Catalan Number

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1 Introduction

Catalan number has a form:

$$\begin{aligned} C_0 &= 1 \\ C_{n+1} &= \sum_{i=0}^{n} C_i C_{n-i} \end{aligned} \tag{1}$$

Catalan numbers happened to be the answer to many counting problems that can split into two small problems. For example:

1. the number of binary trees with n nodes.
2. each polygon can be cut into triangles, the number of cuts with n vertices.
3. the number of dick words with length $2n$. Since each word is $(w_1)w_2$.

The closed form of the Catalan number is:

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{2}$$

There are multiple proofs that show the equivalent between the recurrence form and the closed form. Here we provide one proof using generative function and power series. Since this is a more general way to get the closed form of a recurrence form.

2 Background

The binomial theorem:

$$(1+u)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} u^n \tag{3}$$

$$\sum_{n=0}^{\infty}\sum_{i=0}^{n} a_i b_{n-i} x^n = (\sum_{s=0}^{\infty} a_i x^s)(\sum_{t=0}^{\infty} b_t x^t) \tag{4}$$

$$\binom{\frac{1}{2}}{n} = \frac{(-1)^{n+1}}{4^n(2n-1)}\binom{2n}{n} \tag{5}$$

3 Proof

Let $c(x)$ be the generative function where C_i is the coefficient of x^i.

$$
\begin{aligned}
c(x) &= \sum_{n=0}^{\infty} C_n x^n \\
&= 1 + x\sum_{n=1}^{\infty} C_n x^n \\
&= 1 + x\sum_{n=0}^{\infty} C_{n+1} x^n \\
&= 1 + x\sum_{n=0}^{\infty}\sum_{i=0}^{n} C_i C_{n-i} x^n \\
&= 1 + x(\sum_{s=0}^{\infty} C_i x^s)(\sum_{t=0}^{\infty} C_t x^t) \\
&= 1 + xc^2(x)
\end{aligned} \tag{6}
$$

This is a quadratic function, so,

$$c(x) = \frac{1 \pm \sqrt{1-4x}}{2x} \tag{7}$$

But we must choose one of them because $c(x)$ must satisfy $1 = C_0 = \lim_{x\to 0} c(x)$:

$$\lim_{x\to 0} \frac{1+\sqrt{1-4x}}{2x} = DNE \tag{8}$$

$$\lim_{x\to 0} \frac{1-\sqrt{1-4x}}{2x} = 1 \tag{9}$$

So:

$$
\begin{aligned}
(1-4x)^{\frac{1}{2}} &= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4x)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{4^n(2n-1)} \binom{2n}{n} (-4x)^n \\
&= \sum_{n=0}^{\infty} \frac{-1}{2n-1} \binom{2n}{n} x^n
\end{aligned} \tag{10}
$$

Thus,

$$
\begin{aligned}
c(x) &= \frac{1-\sqrt{1-4x}}{2x} \\
&= \frac{1-\sum_{n=0}^{\infty} \frac{-1}{2n-1}\binom{2n}{n} x^n}{2x} \\
&= \frac{\sum_{n=1}^{\infty} \frac{1}{2n-1}\binom{2n}{n} x^n}{2x} \\
&= \sum_{n=1}^{\infty} \frac{1}{2(2n-1)} \binom{2n}{n} x^{n-1} \\
&= \sum_{n=0}^{\infty} \frac{1}{2(2n+1)} \binom{2n+2}{n+1} x^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n
\end{aligned} \tag{11}
$$

QED.