Elements of Information Theory

Lecture 4
Data Compression (Source Coding)

Instructor: Yichen Wang

Ph.D./Professor

School of Information and Communications Engineering
Division of Electronics and Information
Xi’an Jiaotong University

Outlines

- AEP for Data Compression (Source Coding)
- Fixed-Length Coding
- Variable-Length Coding
- Optimal Codes
- Huffman Code

AEP for Data Compression

Let X_1, X_2, … , X_n be independent, identically distributed (i.i.d.) random variables with probability mass function p(x).

⇩

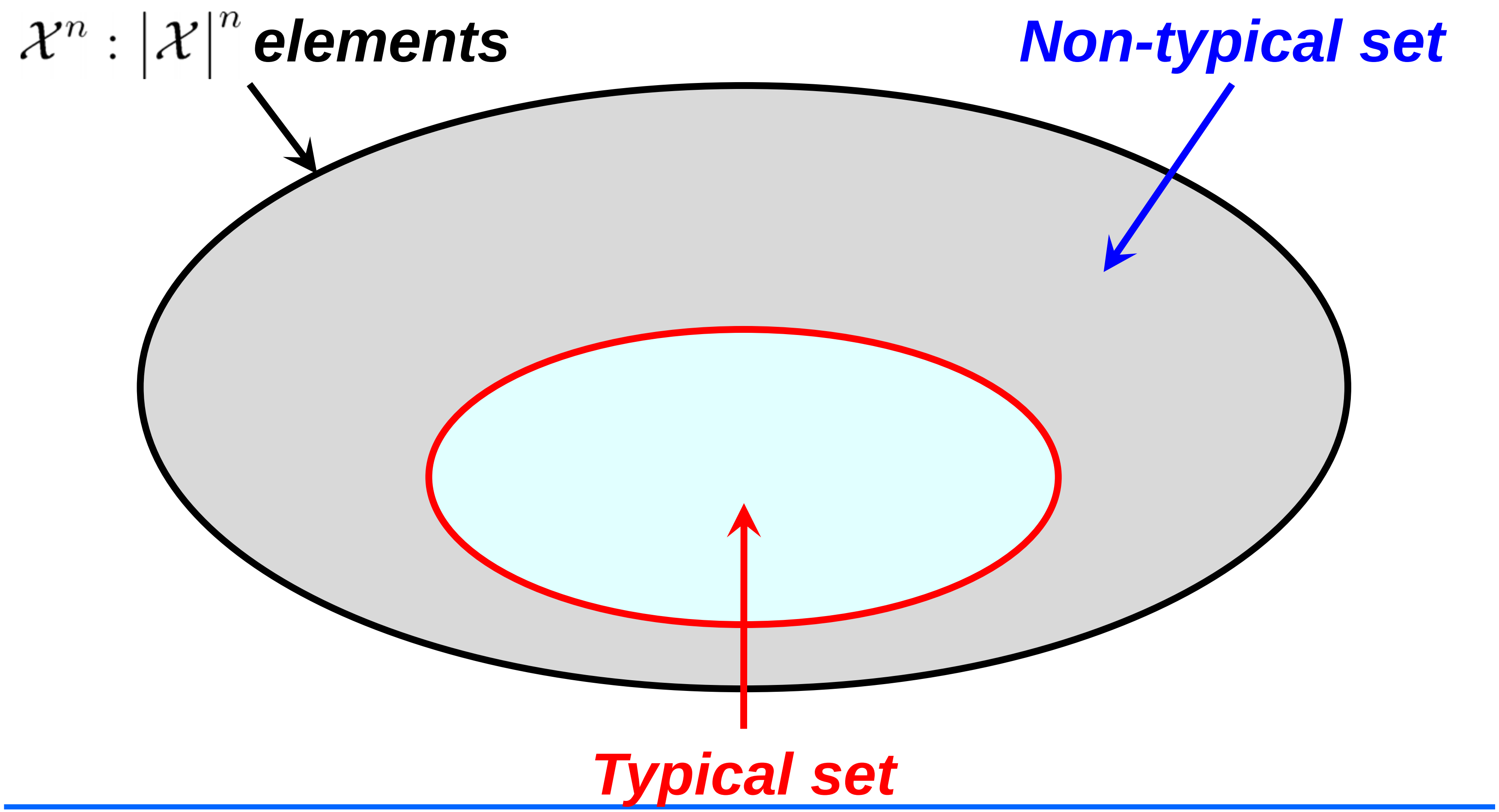
How many bits are enough for describe the sequence of random variables X_1, X_2, … , X_n?

⇩

Typical Sequence and Typical Set

AEP for Data Compression

Divide all sequences in $\mathcal{X}^n$ into two sets


$\mathcal{X}^n : |\mathcal{X}|^n$ elements

Non-typical set

Typical set

$A_\epsilon^{(n)} : \; \leq 2^{n(H+\epsilon)}$ elements

AEP for Data Compression

Indexing sequences in typical set

- ***No more than n(H+ε)+1 bits (Why having 1 extra bit?)***
- ***If prefixing all these sequences by a 0, the required total length is no more than n(H+ε)+2 bits***

Indexing sequences in atypical set

- ***No more than*** $n\log|\mathcal{X}|+1$ ***bits (Why?)***
- ***If prefixing all these sequences by a 1, the required total length is no more than*** $n\log|\mathcal{X}|+2$ ***bits***

AEP for Data Compression

- Use x^n to denote a sequence $x_1, x_2, \dots, x_n$
- Let $l(x^n)$ be the length of the codeword corresponding to x^n
- The expected length of the codeword is

$$
\begin{aligned}
\mathbb{E}\{l(x^n)\} &= \sum_{x^n} p(x^n) l(x^n) \\
&= \sum_{x^n \in A_\epsilon^{(n)}} p(x^n) l(x^n) + \sum_{x^n \in A_\epsilon^{(n)c}} p(x^n) l(x^n) \\
&\le \sum_{x^n \in A_\epsilon^{(n)}} p(x^n) \left[n(H+\epsilon) + 2 \right] + \sum_{x^n \in A_\epsilon^{(n)c}} p(x^n) \left[n\log|\mathcal{X}| + 2 \right] \\
&= \Pr\left\{ A_\epsilon^{(n)} \right\} \left[n(H+\epsilon) + 2 \right] + \Pr\left\{ A_\epsilon^{(n)^c} \right\} \left[n\log|\mathcal{X}| + 2 \right] \\
&\le n(H+\epsilon) + 2 + \epsilon \left[n\log|\mathcal{X}| + 2 \right] \\
&= n\left(H + \underbrace{\epsilon + \epsilon\log|\mathcal{X}| + \frac{2}{n} + \frac{2\epsilon}{n}}_{\text{Can be arbitrary small}} \right)
\end{aligned}
$$

AEP for Data Compression

Theorem

Let X^n be i.i.d. ~ $p(x)$. Let $\varepsilon > 0$. Then there exists a code that maps sequences x^n of length n into binary strings such that the mapping is one-to-one (and therefore invertible) and

$$\mathbb{E}\left\{\frac{1}{n}l\left(x^n\right)\right\} \leq H(X) + \epsilon$$

for n sufficiently large.

⇩

We can represent sequences X^n using $nH(X)$ bits on average.

Outlines

- AEP for Data Compression (Source Coding)
- Fixed-Length Coding
- Variable-Length Coding
- Optimal Codes
- Huffman Code

Fixed-Length Coding

Consider a sequence of *L* consecutive letters from the discrete source

$$\boldsymbol{u}^L = \left(u_1, u_2, \cdots, u_L\right),\ u_l \in \left\{a_1, a_2, \cdots, a_K\right\},\ \forall l$$

Encode such sequences into *fixed-length* code words. Moreover, we assume that the *code alphabet contains D symbols* and the *length of each code word is N*.

