | x_1 \dots x_{i-1})$
- ❖ Assume conditional independences: $P(x_i | x_1, \dots, x_{i-1}) = P(x_i | \text{parents}(X_i))$

→ Consequence:

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$

- ❖ Not every BN can represent every joint distribution
 - ❖ The topology enforces certain conditional independencies

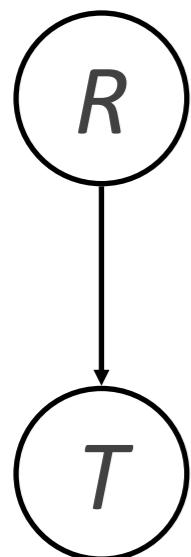
Example: Coin Flips



$$P(h, h, t, h) =$$

Only distributions whose variables are absolutely independent can be represented by a Bayes' net with no arcs.

Example: Traffic



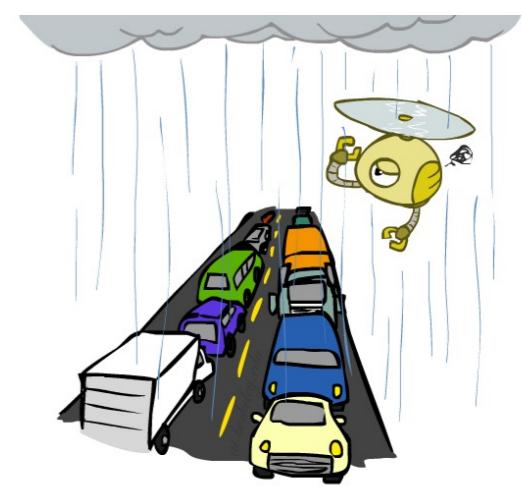
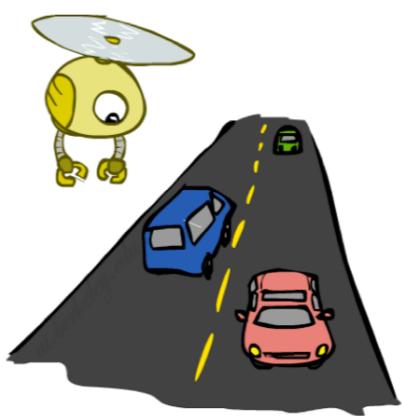
$P(R)$

+r	1/4
-r	3/4

$P(+r, -t) =$

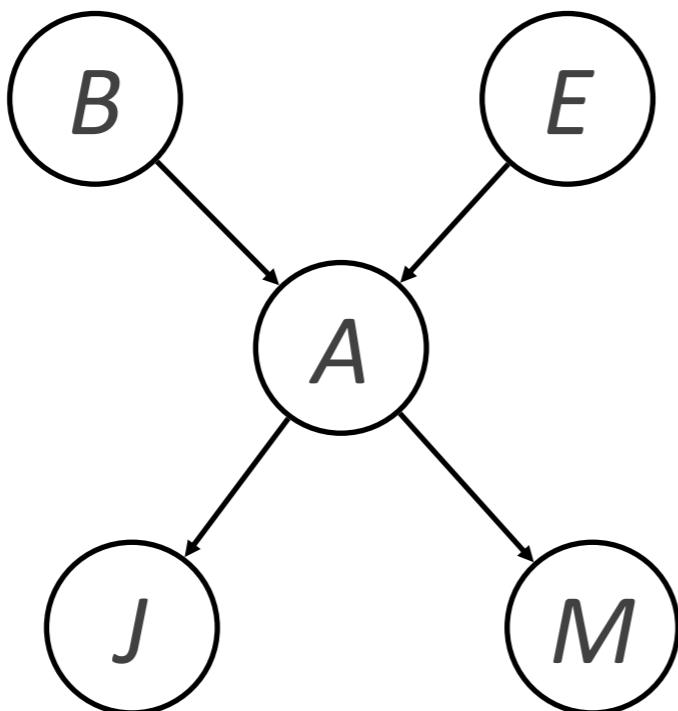
$P(T|R)$

+r	3/4
-t	1/4
+r	1/2
-t	1/2



Quiz: Alarm Network

B	P(B)
+b	0.001
-b	0.999

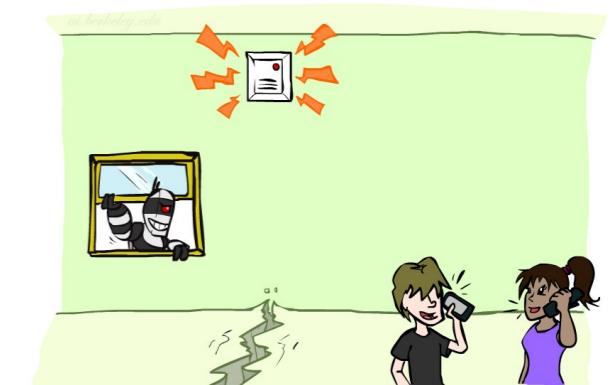


E	P(E)
+e	0.002
-e	0.998

A	J	P(J A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

$$P(+b, -e, +a, -j, +m) =$$

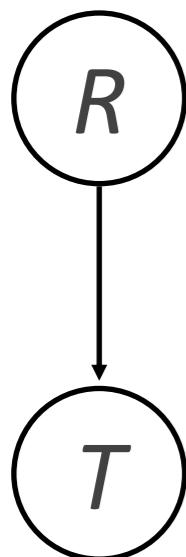
A	M	P(M A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99



B	E	A	P(A B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

Example: Traffic

❖ Causal direction



$P(R)$

+r	1/4
-r	3/4

$P(T|R)$

+r	+t	3/4
	-t	1/4
-r	+t	1/2
	-t	1/2

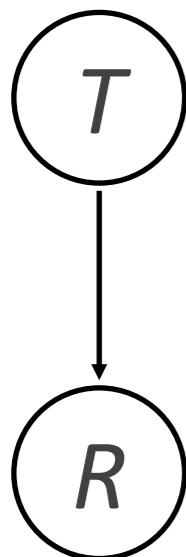


$P(T, R)$

+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16

Example: Reverse Traffic

- ❖ Reverse causality?



$$P(T)$$

+t	9/16
-t	7/16

$$P(R|T)$$

+t	+r	1/3
	-r	2/3
-t	+r	1/7
	-r	6/7

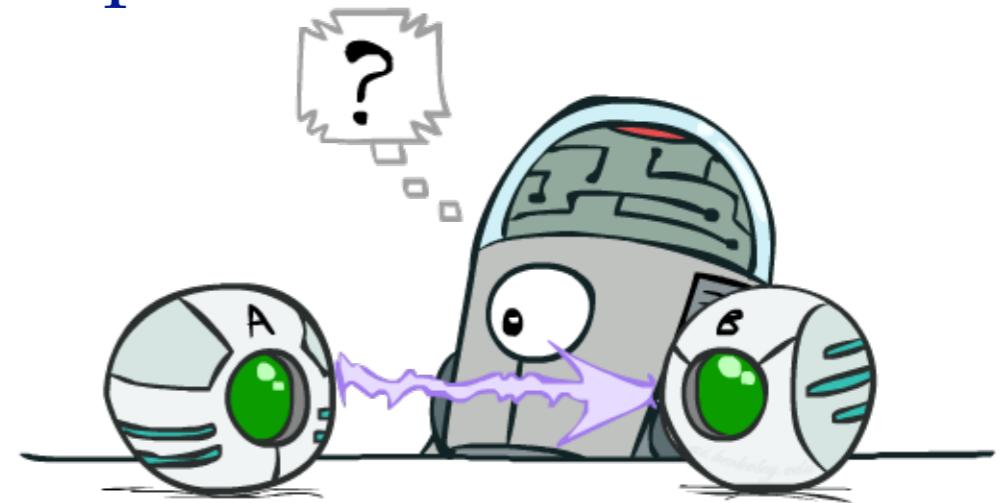


$$P(T, R)$$

+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16

Causality?

