Image Classification w/ SVD

Image $\quad I$: $k \times k$ matrix of pixels

$I_{ij} = 0, 1, \ldots, 255$ (for example)

e.g. greyscale

Example

$$I \equiv \begin{bmatrix} 12 & 27 & 150 & 33 & \cdot & \cdot & \cdot & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & \cdot & \cdot & \cdot & \cdot & 18 & 79 \end{bmatrix} \qquad \left.\begin{array}{l} I_{11} = 12, \\ I_{12} = 27 \\ \vdots \\ I_{kk} = 79 \end{array}\right\} \text{Pixel Values}$$

Data $\quad n$ = # of pixels k^2

p = # of images

$$X \atop n \times p$$

Each column is an Image

k^2
# of pixels

# of images

Each row is a pixel

SVD $\quad X = \underset{n \times p}{U} \, \underset{p \times p}{\Sigma} \, \underset{p \times p}{V^T}$

$UU^T = VV^T = I \qquad \Sigma = \text{diag}(\sigma_1, \dots, \sigma_r)$

$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$

Aside

$\boxed{\text{Image Compression (Approximation)}}$

Perform SVD on $k \times k$ image I

$$\underset{k \times k}{I} = \underset{k \times k}{U} \, \underset{k \times k}{\Sigma} \, \underset{k \times k}{V^T}$$

Let $\ell = 1, 2, \dots, k$

$$\tilde{U} = [u_1, \dots u_\ell] \qquad k \times \ell$$
$$\tilde{\Sigma} = \text{diag}(\sigma_1, \dots, \sigma_\ell) \qquad \ell \times \ell$$
$$\tilde{V} = [v_1, \dots v_\ell] \qquad k \times \ell$$

Build $\quad \underset{k \times k}{\tilde{I}} = \underset{k \times \ell}{\tilde{U}} \, \underset{\ell \times \ell}{\tilde{\Sigma}} \, \underset{\ell \times k}{\tilde{V}^T}$

$\tilde{I}$ "best rank ℓ approx to I"

Amount of Data $\quad I \quad k \times k = \underline{\underline{k^2}}$

$$\tilde{I} = \underset{k\times \ell}{\tilde{U}} \ \underset{\ell \times \ell}{\Sigma} \ \underset{\ell \times k}{\tilde{V}^T}$$

$$k\ell + \ell + k\ell = 2k\ell + \ell$$

Example $\quad k = 1024 \qquad k^2 \approx 10^6$

$\qquad \ell = 50 \qquad 2k\ell + \ell \simeq 10^5$

Question How much does $\tilde{I}$ resemble I?

Answer It depends on how much variability is contained in first ℓ singular values

i.e. $$\rho = \frac{\sigma_1 + \sigma_2 + \cdots + \sigma_\ell}{\sigma_1 + \sigma_2 + \cdots + \sigma_\ell + \cdots + \sigma_k}$$

If $\rho \approx 1$, then $\tilde{I} \approx I$

Effective Compression.

Back to Image Recognition

SVD $\quad X = \underset{n\times p}{U} \; \underset{p\times p}{\Sigma} \; \underset{p\times p}{V^T}$

