Optimize over $I[f]=\int ds_1\int ds_2\frac{f(s_1)+f(s_2)}{2}\ln\left(\frac{f(s_1)+f(s_2)}{2}\right)$, $\int f(x)dx=1$
$\ln(\frac{f(s_1)+f(s_2)+h(s_1)+h(s_2)}{2})=\ln(\frac{f(s_1)+f(s_2)}{2})+\frac{h(s_1)+h(s_2)}{f(s_1)+f(s_2)}+O(h^2)$

$$\begin{aligned}I[f+h] &= I[f]+\int ds_1\int ds_2\frac{f(s_1)+f(s_2)}{2}\cdot\frac{h(s_1)+h(s_2)}{f(s_1)+f(s_2)}+\int ds_1\int ds_2\frac{h(s_1)+h(s_2)}{2}\ln\left(\frac{f(s_1)+f(s_2)}{2}\right)+O(h^2)\\ &= I[f]+\int h(s)ds+\int ds_1 h(s_1)\int ds_2\ln\left(\frac{f(s_1)+f(s_2)}{2}\right)+O(h^2)\end{aligned}$$

$$\nabla_f I(x)=1+\int dy\ln\left(\frac{f(x)+f(y)}{2}\right)$$

$\eta=s+\epsilon\phi(s)$
$ds=\frac{1}{1+\epsilon\phi'(s)}d\eta=\left(1-\epsilon\phi'(\eta)+O(\epsilon^2)\right)d\eta$
$\int f(x)\phi'(x)dx=0$

$$\begin{aligned}I[f(x+\epsilon\phi(x))] &= \int ds_1\int ds_2\frac{f(s_1+\epsilon\phi(s_1))+f(s_2+\epsilon\phi(s_2))}{2}\ln\left(\frac{f(s_1+\epsilon\phi(s_1))+f(s_2+\epsilon\phi(s_2))}{2}\right)\\ &= \int(1-\epsilon\phi'(\eta_1))\,d\eta_1\int(1-\epsilon\phi'(\eta_2))\,d\eta_2\frac{f(\eta_1)+f(\eta_2)}{2}\ln\left(\frac{f(\eta_1)+f(\eta_2)}{2}\right)+O(\epsilon^2)\\ &= I[f]-\epsilon\int\phi'(\eta_1)d\eta_1\int d\eta_2\,(f(\eta_1)+f(\eta_2))\ln\left(\frac{f(\eta_1)+f(\eta_2)}{2}\right)+O(\epsilon^2)\\ &= I[f]+\epsilon\int d\eta_1\phi(\eta_1)\frac{d}{d\eta_1}\left[\int d\eta_2\,(f(\eta_1)+f(\eta_2))\ln\left(\frac{f(\eta_1)+f(\eta_2)}{2}\right)\right]+O(\epsilon^2)\end{aligned}$$

Thus

$$\nabla_\phi I(x)=\int dy\,(f_c'(x)+f(y))\ln\left(\frac{f_c'(x)+f(y)}{2}\right)+\sum_{x_i\in D}\triangle_i\tilde{f}\cdot\delta_{x_i}(x)$$

where δ is the dirac delta function,

$$\triangle_i\tilde{f}\coloneqq\int(f(x_i+)+f(y))\,dy\ln\left(\frac{f(x_i+)+f(y)}{2}\right)-\int(f(x_i-)+f(y))\,dy\ln\left(\frac{f(x_i-)+f(y)}{2}\right).$$

Assume that f is piece-wise continuous with only a finite number of jump discontinuities.
KKT conditions:

$$\nabla_f I(x) = \alpha_+(x)-\alpha_-(x)+\mu$$

When f does not take the upper/lower bounds:

$$\begin{aligned}\nabla_f I(x) &= \mu\\ \int dy\ln\left(\frac{f(x)+f(y)}{2}\right) &= \mu-1\end{aligned}$$

Assume f is continuously differentiable at x,

$$\begin{aligned}\frac{d}{dx}\int dy\ln\left(\frac{f(x)+f(y)}{2}\right) &= 0\\ f_c'(x)\int dy\frac{1}{f(x)+f(y)} &= 0\\ f_c'(x) &= 0\end{aligned}$$

where the last equation follows from $f \geq f_- > 0$.

Thus f is a piece-wise constant function. with only at most two possible values? need to be proved below.