- ❖ When Bayes' nets reflect the true causal patterns:
 - ❖ Often simpler (nodes have fewer parents)
 - ❖ Often easier to think about
 - ❖ Often easier to elicit from experts
- ❖ BNs need not actually be causal
 - ❖ Sometimes no causal net exists over the domain (especially if variables are missing)
 - ❖ E.g. consider the variables *Traffic* and *Drips*
 - ❖ End up with arrows that reflect correlation, not causation
- ❖ What do the arrows really mean?
 - ❖ Topology may happen to encode causal structure
 - ❖ **Topology really encodes conditional independence**



$$P(x_i|x_1, \dots x_{i-1}) = P(x_i|\text{parents}(X_i))$$

Size of a Bayes' Net

- ❖ How big is a joint distribution over N Boolean variables?

$$2^N$$

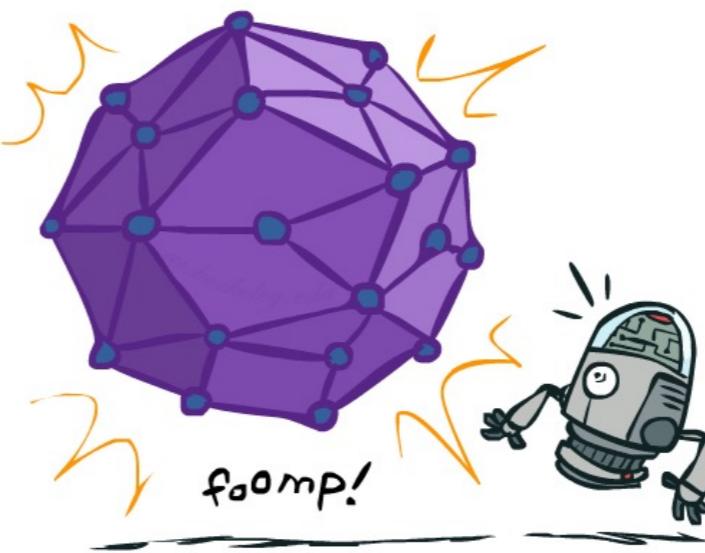
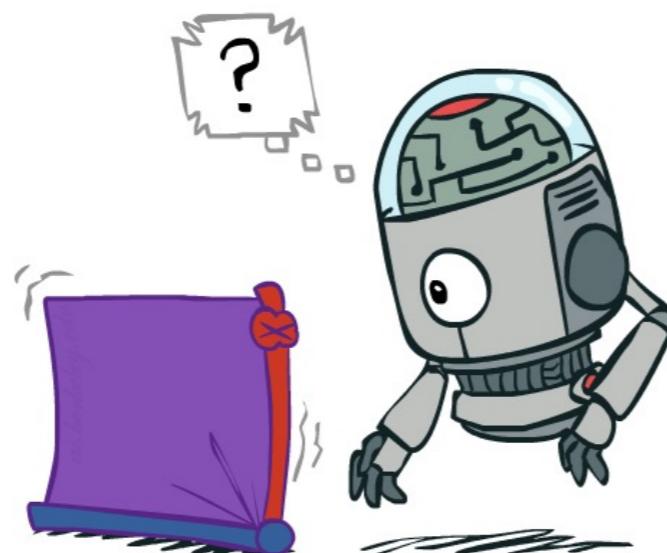
- ❖ How big is an N-node net if nodes have up to k parents?

$$O(N * 2^{k+1})$$

- ❖ Both give you the power to calculate

$$P(X_1, X_2, \dots, X_n)$$

- ❖ BNs: Huge space savings!
- ❖ Also easier to elicit local CPTs
- ❖ Also faster to answer queries (coming)

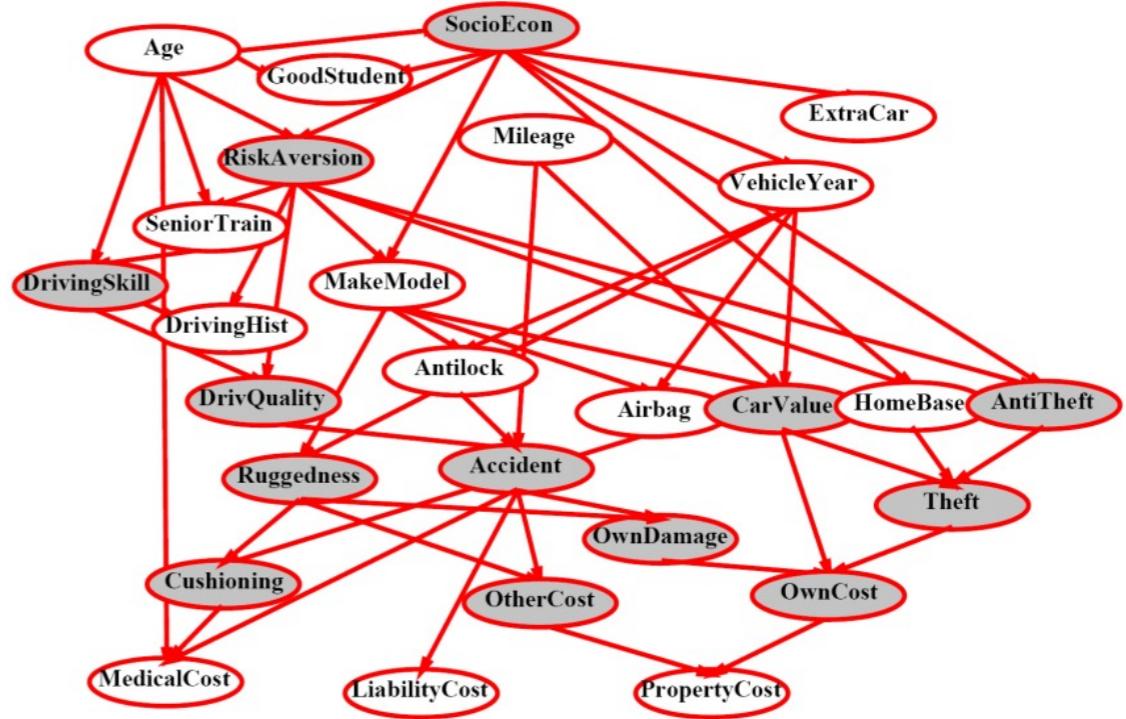


Bayes' Nets

- ❖ A Bayes' net is an efficient encoding of a probabilistic model of a domain

- ❖ Questions we can ask:

- ❖ Inference: given a fixed BN, what is $P(X \mid e)$?
- ❖ Representation: given a BN graph, what kinds of distributions can it encode?
- ❖ Modeling: what BN is most appropriate for a given domain?