If we wish to *provide a separate code word for each source sequence*, we mush have

$$\frac{N}{L} \geq \frac{\log K}{\log D}$$

What does it mean?

Fixed-Length Coding

Question 1:

If we wish to use fewer than logK/logD code letters per source symbol, what should we do?

------ We must relax our insistence on always being able to decode the source sequence from the code sequence.

Question 2:

If we assign the code words only to a subset of the source sequences of length L, what will happen when L is sufficient large?

Fixed-Length Coding

We also consider the sequence of *L* consecutive letters from a discrete source

$$\boldsymbol{u}^L = (u_1, u_2, \cdots, u_L), \ u_l \in \{a_1, a_2, \cdots, a_K\}$$

For the *discrete memoryless source*, the probability of a given sequence $\boldsymbol{u}^L$ is

$$\Pr\{\boldsymbol{u}^L\} = \prod_{l=1}^{L} P(u_l)$$

Moreover, the *self-information* of the sequence is

$$I(\boldsymbol{u}^L) = \sum_{l=1}^{L} I(u_l)$$

Fixed-Length Coding

The law of large numbers tells us that

$$\frac{I\left(\boldsymbol{u}^L\right)}{L} \longrightarrow H(U), \text{ if } L \text{ is sufficiently large}$$

The probability of a given *L*-length sequence

$$\Pr\{\boldsymbol{u}^L\} \longrightarrow 2^{-LH(U)}$$

The number of such *L*-length sequences

$$M_T \longrightarrow 2^{LH(U)}$$

Typical sequence and Typical set

Fixed-Length Coding

By employing the *weak law of large numbers*, we have

$$\Pr\left\{\left|\frac{I\left(\boldsymbol{u}^L\right)}{L} - H(U)\right| > \delta\right\} \le \epsilon$$

Let *T* be the set of sequences u^L, for which

$$\left|\frac{I\left(\boldsymbol{u}^L\right)}{L} - H(U)\right| \le \delta,\ \boldsymbol{u}^L \in T$$

Then, we have that the probability of event *T* satisfies

$$\Pr\{T\} \ge 1 - \epsilon$$

The number of sequences in set *T* satisfies the following two constraints

$$\begin{cases} M_T \le 2^{L[H(U)+\delta]} \\ M_T \ge [1-\epsilon]2^{L[H(U)-\delta]} \end{cases}$$

Fixed-Length Coding

Theorem (Fixed-Length Source Coding Theorem)

Let a discrete memoryless source have finite entropy H(U) and consider a coding from sequences of L source letters into sequences of N code letters from a code alphabet of size D. Only one source sequence can be assigned to each code sequence and we let P_e be the probability of occurrence of a source sequence for which no code sequence has been provided.

Then, for any δ>0, if

$$\frac{N}{L} \geq \frac{H(U)+\delta}{\log D}$$

P_e can be made arbitrarily small by making L sufficiently large.

Conversely, if

$$\frac{N}{L} \leq \frac{H(U)-\delta}{\log D}$$

then P_e must become arbitrarily close to 1 as L is made sufficiently large.

Fixed-Length Coding

Question:

Is fixed-length coding good enough?

Coding Efficiency

$$\eta = \frac{H(U)}{\frac{N}{L}\log D} \longrightarrow$$

The number of bits for describing each source symbol after source coding

We hope that the coding efficiency can reach 1, i.e., $\eta \to 1$

For fixed-length coding, we have

$$\eta = \frac{H(U)}{H(U)+\delta} \longrightarrow \delta = \frac{1-\eta}{\eta}H(U)$$

Fixed-Length Coding

The error probability *P_e*

$$P_e \leq \epsilon(L,\delta) = \frac{D\left[I(u_l)\right]}{L\delta^2}$$

For the given variance and parameter δ, the error probability P_e can be smaller than arbitrary positive number if L is sufficiently large.

$$L \geq \frac{D\left[I(u_l)\right]}{H^2(U)} \frac{\eta^2}{(1-\eta)^2\epsilon}$$

$$D\left[I(u_l)\right] = \sum_{u_l \in \mathcal{U}} P(u_l)\left[\log P(u_l)\right]^2 - H^2(U)$$

The required length L increases as P_e decreases!

Fixed-Length Coding

Example

The alphabet for the source is {a_1, a_2, a_3, a_4} and the corresponding probability distribution is 1/2, 1/4, 1/8, and 1/8, respectively. If we require the coding efficiency is no smaller than 95% and the error probability P_e cannot larger than 10^{-6}, please calculate the required minimum length of source letter sequence.

$$H(U) = -\sum_{i=1}^{4} P(a_i)\log P(a_i) = 1.75 \text{ bits}$$

$$D[I(u)] = \sum_{i=1}^{4} P(a_i)\left[\log P(a_i)\right]^2 - H^2(U) = 0.6875 \text{ bits}^2$$

Fixed-Length Coding

Example

The alphabet for the source is {a$_1$, a$_2$, a$_3$, a$_4$} and the corresponding probability distribution is 1/2, 1/4, 1/8, and 1/8, respectively. If we require the coding efficiency is no smaller than 95% and the error probability P$_e$ cannot larger than 10^{-6}, please calculate the required minimum length of source letter sequence.

$$L \geq \frac{D\left[I(u)\right]}{H^2(U)} \frac{\eta^2}{(1-\eta)^2\varepsilon} = 0.81 \times 10^8$$

Fixed-length coding: Storage and Delay

Outlines

- AEP for Data Compression (Source Coding)
- Fixed-Length Coding
- Variable-Length Coding
- Optimal Codes
- Huffman Code

Variable-Length Coding

Example

The alphabet for the source is {a_1, a_2, a_3, a_4} and the corresponding probability distribution is 1/2, 1/4, 1/8, and 1/8, respectively. If we encode source symbols as follows:

a_1	a_2	a_3	a_4
0	10	110	111

Then, we have

$$\overline{n} = \sum_{i=1}^{4} p_i n_i = \frac{1}{2} \times 1 + \frac{1}{4} \times 2 + 2 \times \frac{1}{8} \times 3 = 1.75 \text{ bits} \longrightarrow \eta = \frac{H(U)}{\overline{n}} = 1$$

Variable-Length Coding

Most source coding is variable-length in the reality.

Basic Principles for Variable-Length Coding Design

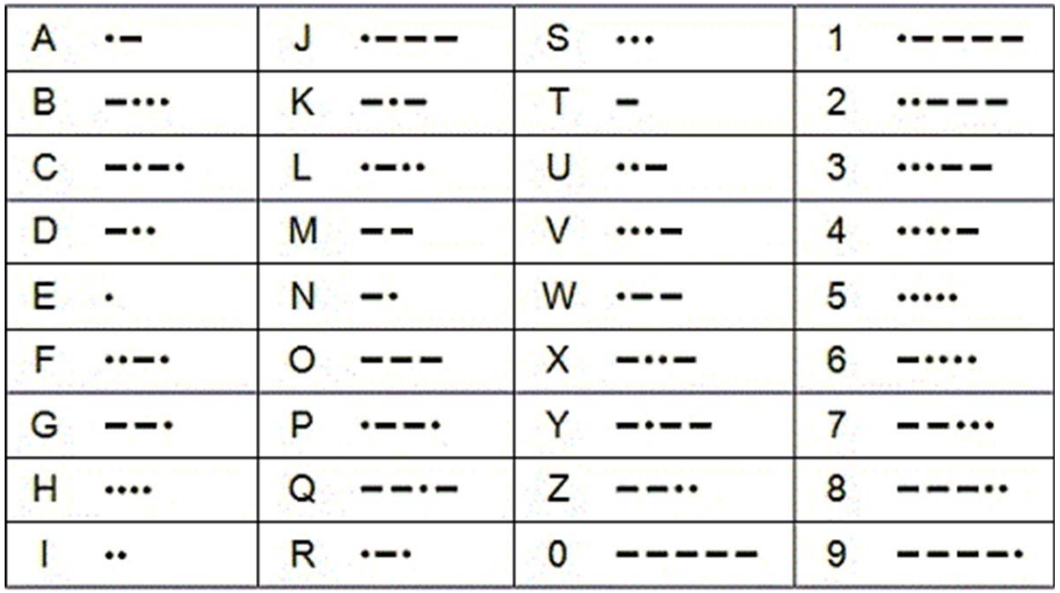
1. ***Assign short descriptions to the most frequent outcomes of the data source;***
2. ***Assign longer descriptions to the less frequent outcomes.***

Morse Code

Morse code is a reasonably efficient code for English alphabet using an alphabet of four symbols: dot, dash, letter space, and word space.

