#plasticity

❖ Return mapping algorithm

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Perfect plasticity

Consider an associative material following a specific yield function f, its plastic strain is given by

$$\dot{\varepsilon}_{ij}^{p} = \dot{\gamma}\frac{\partial f}{\partial \sigma_{ij}}$$

where $\frac{\partial f}{\partial \sigma_{ij}}$ is the direction of plastic strain, and $\dot{\gamma}$ is the counterpart magnitude can be computed from consistency condition

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} = 0$$

the increased stress is stated as

$$\dot{\sigma}_{ij} = C_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}_{kl}^{p}) = C_{ijkl}^{ep}\dot{\varepsilon}_{kl}$$

substituting it into consistency condition yields

$$\dot{f} = \frac{\partial f}{\sigma_{ij}}C_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\gamma}\frac{\partial f}{\sigma_{kl}}) = 0$$

$$\dot{\gamma} = \frac{\frac{\partial f}{\sigma_{ij}}C_{ijkl}}{\frac{\partial f}{\partial \sigma_{mn}}C_{mnpq}\frac{\partial f}{\sigma_{pq}}}\dot{\varepsilon}_{kl}$$

Consequently, the elastoplastic tangential modulus can be attained by

$$\begin{aligned}
\dot{\sigma}_{ij} &= C_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}_{kl}^{p}) \\
&= C_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\gamma}\frac{\partial f}{\partial \sigma_{kl}}) \\
&= C_{ijkl}(\dot{\varepsilon}_{kl} - \frac{\frac{\partial f}{\partial \sigma_{ab}}C_{abcd}\dot{\varepsilon}_{cd}}{\frac{\partial f}{\partial \sigma_{efgh}}C_{efgh}\frac{\partial f}{\partial \sigma_{gh}}}\frac{\partial f}{\partial \sigma_{kl}}) \\
&= C_{ijkl}(\delta_{ck}\delta_{dl} - \frac{\frac{\partial f}{\partial \sigma_{ab}}C_{abcd}}{\frac{\partial f}{\partial \sigma_{efgh}}C_{efgh}\frac{\partial f}{\partial \sigma_{gh}}}\frac{\partial f}{\partial \sigma_{kl}})\dot{\varepsilon}_{cd} \\
&= (C_{ijcd} - \frac{\frac{\partial f}{\partial \sigma_{ab}}C_{abcd}C_{ijkl}\frac{\partial f}{\partial \sigma_{kl}}}{\frac{\partial f}{\partial \sigma_{efgh}}C_{efgh}\frac{\partial f}{\partial \sigma_{gh}}})\dot{\varepsilon}_{cd} \\
&= (C_{ijkl} - \frac{C_{ijab}\frac{\partial f}{\partial \sigma_{ab}}\frac{\partial f}{\partial \sigma_{cd}}C_{cdkl}}{\frac{\partial f}{\partial \sigma_{efgh}}C_{efgh}\frac{\partial f}{\partial \sigma_{gh}}})\dot{\varepsilon}_{kl} \\
&= C_{ijkl}^{ep}\dot{\varepsilon}_{kl}
\end{aligned}$$

$$C_{ijkl}^{ep} = C_{ijkl} - \frac{C_{ijab}\frac{\partial f}{\partial \sigma_{ab}}\frac{\partial f}{\partial \sigma_{cd}}C_{cdkl}}{\frac{\partial f}{\partial \sigma_{ef}}C_{efgh}\frac{\partial f}{\partial \sigma_{gh}}}$$

Discrete form

$$\boldsymbol{\varepsilon}^{p(n+1)} = \boldsymbol{\varepsilon}^{p(n)} + \Delta\gamma \frac{\partial f}{\partial \boldsymbol{\sigma}}$$

$$\begin{aligned}
\boldsymbol{\sigma}^{(n+1)} &= \boldsymbol{C} : (\boldsymbol{\varepsilon}^{(n+1)} - \boldsymbol{\varepsilon}^{p(n+1)}) \\
&= \boldsymbol{C} : (\boldsymbol{\varepsilon}^{(n+1)} - \boldsymbol{\varepsilon}^{p(n)}) - \boldsymbol{C} : (\boldsymbol{\varepsilon}^{p(n+1)} - \boldsymbol{\varepsilon}^{p(n)}) \\
&= \boldsymbol{\sigma}^{tr} - \boldsymbol{C} : \Delta\boldsymbol{\varepsilon}^{p} \\
&= \boldsymbol{\sigma}^{tr} - \Delta\gamma \boldsymbol{C} : \frac{\partial f}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{(n+1)}) \\
&= \boldsymbol{\sigma}^{tr} - \Delta\gamma \boldsymbol{C} : \frac{\partial f}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{tr}) \\
&= \boldsymbol{\sigma}^{tr} - \Delta\gamma \boldsymbol{C} : \frac{\partial f^{tr}}{\partial \boldsymbol{\sigma}}
\end{aligned}$$

