9.1 and 9.5 Likelihood Ratio Test

Recall our t-test example. We reject H_0 if $|U| = \left|\frac{\bar{X}_n - \mu_0}{\sigma'/\sqrt{n}}\right| \geq c$

Our test statistic: $U = \frac{\bar{X}_n - \mu_0}{\sigma'/\sqrt{n}} \sim t_{(n-1)}$ if $H_0 : \mu = \mu_0$ is true.

But $U = \frac{\bar{X}_n - \mu_0}{\sigma'/\sqrt{n}} \sim t_{(n-1),\psi}$ if H_0 is false.

Where $\psi = \frac{\mu - \mu_0}{\sigma/\sqrt{n}}$ and μ is the true mean.

Is the one sample t-test optimal in any sense?

We covered many properties of parameter etimates/estimators that are desirable (maximum likelihood, sufficiecy, efficiency, unbiasedness, etc). Are there similar ways we can evaluate test-procedures?

Evaluating Test Procedures

Let's consider a general framework for suggesting and evaluating test procedures (minimize Type II for fixed Type I error).

$$H_0 : \theta \in \Omega_0$$

$$H_A : \theta \in \Omega_1$$

The likelihood function is highest near true values of θ. We will consider tests based on the likelihood function $f_n(x|\theta)$.

So if the likelihood function is greater for values of θ under H_1 vis a via H_0 we should reject H_0.

If $sup_{\theta \in \Omega_1} f_n(x|\theta) > \sup_{\theta \in \Omega_0} f_n(x|\theta)$ then we should reject H_0.

Likelihood Ratio Test Statistic (Def 9.1.11)

Alternatively, we can consider, $\sup_{\theta \in \Omega} f_n(x|\theta)$ rather than $\sup_{\theta \in \Omega_0} f_n(x|\theta)$.

The Likelihood Ratio Statistic is defined to be:

$$\Lambda(x) = \frac{\sup_{\theta \in \Omega_0} f_n(x|\theta)}{\sup_{\theta \in \Omega} f_n(x|\theta)}$$

Using the Likelihood Ratio Test Statistic $\Lambda(x)$ we reject H_0 if $\Lambda(x) \leq k$.

Intuitively, if the max likelihood under H_0 is far below the overall max then we should reject H_0.

The t-test as a LRT: Setup

Consider once again $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$ where $X's \sim N(\mu, \sigma^2)$, μ, σ^2 both are unknown.

$$\Omega_0 = \{(\mu, \sigma^2) : \mu = \mu_0, \sigma^2 > 0\}$$

$$\Omega = \Omega_0 \cup \Omega_1 = \{(\mu, \sigma^2) : \mu = (-\infty, \infty), \sigma^2 > 0\}$$

$$f_n(x|\theta) = f_n(x|(\mu, \sigma^2) = (2\pi\sigma^2)^{-n/2} \exp\{-\frac{1}{2} \sum \frac{(x_i \mu)^2}{\sigma^2}\}$$

In order to maximize our likelihood function we essentially want to find maximum likelihood estimators of μ and σ^2 (subject to contraints).

The t-test as a LRT

Under $\Omega_0 : \hat{\mu}_0 = \mu_0$; to find $\hat(\sigma)_0^2$ take deriative and set to zero $\rightarrow \hat{\sigma}_0^2 = \frac{1}{n}\sum(x_i - \mu_0)^2$

Under Ω : $\hat{\mu}$ and $\hat{\sigma}^2$ are the usual MLEs:
$\hat{\mu} = \bar{X}_n; \hat{\sigma}^2 = \frac{1}{n}\sum(x_i - \bar{X}_n)^2$.

$$\begin{aligned}
\Lambda(x) &= \frac{f_n(x|\hat{\mu}_0, \hat{\sigma}_0^2)}{f_n(x|\hat{\mu}, \hat{\sigma}^2)} \\
&= \frac{(2\pi\frac{1}{n}\sum(x_i - \mu_0)^2)^{-n/2}\exp\{-\frac{1}{2}\sum\frac{(x_i-\mu)^2}{\frac{1}{n}\sum(x_i-\mu_0)^2}\}}{(2\pi\frac{1}{n}\sum(x_i - \bar{X}_n)^2)^{-n/2}\exp\{-\frac{1}{2}\sum\frac{(x_i-\bar{X}_n)^2}{\frac{1}{n}\sum(x_i-\bar{X}_n)^2}\}} \\
&= \left[\frac{\sum(x_i - \mu_0)^2}{\sum(x_i - \bar{X}_n)^2}\right]^{-n/2}
\end{aligned}$$

The t-test as LRT: Example 9.5.12

So Reject H_0 if $\Lambda(x) \leq k$,

$$\left[\frac{\sum(x_i - \mu_0)^2}{\sum(x_i - \bar{X}_n)^2}\right]^{-n/2} \leq k$$

Is this something we recognize?

1. Show that this LRT is equivalent to a one sided t-test. Follow set up on p584 – what do you need to fill in to show this is true?
2. Exercise 9.5.17 – Can also show this is the case for the two-sided hypothesis test.