Bayes' Nets

✓ Representation

- ❖ Conditional Independences
- ❖ Probabilistic Inference

Conditional Independence

- ❖ X and Y are independent if

$$\forall x, y \ P(x, y) = P(x)P(y) \dashrightarrow X \perp\!\!\!\perp Y$$

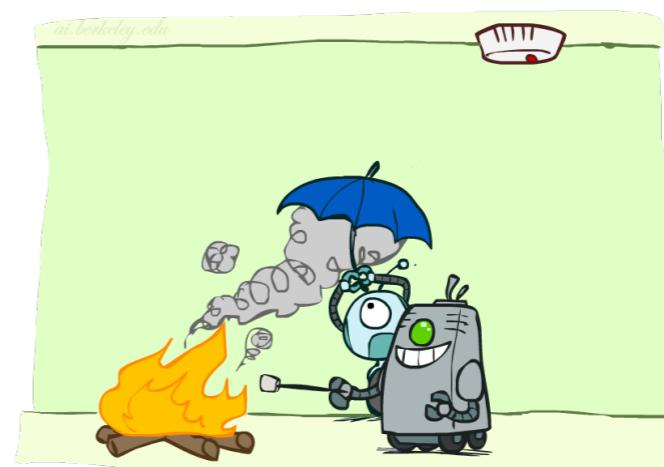
- ❖ X and Y are conditionally independent given Z

$$\forall x, y, z \ P(x, y|z) = P(x|z)P(y|z) \dashrightarrow X \perp\!\!\!\perp Y|Z \quad \text{💡}$$

- ❖ (Conditional) independence is a property of a joint distribution

- ❖ Example:

$$Alarm \perp\!\!\!\perp Fire|Smoke$$

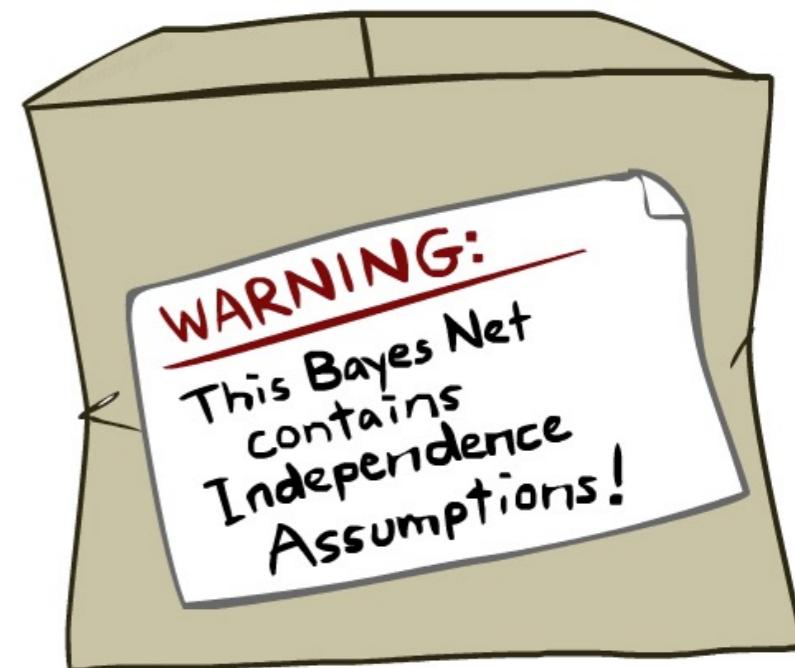


Bayes' Nets: Assumptions

- ❖ Assumptions we are required to make to define the Bayes' net when given the graph:

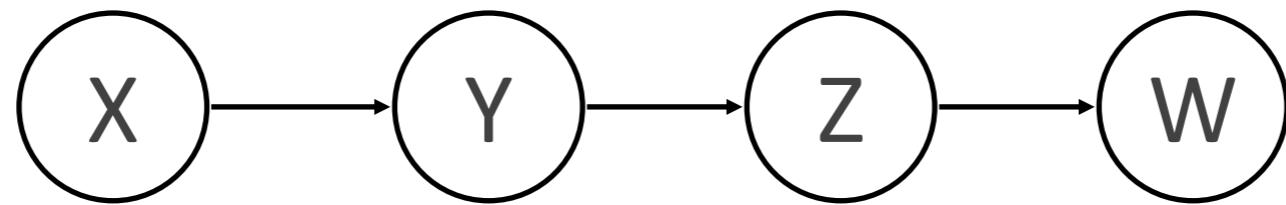
$$P(x_i | x_1 \cdots x_{i-1}) = P(x_i | \text{parents}(X_i))$$

- ❖ Beyond the above conditional independence assumptions
 - ❖ Often additional conditional independences
 - ❖ They can be read off the graph
- ❖ Important for modeling: understand assumptions made when choosing a Bayes' net graph



Example

- ❖ Conditional independence assumptions directly from simplifications in chain rule:

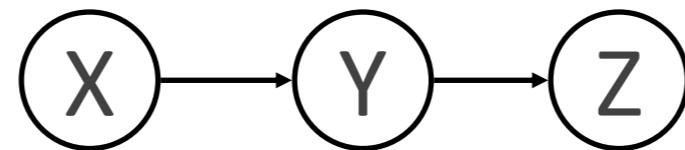


$$1. Z \perp\!\!\!\perp X | Y \quad 2. W \perp\!\!\!\perp X, Y | Z$$

- ❖ Additional implied conditional independence assumptions?

