9.1 Testing Hypotheses: Uniform Example

$$X \sim Unif(0, \theta)$$

We want to test

$$H_0 : \theta \geq 2$$

$$H_A : \theta < 2$$

with $\alpha_\theta = 0.10$ based on a random sample of size 8. We'll base our test on $\hat{\theta}_{MLE} = max(X_1, ..., X_8) = X_{(8)}$. We want to find $\pi(\theta|\delta)$.

Reject H_0 if $X_{(8)} \leq C$ (since we are looking for evidence that $\theta < 2$).

We want Pr(Reject $H_0|H_0$ true)=.10 or $Pr(X_{(8)} \leq C|\theta = 2) = .10$

Finding the pdf for an order statistic

First we need the pdf for $X_{(8)}$. Begin with cdf of $X_{(8)}$.

$$
\begin{aligned}
F_{X_{(8)}}(x) &= [Pr(X \leq x)]^8 \\
f_{X_{(8)}}(x) &= 8F(x)^{n-1}f(x) \\
&= 8\left(\frac{x}{\theta}\right)^7 \frac{1}{\theta} \\
&= \frac{8\theta^7}{\theta^8}
\end{aligned}
$$

If $\theta = 2$ then $f_{X_{(8)}}(x) = \frac{1}{32}x^7$ for $0 \leq x \leq 2$.

Finding the Critical Region

So

$$
\begin{aligned}
Pr(X_{(8)} \leq C | H_0(true) = .10 &\rightarrow \int_0^c \frac{1}{32} x^7 dx = .10 \\
&\rightarrow \frac{1}{256} x^8 \Big|_0^c \\
&\rightarrow c = (256)^{1/8} = 1.50
\end{aligned}
$$

So the Critical Region (Rejection Region) is $\{x : x_{(8)} \leq 1.50\}$.

What is the power of the test when $\theta = 1.7$?

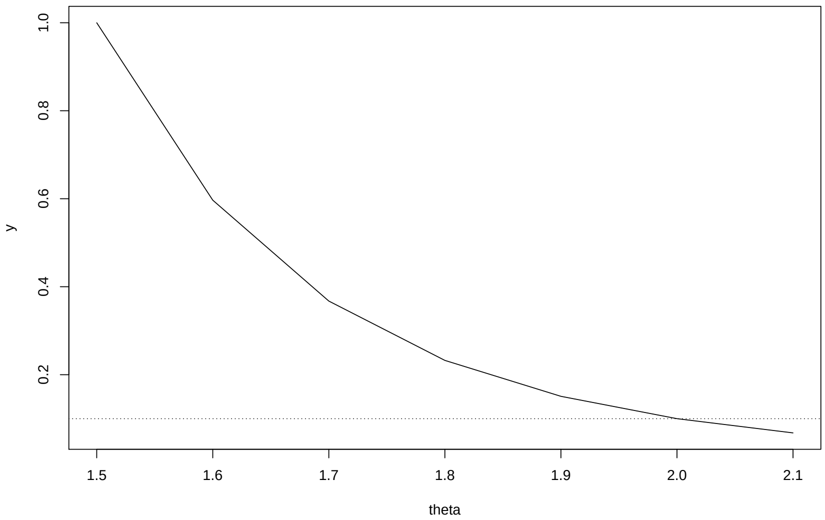
$$Pr(\text{Reject} H_0 | \theta = 1.7) = \int_0^{1.5} \frac{8}{1.7^8} x^7 dx =$$

```
## [1] 0.3673996
```

```
## [1] 1.00000000 0.59671947 0.36739962 0.23256804 0.150904
## [7] 0.06776036
```

θ	Power
1.5	1.00
1.6	0.597
1.7	0.367
1.8	0.233
1.9	0.151
2.0	0.100
2.1	0.068

Plot of Power vs θ



Normal Example

$X \sim N(\mu, \sigma^2)$ with σ^2 known.

