9.8 Bayes Test Procedures

```r
# The data - TV radiation example
x <- c(0.65,0.28,0.47,0.44,0.25,1.03,0.28,0.64,0.71,0.49)
n <- length(x)
xbar <- mean(x)
snsq <- (n-1)*var(x)
```

TV Radiation Example (one-sample t-test)

$H_0 : \mu \leq 0.50$ assume safe levels vs $H_1 : \mu > 0.5$

$U = \frac{\bar{X}_n - \mu_0}{\sigma'/\sqrt{n}}$

Reject H_0 if $U > t^{-1}_{(9)}(.95) = 1.833$

P-value Interpretation (Frequentist)

```
t.test(x,mu=.5,var.equal=TRUE,alternative="greater")
```

```
##
##  One Sample t-test
##
## data:  x
## t = 0.31416, df = 9, p-value = 0.3803
## alternative hypothesis: true mean is greater than 0.5
## 95 percent confidence interval:
##  0.3839617       Inf
## sample estimates:
## mean of x
##     0.524
```

Interpretation of p-value: There is a 38% chance of getting a sample mean as larger or larger than 0.52 just by chance when the true mean radiation level is 0.50 mR/hr.

Loss Functions

Consider two simple hypotheses:

$$H_0 : \theta = \theta_0$$

$$H_1 : \theta = \theta_1$$

d_0: decision to **not** reject H_0

d_1: decision to reject H_0

The loss, $L(\theta_i, d_i)$, that occurs when θ_i is true and decision d_j is chosen is given in the table:

	d_0	d_1
θ_0	0	ω_0
θ_1	ω_1	0

Expected Loss

- Prior probability that H_0 is true: ξ_0
- Prior probability that H_1 is true: $\xi_1 = 1 - \xi_0$
- Expected loss of each test procedure δ: $r(\delta)$

$$r(\delta) = \xi_0 E(Loss|\theta = \theta_0) + \xi_1 E(Loss|\theta = \theta_1)$$

$$r(\delta) = \xi_0 \omega_0 \alpha(\delta) + \xi_1 \omega_1 \beta(\delta)$$

- $\alpha(\delta)$: Pr(Type I error) $= Pr(d_1|\theta = \theta_0)$
- $\beta(\delta)$: Pr(Type II error) $= Pr(d_0|\theta = \theta_1)$

Bayes Test Procedure

A procedure δ for which $r(\delta)$ is minimized is called a **Bayes test procedure**.

E.g Reject H_0/choose d_1 if

$$\begin{aligned} r(d_0|x) &> r(d_1|x) \\ \omega_1 Pr(H_1 \ true|x) &> Pr(H_0 \ true|x) \\ \omega_1[1 - Pr(H_0 \ ture|x)] &> \omega_0 Pr(H_0 \ true|x) \\ Pr(H_0 \ true|x) &< \frac{\omega_1}{\omega_0 + \omega_1} \\ \int_{\Omega_0} \xi(\theta|x) d\theta &\leq \frac{\omega_1}{\omega_0 + \omega_1} \end{aligned}$$

Choose d_1 if $r(d_0|x) > r(d_1|x)$, where $r(d_i|x) = \int L(\theta, d_i)\xi(\theta|x)d\theta$

Reject H_0 if $\frac{\omega_1}{\omega_0+\omega_1} \geq Pr(H_0 \ true|x) = \int_\Omega \xi(\theta|x)d\theta$

TV Example

We need $\xi(\mu|x)$ for Bayesian Inference

- Likelihood: $f_n(x|\mu,\tau) = \prod N(\mu,\tau)$
- prior: $\xi(\mu,\tau) = \xi(\tau)\xi(\mu|\tau) = gamma(\alpha_0,\beta_0)N(\mu_0,\lambda_0\tau)$

$$\xi(\mu,\tau|x) \propto f_n(x|\mu,\tau)\xi(\mu,\tau)$$

Posterior marginal of μ (Theorem 8.6.1 p496)

$U = \left(\frac{\lambda_1\alpha_1}{\beta_1}\right)^{1/2}(\mu - \mu_1) \sim t_{2\alpha_1}$

- $\mu_1 = \left(\frac{\lambda_0}{\lambda_0+n}\right)\mu_0 + \left(\frac{n}{\lambda_0+n}\right)\bar{x}_n$
- $\lambda_1 = \lambda_0 + n$
- $\beta_1 = \beta_0 + \frac{1}{2}s_n^2 + \frac{n\lambda_0(\bar{x}_n-\mu_0)^2}{2(\lambda_0+n)}$
- $\alpha_1 = \alpha_0 + n/2$

Choosing Non-Informative Prior

Recall: When $X_i \sim N(\theta, \sigma^2)$ with σ^2 known $\xi(\theta) \sim N(\mu, \nu^2)$. We let $\nu^2 \to \infty$ to get the non informative prior. Then $\xi(\theta|x) \sim N(\bar{X}_n, \sigma^2/n)$

How do we pick a non-informative prior for μ and σ^2 jointly?

- for mean μ consider uniform, e.g. all intervals (a, a+h) having same prior probability
- for scale parameter $\sigma = \sqrt{\frac{1}{\tau}}$ all intervals (a, ka) having same prior probability

$$\xi(\mu) = 1; \xi(\tau) = \frac{1}{\tau} \to \xi(\mu, \tau) = 1 \times \frac{1}{\tau} = \frac{1}{\tau}$$

Jeffreys Prior:Jeffreys prior is proportional to the square root of the determinant of the Fisher Information matrix and is invariant to reparameterization.

Back to Bayesian Analysis of TV Radiation

$H_0 : \mu \leq \mu_0$ vs $H_1 : \mu > \mu_0$

Consider Pr(H_0 true | data) and Reject H_0 if small.
$Pr(H_0\ true|x) = \int_{\Omega_0} \xi(\mu|x)d\mu$

1. Non-informative Prior:
   $\xi(\mu, \tau) = \xi(\mu|\tau)\xi(\tau) = N(\mu_0, \lambda_0\tau)gamma(\alpha_0, \beta_0)$

For what values of $\mu_0, \lambda_0, \alpha_0, \beta_0$ would $\xi(\mu, \tau) = 1/\tau$ (Jeffreys Prior)?

2. Find the values of $\mu_1, \lambda_1, \alpha_1, \beta_1$ using the non informative prior.
3. Using these posterior values show the equivalence of the Bayes Test Procedure using Jeffreys prior to the frequentist result ($U \sim t_{(n-1)}$)
4. Compare the posterior probability to p-value (using R).

And More

5. Repeat using informative hyper parameters for the conjugate prior relationship.

6. Show equivalence with non informative prior to the two-sided t-test. $(\mu_0 = .25; \lambda_0 = 4; \alpha_0 = 2; \beta_0 = 0.5)$