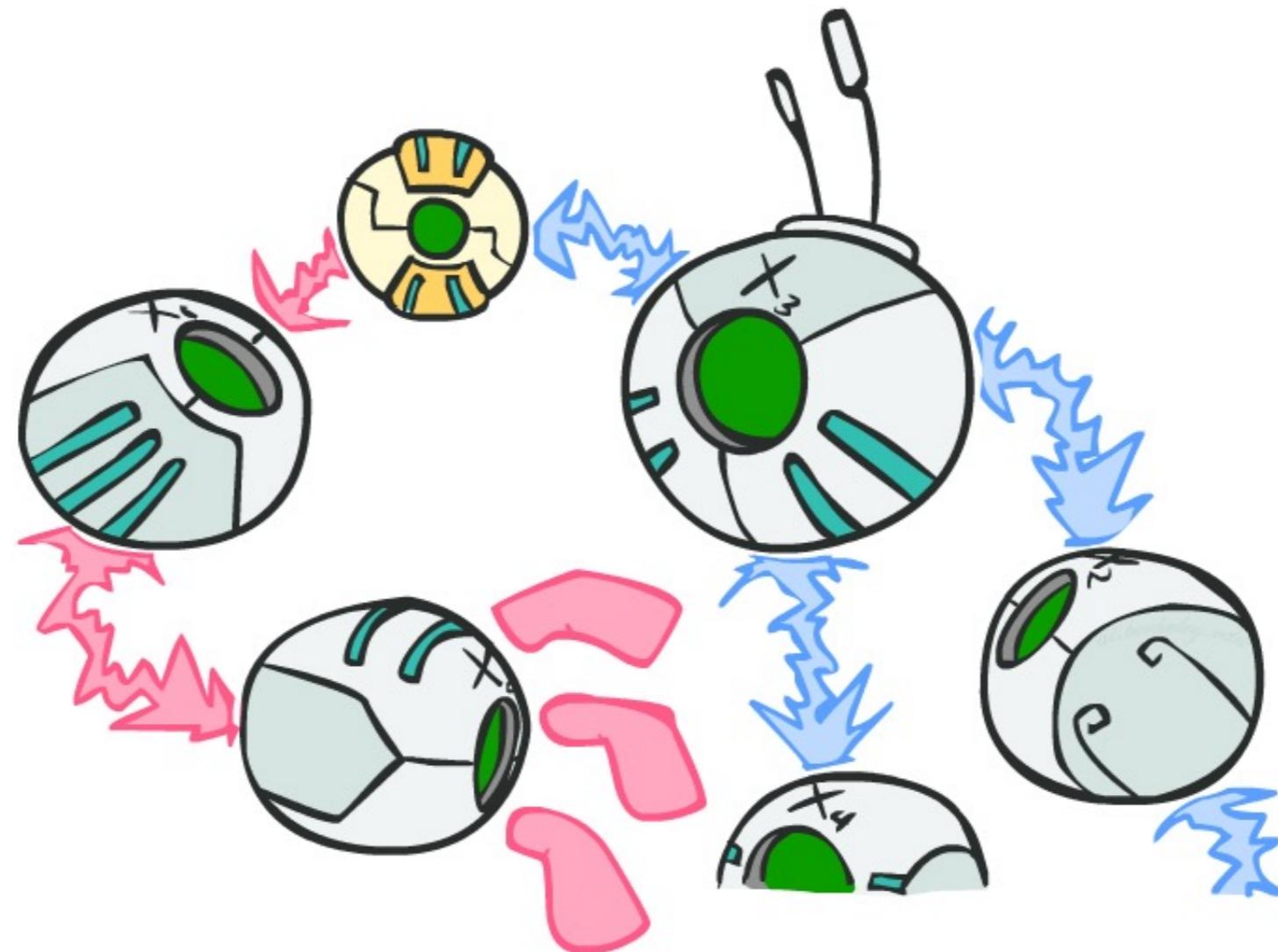
Independence in a BN

- ❖ Important question about a BN:
 - ❖ Are two nodes independent given certain evidence?
 - ❖ If yes, can prove using algebra (tedious in general)
 - ❖ If no, can prove with a counter example
 - ❖ Example:



- ❖ Question: are X and Z necessarily independent?
 - ❖ Answer: No, e.g., low pressure causes rain, which causes traffic.
 - ❖ X can influence Z, Z can influence X (via Y)
 - ❖ Addendum: they *could* be independent: how?

D-separation: Outline



D-separation: Outline

- ❖ Study independence properties for triples
- ❖ Analyze complex cases in terms of member triples
- ❖ D-separation: a condition / algorithm for answering such queries

Serial Chains

- ❖ This configuration is a “serial chain”



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- ❖ Is X always independent of Z ?

❖ *No!*

❖ One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.

❖ Counter-example:

❖ Low pressure => rain => traffic,
high pressure => no rain => no traffic

❖ In numbers:

$$\begin{aligned} P(+y \mid +x) &= 1, & P(-y \mid -x) &= 1, \\ P(+z \mid +y) &= 1, & P(-z \mid -y) &= 1 \end{aligned}$$

Serial Chains

- ❖ This configuration is a “serial chain”
- ❖ Is X always independent of Z given Y?



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

$$\begin{aligned} P(z|x, y) &= \frac{P(x, y, z)}{P(x, y)} \\ &= \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)} \\ &= P(z|y) \end{aligned}$$

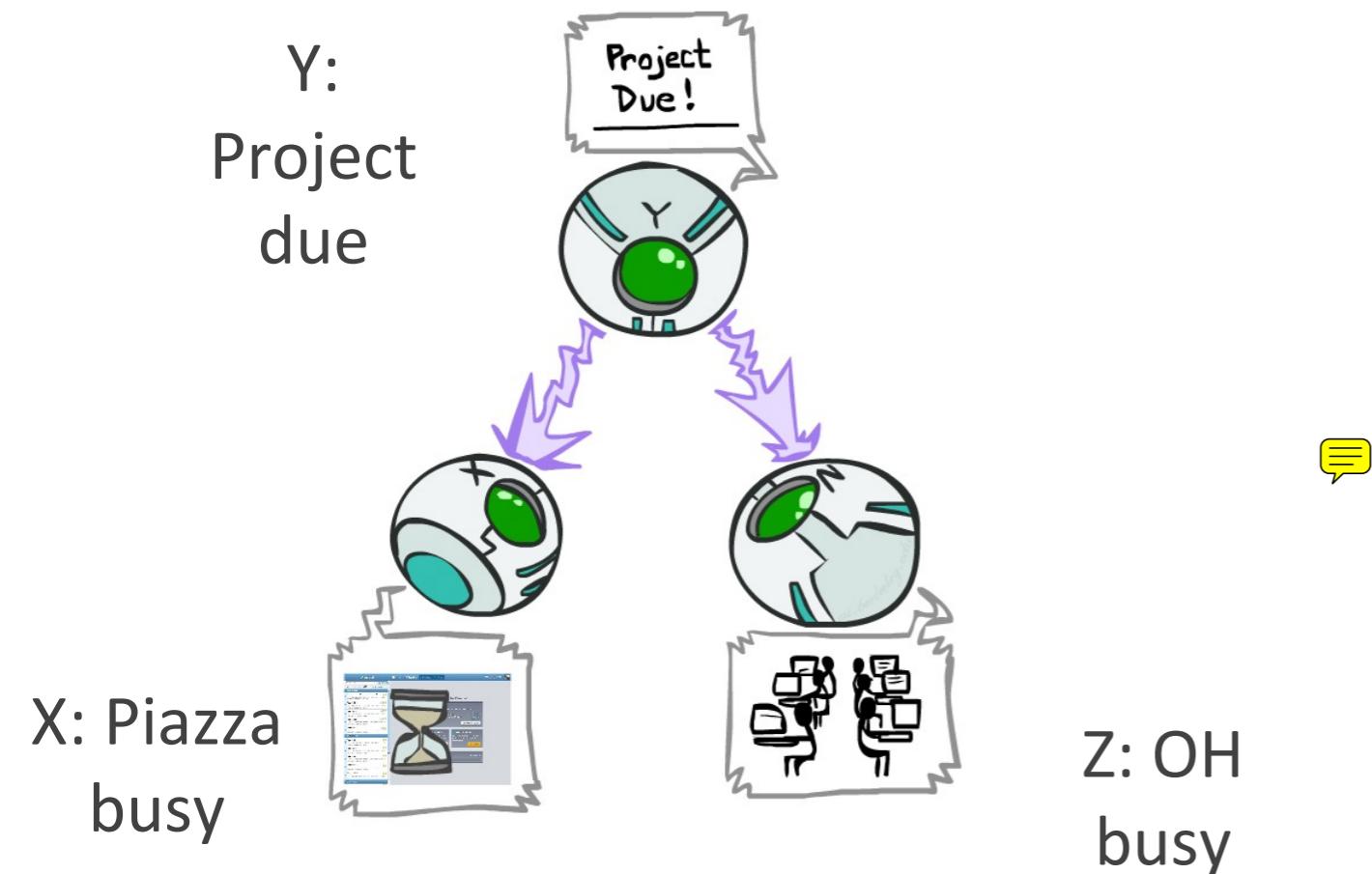
Yes!