#! $\frac{\partial f}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}^{(n+1)}) = \frac{\partial f^{tr}}{\partial \boldsymbol{\sigma}}$

consistency condition, the yield function at time step $n+1$ can be deduced as

$$\begin{aligned}
f^{(n+1)} &= f(\boldsymbol{\sigma}^{(n+1)}) \\
&= f(\boldsymbol{\sigma}^{tr}) + (\boldsymbol{\sigma}^{(n+1)} - \boldsymbol{\sigma}^{tr}) : \frac{\partial f^{tr}}{\partial \boldsymbol{\sigma}} \\
&= f^{tr} - \Delta\gamma \frac{\partial f^{tr}}{\partial \boldsymbol{\sigma}} : \boldsymbol{C} : \frac{\partial f^{tr}}{\partial \boldsymbol{\sigma}} \\
&= 0
\end{aligned}$$

$$\Delta\gamma = \frac{f^{tr}}{\frac{\partial f^{tr}}{\partial \boldsymbol{\sigma}} : \boldsymbol{C} : \frac{\partial f^{tr}}{\partial \boldsymbol{\sigma}}}$$

Flowchart

For time step n, input: $\boldsymbol{\sigma}^{(n)}$, $\boldsymbol{\varepsilon}^{p(n)}$, $\Delta\boldsymbol{d}_I^{(n)}$

1. calculate follows:
   - $\Delta\boldsymbol{\varepsilon}^{(n)}$:

     $$\Delta\varepsilon_{ij}^{(n)} = \sum_{I=1}^{n_p} \frac{1}{2}(\Psi_{I,j} d_{iI}^{(n)} + \Psi_{I,i} d_{jI}^{(n)})$$

   - $\boldsymbol{\sigma}^{tr}$:

     $$\sigma_{ij}^{tr} = \sigma_{ij}^{(n)} + C_{ijkl}\Delta\varepsilon_{kl}^{(n)}$$

   - f^{tr}:

     $$f^{tr} = f(\boldsymbol{\sigma}^{tr})$$

2. return mapping algorithm:
   1. if $f^{tr} \leq 0$
      - $\boldsymbol{\sigma}^{(n+1)} = \boldsymbol{\sigma}^{tr}$
      - $\boldsymbol{\varepsilon}^{p(n+1)} = \boldsymbol{\varepsilon}^{p(n)}$
      - $\boldsymbol{C}^T = \boldsymbol{C}$
   2. else
      - $\Delta\gamma$:

        $$\Delta\gamma = \frac{f^{tr}}{\frac{\partial f^{tr}}{\partial \boldsymbol{\sigma}} : \boldsymbol{C} : \frac{\partial f^{tr}}{\partial \boldsymbol{\sigma}}}$$

      - $\boldsymbol{\sigma}^{(n+1)}$:

$$\sigma_{ij}^{(n+1)} = \sigma_{ij}^{tr} - \Delta\gamma C_{ijkl} \frac{\partial f^{tr}}{\partial \sigma_{kl}}$$

- $\boldsymbol{\varepsilon}^{p(n+1)}$:

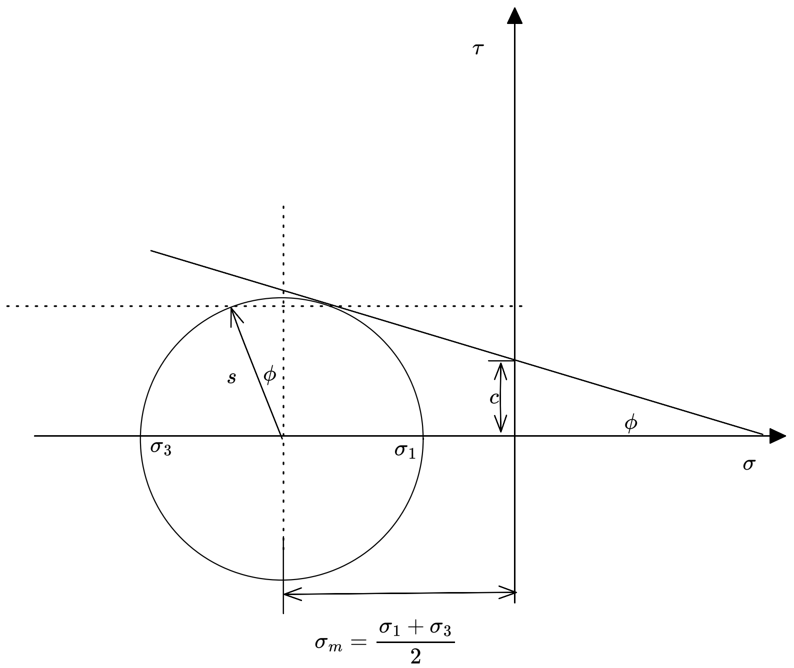
$$\varepsilon_{ij}^{p(n+1)} = \varepsilon_{ij}^{p(n)} + \Delta\gamma \frac{\partial f^{tr}}{\partial \sigma_{ij}}$$