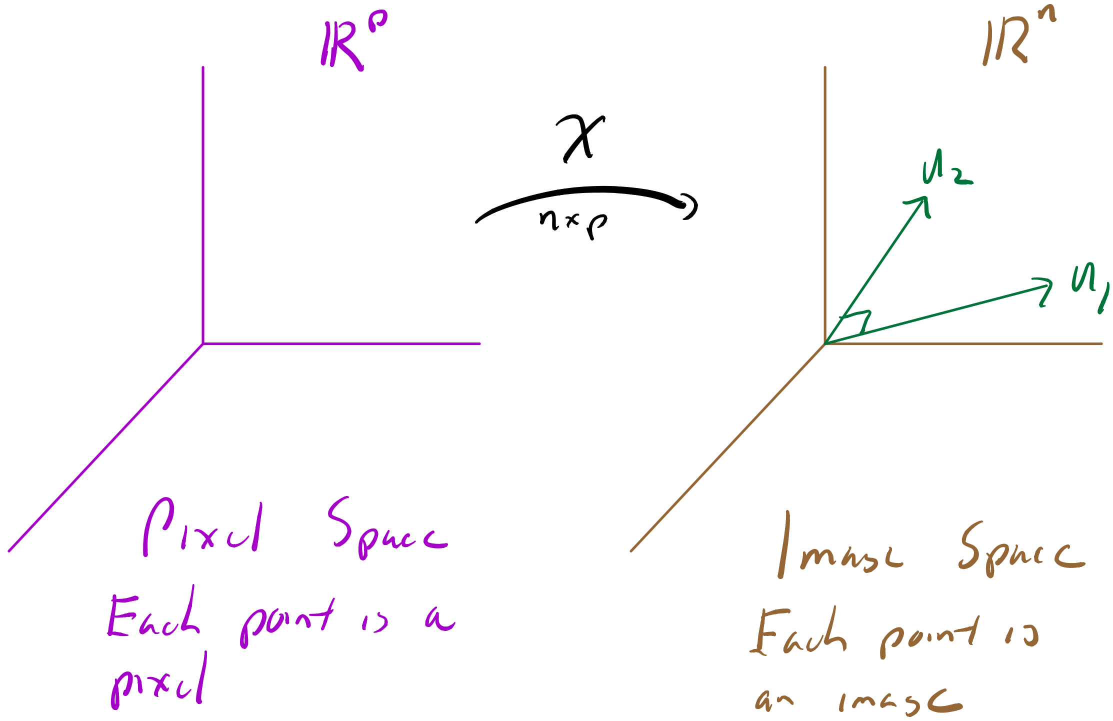
$UU^T = VV^T = I \qquad \Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_r)$

$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$

<u>Recall</u> $\quad U = [\underbrace{u_1, \dots, u_p}_{\text{eigenvectors}}]$ $\quad$ Span Column Space

"Eigen Images"

Geometry



- Column Space Spanned by $u_1, \ldots u_p$
- For any $Y \in \mathbb{R}^n$ $\quad \hat{Y} = UU^T Y$

  Projection onto Col. Sp.

- Error = Distance from $\hat{Y}$ to Y

$$\| Y - UU^T Y \|$$

Theory → Practice

Suppose images all from the same "Class"

e.g. Cats

Space of cat images is infinite dimensional!

But... we have enough cats to almost "span" the cat space, i.e.

Any Cat Image can be approximated with linear combination eigenvectors

Eigen Cats

Any Cat $\equiv Y$

Best Approx $\equiv \hat{Y} = UU^TY$

Closeness $\equiv \|Y - UU^TY\|$

$u_1, u_2 \ldots u_p$ "Eigen Cats"

Alt

Given any image Y

$\delta_Y = \|Y - UU^T Y\| \equiv$ measure of "catness"

If δ_Y small enough, classify Y as "cat"

