第七讲 正交矩阵

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内容提要

- 标准正交向量与正交矩阵
- Gram-Schmidt正交化
- 矩阵的QR分解

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标准正交向量组

➢ 定义

- 当向量$\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$满足如下条件时，称这$n$个向量是标准正交的：

$$\mathbf{q}_i^T\mathbf{q}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

正交性（$i \neq j$）

单位向量（$i = j$）

- 若矩阵的各列由标准正交向量构成，则将该矩阵记作Q，它满足如下性质：

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$$

标准正交向量组

➢ 两条重要性质

若矩阵Q的各列为标准正交向量，则对于任意向量x和y，有：

- 向量长度保持不变：

$$\|\mathbf{Qx}\| = \|\mathbf{x}\|$$

- 向量内积保持不变：

$$(\mathbf{Qx})^T(\mathbf{Qy}) = \mathbf{x}^T\mathbf{y}$$

$$\|\mathbf{Qx}-\mathbf{Qy}\| = \|\mathbf{Q}(\mathbf{x}-\mathbf{y})\| = \|\mathbf{x}-\mathbf{y}\|$$

几何结构不变性

正交矩阵

定义

若矩阵Q的各列为标准正交向量且Q为方阵，则称Q为正交矩阵。正交矩阵满足如下性质：

- Q的各行向量也是标准正交的向量

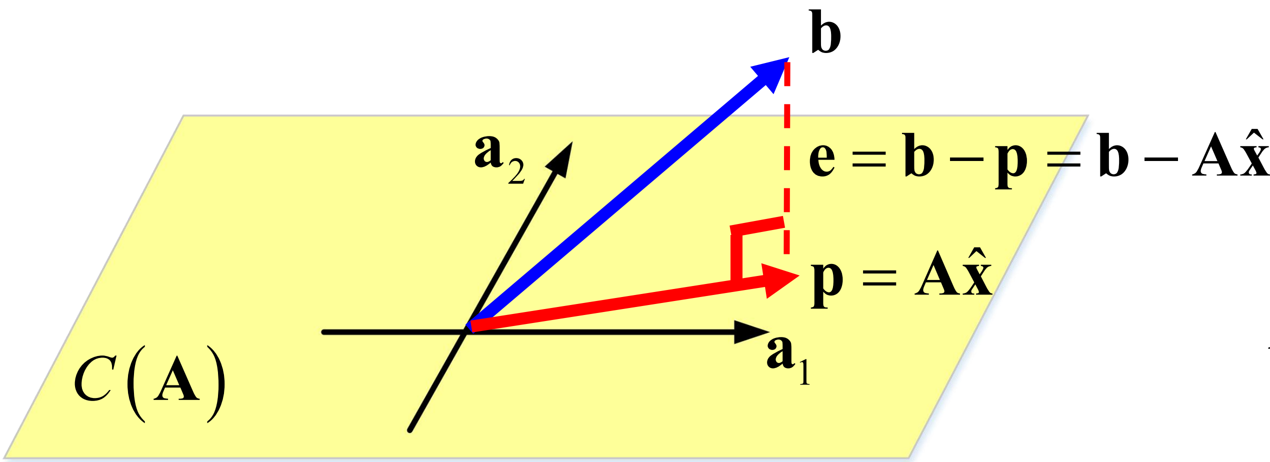
$$\mathbf{Q}^T\mathbf{Q}=\mathbf{Q}\mathbf{Q}^T=\mathbf{I}$$

- Q的逆等于其转置

$$\mathbf{Q}^T=\mathbf{Q}^{-1}$$

为什么?

正交矩阵与子空间投影



假设各列线性无关

$$\mathbf{A}=\left[\mathbf{a}_1\cdots\mathbf{a}_n\right]$$

$$\hat{\mathbf{x}}=\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b} \qquad \hat{\mathbf{x}}=\mathbf{Q}^T\mathbf{b}$$

$$\mathbf{p}=\mathbf{A}\hat{\mathbf{x}}=\mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b} \xRightarrow{\mathbf{A}=\mathbf{Q}} \mathbf{p}=\mathbf{Q}\hat{\mathbf{x}}=\mathbf{Q}\mathbf{Q}^T\mathbf{b}$$

$$\mathbf{P}=\mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T \qquad \mathbf{P}=\mathbf{Q}\mathbf{Q}^T$$

正交矩阵与子空间投影

$$\mathbf{p} = \mathbf{Q}\hat{\mathbf{x}} = \mathbf{Q}\mathbf{Q}^T\mathbf{b}$$

$$\mathbf{p} = \mathbf{Q}\hat{\mathbf{x}} = \mathbf{Q}\mathbf{Q}^T\mathbf{b} = \left[\mathbf{q}_1\ \mathbf{q}_2 \cdots \mathbf{q}_n\right]\begin{bmatrix} \mathbf{q}_1{}^T \\ \mathbf{q}_2{}^T \\ \vdots \\ \mathbf{q}_n{}^T \end{bmatrix}\mathbf{b}$$