Variable-Length Coding

International Morse Code



A	·−	J	·−−−	S	···	1	·−−−−
B	−···	K	−·−	T	−	2	··−−−
C	−·−·	L	·−··	U	··−	3	···−−
D	−··	M	−−	V	···−	4	····−
E	·	N	−·	W	·−−	5	·····
F	··−·	O	−−−	X	−··−	6	−····
G	−−·	P	·−−·	Y	−·−−	7	−−···
H	····	Q	−−·−	Z	−−··	8	−−−··
I	··	R	·−·	0	−−−−−	9	−−−−·

1. *The length of a dot is one unit;*
2. *A dash is three units;*
3. *The space between parts of the same letter is one unit;*
4. *The space between letters is three units;*
5. *The space between words is seven units;*

Short sequences represent frequent letters (e.g., E); long sequences represent infrequent letters (e.g., Q).

Variable-Length Coding

For a given source, we can design many source coding schemes.

Source Letters	$P(a_k)$	*Code 1*	*Code 2*	*Code3*	*Code 4*
a_1	0.5	0	0	0	0
a_2	0.25	0	1	01	10
a_3	0.125	1	00	011	110
a_4	0.125	10	11	0111	1110

Question:

Which coding scheme is the best?

Variable-Length Coding

Definition (Source Code)

A source code C for a random variable X is a mapping from $\mathcal{X}$, the range of X, to $\mathcal{D}^\star$, the set of finite-length strings of symbols from a D-ary alphabet.

Let C(x) denote the codeword corresponding to x.

Let l(x) denote the length of C(x).

For example

C(red) = 00 and C(blue) = 11 are source codes for $\mathcal{X} = \{red, blue\}$ with alphabet $\mathcal{D} = \{0,1\}$.

Variable-Length Coding

Definition

The expected length L(C) of a source code C(x) for a random variable X with probability mass function p(x) is given by

$$L(C) = \sum_{x \in \mathcal{X}} p(x) l(x)$$

where l(x) is the length of the codeword associated with x.

The expected length is a critically important metric to judge the source coding scheme!!!

Variable-Length Coding

Definition

*A code is said to be **nonsingular** if every element of the range of X maps into a different string in $\mathcal{D}^\star$; that is,*

$$x \neq x' \Longrightarrow C(x) \neq C(x')$$

Sending sequences of symbols X

Definition

*The **extension** C* of a code C is the mapping from finite-length strings of $\mathcal{X}$ to finite-length of $\mathcal{D}$, defined by*

$$C(x_1 x_2 \cdots x_n) = C(x_1)C(x_2)\cdots C(x_n)$$

where $C(x_1)C(x_2)\cdots C(x_n)$ indicates concatenation of the corresponding codewords.

Variable-Length Coding

Definition

A code is called uniquely decodable if its extension is nonsingular.

1. *Any encoded strings comes from an unique sequence of source symbols;*
2. *For any sequence of source symbol, the sequence can be reconstructed unambiguously from the encoded bit sequence.*

Variable-Length Coding

Challenge for Uniquely Decodable Code Design

Decode the codewords from the encoded sequence as fast as possible ---- reducing decoding delay.

Definition

A code is called a prefix code or an instantaneous code if no codeword is a prefix of any other codeword.

⇩

The decoder can decode each codeword of a prefix code immediately on the arrival of the last bit in that word.

Variable-Length Coding

Example

Source Letters	*$P(a_k)$*	*Code 1*	*Code 2*	*Code3*	*Code 4*
a_1	*0.5*	*0*	*0*	*0*	*0*
a_2	*0.25*	*0*	*1*	*01*	*10*
a_3	*0.125*	*1*	*00*	*011*	*110*
a_4	*0.125*	*10*	*11*	*0111*	*1110*

Code 1: Singular

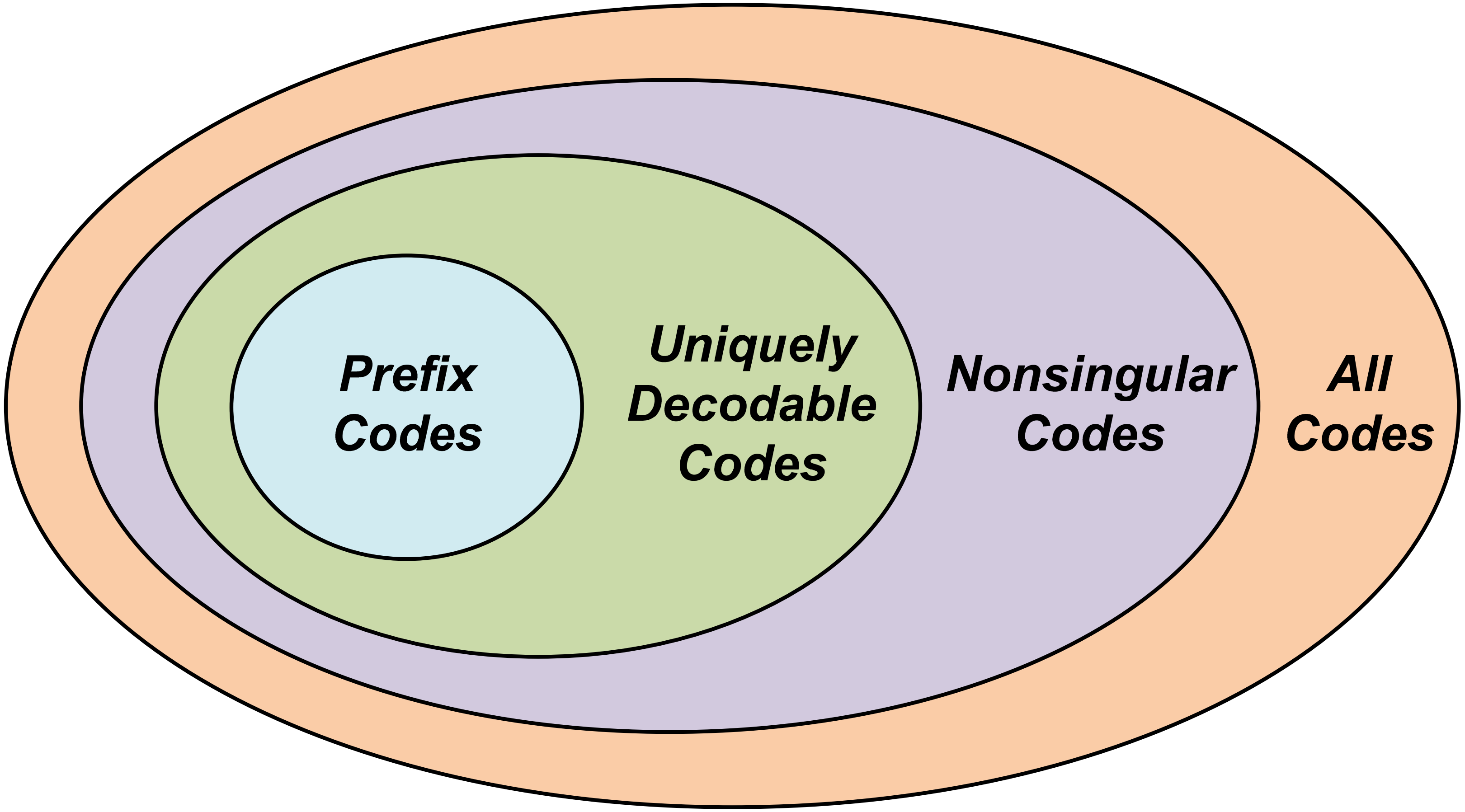
Code 2: Nonsingular, but not uniquely decodable

