9.1 (continued) P-value

P-value: The smallest level α_0 such that we would reject the null hypothesis at a level α_0 with the observed data.

If $\delta =$ Reject H_0 when $T \geq c$ (δ_t : test that rejects H_0 if $T \geq t$.

$$\text{pvalue} = \sup_{\theta \in \Omega_0} \pi(\theta|\delta_t) = \sup_{\theta \in \Omega_0} Pr(T \geq t|\theta)$$

Bernoulli Example

If we observe $Y = 8$ then

$$\begin{aligned}
\text{pvalue} &= \sup_{p \leq .3} Pr(Y \geq 8|p) \\
&= Pr(Y \geq 8|p = .3) \\
&= 0.0016
\end{aligned}$$

Equivalence of Hypthesis test and Confidence Intervals

A coefficient γ confidence set can be thought of as a set of H_0s that would be accepted at significance level $1 - \gamma$.

Example 9.1.15

Coefficient γ Confidence Interval:

$$\left(\bar{X}_n \pm c\sigma'/\sqrt{n}\right)$$

where c is the $\frac{1+\gamma}{2}$ quantile of t_{n-1} distribution.

Normal CIs

We can use the CI to find a level $\alpha_0 = 1 - \gamma$ test of $H_0 : \mu = \mu_0; H_A : \mu \neq \mu_0$

Reject H_0 if $\mu_0 \notin \left(\bar{X}_n \pm c\sigma'/\sqrt{n}\right)$

Which is equivalent to $\left(|\bar{X}_n - \mu_0| \geq \pm c\sigma'/\sqrt{n}\right)$

Or

$$\left|\frac{\bar{X}_n - \mu_0}{\sigma'/\sqrt{n}}\right| \geq c$$

Which looks like a t-test!

Chapter 9.5 The t-test

Again, consider $X_1 ..., X_n \sim N(\mu, \sigma^2)$ with μ, σ^2 unknown.

$$H_0 : \mu \leq \mu_0; \{\Omega_0 : \mu \leq \mu_0 \text{ and } \sigma^2 > 0\}$$
$$H_A : \mu > \mu_0; \{\Omega_1 : \mu > \mu_0 \text{ and } \sigma^2 > 0\}$$

Reject H_0 if $U \geq c$ where $U = \frac{\bar{X}_n - \mu_0}{\sigma'/\sqrt{n}}$ and $c = T'_{(n-1)(1-\alpha_0)}$ since $U \sim t_{(n-1)}$ when $H_0 : \mu = \mu_0$ is true.

Example TV Radition

It's 1980 and we are interested in measuring the radiation level of TV displays in a department store. The safety limit established by the FDA is 0.5 milliroentgen per hour (mR/hr). We collect data.

$$n = 10$$
$$\bar{x}_n = 0.52$$
$$\sigma' = .241$$

Confidence Interval for μ

$$\bar{x}_n \pm t^{-1}_{(n-1)}(.975)\sigma'/\sqrt{n}$$
$$.52 \pm 2.2622 \times .241/\sqrt{10}$$
$$(0.35, 0.69)$$

Hypothesis Test for TV radiation

$H_0 : \mu \leq 0.50$ assume safe levels vs $H_1 : \mu > 0.5$

$U = \frac{\bar{X}_n - \mu_0}{\sigma'/\sqrt{n}} = \frac{.52 - .50}{.241/\sqrt{10}} = .26$

Reject H_0 if $U > t^{-1}_{(9)}(.95) = 1.833$

$$\begin{aligned} \text{p-value} &= \sup_{\mu \leq 0.50} P(U \geq .26 | \mu) \\ &= P(U \geq .26 | \mu = 0.50) = .40 \end{aligned}$$

Do not Reject H_0. No significant evidence that unsafe levels exist.

```r
qt(.975,9)
```

```
## [1] 2.262157
```

```r
1-pt(.26,9)
```

```
## [1] 0.400357
```

Assumptions

- Population is Normally Distributed
- Random sample (iid)
- Normal distribution (or n is "large enough")

Question

How much power did this test have to reject H_0 when, say, $\mu = 0.60$?
In general, was this test powerful enough to detect meaningful safety violations?
test statistic: $U = \frac{\bar{X}_n - \mu_0}{\sigma'/\sqrt{n}} \sim t_{(n-1)}$ if $H_0 : \mu = \mu_0$ is true
But $U = \frac{\bar{X}_n - \mu_0}{\sigma'/\sqrt{n}} \not\sim t_{(n-1)}$ if H_0 is false.