KKT condition for $\nabla_\phi I$ (the RHS comes from $\int f(x)\phi'(x) = -\int \phi(x)\left(f_c'(x) + \sum_{x_i\in D} \triangle_i f \cdot \delta_{x_i}(x)\right) = 0$):

$$\begin{aligned}
\nabla_\phi I(x) &= \nu\left(f_c'(x) + \sum_{x_i\in D} \triangle_i f \cdot \delta_{x_i}(x)\right) \\
\int dy\,(f_c'(x) + f(y)) \ln\left(\frac{f_c'(x) + f(y)}{2}\right) + \sum_{x_i\in D} \triangle_i \tilde{f} \cdot \delta_{x_i}(x) &= \nu\left(f_c'(x) + \sum_{x_i\in D} \triangle_i f \cdot \delta_{x_i}(x)\right)
\end{aligned}$$

At continous point x: since $f_c'(x) = 0$, we have

$$\begin{aligned}
\int dy f(y) \ln\left(\frac{0 + f(y)}{2}\right) &= \nu \\
\int f(y)\ln(f(y))dy - \ln 2 &= \nu
\end{aligned}$$

At discontinuity x_i

$$\begin{aligned}
\triangle_i \tilde{f} &= \nu \triangle_i f \\
\int (f(x_i+) + f(y))\,dy \ln\left(\frac{f(x_i+) + f(y)}{2}\right) - \int (f(x_i-) + f(y))\,dy \ln\left(\frac{f(x_i-) + f(y)}{2}\right) &= \nu\left(f(x_i+) - f(x_i-)\right) \\
\frac{\triangle_i \tilde{f}}{\triangle_i f} &= \nu
\end{aligned}$$

$g(t) = \int (t + f(y)) \ln\left(\frac{t+f(y)}{2}\right) dy = \int (t + f(y))\ln(t + f(y))dy - (t+1)\ln 2$ is a convex function on $t \in [f_-, f_+]$: $g''(t) = \int \frac{1}{t+f(y)} dy > 0$.

Assume f jumps between f_+ and f_- at some point, then there should be no other jumps between values in (f_-, f_+), otherwise contradicts the convexity of $g(t)$.

Therefore, when f only takes f_+ or f_-, $I[f]$ reaches a stationary point: $\int f(x)dx = 1$, $|\{f = f_+\}| = \triangle$, $|\{f = f_-\}| = 1 - \triangle$, and $\triangle = \frac{1-f_-}{f_+ - f_-}$.

$$\begin{aligned}
\max I[f] &= \int ds_1 \int ds_2 \frac{f(s_1) + f(s_2)}{2} \ln\left(\frac{f(s_1) + f(s_2)}{2}\right) \\
&= \int ds_1 \int_{\{f(s_2) = f_+\}} ds_2 \frac{f(s_1) + f_+}{2} \ln\left(\frac{f(s_1) + f_+}{2}\right) \\
&\quad + \int ds_1 \int_{\{f(s_2) = f_-\}} ds_2 \frac{f(s_1) + f_-}{2} \ln\left(\frac{f(s_1) + f_-}{2}\right) \\
&= \triangle \int ds_1 \frac{f(s_1) + f_+}{2} \ln\left(\frac{f(s_1) + f_+}{2}\right) \\
&\quad + (1 - \triangle) \int ds_1 \frac{f(s_1) + f_-}{2} \ln\left(\frac{f(s_1) + f_-}{2}\right) \\
&= \triangle^2 \frac{f_+ + f_+}{2} \ln\left(\frac{f_+ + f_+}{2}\right) + 2\triangle(1-\triangle)\frac{f_- + f_+}{2} \ln\left(\frac{f_- + f_+}{2}\right) \\
&\quad + (1-\triangle)^2 \frac{f_- + f_-}{2} \ln\left(\frac{f_- + f_-}{2}\right) \\
&= \sum_{m=0}^{2} \binom{2}{m} \triangle^m (1-\triangle)^{2-m} \frac{mf_+ + (2-m)f_-}{2} \ln\left(\frac{mf_+ + (2-m)f_-}{2}\right)
\end{aligned}$$

—-

More generally, for k positions:
Optimize over $I_k[f] = \int \cdots \int \prod_{l=1}^{k} ds_l \frac{\sum_{l=1}^{k} f(s_l)}{k} \ln\left(\frac{\sum_{l=1}^{k} f(s_l)}{k}\right)$, $\int f(x)dx = 1$