Code 3: Uniquely decodable, but not instantaneous

Code 4: Instantaneous, also prefix codes

Variable-Length Coding

Relationships among Prefix Code, Uniquely decodable codes, and Other Codes



Variable-Length Coding

Code Tree (Binary Case)

- *A corresponding binary code tree can be defined which grows from a root on the left to leaves on the right representing codewords.*
- *Each branch is labeled 0 or 1 and each string corresponding to the branch labels from the root to that node.*
- *The tree is extended just enough to include each codeword.*

Binary Code Tree → D-ary Code Tree

Variable-Length Coding

Relationship between Prefix Code and Code Tree

- ***Root → Codeword beginning***
- ***Number of branches → Number of symbols (#-ary)***
- ***Intermedium node → Part of a codeword***
- ***Leaves → A codeword***
- ***Full tree → Fixed-length code***
- ***Not full tree → Variable-length code***

The prefix condition implies that no codeword is an ancestor of any other codeword on the tree. Hence, each codeword eliminates its descendants as possible codewords.

Variable-Length Coding

What is the ideal source code for us?

- ***Prefix code (Instantaneous code)***
- ***Minimize the expected length as much as possible***

Theorem (Kraft Inequality)

For any instantaneous code (prefix code) over an alphabet of size D, the codeword lengths l_1, l_2, … , l_m must satisfy the inequality

$$\sum_{i=1}^{m} D^{-l_i} \leq 1$$

Conversely, given a set of codeword lengths that satisfy this inequality, there exists an instantaneous code with these word lengths.

Variable-Length Coding

Some Discussions

- ***Any codeword set that satisfies the prefix condition has to satisfy the Kraft inequality. (necessary condition)***
- ***Kraft inequality is a sufficient condition for the existence of codeword set with the specified set of codeword lengths.***

Variable-Length Coding

Theorem (Extended Kraft Inequality)

For any countably infinite set of codewords that form a prefix code, the codeword lengths satisfy the extended Kraft inequality,

$$\sum_{i=1}^{\infty} D^{-l_i} \leq 1$$

Conversely, given any l_1, l_2, … satisfying the extended Kraft inequality, we can construct a prefix code with these codeword lengths.

Outlines

- AEP for Data Compression (Source Coding)
- Fixed-Length Coding
- Variable-Length Coding
- Optimal Codes
- Huffman Code

Optimal Codes

Prefix code with the minimum expected length

- *The codeword length of prefix code, l_1, l_2, … , l_m, satisfy the Kraft equality*
- *The expected length is less than the expected length of any other prefix code*

$$\underset{l_1,l_2,\cdots,l_m}{\text{Minimize}} \quad L = \sum_{i=1}^{m} p_i l_i$$

$$\text{s.t.} \quad \sum_{i=1}^{m} D^{-l_i} \leq 1$$

$$l_1, l_2, \cdots, l_m \text{ are integers}$$

Optimal Codes

We first neglect the integer constraint on l_i

$$\underset{l_1, l_2, \cdots, l_m}{\text{Minimize}} \quad L = \sum_{i=1}^{m} p_i l_i$$

$$\text{s.t.} \qquad \sum_{i=1}^{m} D^{-l_i} \leq 1$$

Lagrangian Multiplier

$\lambda \geq 0$

Construct the Lagrangian function:

$$\mathcal{J}\left(\{l_i\}_{i=1}^{m}, \lambda\right) = \sum_{i=1}^{m} p_i l_i + \lambda \left(\sum_{i=1}^{m} D^{-l_i} - 1 \right)$$

Differentiating with respect to l_i , we can obtain

$$\frac{\partial \mathcal{J}}{\partial l_i} = p_i - \lambda D^{-l_i} \log_e D$$

Optimal Codes

Setting the derivative to 0, we can obtain

$$D^{-l_i} = \frac{p_i}{\lambda \log_e D}$$

We hope the following equality can be held:

$$\sum_{i=1}^{m} D^{-l_i} = 1$$

Then, we derive that the optimal Lagrangian multiplier is

$$\lambda^* = 1/\log_e D$$

Consequently, we have

$$l_i^* = -\log_D p_i$$

Optimal Codes

This noninteger choice of codeword lengths yields expected codeword length

$$L^* = \sum_{i=1}^{m} p_i l_i^* = -\sum_{i=1}^{m} p_i \log_D p_i = H_D(X)$$

Choose a set of codeword lengths l_i "close" to the optimal set.

Theorem

The expected length L of any instantaneous D-ary code for a random variable X is greater than or equal to the entropy $H_D(X)$; that is

$$L \geq H_D(X)$$

with equality if and only if $D^{-l_i} = p_i$ **.**

Optimal Codes

We have obtained the lowerbound of the optimal expected length. How about the upperbound?

Choose the set of codeword length {l_1, l_2, … , l_m} such that

$$l_i = \left\lceil \log_D \frac{1}{p_i} \right\rceil, \quad i = 1, 2, \cdots, m$$

$$\sum_{i=1}^{m} D^{-l_i} = \sum_{i=1}^{m} D^{-\left\lceil \log_D \frac{1}{P_i} \right\rceil} \le \sum_{i=1}^{m} D^{-\log_D \frac{1}{P_i}} = 1$$

$$\Bigg\downarrow \quad \log_D \frac{1}{p_i} \le l_i < \log_D \frac{1}{p_i} + 1$$

$$H_D(X) \le L < H_D(X) + 1$$

Optimal Codes

Theorem

Let $l_1^\star, l_2^\star, \cdots, l_m^\star$ ***be optimal codeword lengths for a source distribution*** $\mathbf{p}$ ***and a D-ary alphabet, and let be the associated expected length of an optimal code***

$$L^\star = \sum_{i=1}^{m} p_i l_i^\star$$

Then, we have

$$H_D(X) \leq L^\star < H_D(X) + 1$$

- ***There is an overhead that is at most 1 bit due to the integer constraint***
- ***We can reduce the overhead per symbol by spreading it out over many symbols.***

Optimal Codes

Consider n consequent symbols, we have

$$L_n = \frac{1}{n}\sum p(x_1, x_2, \cdots, x_n) l(x_1, x_2, \cdots, x_n) = \frac{1}{n}\mathbb{E}\Big\{l(X_1, X_2, \cdots, X_n)\Big\}$$

$$H(\underbrace{X_1, X_2, \cdots, X_n}_{\textit{i.i.d.}}) \le \mathbb{E}\Big\{l(X_1, X_2, \cdots, X_n)\Big\} < H(X_1, X_2, \cdots, X_n) + 1$$

$$\longrightarrow H(X) \le L_n < H(X) + \frac{1}{n}$$

By using large block lengths we can achieve an expected code-length per symbol arbitrarily close to the entropy.