- ❖ Evidence along the chain “blocks” the influence



Divergent Chain

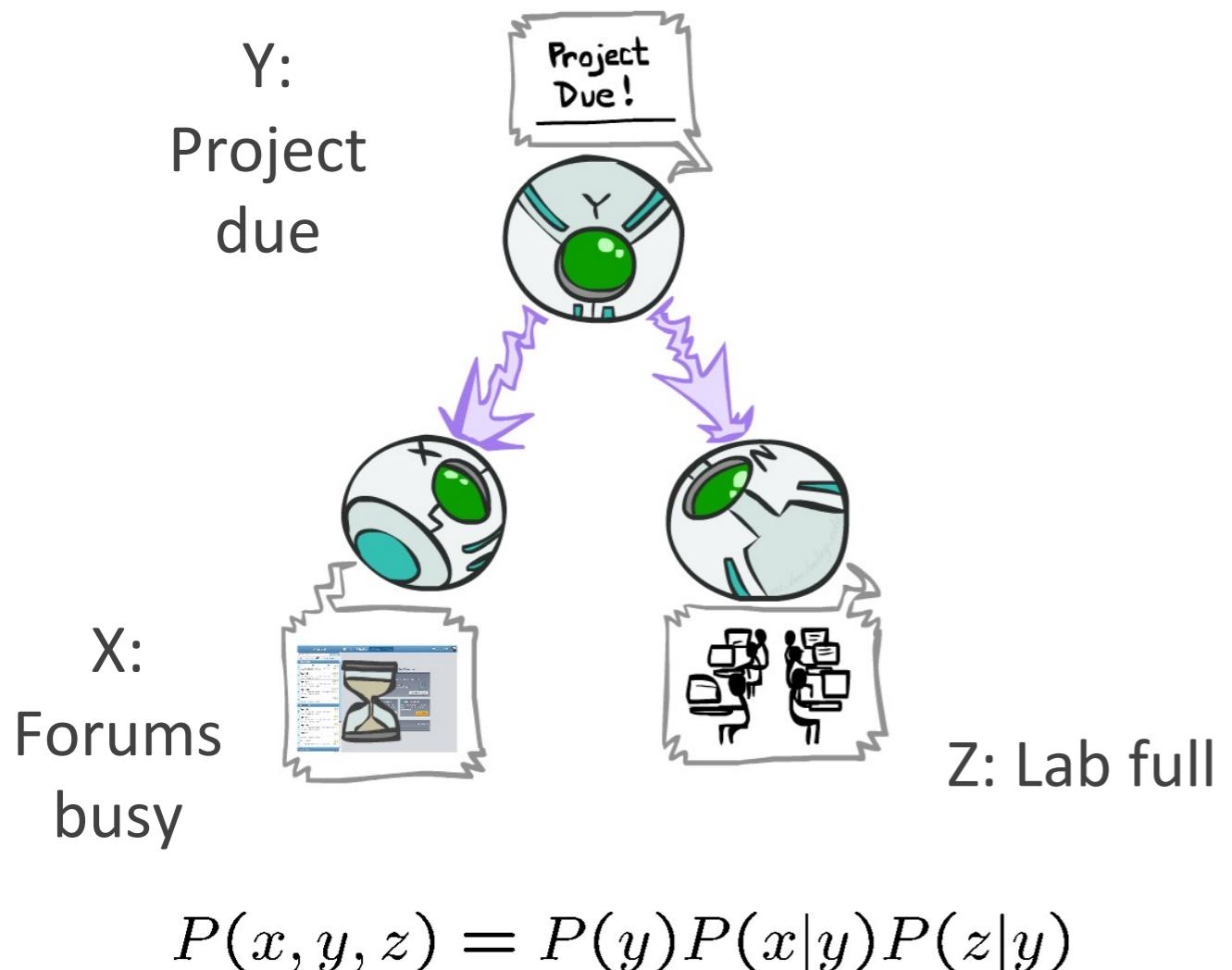
- ❖ This configuration is a “divergent chain”
- ❖ Is X always independent of Z ?
 - ❖ No!
 - ❖ One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.
 - ❖ Counter-example:
 - ❖ Project due => Piazza busy and OH busy
 - ❖ In numbers:
$$P(+x | +y) = 1, P(-x | -y) = 1,$$
$$P(+z | +y) = 1, P(-z | -y) = 1$$



$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

Divergent Chain

- ❖ This configuration is a “divergent chain”
- ❖ Guaranteed X and Z independent given Y?



$$\begin{aligned} P(z|x, y) &= \frac{P(x, y, z)}{P(x, y)} \\ &= \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)} \\ &= P(z|y) \end{aligned}$$

Yes!

- ❖ Observing the cause blocks influence between effects.

Convergent Chain

- ❖ Last configuration: “convergent chain”
(v-structures)

X: Raining



Y: Ballgame



- ❖ Are X and Y independent?

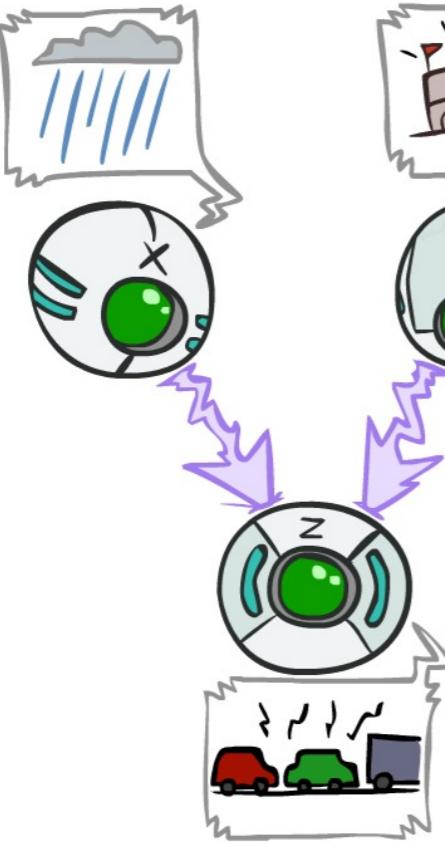
- ❖ Yes: the ballgame and the rain cause traffic, but they are not correlated
- ❖ Still need to prove they must be (try it!)

