8.8 Fisher Information

Consider X_1 vs $\bar{X}_n$ as estimators for the mean μ.

$\bar{X}_n$ is more *efficient* than X_1 (smaller variance among unbiased estimators)

Relative Efficiency

Relative Efficiency of $\bar{X}_n$ with respect to X_1

$$\frac{Var[X_1]}{Var[\bar{X}_n]} = \frac{\sigma^2}{\sigma^2/n} = n$$

There are an infinite number of unbiased estimators for μ:
$\delta(X_1, ..., X_n) = \alpha_1 X_1 + \alpha_2 X_2 + ... + \alpha_n X_n$ where $\sum \alpha_i = 1$

Which is the “Best” unbiased estimator?

Fisher Information

- We want to describe the amout of information contained in sample data about the unknown parameter.
- Intuititvely:
  - The more data, the more information,
  - The more precise the data, the more information.

Assume:

1. $X \sim f(x|\theta)$
2. $X : f(x|\theta) > 0$ doesn't depend on θ (e.g. no $X \sim Unif(0, \theta)$)
3. $f(x|\theta)$ is a twice differnetiable function of θ

Define:

$\lambda(x|\theta) = log(f(x|\theta))$; $\lambda'(x|\theta) = \frac{d}{d\theta} log(f(x|\theta))$;
$\lambda''(x|\theta) = \frac{d^2}{d\theta^2} log(f(x|\theta))$

Fisher Information

Then, the Fisher Information, $I(\theta)$, in the random varible X is

$$
\begin{aligned}
I(\theta) &= E_\theta\{\left[\lambda'(x|\theta)\right]^2\} \\
&= \int_S \left[\lambda'(x|\theta)\right]^2 f(x|\theta)dx
\end{aligned}
$$

If we also assume:

4. $\int f'(x|\theta)dx = 0 \quad \forall\theta$
5. $\int f''(x|\theta)dx = 0 \quad \forall\theta$

Then

a. $I(\theta) = Var_\theta\left[\lambda'(x|\theta)\right]$

b. $I(\theta) = -E_\theta\left[\lambda''(x|\theta)\right]$

Example

Assume $X \sim N(\mu, \sigma^2)$ where σ^2 is known

1. Find Fisher Information for μ using (a) and (b)
2. Confirm assumptions (4) and (5)