Theorem

The minimum expected codeword length per symbol satisfies

$$\frac{H(X_1, X_2, \cdots, X_n)}{n} \le L_n^\star < \frac{H(X_1, X_2, \cdots, X_n)}{n} + \frac{1}{n}$$

Outlines

- AEP for Data Compression (Source Coding)
- Fixed-Length Coding
- Variable-Length Coding
- Optimal Codes
- Huffman Code

Huffman Code

- ***We have discussed several properties of the optimal codes***

- ***The problem is, how to construct an optimal code?***

- ***Huffman code is an optimal (shortest expected length) prefix code for a given distribution***

- ***There are many optimal codes***

- ***Huffman procedure constructs one such optimal code by a simple algorithm***

Huffman Code

Procedure of Huffman Codes (Binary Case)

Source X	x_1	x_2	……	x_m
Probability	p_{x1}	p_{x2}	……	p_{xm}

x_i *(i = 1, 2, … , m) ------ Binary strings*

1. *Rearrange the m source symbols in a descent order according to their probabilities, i.e.,* $p_1 \geq p_2 \geq \cdots \geq p_m$ *;*
2. *Assign codeword 0 and 1 to the two source symbols with the least probabilities, and combine these two messages into a new symbol with the probability that is the sum of the least two probabilities;*

Huffman Code

Procedure of Huffman Codes (Binary Case)

Source X	x_1	x_2	……	x_m
Probability	p_{x1}	p_{x2}	……	p_{xm}

x_i *(i = 1, 2, … , m) ------ Binary strings*

3. *Rearrange the new source symbols in descent order according to their new probabilities;*
4. *Take the procedure iteratively until all messages are combined.*

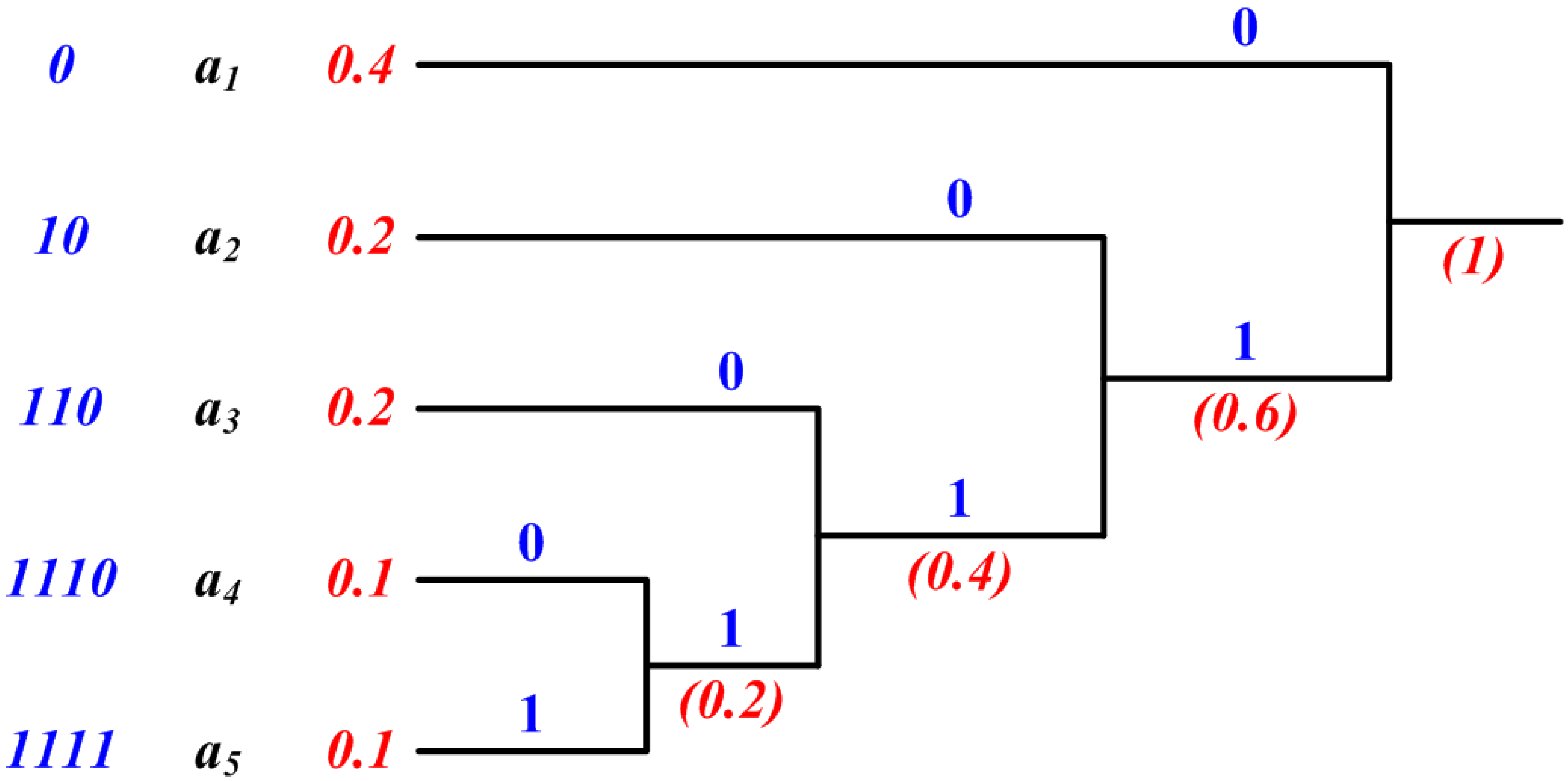
Huffman Code

Example

Please find the Huffman code of the following source, and the coding efficiency.

