8.7 Unbiased Estimators

Unbiased Estimators – Variance Example

Consider two estimators of σ^2

1. $\hat{\sigma}^2 = \frac{1}{n}\sum(X_i - \bar{X}_n)^2$ the MLE
2. $\hat{\sigma'}^2 = \frac{1}{n-1}\sum(X_i - \bar{X}_n)^2$ our "usual" variance estimator.

Definition

An estimator $\delta(X_1, ...X_n)$ is an unbaised estimator of a parameter θ if

$$E_\theta\left[\delta(X_1, ..., X_n)\right] = \theta \quad \forall\theta$$

Examples

1. $X_1, ...X_n \sim^{iid} N(\theta, \sigma^2)$ where σ^2 known. Is $\bar{X}_n$ unbiased for θ?

$$E[\bar{X}_n] = E[\frac{1}{n}\sum X_i = ...$$

2. $X_1, ...X_n$ come from a distribution with unknown mean μ and unknown variance σ^2. is $\hat{\sigma}^2_{MLE}$ unbiased for σ^2?

$$E[\hat{\sigma}^2_{MLE}] = E[\frac{1}{n}\sum (X_i - \bar{X}_n)^2] = ...$$

Some helpful properties

1. $Var(X_i) = E(X_i^2) - E(X_i)^2$
2. $Var(\bar{X}_n) = Var(\frac{1}{n}\sum X_i)$
3. $E(\bar{X}_n^2) = Var(\bar{X}_n) + E(\bar{X}_n)^2$

Brief Solution

Example 1

1. $X_1, ...X_n \sim^{iid} N(\theta, \sigma^2)$ where σ^2 known. Is $\bar{X}_n$ unbiased for θ?

$$\begin{aligned} E[\bar{X}_n] &= E[\frac{1}{n}\sum X_i] \\ &= \frac{1}{n}E[\sum X_i] \\ &= \frac{1}{n}\sum E[X_i] \\ &= \frac{1}{n}n\theta \\ &= \theta \end{aligned}$$

In Conclusion

So, YES!, $\bar{X}_n$ is unbiased for the mean.

(Recall that you can show $E[X_1] = \theta = \int xf(x)dx$ or by using moment generating functions.)

In fact $\bar{X}_n$ is an unbiased estimator for μ for any random sample $X_1, ...X_n$ from any distribution with unknown mean μ (a function of θ).

Example 2

2. $X_1, ...X_n$ come from a distribution with unknown mean μ and unknown variance σ^2. is $\hat{\sigma}^2_{MLE}$ unbiased for $sigma^2$?

$$
\begin{aligned}
E[\hat{\sigma}^2_{MLE}] &= E\left[\frac{1}{n}\sum(X_i - \bar{X}_n)^2\right] \\
&= \frac{1}{n}E\left[\sum(X_i - \bar{X}_n)^2\right] \\
&= E\left[\frac{1}{n}\sum(X_i^2 - 2\bar{X}_n X_i + \bar{X}_n^2)\right] \\
&= E\left[\frac{1}{n}\sum(X_i^2\right] - E\left[\frac{2}{n}\bar{X}_n\sum X_i\right] + E\left[\frac{1}{n}n\bar{X}_n^2\right] \\
&= \frac{1}{n}E\left[\sum X_i^2\right] - E\left[2\bar{X}_n^2\right] + E\left[\bar{X}_n^2\right] \\
&= \frac{1}{n}E\left[\sum X_i^2\right] - E\left[\bar{X}_n^2\right] \\
&= \frac{1}{n}\sum E\left[X_i^2\right] - E\left[\bar{X}_n^2\right]
\end{aligned}
$$

Employing some of the helpful properties

1. $Var(X_i) = E(X_i^2) - E(X_i)^2 \rightarrow E(X_i^2) = Var(X_i) + E(X_i)^2$
2. $Var(\bar{X}_n) = Var(\frac{1}{n}\sum X_i) = \frac{1}{n^2} Var \sum X_i = \frac{1}{n^2}\sum \sigma^2 = \frac{\sigma^2}{n}$
3. $E(\bar{X}_n^2) = Var(\bar{X}_n) + E(\bar{X}_n)^2 = \frac{\sigma^2}{n} + \mu^2$

So

$$
\begin{aligned}
\frac{1}{n}\sum E\left[X_i^2\right] - E\left[\bar{X}_n^2\right] &= \frac{1}{n}\sum\left[Var(X_i) + E(X_i^2)\right] - \frac{\sigma^2}{n} - \mu^2 \\
&= \frac{1}{n}\left[n\sigma^2 + n\mu^2\right] - \frac{\sigma^2}{n} - \mu^2 \\
&= \sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2 \\
&= \frac{n-1}{n}\sigma^2
\end{aligned}
$$

So $E\left[\frac{n}{n-1}\hat{\sigma}^2\right] = E\left[\frac{\sum(X_i - \bar{X}_n)^2}{n-1}\right] = \sigma^2$

Great news. Our usual estimate of the variance is unbiased!

In conclusion

- $\hat{\sigma}^2$ is not an unbiased estimator of σ^2. The limit does go to σ^2, so the MLE is *asymptotocally unbiased*.
- $\hat{\sigma'}^2$ is an unbiased estimator for σ^2

Unbiasedness is only one quality of an estimator

Consider $X_1, ...X_n \sim^{iid} N(\theta, \sigma^2)$ where σ^2 is know.

$$E[X_1] = \mu; E[\bar{X}_n] = \mu$$

$$Var[X_1] = \sigma^2; Var[\bar{X}_n] = \frac{\sigma^2}{n}$$

Unbiasedness only tells you where you end up on average, not anything about the sampling distribution or the precision of your estimate.
Ideally an estimator would be unbaised and have small variance.

Definition: Mean Square Error (MSE)

$$
\begin{aligned}
MSE &= E[(\delta - \theta)^2] \\
&= E[(\delta - \theta)]^2 + Var(\delta) \\
&= Bias^2 + Var(\delta)
\end{aligned}
$$

Later we'll talk about the bias-variance trade off. However, if we consider the variance estimate $\frac{\sum (X_i - \bar{X}_n)^2}{n+1}$ this esitmator minimizes the MSE.