#1 Cats: $C_1, \dots C_{p_1}$

$u_1, u_2 \dots u_{p_1}$: Eigen Cats

Dogs: $D_1 \dots D_{p_2}$

$\tilde{u}_1\ \tilde{u}_2 \dots \hat{u}_{p_2}$: Eigen Dogs

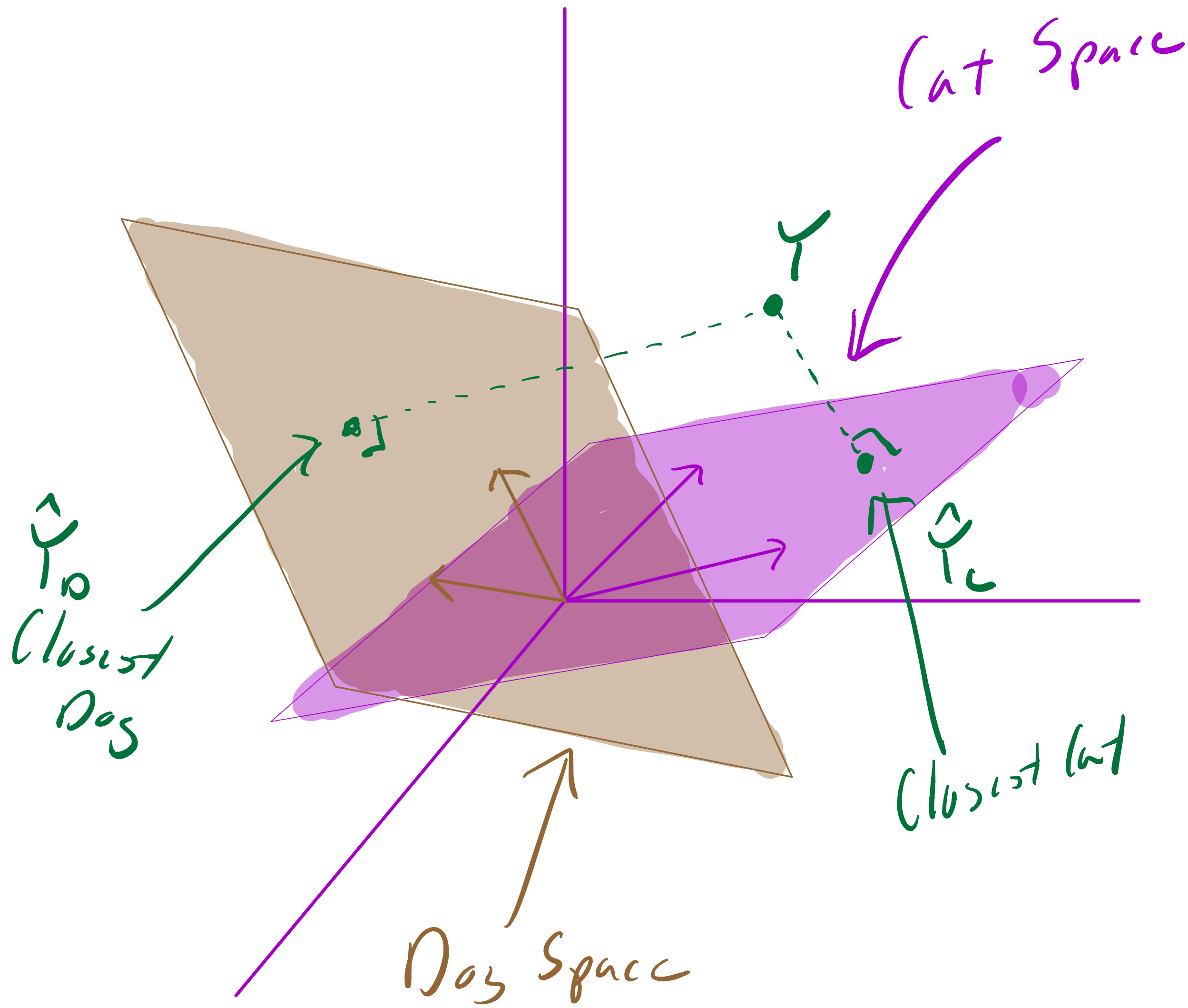
Image Y: Cat or Dog?

Compute $\delta_c = \| Y - U_1 U^T Y \|$ ← Distance to Cat Space

$\delta_D = \| Y - \tilde{U} \hat{U}^T Y \|$ ← Distance to Dog Space

If $\delta_c < \delta_D$, then Y is "Cat"

If $\delta_D < \delta_c$, then Y is "Dog"



Since $\|Y-\hat{Y}_C\| < \|Y-\hat{Y}_D\|$

Y must be a cat!