Source X	x_1	x_2	x_3	x_4	x_5
Probability p(X)	*0.4*	*0.2*	*0.2*	*0.1*	*0.1*

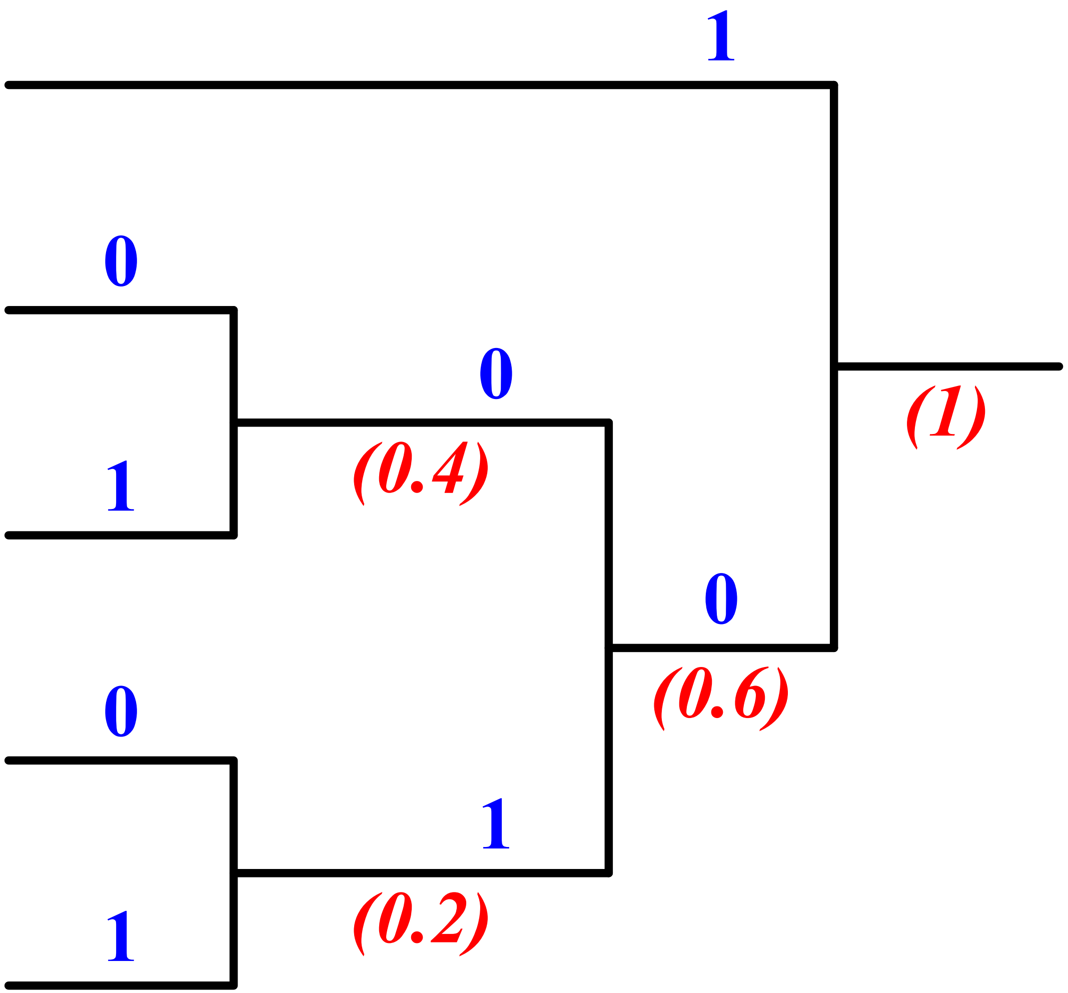
Huffman Code



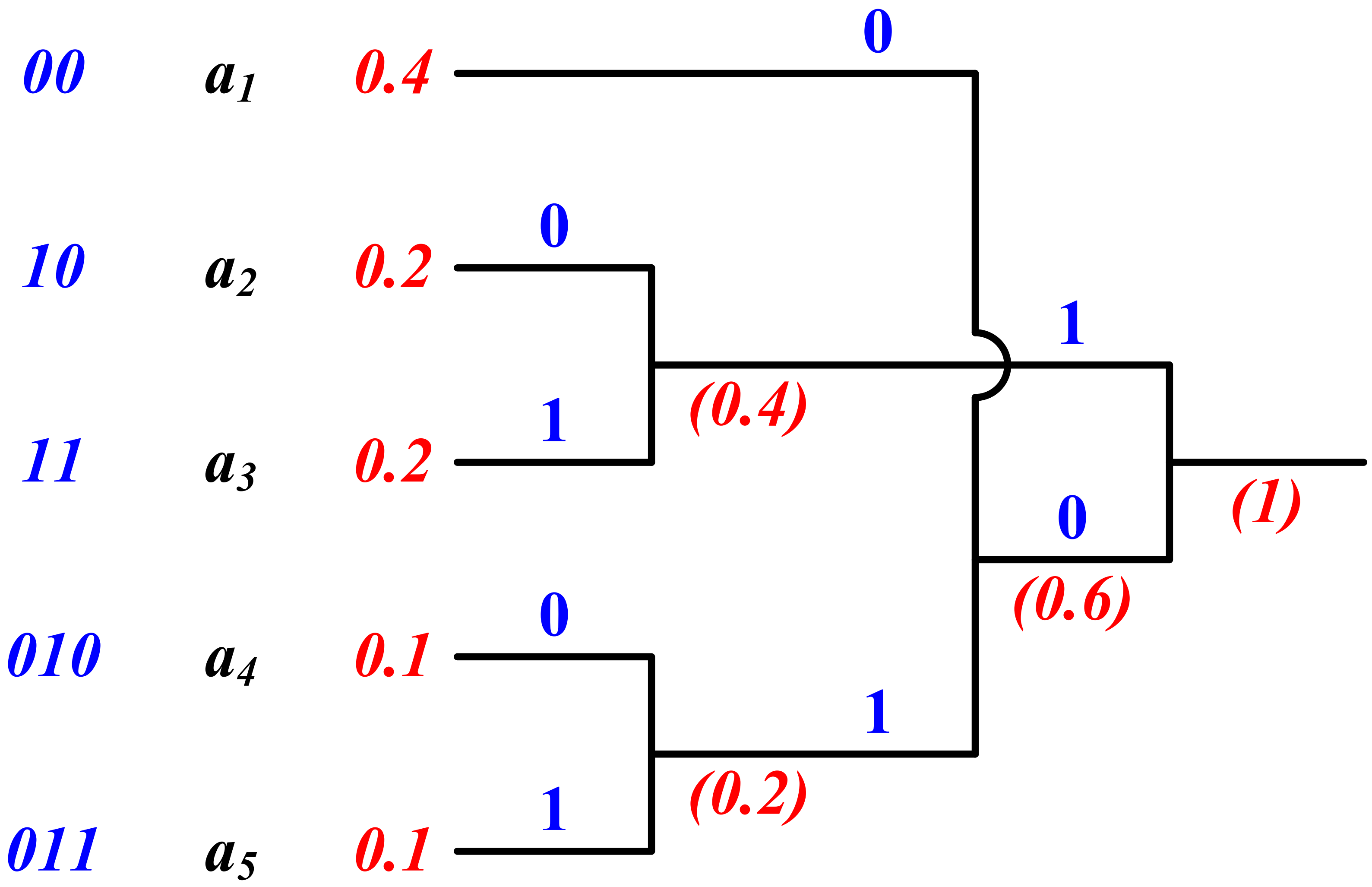
0	a_1	*0.4*
10	a_2	*0.2*
110	a_3	*0.2*
1110	a_4	*0.1*
1111	a_5	*0.1*

Huffman Code

1	a_1	*0.4*
000	a_2	*0.2*
001	a_3	*0.2*
010	a_4	*0.1*
011	a_5	*0.1*



Huffman Code



00	a_1	*0.4*
10	a_2	*0.2*
11	a_3	*0.2*
010	a_4	*0.1*
011	a_5	*0.1*

Huffman Code

The average length

$$\overline{L}_1 = 0.4\times 1 + 0.2\times 2 + 0.2\times 3 + 0.1\times 4 + 0.1\times 4 = 2.2 \text{ bits}$$

$$\overline{L}_2 = 0.4\times 1 + 0.2\times 3 + 0.2\times 3 + 0.1\times 3 + 0.1\times 3 = 2.2 \text{ bits}$$

$$\overline{L}_3 = 0.4\times 2 + 0.2\times 2 + 0.2\times 2 + 0.1\times 3 + 0.1\times 3 = 2.2 \text{ bits}$$

The entropy of the source

$$H(X) = -\sum_{i=1}^{5} p(x_i)\text{log}p(x_i) = 2.122 \text{ bits}$$

Coding efficiency

$$\eta = \frac{H(X)}{L} = \frac{2.122}{2.2} = 96.4\%$$

Huffman Code

Question:

The above three obtained Huffman codes have the same average codeword length. However, which one is better?

$$\sigma^2 = \mathbb{E}\left\{\left(l_i - \overline{L}\right)^2\right\} = \sum_{i=1}^{m} p_i\left(l_i - \overline{L}\right)^2$$

$$\sigma_1 = 0.4 \times 1.2^2 + 0.2 \times 0.2^2 + 0.2 \times 0.8^2 + 0.1 \times 1.8^2 \times 2 = 1.36$$

$$\sigma_2 = 0.4 \times 1.2^2 + 0.2 \times 0.8^2 \times 2 + 0.1 \times 0.8^2 \times 2 = 0.96$$

$$\sigma_3 = 0.4 \times 0.2^2 + 0.2 \times 0.2^2 \times 2 + 0.1 \times 0.8^2 \times 2 = 0.16$$

The third one is the best!!!