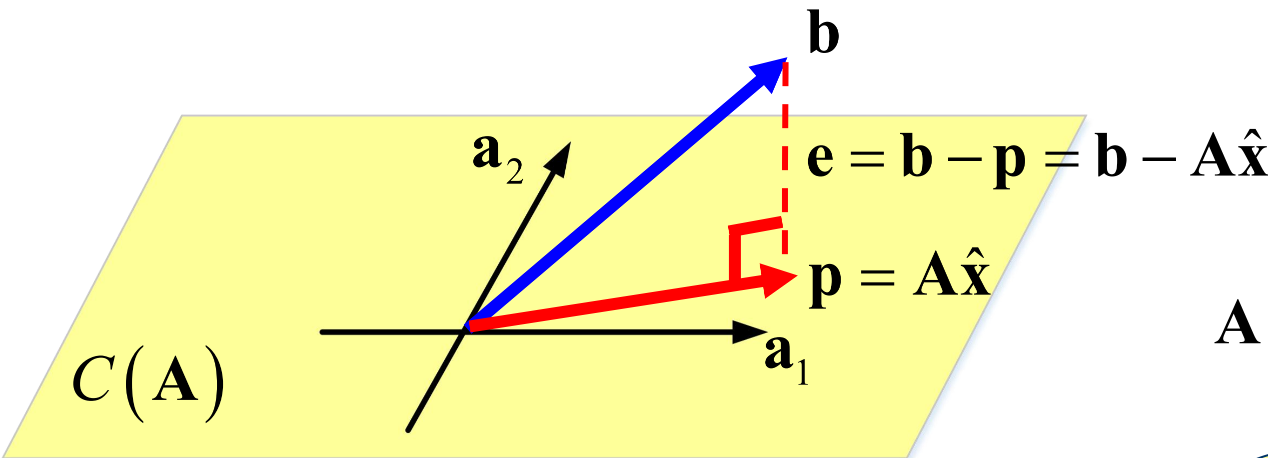
$$= \left[\mathbf{q}_1\ \mathbf{q}_2 \cdots \mathbf{q}_n\right]\begin{bmatrix} \mathbf{q}_1{}^T\mathbf{b} \\ \mathbf{q}_2{}^T\mathbf{b} \\ \vdots \\ \mathbf{q}_n{}^T\mathbf{b} \end{bmatrix} = \mathbf{q}_1\left(\mathbf{q}_1{}^T\mathbf{b}\right) + \cdots + \mathbf{q}_n\left(\boxed{\mathbf{q}_n{}^T\mathbf{b}}\right)$$

投影

$$\boxed{\mathbf{q}_n{}^T\mathbf{b} = \frac{\mathbf{q}_n{}^T\mathbf{b}}{\mathbf{q}_n{}^T\mathbf{q}}}$$

正交基的作用：解耦

正交矩阵与子空间投影



标准正交基

$$\mathbf{A}=\mathbf{Q}=[\mathbf{q}_1\cdots\mathbf{q}_n]$$

解释了正交矩阵的性质

$$\hat{\mathbf{x}}=\mathbf{Q}^T\mathbf{b}$$

$$\mathbf{p}=\mathbf{Q}\hat{\mathbf{x}}=\mathbf{Q}\mathbf{Q}^T\mathbf{b} \xRightarrow{m=n} C(\mathbf{A})=\mathbf{R}^m \xRightarrow[\mathbf{Pb}=\mathbf{b}]{\mathbf{b}\text{在列空间内}} \mathbf{Q}\mathbf{Q}^T=\mathbf{I}$$

$$\mathbf{P}=\mathbf{Q}\mathbf{Q}^T$$

$$\hat{\mathbf{x}}=\mathbf{Q}^T\mathbf{b} \Longleftrightarrow \mathbf{x}=\mathbf{Q}^{-1}\mathbf{b}$$

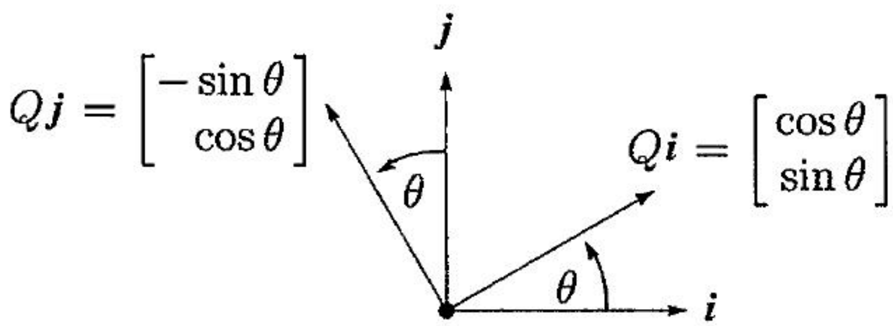
$$\mathbf{b}=\mathbf{q}_1\left(\mathbf{q}_1^T\mathbf{b}\right)+\mathbf{q}_2\left(\mathbf{q}_2^T\mathbf{b}\right)+\cdots+\mathbf{q}_n\left(\mathbf{q}_n^T\mathbf{b}\right)$$