Non-Central t-distribution

Definition (9.5.1): If Y and W are independent random variables, with $W \sim N(\psi, 1)$ and $Y \sim \chi^2_{(m)}$, then the distribution of

$$X = \frac{W}{\sqrt{Y/m}}$$

is called a non-central t distribution with m degrees of freedom and non-centrality parameter ψ

Consider U when H_0 is false.

$$U = \frac{\bar{X}_n - \mu_0}{\sigma'/\sqrt{n}} = \frac{\frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)\sigma'^2}{\sigma^2}/(n-1)}}$$

$$= \frac{\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} + \frac{\mu - \mu_0}{\sigma/\sqrt{n}}}{\sqrt{\frac{\chi^2_{(n-1)}}{(n-1)}}} = \frac{N(\psi, 1)}{\sqrt{\chi^2_{(n-1)}/(n-1)}} \sim t_{(n-1),\psi}$$

Calculating Power

Then the power at μ is

$$1 - T_{(n-1)(c|\psi)} = \int_{1.833}^{\infty} f_T(t)dt$$

- $P(X \leq t) = T_n(t|\psi)$
- $c =$ critical value
- Reject H_0 if $U \geq c = 1.833$
- $\psi = \frac{\mu-\mu_0}{\sigma/\sqrt{n}} = \frac{.60-.50}{\sigma/\sqrt{10}}$

But we need σ...

Power Using Non-Central t

1. Find power at $\mu = .6$ assuming $\sigma = s$ (sample standard deviation)
2. Find power as a function of effect size in standard deviations (express μ in terms of σ and μ_0)
3. Assuming $\mu = \mu_0 + 1 \times \sigma$, plot the following
    a. Power vs ψ
    b. Power vs n

Find power at $\mu = .6$ assuming $\sigma = s$ (sample standard deviation)

```
psi=(.6-.5)/(.241/sqrt(10))
1-pt(1.833,df=9,ncp=psi)
```

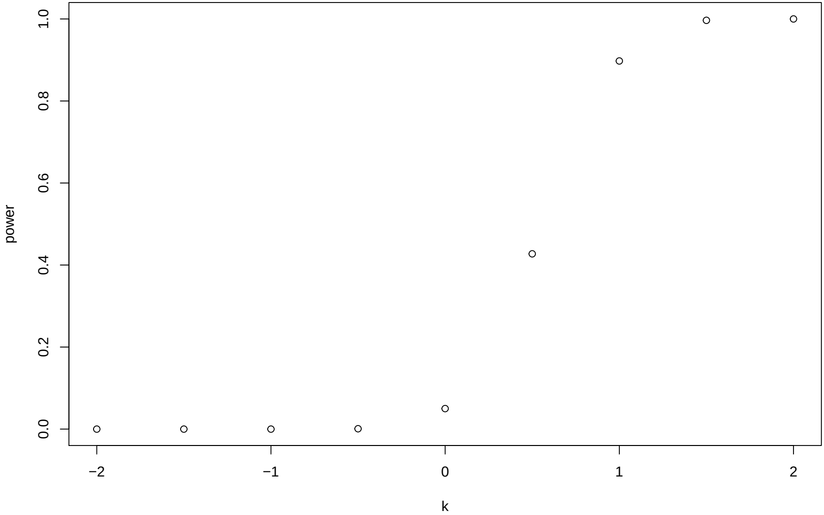
```
## [1] 0.3332105
```

Find power as a function of effect size in standard deviations (express μ in terms of σ and μ_0)

```
psi=1*sqrt(10)
1-pt(1.833,df=9,ncp=psi)
```

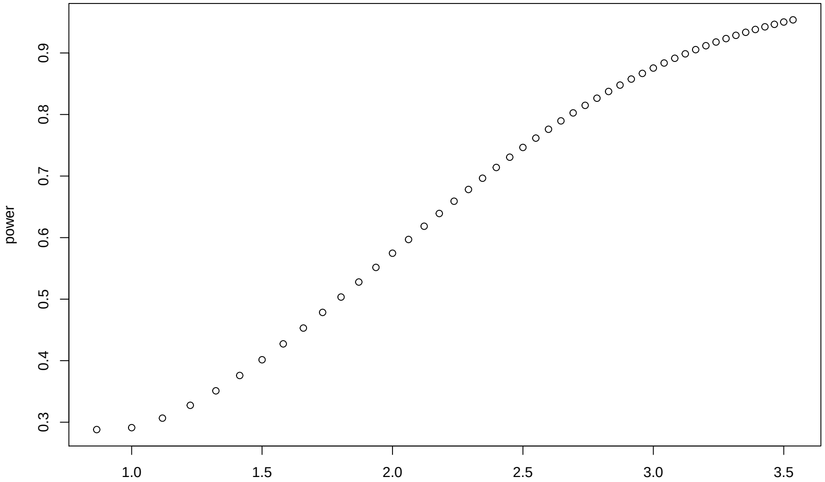
```
## [1] 0.8975371
```

Bad plot trying to look at different number of SDs away from null



Assuming $\mu = \mu_0 + 1 \times \sigma$: Power vs ψ

```
n=c(seq(3,50,by=1))
psi=.5*sqrt(n)
power=1-pt(1.833,df=(n-1),ncp=psi)
plot(power~psi)
```



Power vs n

```
plot(power~n)
```