$$H_0 : \mu = \mu_0; H_A : \mu \neq \mu_0$$

$\pi(\mu|\delta) = Pr(\text{Reject } H_0|\mu)$

Let δ be the test that rejects when $|Z| > 1.96$

$$
\begin{aligned}
&= Pr\left(\frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} > 1.96|\mu\right) + Pr\left(\frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} < -1.96|\mu\right) \\
&= Pr\left(\bar{X} > \mu_0 + 1.96 \times \sigma/\sqrt{n}|\mu\right) + Pr\left(\bar{X} < \mu_0 - 1.96 \times \sigma/\sqrt{n}|\mu\right) \\
&= Pr\left(Z > \frac{\mu_0 + 1.96 \times \sigma/\sqrt{n} - \mu}{\sigma/\sqrt{n}}\right) + Pr\left(Z < \frac{\mu_0 - 1.96 \times \sigma/\sqrt{n} - \mu}{\sigma/\sqrt{n}}\right) \\
&= Pr\left(Z > \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + 1.96\right) + Pr\left(Z < \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} - 1.96\right)
\end{aligned}
$$

Bernoulli Example (Example 9.1.9)

$$X_1, ..., X_{10} \sim \text{Bernoulli}(p)$$

$$H_0 : p \leq .3$$

$$H_A : p > .3$$

Test statistic: $Y = \sum X_i \rightarrow$ Reject H_0 if $Y > C$ where

$$Y \sim \mathit{Binomial}(n, p); \; n = 10$$

Size of Binomial Test

$$\alpha(\delta) = \sup_{p \in \Omega_0} \pi(p|\delta) = \sup_{pin\Omega_0} Pr(Y \geq c|p) = Pr(Y \geq c|p = 0.3)$$

In general $Pr(Y = y|p = .3) = \binom{10}{y}.3^y.7^{10-y}$

$$Pr(Y = 10|p = .3) = \binom{10}{10}.3^{10}.7^0 = 0.0000059$$

$$Pr(Y = 9|p = .3) = \binom{10}{9}.3^9.7^1 = 0.00014$$

Using R

```r
y=seq(5,10)
d=dbinom(y,10,.3)
p=1-pbinom(y-1,10,.3)
cbind(y,d,p)
```

```
##       y            d            p
## [1,]  5 0.1029193452 0.1502683326
## [2,]  6 0.0367569090 0.0473489874
## [3,]  7 0.0090016920 0.0105920784
## [4,]  8 0.0014467005 0.0015903864
## [5,]  9 0.0001377810 0.0001436859
## [6,] 10 0.0000059049 0.0000059049
```

In order to keep the size of the test to at most .1 (or 0.05 for that matter) we must choose c>5.

P-value

P-value: The smallest level α_0 such that we would reject the null hypothesis at a level α_0 with the observed data.

If $\delta =$ Reject H_0 when $T \geq c$ (δ_t : test that rejects H_0 if $T \geq t$.

$$\text{pvalue} = \sup_{\theta \in \Omega_0} \pi(\theta | \delta_t) = \sup_{\theta \in \Omega_0} Pr(T \geq t | \theta)$$

Bernoulli Example

If we observe $Y = 8$ then

$$\begin{aligned} \text{pvalue} &= \sup_{p \leq .3} Pr(Y \geq 8 | p) \\ &= Pr(Y \geq 8 | p = .3) \\ &= 0.0016 \end{aligned}$$

Equivalence of Hypthesis test and Confidence Intervals

A coefficient γ confidence set can be thought of as a set of H_0s that would be accepted at significance level $1-\gamma$.

Example 9.1.15

$X_1 ..., X_n \sim N(\mu, \sigma^2)$ with μ, σ^2 unknown.

$$H_0 : \mu \leq \mu_0; \{\Omega_0 : \mu \leq \mu_0 \text{ and } \sigma^2 > 0\}$$

$$H_A : \mu > \mu_0; \{\Omega_1 : \mu > \mu_0 \text{ and } \sigma^2 > 0\}$$

Coefficient γ Confidence Interval:

$$\left(\bar{X}_n \pm c\sigma'/\sqrt{n}\right)$$

where c is the $\frac{1+\gamma}{2}$ quantile of t_{n-1} distribution.

Normal CIs

We can use the CI to find a level $\alpha_0 = 1 - \gamma$ test of $H_0 : \mu = \mu_0; H_A : \mu \neq \mu_0$

Reject H_0 if $\mu_0 \notin \left(\bar{X}_n \pm c\sigma'/\sqrt{n}\right)$

Which is equivalent to $\left(|\bar{X}_n - \mu_0| \geq \pm c\sigma'/\sqrt{n}\right)$

Or

$$\left|\frac{\bar{X}_n - \mu_0}{\sigma'/\sqrt{n}}\right| \geq c$$

Which looks like a t-test!