- ❖ Are X and Y independent given Z?

- ❖ No: seeing traffic puts the rain and the ballgame in competition as explanation.

- ❖ This is backwards from the other cases

❖ Observing an effect activates influence between possible causes.



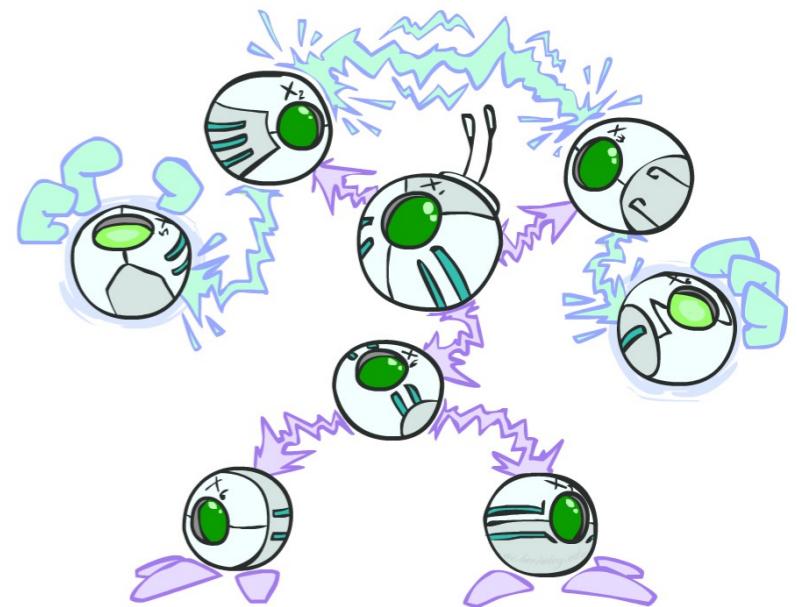
$$P(x, y, z) = P(x)P(y)P(z|x, y)$$

The General Case



The General Case

- ❖ **General question:** in a given BN, are two variables independent (given evidence)?
- ❖ **Solution:** analyze the graph
- ❖ Any complex example can be broken into repetitions of the three canonical cases



Active / Inactive Paths

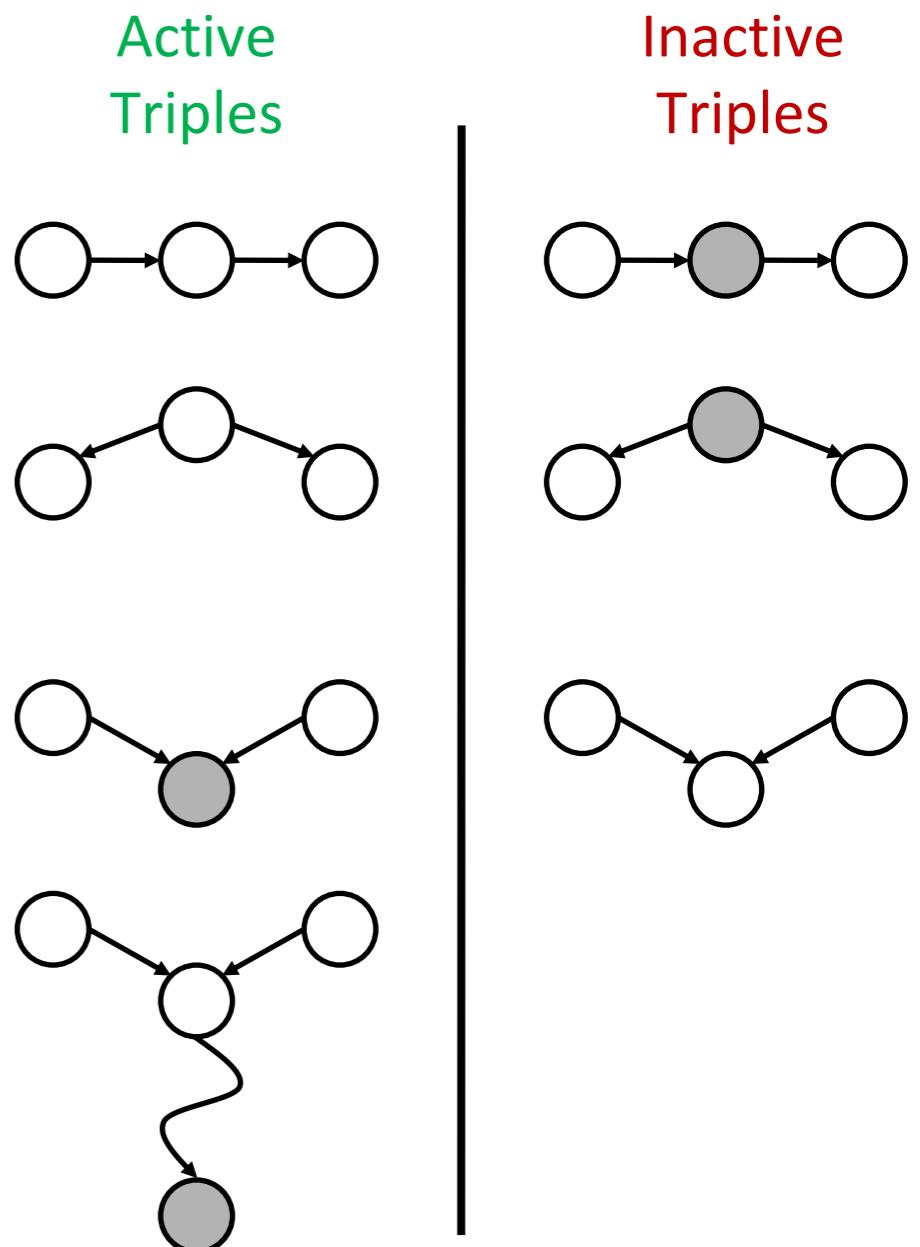
- ❖ Question: Are X and Y conditionally independent given evidence variables $\{Z\}$?

- ❖ Yes, if X and Y “d-separated” by Z
- ❖ Consider all (undirected) paths from X to Y
- ❖ No active paths = independence!