正交基的分量之和

几个典型的正交矩阵

- 旋转

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



- 置换

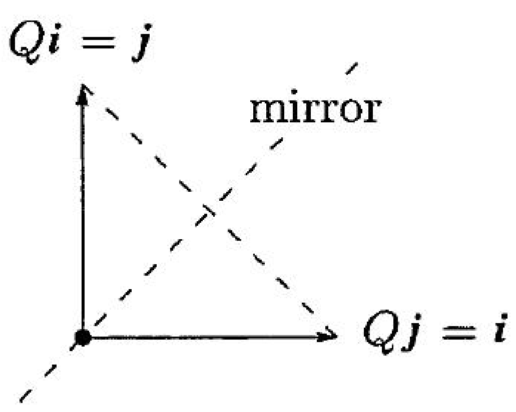
$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- 反射

$$Q = I - 2uu^{\mathrm{T}}$$

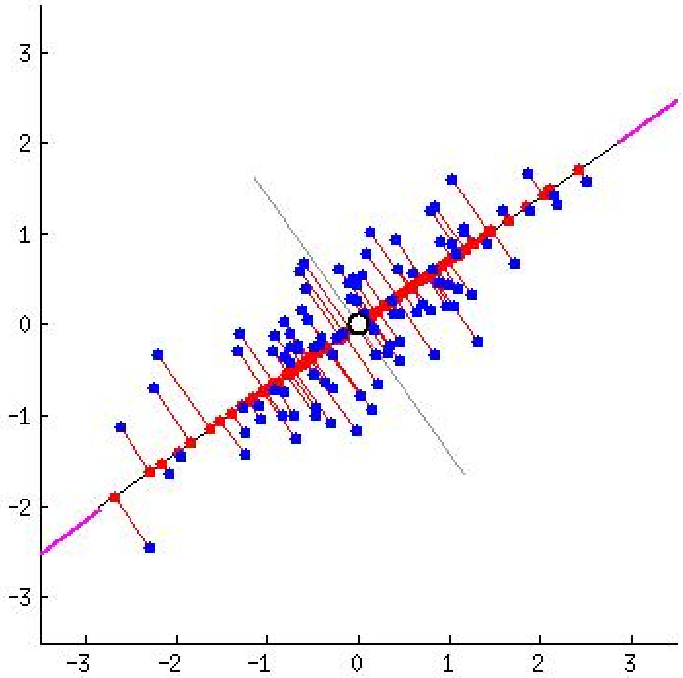
$$\mathbf{Qx} = \left(\mathbf{I} - \mathbf{uu}^T - \mathbf{uu}^T\right)\mathbf{x}$$

$$= \left(\mathbf{I} - \mathbf{uu}^T\right)\mathbf{x} - \mathbf{uu}^T\mathbf{x}$$

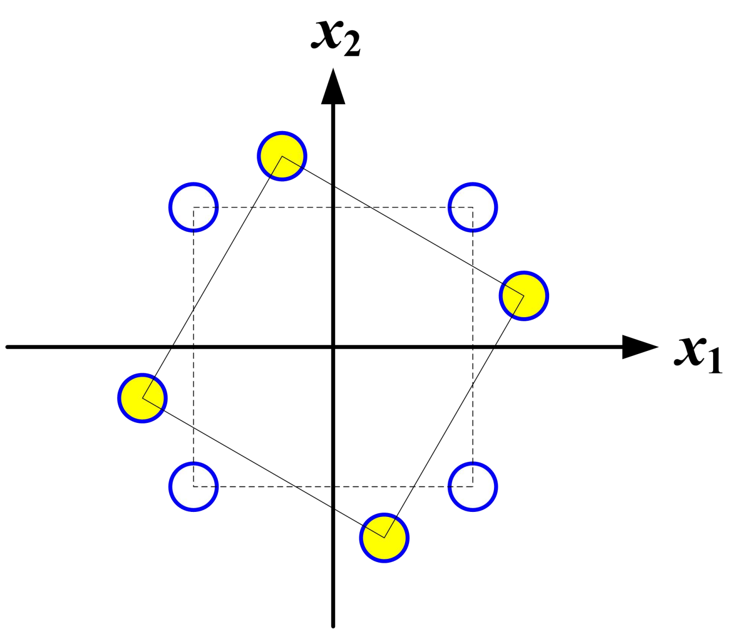


旋转矩阵的应用

- 数据降维



- 信号星座设计



内容提要

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- Gram-Schmidt正交化
- 矩阵的QR分解

Gram-Schmidt正交化

问题：如何由一组给定的向量$\mathbf{a}_1$、$\mathbf{a}_2$、$\mathbf{a}_3$，构造一组标准正交向量$\mathbf{q}_1$、$\mathbf{q}_2$、$\mathbf{q}_3$?