Huffman Code

- ***There are many optimal source codes. Huffman procedure constructs one such optimal code.***

- ***Can we find some properties that at least one optimal code satisfies?***

- ***We restrict our attention to prefix codes (instantaneous codes)***

- ***Without loss of generality, we order the probability masses as follows:***

$$p_1 \geq p_2 \geq \cdots \geq p_m$$

Huffman Code

Lemma (Binary Case)

For any distribution, there exists an optimal instantaneous code (with minimum expected length) that satisfies the following properties:

1. ***The lengths are ordered inversely with the probabilities (i.e., if*** $p_j > p_k$***, then*** $l_j \le l_k$***).***
2. ***The two longest codewords have the same length.***
3. ***Two of the longest codewords differ only in the last bit and correspond to the two least likely symbols.***

Huffman Code

Proof for Property 1

Consider an optimal code C_m in which $p_j > p_k$.

Consider another code C'_m where the codewords j and k of C_m interchanged.

$$
\begin{aligned}
L(C'_m) - L(C_m) &= \sum p_i l'_i - \sum p_i l_i \\
&= p_j l_k + p_k l_j - p_j l_j - p_k l_k \\
&= (p_j - p_k)(l_k - l_j) \geq 0
\end{aligned}
$$

Consequently, property 1 holds.

Huffman Code

- ***The lemma tells us that if $p_1 \geq p_2 \geq \ldots \geq p_m$, there exists an optimal code with $l_1 \leq l_2 \leq \ldots \leq l_{m-1} = l_m$, and codewords $C(x_{m-1})$ and $C(x_m)$ that differ only in the last bit.***

- ***We call codes that satisfy the properties in the lemma as canonical codes.***

- ***By trimming, rearranging, and swapping, we can convert any prefix code to the canonical form.***

Huffman Code

Optimality of Huffman Code

Source X	x_1	x_2	……	x_m
Probability	p_1	p_2	……	p_m
Codeword-length	l_1	l_2	……	l_m

$$\mathbf{p} = (p_1, p_2, \cdots, p_m)$$

Huffman reduction ↓ ↑ *Extension*

$$\mathbf{p}' = (p_1, p_2, \cdots, p_{m-2}, p_{m-1} + p_m)$$

Huffman Code

$$p'_i = \begin{cases} p_i, & i \leq m-2 \\ p_{m-1} + p_m, & i = m-1 \end{cases} \qquad l_i = \begin{cases} l'_i, & i \leq m-2 \\ l'_{m-1} + 1, & i = m-1, m \end{cases}$$

↓

$$\overline{L}(\mathbf{p}) = \overline{L}(\mathbf{p}') + p'_{m-1}$$

- ***If a prefix code is optimal for X, then the corresponding prefix code for its Huffman reduction is also optimal.***

- ***If a prefix code is optimal for X, then the corresponding prefix code for its extension is also optimal.***

Huffman Code

Optimal for ($x_1, x_2, \ldots, x_{m-1}$)

$$\overline{L}(\mathbf{p}) = \overline{L}^*(\mathbf{p}') + p_{m-1} + p_m$$

$$\overline{L}(\mathbf{p}') = \overline{L}^*(\mathbf{p}) - p_{m-1} - p_m$$

Optimal for ($x_1, x_2, \ldots, x_m$)

$$\Downarrow$$

$$\left[\overline{L}(\mathbf{p}') - \overline{L}^*(\mathbf{p}')\right] + \left[\overline{L}(\mathbf{p}) - \overline{L}^*(\mathbf{p})\right] = 0$$

Theorem (Arbitrary D-ary coding)

Huffman coding is optimal; that is, if C^* is a Huffman code and C' is any other uniquely decodable code, we have

$$\overline{L}(C^*) \leq \overline{L}(C')$$

Huffman Code

Question:

How to design Huffman code for any arbitrary D-ary source coding?

For D-ary source coding, define a complete code tree as a finite tree in which each intermediate node has D nodes of the next higher order stemming from it.

Lemma

The number of terminal nodes in a complete code tree with alphabet size D must be of the form D+m(D-1) for some integer m.

Huffman Code

Binary case:

As D=2, the number of terminal nodes in the complete code tree can be written as m+2

If the number of source symbols is larger than 1, the source symbols can be represented by a complete code tree.

The above conclusion does not hold if D>2.

Huffman Code

D-ary case (D>2):

For an optimal code, all the unused terminal nodes must have the same length as the longest code word.

All unused terminal nodes can be arranged to differ in only the last digit.

An optimal code tree must have, at most, D-2 unused terminal nodes.

Huffman Code

D-ary case (D>2):

Denote the number of unused terminals by B. The source has K symbols. Then, we have

$$B + K = D + m(D-1)$$

$$K - 2 = m(D-1) + (D-2-B)$$

The remainder upon dividing K-2 by D-1 →

$$R_{D-1}(K-2) = D - 2 - B$$

$$D - B = 2 + R_{D-1}(K-2)$$

The first step is to group together the least probable $D - B = 2 + R_{D-1}(K-2)$ ***nodes.***

Huffman Code

Example

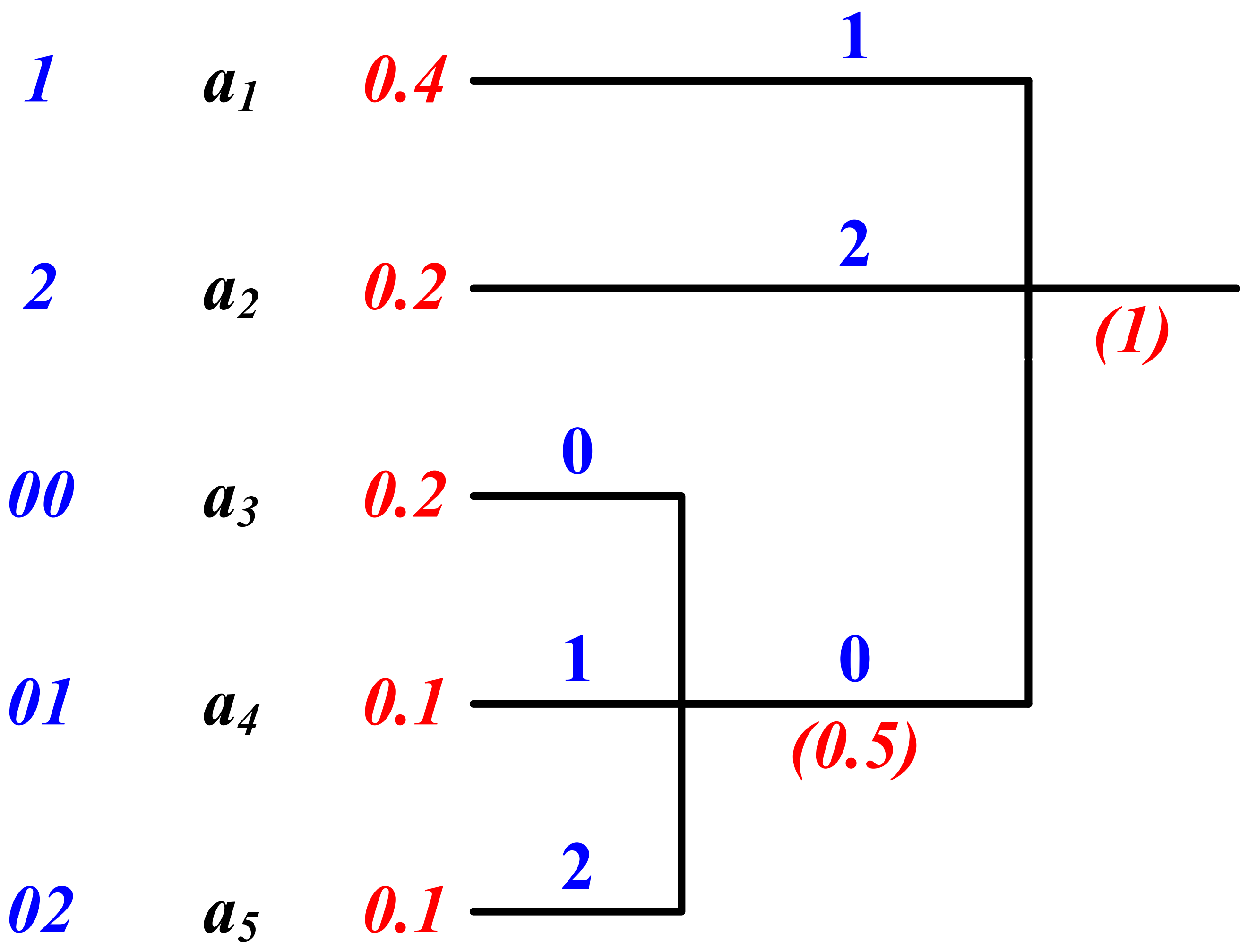
Please find the Huffman code of the following source when D=3