- $\boldsymbol{C}^T = \boldsymbol{C}^{ep}$

3. calculate increased displacement
   - $\Delta \boldsymbol{d}^{(n+1)} = \boldsymbol{K} \setminus (\boldsymbol{f}^{ext(n+1)} - \boldsymbol{f}^{int(n)})$

]

Yield function 1



Yield function:

$$f := \tau + \sigma \tan\phi - c$$

with

$$\begin{aligned} \tau &= s\cos\phi \\ \sigma &= \sigma_m + s\sin\phi \\ s &= \frac{\sigma_1 - \sigma_3}{2} \\ \sigma_m &= \frac{\sigma_1 + \sigma_3}{2} \end{aligned}$$

where σ_α is the α th principal stress evaluating by

$$\sigma_\alpha = \sigma_{ij} n_i^{(\alpha)} n_j^{(\alpha)}$$

and $n_i^{(\alpha)}$'s are the components of direction relating to σ_α.
In accordance with consistency condition, the elastoplastic modulus is attained by

$$C_{ijkl}^{ep} = C_{ijkl} - \frac{C_{ijab} \frac{\partial f}{\partial \sigma_{ab}} \frac{\partial f}{\partial \sigma_{cd}} C_{cdkl}}{\frac{\partial f}{\partial \sigma_{ef}} C_{efgh} \frac{\partial f}{\partial \sigma_{gh}}}$$

with

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$\begin{aligned}
\frac{\partial f}{\partial \sigma_{ij}} &= \frac{\partial \tau}{\partial \sigma_{ij}} + \frac{\partial \sigma}{\partial \sigma_{ij}} \tan\phi \\
&= \frac{\partial s}{\partial \sigma_{ij}} \frac{1}{\cos\phi} + \frac{\partial \sigma_m}{\sigma_{ij}} \tan\phi \\
&= \frac{1}{2} n_i^{(1)} n_j^{(1)} \frac{\sin\phi + 1}{\cos\phi} + \frac{1}{2} n_i^{(3)} n_j^{(3)} \frac{\sin\phi - 1}{\cos\phi}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial s}{\partial \sigma_{ij}} &= \frac{1}{2}(\frac{\partial \sigma_1}{\partial \sigma_{ij}} - \frac{\partial \sigma_3}{\partial \sigma_{ij}}) \\
&= \frac{1}{2}(n_i^{(1)} n_j^{(1)} - n_i^{(3)} n_j^{(3)})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \sigma_m}{\partial \sigma_{ij}} &= \frac{1}{2}(\frac{\partial \sigma_1}{\partial \sigma_{ij}} + \frac{\partial \sigma_3}{\partial \sigma_{ij}}) \\
&= \frac{1}{2}(n_i^{(1)} n_j^{(1)} + n_i^{(3)} n_j^{(3)})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial \sigma_{ij}} C_{ijkl} \frac{\partial f}{\partial \sigma_{kl}} &= \frac{1}{4\cos^2\phi} C_{ijkl} n_i^{(1)} n_j^{(1)} n_k^{(1)} n_l^{(1)} (\sin\phi + 1)^2 \\
&+ \frac{1}{4\cos^2\phi} C_{ijkl} n_i^{(3)} n_j^{(3)} n_k^{(3)} n_l^{(3)} (\sin\phi - 1)^2 \\
&+ \frac{1}{4\cos^2\phi} C_{ijkl} (n_i^{(1)} n_j^{(1)} n_k^{(3)} n_l^{(3)} + n_i^{(3)} n_j^{(3)} n_k^{(1)} n_l^{(1)}) (\sin^2\phi - 1) \\
&= \frac{1}{4\cos^2\phi} (\lambda + 2\mu)(\sin^2\phi + 2\sin\phi + 1) \\
&+ \frac{1}{4\cos^2\phi} (\lambda + 2\mu)(\sin^2\phi - 2\sin\phi + 1) \\
&- \frac{1}{2}\lambda \\
&= \tan^2\phi\lambda + \frac{1 + \sin^2\phi}{\cos^2\phi}\mu
\end{aligned}$$