假定$\mathbf{a}_1$、$\mathbf{a}_2$、$\mathbf{a}_3$线性无关：

线性相关的情况?

$$\mathbf{a}_1 \Rightarrow \mathbf{q}_1 = \mathbf{a}_1 / \|\mathbf{a}_1\|$$

$$\mathbf{a}_2 \Rightarrow \mathbf{B} = \mathbf{a}_2 - \left(\mathbf{q}_1^T \mathbf{a}_2\right)\mathbf{q}_1 \Rightarrow \mathbf{q}_2 = \mathbf{B} / \|\mathbf{B}\|$$

$$\mathbf{a}_3 \Rightarrow \mathbf{C} = \mathbf{a}_3 - \left(\mathbf{q}_1^T \mathbf{a}_3\right)\mathbf{q}_1 - \left(\mathbf{q}_2^T \mathbf{a}_3\right)\mathbf{q}_2$$

$$\mathbf{q}_3 = \mathbf{C} / \|\mathbf{C}\|$$

波形信号的Gram-Schmidt正交化

假设存在如下的信号集合：

$$\{s_1(t),\ldots,s_M(t)\}$$

如何由这组信号构造一个标准正交函数集？

$$\{\phi_1(t),\ldots,\phi_K(t)\}$$

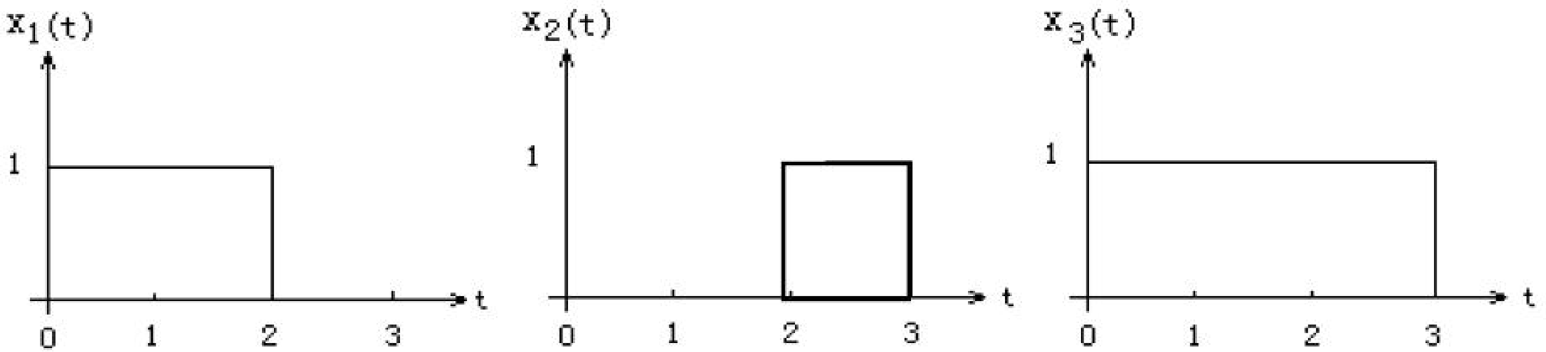
$$\varphi_1(t)=\frac{1}{\sqrt{E_1}}s_1(t) \qquad E_1=\int_T\left|s_1(t)\right|^2dt$$

$$\theta_2(t)=s_2(t)-\langle s_2,\phi_1\rangle\phi_1(t) \qquad \phi_2(t)=\frac{1}{\sqrt{E_2}}\theta_2(t) \qquad E_2=\int_T\left|\theta_2(t)\right|^2dt$$

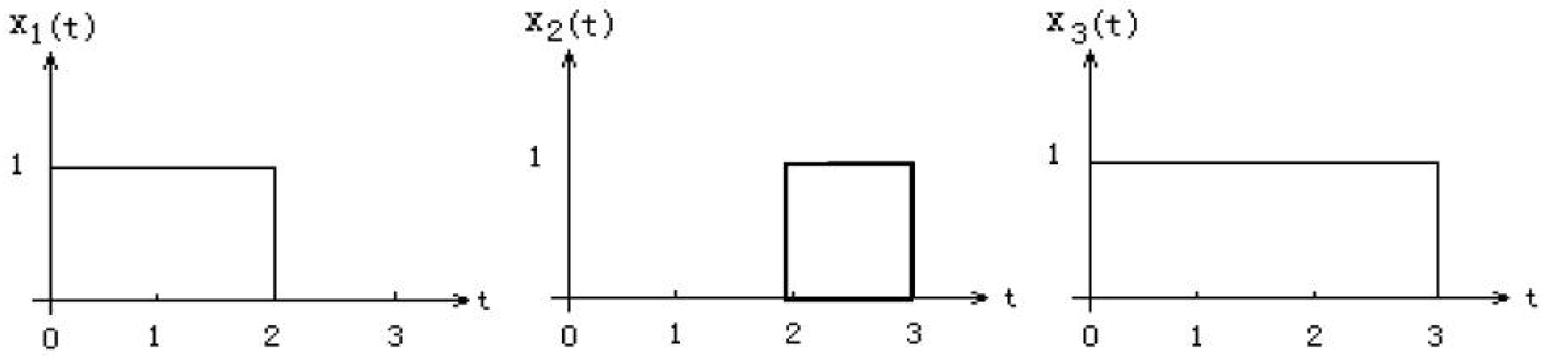
$$\theta_j(t)=s_j(t)-\sum_{k=1}^{j-1}\langle s_j,\phi_k\rangle\phi_k(t) \qquad \phi_j(t)=\frac{1}{\sqrt{E_j}}\theta_j(t) \qquad E_j=\int_T\left|\theta_j(t)\right|^2dt$$