- ❖ A path is active if each triple is active:

- ❖ Serial chain $A \rightarrow B \rightarrow C$ where B is unobserved (either direction)
- ❖ Divergent chain $A \leftarrow B \rightarrow C$ where B is unobserved
- ❖ Convergent chain (aka v-structure)
 $A \rightarrow B \leftarrow C$ where B or one of its descendants is observed

- ❖ All it takes to block a path is a single inactive segment



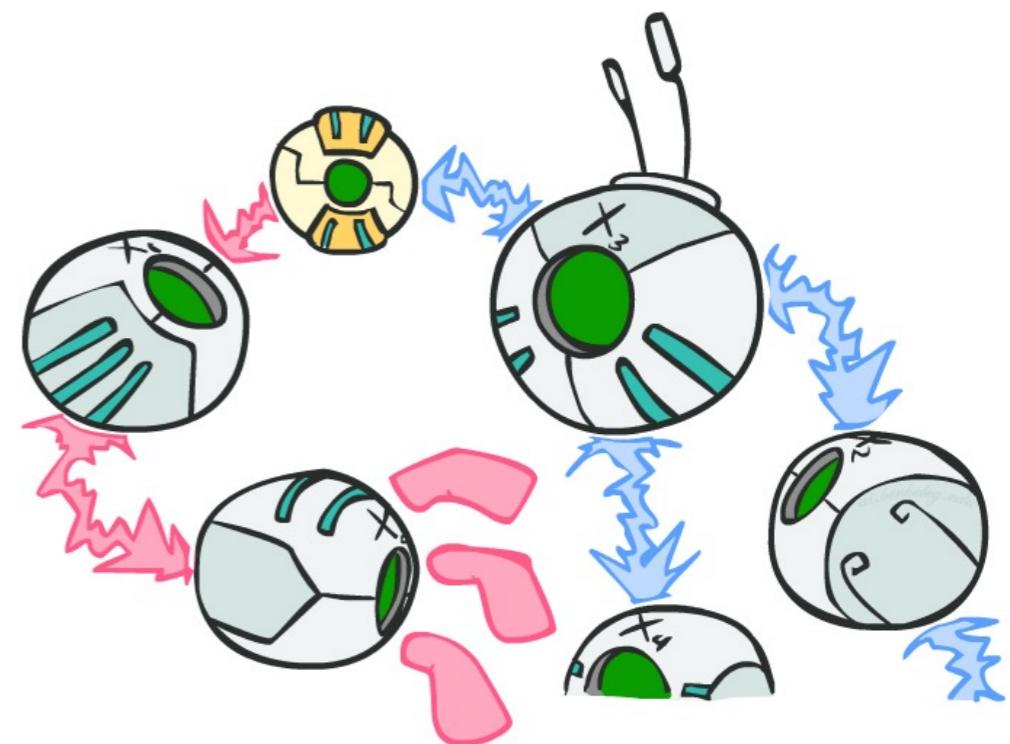
D-Separation

- ❖ Query: $X_i \perp\!\!\!\perp X_j | \{X_{k_1}, \dots, X_{k_n}\}$?
- ❖ Check all (undirected!) paths between X_i and X_j
 - ❖ If one or more active, then independence not guaranteed

$$X_i \not\perp\!\!\!\perp X_j | \{X_{k_1}, \dots, X_{k_n}\}$$

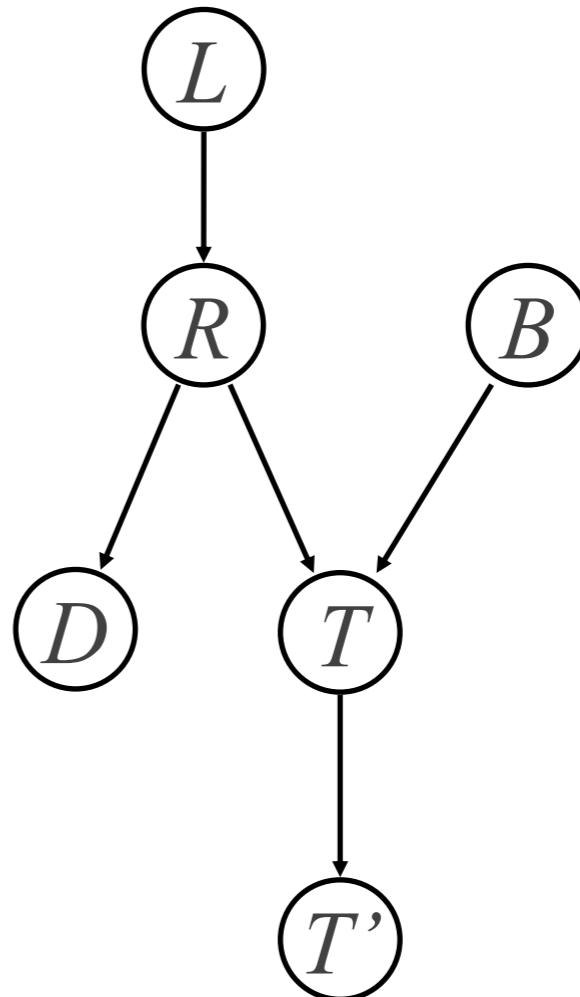
- ❖ Otherwise (i.e. if all paths are inactive), then independence is guaranteed

$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, \dots, X_{k_n}\}$$



Example

$L \perp\!\!\!\perp T' T$	Yes
$L \perp\!\!\!\perp B$	Yes
$L \perp\!\!\!\perp B T$	No
$L \perp\!\!\!\perp B T'$	No
$L \perp\!\!\!\perp B T, R$	Yes



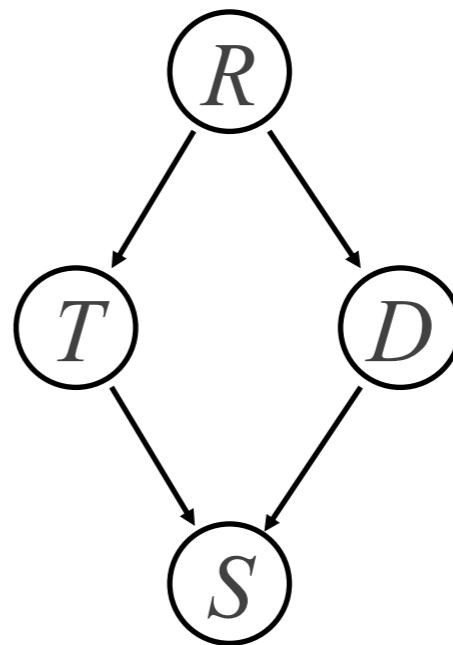
Quiz: Conditional Independence

- ❖ Variables:

- ❖ R: Raining
- ❖ T: Traffic
- ❖ D: Roof drips
- ❖ S: I'm sad

- ❖ Questions:

- $T \perp\!\!\!\perp D$ ✗
- $T \perp\!\!\!\perp D|R$ ✓
- $T \perp\!\!\!\perp D|R, S$ ✗

