8.8 Fisher Information: Part 2

Assuming $X \sim N(\mu, \sigma^2)$ where σ^2 is known, $I(\theta) = \frac{1}{\sigma^2}$

$Var_\mu = \sigma^2$ so as $Var(X)$ increases, $I(\theta)$ decreases.

$I(\theta)$ measures the information about a parameter that a random variable contains.

Theorem 8.8.2

$$I_n(\theta) = nI(\theta)$$

$I_n(\theta)$ is the information in the entire sample, where $I(\theta)$ is the information in a single observation.

Question

How can we use $I_n(\theta)$ to evaluate estimates like $\bar{X}_n$?

Cramer-Rao Lower Bound (CRLB)

For any arbitrary estimator of θ, $T = r(x)$ the **Cramer-Rao** (information) Inequality States:

$$Var_\theta(T) \geq \frac{m'(\theta)^2}{nI(\theta)}$$

where $m(\theta) = E_\theta(T)$

So for an unbiased estimator $m(\theta) = \theta$ and $Var_\theta(T) \geq \frac{1}{nI(\theta)}$

No unbiased estimator of θ can have lower variance than the CRLB.
If equality is achieved in the inequality T is said to be an efficient esitmator.
Minimum Variance Unbiased Estimators (MVUE)
#Example
Assume $X_1, ..., X_n \sim N(\mu, \sigma^2)$ where σ^2 is known, and $\alpha_1 X_1 + \alpha_2 X_2 + ... + \alpha_n X_n$ is unbiased for μ if $\sum \alpha_i = 1$

$$\lambda(x|\mu)' = \frac{(x-\mu)}{}$$

Theorem 8.8.3 (continued)

T will be an efficient estimator if and only if there exist functios $u(\theta)$ and $v(\theta)$ that may depends on θ but not $X_1, ..., X_n$ and that satisfy the relation

$$T = u(\theta)\lambda'_n(x|\theta) + v(\theta)$$

Example

$T = \bar{X}_n$

$$\bar{X}_n = u(\mu) \sum \left(\frac{x_i - \mu}{\sigma} \right) + v(\mu)$$

where $u(\mu) = \frac{\sigma}{n}$ and $v(\mu) = \mu$