Gram-Schmidt正交化的例子

- 例：利用Gram-Schmidt过程得到下列信号集的一组标准正交基。



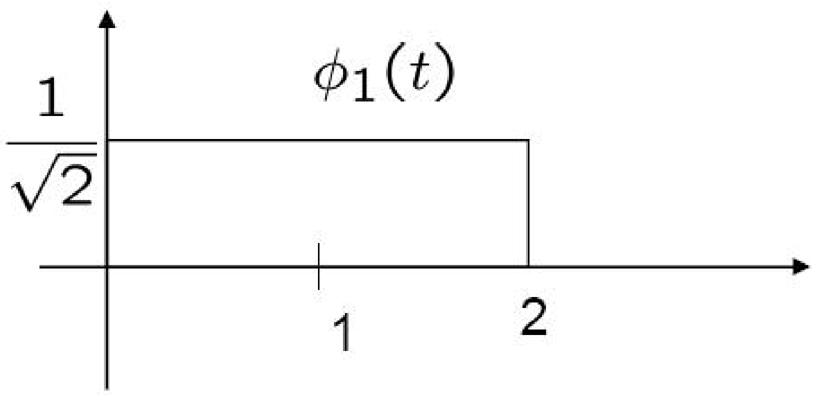
Gram-Schmidt正交化的例子



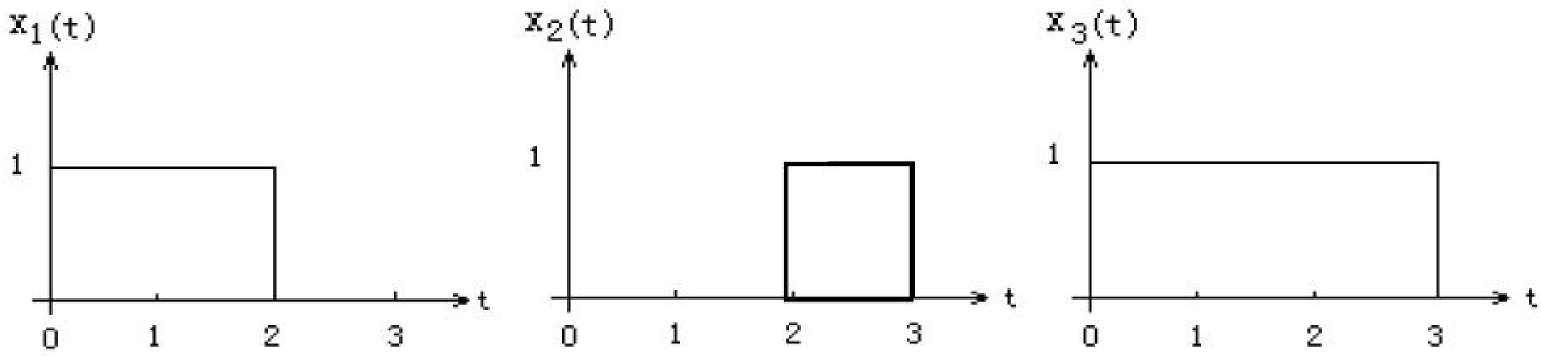
- 步骤一：

$$E_1 = \int_{-\infty}^{\infty} x_1^2(t)dt = 2$$

$$\phi_1(t) = \frac{1}{\sqrt{2}} x_1(t)$$



Gram-Schmidt正交化的例子

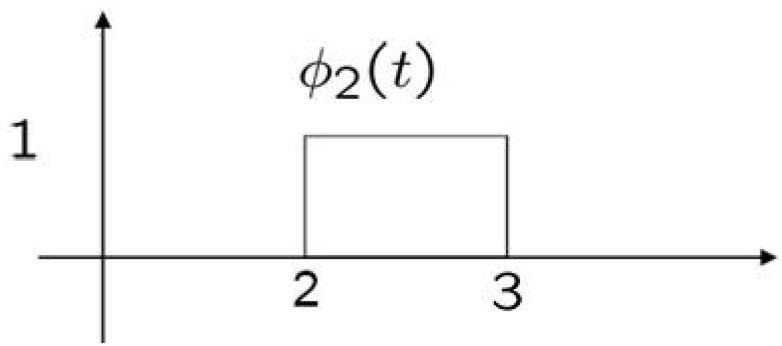


- 步骤二：

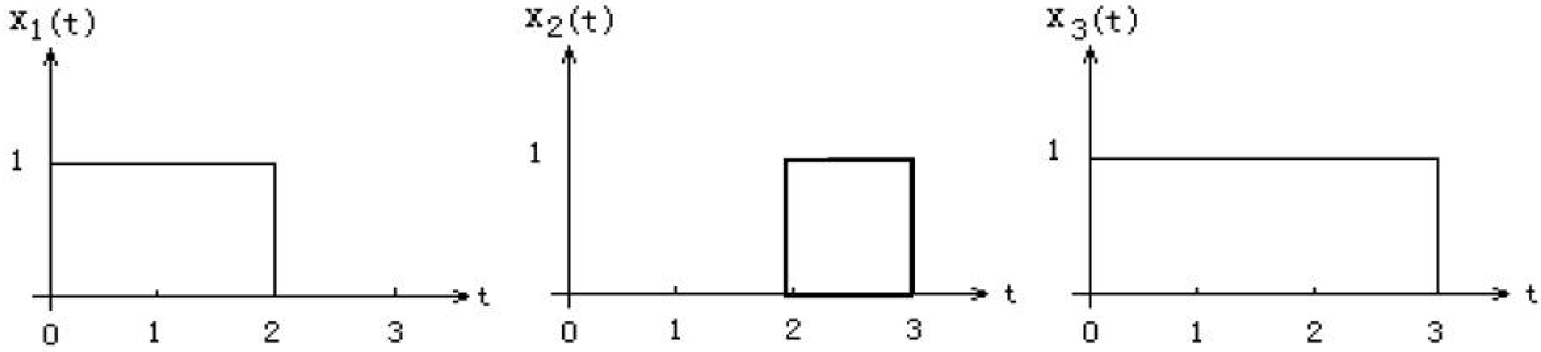
$$x_{21} = \int_{-\infty}^{\infty} x_2(t)\phi_1(t)dt = 0$$

$$g_2(t) = x_2(t) \quad E_{g_2} = 1$$

$$\phi_2(t) = x_2(t)$$



Gram-Schmidt正交化的例子



- 步骤三：

