Bayesian Analysis of Samples from a Normal Distribution

8.6 Bayesian Analysis of Samples from a Normal Distribution

Consider $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with μ, σ^2 unknown.

$U = \frac{\bar{X}_n - \mu_0}{s/\sqrt{n}} \sim t_{(n-1)}$ gives us the Frequentist approach.

So how would a Bayesian approach this problem?

First, we need a likelihood:

$$f_n(x, \mu, \tau) = (\frac{\tau}{2\pi})^{n/2} \exp\{\frac{\tau}{2}\sum_{i=1}^{n}(x_i - \mu)^2\}$$

($\tau = 1/\sigma^2$, precision)

Prior for μ and τ: $\xi(\mu, \tau) =??$

So, we need a joint distribution to make inferences for μ and σ^2 jointly. Can we find a conjugate family of priors?

Well.. yes.

Normal*gamma

Joint distribution = conditional*marginal

$$\xi(\mu, \tau) = \xi_1(\mu|\tau) \times \xi_2(\tau)$$

Where $\xi_1(\mu|\tau) \sim$ *Normal* and $\xi_2(\tau) \sim$ *Gamma*

$$\xi_1(\mu|\tau) \sim N(\mu_0, \lambda_0\tau)$$

$$\xi_2(\tau) \sim \mathit{gamma}(\alpha_0, \beta_0)$$

(i.e. $\xi(\sigma^2) \sim$ Inverse Gamma)

Theorem 8.6.1, p496

If the above is assumed:

$$\xi_1(\mu|\tau, x) \sim N(\mu_1, \lambda_1\tau)$$

and

$$\xi_2(\tau|x) \sim \textit{gamma}(\alpha_1, \beta_1)$$

Prior: Normal-gamma with hyperparameters $\mu_0, \lambda_0, \alpha_0, \beta_0$

Posterior: Normal-gamma with hyperparameters $\mu_1, \lambda_1, \alpha_1, \beta_1$

Note: μ and τ are **NOT** independent

Posterior Parameters

$$\mu_1 = \frac{\lambda_0 \mu_0 + n\bar{x}_n}{\lambda_0 + n}$$

$$\lambda_1 = \lambda_0 + n$$

$$\alpha_1 = \alpha_0 + n/2$$

$$\beta_1 = \beta_0 + \frac{1}{2} S_n^2 + \frac{n\lambda_0(\bar{x}_n - \mu_0)^2}{2(\lambda_0 + n)}$$

Interpretations:

$$\mu_1 = \frac{\lambda_0}{\lambda_0 + n}\mu_0 + \frac{n}{\lambda_0 + n}\bar{x}_n$$

μ_0	prior estimate of the mean
λ_0	Amount of information abut the mean (e.g. $\lambda_0 = 5$ you have about as much information as contained in a sample of size 5)
$\frac{\lambda_0}{\lambda_0+n}$	fraction of information in prior
$\frac{\beta_0}{\alpha_0}$	Prior estimate of the variance
$2\alpha_0$	amount of prior info about variance

Goal: conditional distribution

$\xi(\mu|x) =??$

Show (in groups)

1. if $Y = 2\beta_0\tau$ then $Y \sim \chi^2_{2\alpha_0}$ (Use exercise 5.7.1)
2. $U = \sqrt{\frac{\lambda_0\alpha_0}{\beta_0}}(\mu - \mu_0) \sim t_{2\alpha_0}$
3. If $U = \sqrt{\frac{\lambda_0\alpha_0}{\beta_0}}(\mu - \mu_0) \sim t_{2\alpha_0}$ find $E(\mu)$ and $Var(\mu)$
4. If $U = \sqrt{\frac{\lambda_0\alpha_0}{\beta_0}}(\mu - \mu_0) \sim t_{2\alpha_0}$ find 95% CI (credible interval) for μ.

(Key outline posted on Moodle)

All of the previous statements (1-4) are also true for the posterior distribution

1. if $Y = 2\beta_1\tau$ then $Y \sim \chi^2_{2\alpha_1}$

2. $U = \sqrt{\frac{\lambda_1\alpha_1}{\beta_1}}(\mu - \mu_1) \sim t_{2\alpha_1}$

3.$E(\mu|\underline{x}) = \mu_1$ and $Var(\mu|\underline{x}) = \frac{\beta_1}{\lambda_1(\alpha_1-1)}$

4. 95% CI (credible interval) for $\mu|\underline{x}$ is

$$\left(\frac{t_{2\alpha_1,.025}}{\sqrt{\frac{\lambda_1\alpha_1}{\beta_1}}} + \mu_1, \frac{t_{2\alpha_1,.975}}{\sqrt{\frac{\lambda_1\alpha_1}{\beta_1}}} + \mu_1 \right).$$

Note for HW:

$$\Gamma(r) = (r-1)\Gamma(r-1)$$

$$\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma\frac{1}{2}$$