12.5 Markov Chain Monte Carlo

Inference for a mean from a single sample

What we have considered:

- MLE for μ
- Frequentist CI for μ based on t-distribution
- Bayesian Estimates and credible intervals for μ based on Normal-Gamma*
- Efficiency/CRLB (MLE is efficient)
- t-test for $\mu = \mu_0$ (Likelihood Ratio Test)
- Bayes test for μ – $\Pr(H_0 \text{ true})^*$

* We showed equivalence to Frequentist procedures in the case of non informative priors.

We will explore new procedures for simulating posterior distributions. The `expland.grid` approach falls apart with more parameters. We'll consider MCMC methods to simulate posterior distributions, and we'll start with Gibbs Sampling

Gibbs Sampling Algorithm

Assume we want to simulate a joint pdf of (X_1, X_2) where $f(x_1, x_2) = c \times g(x_1, x_2)$ where $g(x_1, x_2)$ is known.

Example: X_1 and X_2 are μ and τ

$g(x_1, x_2) =$ prior $\times$ likelihood $c =$ proportionality constant

Then $g(x_1|x_2) = c_2 h_2(x_1)$ where $h_2(x_1) = g(x_1, x_2)$ with x_2 fixed.
And $g(x_2|x_1) = c_1 h_1(x_2)$ where $h_1(x_2) = g(x_1, x_2)$ with x_1 fixed.

If $g(x_1|x_2)$ and $g(x_2|x_1)$ are recognizable and easily sampled from, we can do this.

Extensions

It can also be extended to problems with n parameters:

$$g(x_1|x_2, x_3, ..., x_n)$$

$$g(x_2|x_1, x_3, ..., x_n)$$

$$...$$

$$g(x_n|x_1, x_2, ..., x_{n-1})$$

Example: $X \sim N(\mu, \tau)$ Normal-Gamma conjugate prior

Algorithm

1. Pick a starting value $x_2^{(0)}$ for x_2; and set $i = 0$.
2. Simulate anew value $x_1^{(i+1)}$ from the conditional distribution of X_1 given $X_2 = x_2^{(i)}$
3. Simulate a new value $x_2^{(i+1)}$ from the conditional distribution of X_2 given $X_1 = x_1^{(i+1)}$
4. Replace i by $i + 1$ and return to Step (2).

Build Gibbs for one mean example (Example 12.5.1 p824)

- likelihood: $X_1, ..., X_n \sim N(\mu, \tau)$
- prior for τ: $\xi(\tau) \sim gamma(\alpha_0, \beta_0)$
- prior for μ (given τ): $\xi(\mu|\tau) \sim N(\mu_0, \lambda_0\tau)$
- posterior: $\xi(\mu, \tau|x) \propto \tau^{\alpha_1+\frac{1}{2}-1} \exp\left(-\tau\left[\frac{1}{2}\lambda_1(\mu-\mu_1)^2+\beta_1\right]\right)$

Consider $\xi(\mu, \tau|x)$ as a function of μ for fixed τ:
$\xi(\mu|\tau, x) \sim N(\mu_1, \tau\lambda_1)$

$$\xi(\mu|\tau, x) \propto ...$$

Consider $\xi(\mu, \tau|x)$ as a function of τ for fixed μ:
$\xi(\tau|\mu, x) \sim gamma(\alpha_1 + 1/2, \lambda_1(\mu-\mu_1)^2/2+\beta_1)$

$$\xi(\tau|\mu, x) \propto ...$$

Build Gibbs for one mean example

1. Choose initial value of μ
2. Sample τ from $\xi(\tau|\mu, x)$
3. Sample μ from $\xi(\mu|\tau, x)$
4. Go to (2) and Repeat many times.