Source X	x_1	x_2	x_3	x_4	x_5
Probability p(X)	*0.4*	*0.2*	*0.2*	*0.1*	*0.1*

$$D - B = 2 + R_{D-1}(K-2) = 2 + R_2(3) = 3$$

The first step is to group together x_3, x_4, and x_5.

Huffman Code



Huffman Code

Example

Please find the Huffman code of the following source when D=3

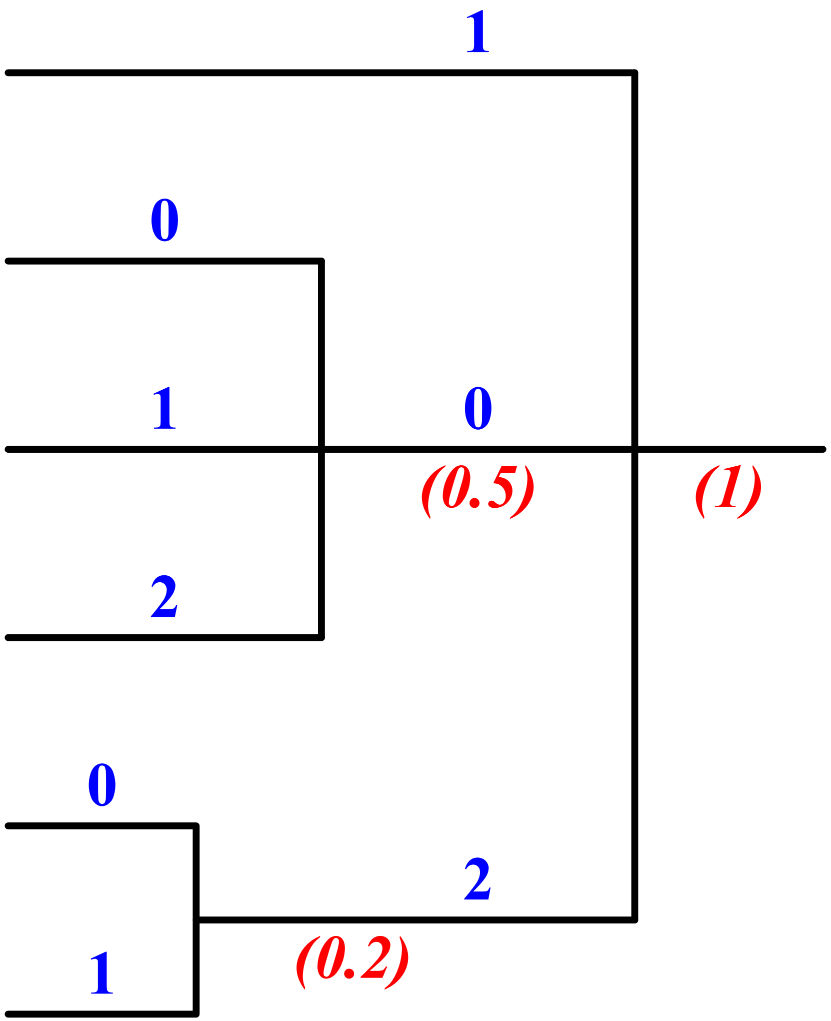
Source X	x_1	x_2	x_3	x_4	x_5	x_6
Probability p(X)	*0.3*	*0.2*	*0.15*	*0.15*	*0.1*	*0.1*

$$D - B = 2 + R_{D-1}(K-2) = 2 + R_2(4) = 2$$

The first step is to group together x_5 and x_6.

Huffman Code

1	x_1	*0.3*
00	x_2	*0.2*
01	x_3	*0.15*
02	x_4	*0.15*
20	x_5	*0.1*
21	x_6	*0.1*



Review

1. AEP for Data Compression (Source Coding)

Basic idea:
Divide all sequences into two sets: Typical set and Atypical set

Theorem

Let X^n be i.i.d. ~ p(x). Let ε > 0. Then there exists a code that maps sequences x^n of length x into binary strings such that the mapping is one-to-one (and therefore invertible) and

$$\mathbb{E}\left\{\frac{1}{n}l\left(x^n\right)\right\} \leq H(X)+\epsilon$$

for n sufficiently large.

Review

2. Fixed-Length Source Coding

Theorem (Fixed-Length Source Coding Theorem)

Let a discrete memoryless source have finite entropy H(U) and consider a coding from sequences of L source letters into sequences of N code letters from a code alphabet of size D. Only one source sequence can be assigned to each code sequence and we let P_e be the probability of occurrence of a source sequence for which no code sequence has been provided.

Then, for any δ>0, if

$$\frac{N}{L} \geq \frac{H(U)+\delta}{\log D}$$

P_e can be made arbitrarily small by making L sufficiently large.

Conversely, if

$$\frac{N}{L} \leq \frac{H(U)-\delta}{\log D}$$

then P_e must become arbitrarily close to 1 as L is made sufficiently large.

Review

2. Fixed-Length Source Coding

Coding Efficiency

$$\eta = \frac{H(U)}{\frac{N}{L}\log D} \longrightarrow$$

The number of bits for describing each source symbol after source coding

How to determine the required codeword length L under the given source distribution, the targeted coding efficiency η, and the error probability P_e?

Review

3. Variable-Length Source Coding

Basic Principles:

Assign short descriptions to the most frequent outcomes of the data source and longer descriptions to the less frequent outcomes

Some Definitions:

Expected Length, Nonsingular, Extension, Uniquely Decodable, Prefix Code/Instantaneous Code

Theorem (Kraft Inequality)

For any instantaneous code (prefix code) over an alphabet of size D, the codeword lengths $l_1, l_2, \ldots, l_m$ ***must satisfy the inequality***

$$\sum_{i=1}^{m} D^{-l_i} \leq 1$$

Conversely, given a set of codeword lengths that satisfy this inequality, there exists an instantaneous code with these word lengths.

Review

4. Optimal Source Coding

Theorem

Let $l_1^\star, l_2^\star, \cdots, l_m^\star$ ***be optimal codeword lengths for a source distribution*** **p** ***and a D-ary alphabet, and let*** $L^\star$ ***be the associated expected length of an optimal code***

$$L^\star = \sum_{i=1}^{m} p_i l_i^\star$$

Then, we have

$$H_D(X) \le L^\star < H_D(X) + 1$$

Theorem

The minimum expected codeword length per symbol satisfies

$$\frac{H(X_1, X_2, \cdots, X_n)}{n} \le L_n^\star < \frac{H(X_1, X_2, \cdots, X_n)}{n} + \frac{1}{n}$$

Review

5. *Huffman Code*

(1) Binary Huffman Code

(2) Optimality of Huffman Code

- *Huffman Reduction*
- *Extension*

(3) Arbitrary D-ary Huffman Code

How many symbols should be combined in the first step?