Yield function 2

$$f := \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi - 2c\cos\phi$$

$$\begin{aligned}
\frac{\partial f}{\partial \sigma_{ij}} &= (\sin\phi + 1)\frac{\partial \sigma_1}{\sigma_{ij}} + (\sin\phi - 1)\frac{\partial \sigma_3}{\partial \sigma_{ij}} \\
&= (\sin\phi + 1) n_i^{(1)} n_j^{(1)} + (\sin\phi - 1) n_i^{(3)} n_j^{(3)}
\end{aligned}$$

$$\begin{aligned}
C_{ijab}\frac{\partial f}{\partial \sigma_{ab}} &= (\lambda\delta_{ij}\delta_{ab} + \mu(\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja}))((\sin\phi + 1) n_a^{(1)} n_b^{(1)} + (\sin\phi - 1) n_a^{(3)} n_b^{(3)}) \\
&= 2\sin\phi\lambda\delta_{ij} + 2((\sin\phi + 1) n_i^{(1)} n_j^{(1)} + (\sin\phi - 1) n_i^{(3)} n_j^{(3)})\mu
\end{aligned}$$

$$\begin{aligned}
C_{ijab}\frac{\partial f}{\partial \sigma_{ab}} \frac{\partial f}{\partial \sigma_{cd}} C_{cdkl} &= 4\sin^2\phi\lambda^2\delta_{ij}\delta_{kl} \\
&+ 4\sin\phi(\sin\phi + 1)\lambda\mu\delta_{ij} n_k^{(1)} n_l^{(1)} \\
&+ 4\sin\phi(\sin\phi - 1)\lambda\mu\delta_{ij} n_k^{(3)} n_l^{(3)} \\
&+ 4\sin\phi(\sin\phi + 1)\lambda\mu n_i^{(1)} n_j^{(1)} \delta_{kl} \\
&+ 4\sin\phi(\sin\phi - 1)\lambda\mu n_i^{(3)} n_j^{(3)} \delta_{kl} \\
&+ 4(\sin\phi + 1)(\sin\phi + 1)\mu^2 n_i^{(1)} n_j^{(1)} n_k^{(1)} n_l^{(1)} \\
&+ 4(\sin\phi - 1)(\sin\phi + 1)\mu^2 n_i^{(3)} n_j^{(3)} n_k^{(1)} n_l^{(1)} \\
&+ 4(\sin\phi + 1)(\sin\phi - 1)\mu^2 n_i^{(1)} n_j^{(1)} n_k^{(3)} n_l^{(3)} \\
&+ 4(\sin\phi - 1)(\sin\phi - 1)\mu^2 n_i^{(3)} n_j^{(3)} n_k^{(3)} n_l^{(3)}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial f}{\partial \sigma_{ij}} C_{ijkl} \frac{\partial f}{\partial \sigma_{kl}} &= C_{ijkl} n_i^{(1)} n_j^{(1)} n_k^{(1)} n_l^{(1)} (\sin\phi + 1)^2 \\
&+ C_{ijkl} n_i^{(3)} n_j^{(3)} n_k^{(3)} n_l^{(3)} (\sin\phi - 1)^2 \\
&+ C_{ijkl} (n_i^{(1)} n_j^{(1)} n_k^{(3)} n_l^{(3)} + n_i^{(3)} n_j^{(3)} n_k^{(1)} n_l^{(1)}) (\sin^2\phi - 1) \\
&= (\lambda + 2\mu)(\sin^2\phi + 2\sin\phi + 1) \\
&+ (\lambda + 2\mu)(\sin^2\phi - 2\sin\phi + 1) \\
&- 2\cos^2\phi\lambda \\
&= 4\sin^2\phi\lambda + 4(1 + \sin^2\phi)\mu
\end{aligned}
$$

Yield function with the terms of invariants

The Mohr-Coulomb criterion can be rephrased in terms of invariants I_1, J_2 and *Lode angle* $0 \leq \theta \leq \frac{\pi}{3}$ as

$$
f := \frac{1}{3} I_1 \sin\phi + \sqrt{\frac{J_2}{3}} ((1 + \sin\phi)\cos\theta - (1 - \sin\phi)\cos(\theta + \frac{2\pi}{3})) - c\cos\phi
$$