$$x_{31} = \int_{-\infty}^{\infty} x_3(t)\phi_1(t)dt = \sqrt{2}$$

$$x_{32} = \int_{-\infty}^{\infty} x_3(t)\phi_2(t)dt = 1$$

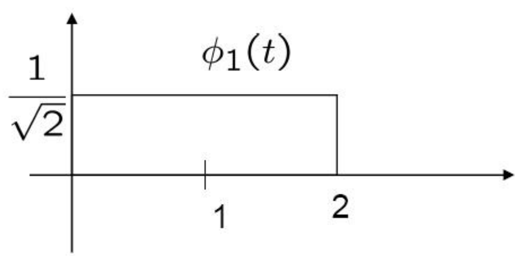
没有新的基函数生成

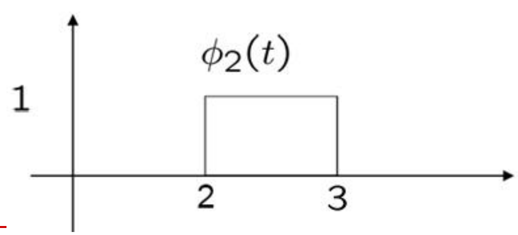
$$g_3(t) = x_3(t) - x_{31}f_1(t) - x_{32}f_2(t) = 0$$

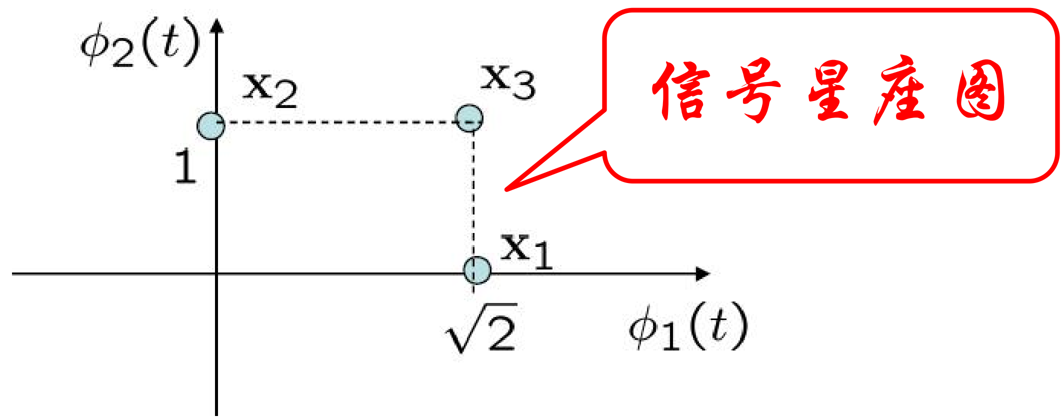
Gram-Schmidt正交化的例子

➢ 基函数：

$$\begin{cases} \phi_1(t) = \dfrac{1}{\sqrt{2}} x_1(t) \\[2em] \phi_2(t) = x_2(t) \end{cases} \Rightarrow \begin{aligned} x_1(t) &= \sqrt{2}\phi_1(t) \\ x_2(t) &= \phi_2(t) \\ x_3(t) &= \sqrt{2}\phi_1(t) + \phi_2(t) \end{aligned}$$