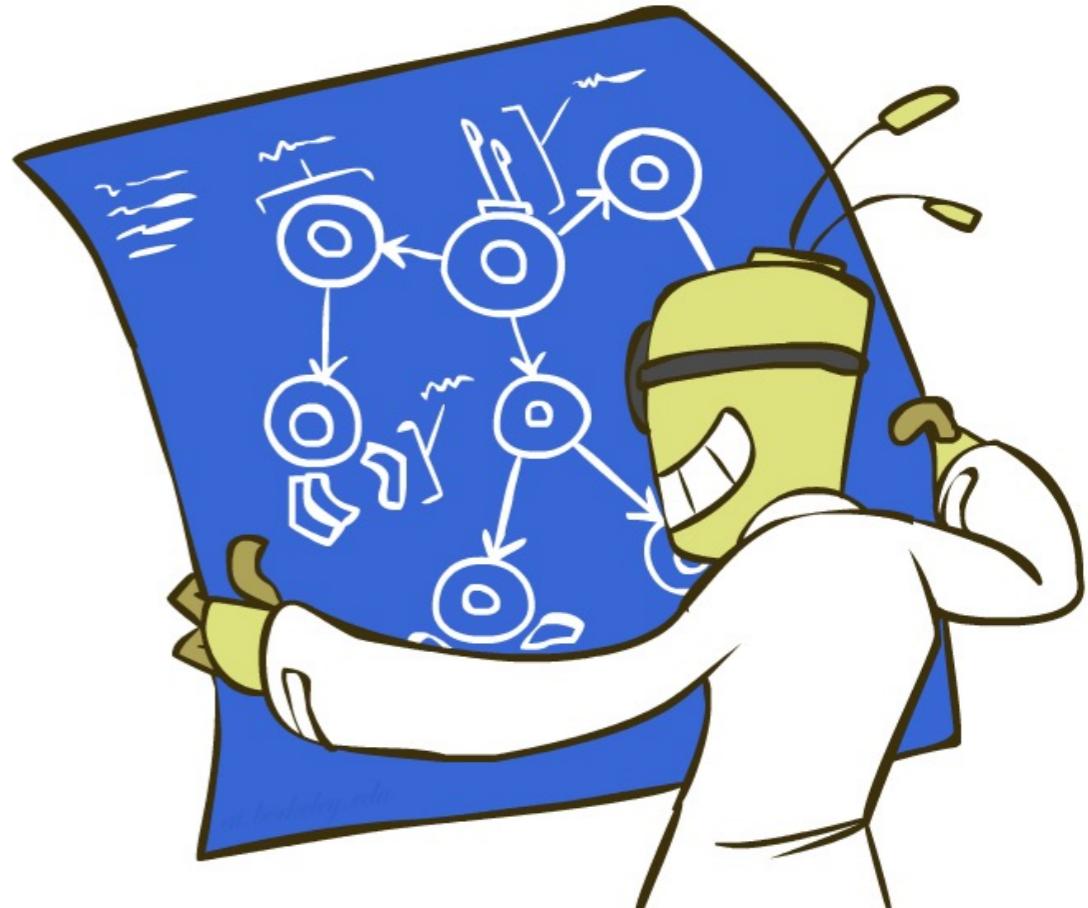


Structure Implications

- Given a Bayes net structure, can run d-separation algorithm to build a complete list of conditional independences that are necessarily true of the form

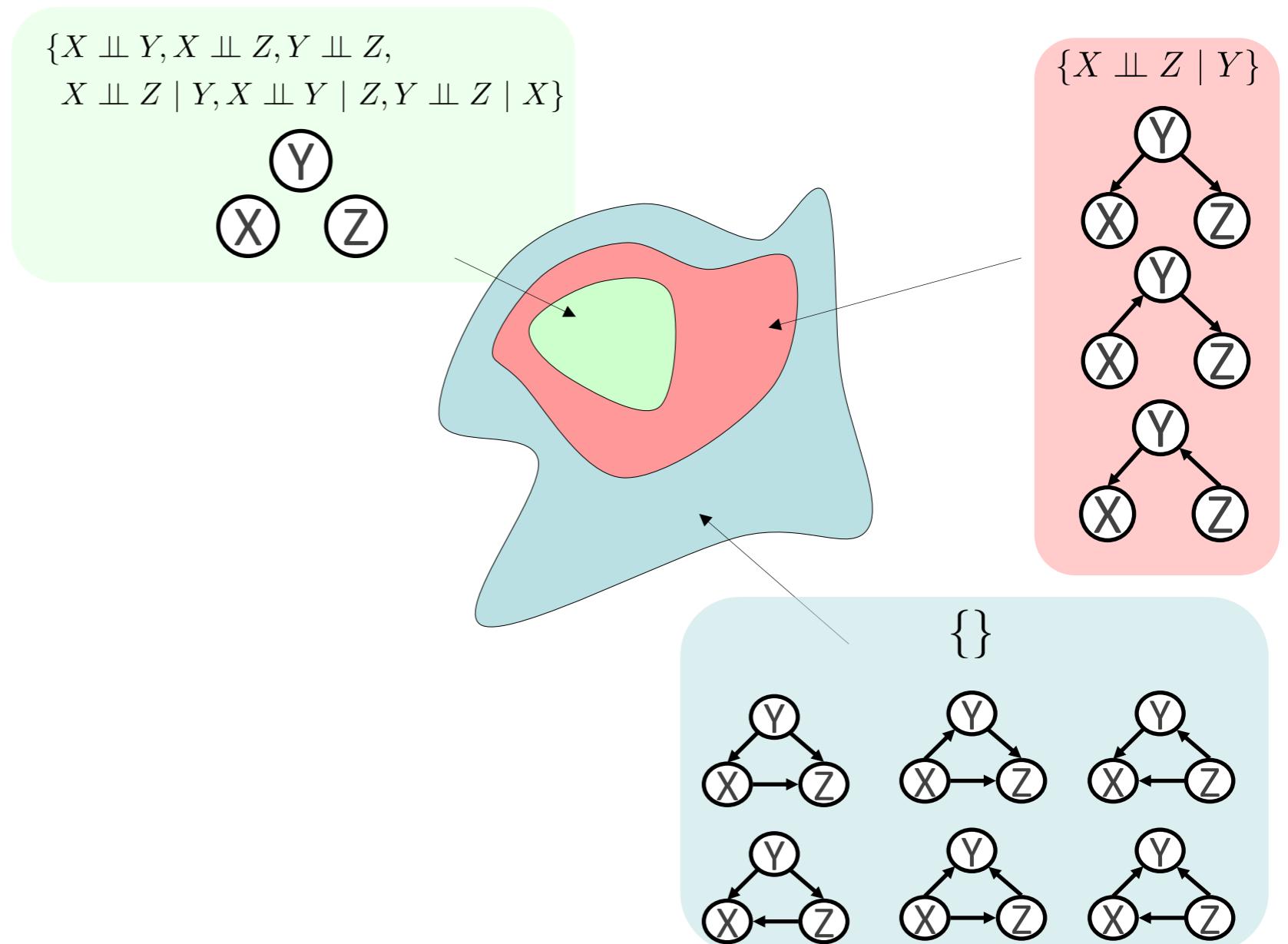
$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, \dots, X_{k_n}\}$$

- This list determines the set of probability distributions that can be represented



Topology Limits Distributions

- Given some graph topology G , only certain joint distributions can be encoded
- The graph structure guarantees certain (conditional) independences
- (There might be more independence)
- Adding arcs increases the set of distributions, but has several costs
- Full conditioning can encode any distribution



Bayes Nets Representation Summary

- ❖ Bayes nets compactly encode joint distributions
- ❖ Guaranteed independencies of distributions can be deduced from BN graph structure
- ❖ D-separation gives precise conditional independence guarantees from graph alone
- ❖ A Bayes' net's joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution

Bayes' Nets

✓ Representation

✓ Conditional Independences

- ❖ Probabilistic Inference

- ❖ Enumeration (exact, exponential complexity)
- ❖ Variable elimination (exact, worst-case exponential complexity, often better)
- ❖ Probabilistic inference is NP-complete
- ❖ Approximate inference (sampling)