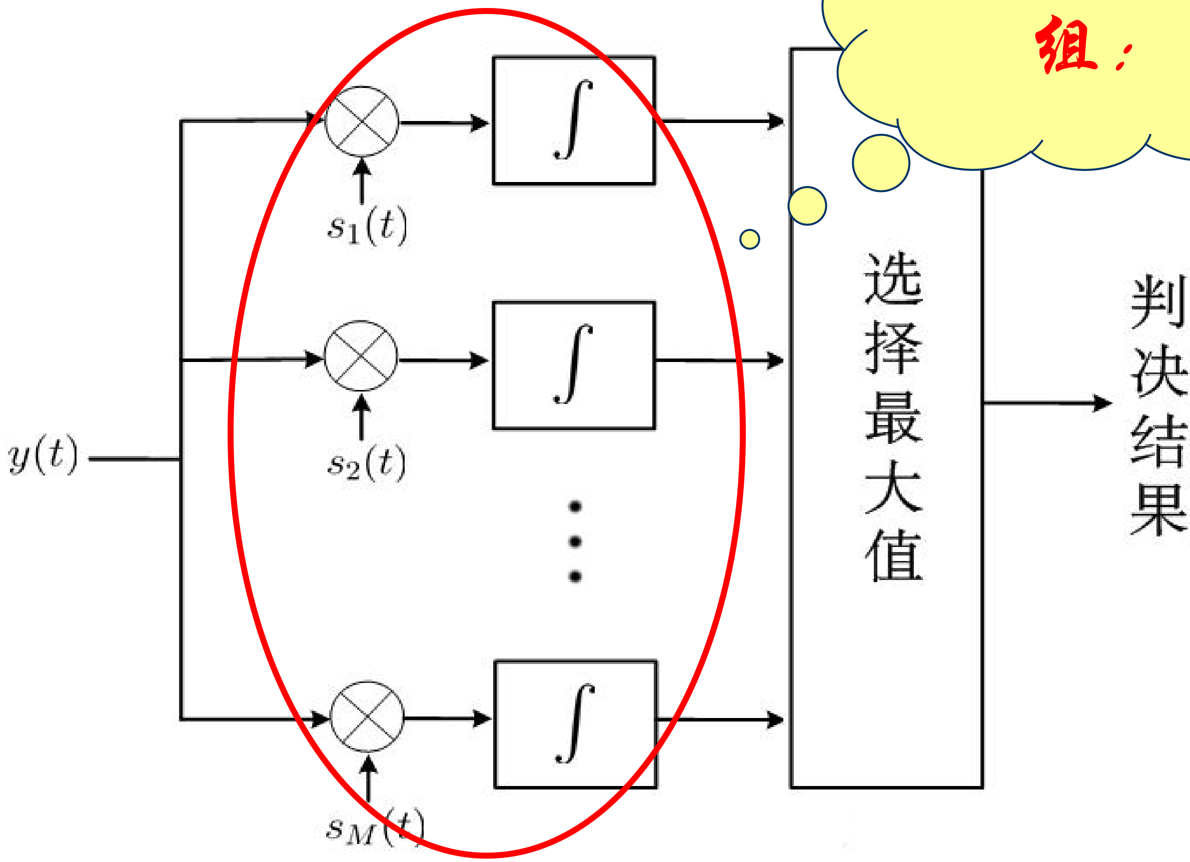






数字通信系统接收机设计

相关接收机组：内积



$$s_i(t) = a_{i,1}\phi_1(t) + a_{i,2}\phi_2(t) + \cdots + a_{i,k}\phi_k(t)$$

$$s_i(t) \leftrightarrow \begin{bmatrix} a_{i,1} & a_{i,2} & \cdots & a_{i,k} \end{bmatrix}$$

数字通信系统接收机设计

$$s_i(t) = a_{i,1}\phi_1(t) + a_{i,2}\phi_2(t) + \cdots + a_{i,k}\phi_k(t)$$

$$s_i(t) \leftrightarrow \left[a_{i,1}\ a_{i,2} \cdots a_{i,k} \right]$$

$$y^s(t) = \beta_{i,1}\phi_1(t) + \beta_{i,2}\phi_2(t) + \cdots + \beta_{i,k}\phi_k(t)$$

$$y^s(t) \leftrightarrow \left[\beta_{i,1}\ \beta_{i,2} \cdots \beta_{i,k} \right]$$

$$\int y^s(t)s_i(t)dt = \left\langle \left[a_{i,1}\ a_{i,2} \cdots a_{i,k} \right], \left[\beta_{i,1}\ \beta_{i,2} \cdots \beta_{i,k} \right] \right\rangle$$

积分 $\Rightarrow$ 向量运算

$$\left\| \mathbf{y}^s - \mathbf{s}_k \right\|^2 = \left(\mathbf{y}^s - \mathbf{s}_k \right)^T \left(\mathbf{y}^s - \mathbf{s}_k \right) = \left\| \mathbf{y}^s \right\|^2 + \left\| \mathbf{s}_k \right\|^2 - 2\left\langle \mathbf{y}^s, \mathbf{s}_k \right\rangle$$

$$\left\langle \mathbf{y}^s, \mathbf{s}_k \right\rangle = \left\| \mathbf{y}^s \right\|^2 + \left\| \mathbf{s}_k \right\|^2 - \frac{\left\| \mathbf{y}^s - \mathbf{s}_k \right\|^2}{2}$$

信号星座

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矩阵的QR分解

假定$\mathbf{a}_1$、$\mathbf{a}_2$、$\mathbf{a}_3$线性无关：

$$\mathbf{a}_1 \Rightarrow \mathbf{q}_1 = \mathbf{a}_1 / \|\mathbf{a}_1\| \Rightarrow \boxed{\mathbf{a}_1 = \mathbf{q}_1 \|\mathbf{a}_1\| = \mathbf{q}_1 \left(\mathbf{q}_1{}^T \mathbf{a}_1\right)}$$

$$\mathbf{a}_2 \Rightarrow \mathbf{B} = \mathbf{a}_2 - \left(\mathbf{q}_1^T \mathbf{a}_2\right)\mathbf{q}_1 \Rightarrow \mathbf{q}_2 = \mathbf{B} / \|\mathbf{B}\|$$

$$\mathbf{a}_3 \Rightarrow \mathbf{C} = \mathbf{a}_3 - \left(\mathbf{q}_1^T \mathbf{a}_3\right)\mathbf{q}_1 - \left(\mathbf{q}_2^T \mathbf{a}_3\right)\mathbf{q}_2$$

$$\mathbf{q}_3 = \mathbf{C} / \|\mathbf{C}\|$$

$$\Rightarrow \boxed{\mathbf{a}_2 = \mathbf{q}_1 \left(\mathbf{q}_1{}^T \mathbf{a}_2\right) + \mathbf{q}_2 \left(\mathbf{q}_2{}^T \mathbf{a}_2\right)} \rightarrow \mathbf{q}_2 \|\mathbf{B}\|$$

$$\Rightarrow \boxed{\mathbf{a}_3 = \mathbf{q}_1 \left(\mathbf{q}_1{}^T \mathbf{a}_3\right) + \mathbf{q}_2 \left(\mathbf{q}_2{}^T \mathbf{a}_3\right) + \mathbf{q}_3 \left(\mathbf{q}_3{}^T \mathbf{a}_3\right)} \rightarrow \mathbf{q}_3 \|C\|$$

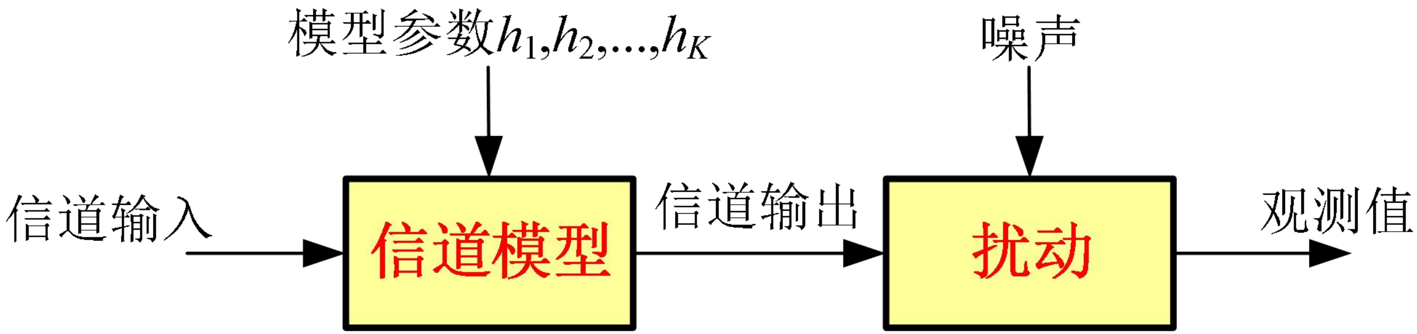
矩阵的QR分解

$$\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}=\begin{bmatrix}\mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3\end{bmatrix}\begin{bmatrix}\mathbf{q}_1^T\mathbf{a}_1 & \mathbf{q}_1^T\mathbf{a}_2 & \mathbf{q}_1^T\mathbf{a}_3\\ & \mathbf{q}_2^T\mathbf{a}_2 & \mathbf{q}_2^T\mathbf{a}_3\\ & & \mathbf{q}_3^T\mathbf{a}_3\end{bmatrix}$$

A **Q** **R**

- 任意列满秩矩阵A都可以分解成一个列向量标准正交的矩阵Q和一个上三角矩阵R的乘积
- 上三角矩阵R的主对角线元素大于零，其数值等于正交化过程中产生的正交向量的范数

QR分解在最小二乘法中的应用



$$\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1K} \\ x_{21} & x_{22} & \cdots & x_{2K} \\ & \vdots & & \\ x_{N1} & x_{N2} & \cdots & x_{NK} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_K \end{bmatrix} + \begin{bmatrix} n_1 \\ \vdots \\ n_N \end{bmatrix}$$

$$\text{minimize } \left\| \mathbf{y} - \mathbf{X}\mathbf{h} \right\|^2$$

$$\left(\mathbf{X}^T\mathbf{X}\right)\hat{\mathbf{h}} = \mathbf{X}^T\mathbf{y}$$

$$\hat{\mathbf{h}} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y}$$

$$\mathbf{X} = \mathbf{Q}\mathbf{R} \Rightarrow \mathbf{X}^T\mathbf{X} = \mathbf{R}^T\mathbf{Q}^T\mathbf{Q}\mathbf{R} = \mathbf{R}^T\mathbf{R} \Rightarrow \mathbf{R}\hat{\mathbf{h}} = \mathbf{Q}^T\mathbf{y}$$

无须求逆，通过反
向迭代快速求解

谢谢大家！