$$\begin{aligned}
I_k[f+h] &= I_k[f] + \int \cdots \int \prod_{l=1}^{k} ds_l \frac{\sum_l f(s_l)}{k} \frac{\sum_l h(s_l)}{\sum_l f(s_l)} + \int \cdots \int \prod_{l=1}^{k} ds_l \frac{\sum_l h(s_l)}{k} \ln\left(\frac{\sum_l f(s_l)}{k}\right) + O(h^2) \\
&= I_k[f] + \int \cdots \int \prod_{l=1}^{k} ds_l \frac{\sum_l h(s_l)}{k} + \int \cdots \int \prod_{l=1}^{k} ds_l h(s_1) \ln\left(\frac{\sum_l f(s_l)}{k}\right) + O(h^2)
\end{aligned}$$

$$\nabla_f I_k(x) = 1 + \int \cdots \int \prod_{l=1}^{k-1} dy_l \ln\left(\frac{f(x) + \sum_{l=1}^{k-1} f(y_l)}{k}\right)$$

$$\begin{aligned}
I_k[f(x+\epsilon\phi(x))] &= \int \cdots \int \prod_{l=1}^{k} (1 - \epsilon\phi'(\eta_l))\, d\eta_l \frac{\sum f(\eta_l)}{k} \ln\left(\frac{\sum f(\eta_l)}{k}\right) + O(\epsilon^2) \\
&= I_k[f] - \epsilon \int k\phi'(\eta_k) d\eta_k \int \cdots \int \prod_{l=1}^{k-1} d\eta_l \frac{\sum f(\eta_l)}{k} \ln\left(\frac{\sum f(\eta_l)}{k}\right) + O(\epsilon^2) \\
&= I_k[f] + \epsilon \int d\eta_k \phi(\eta_k) \frac{d}{d\eta_k}\left[\int \cdots \int \prod_{l=1}^{k-1} d\eta_l \left(\sum_{l=1}^{k} f(\eta_l)\right) \ln\left(\frac{\sum_{l=1}^{k} f(\eta_l)}{k}\right)\right] + O(\epsilon^2)
\end{aligned}$$

Thus

$$\nabla_\phi I_k(x) = \int \cdots \int \prod_{l=1}^{k-1} dy_l \left(f_c'(x) + \sum_{l=1}^{k-1} f(y_l)\right) \ln\left(\frac{f_c'(x) + \sum_{l=1}^{k-1} f(y_l)}{k}\right) + \sum_{x_i \in D} \triangle_i \tilde{f} \cdot \delta_{x_i}(x)$$

where δ is the dirac delta function,

$$\triangle_i \tilde{f} := \int \cdots \int \prod_{l=1}^{k-1} dy_l \left(f(x_i+) + \sum_{l=1}^{k-1} f(y_l)\right) dy \ln\left(\frac{f(x_i+) + \sum_{l=1}^{k-1} f(y_l)}{k}\right) - \int \cdots \int \prod_{l=1}^{k-1} dy_l \left(f(x_i-) + \sum_{l=1}^{k-1} f(y_l)\right) dy \ln\left(\frac{f(x_i-) +\right.$$

1. KKT conditions on $\nabla_f I_k$:
Assume that f is piece-wise continuous with only a finite number of jump discontinuities.

$$\nabla_f I_k(x) = \alpha_+(x) - \alpha_-(x) + \mu$$

When f does not take the upper/lower bounds:

$$\begin{aligned}
\nabla_f I_k(x) &= \mu \\
1 + \int \cdots \int \prod_{l=1}^{k-1} dy_l \ln\left(\frac{f(x) + \sum_{l=1}^{k-1} f(y_l)}{k}\right) &= \mu
\end{aligned}$$

Assume f is continuously differentiable at x,

$$\begin{aligned}
\frac{d}{dx} \int \cdots \int \prod_{l=1}^{k-1} dy_l \ln\left(\frac{f(x) + \sum_{l=1}^{k-1} f(y_l)}{k}\right) &= 0 \\
f_c'(x) \int \cdots \int \prod_{l=1}^{k-1} dy_l \frac{1}{f(x) + \sum_{l=1}^{k-1} f(y_l)} &= 0 \\
f_c'(x) &= 0
\end{aligned}$$

where the last equation follows from $f \geq f_- > 0$.
Thus f is a piece-wise constant function. with only at most two possible values? need to be proved below.
2. KKT condition for $\nabla_\phi I_k$:
(the RHS comes from $\int f(x)\phi'(x) = -\int \phi(x)\left(f_c'(x) + \sum_{x_i \in D} \triangle_i f \cdot \delta_{x_i}(x)\right) = 0$):

$$\begin{aligned}\nabla_\phi I_k(x) &= \nu\left(f_c'(x) + \sum_{x_i \in D} \triangle_i f \cdot \delta_{x_i}(x)\right)\\ \int \cdots \int \prod_{l=1}^{k-1} dy_l \left(f_c'(x) + \sum_{l=1}^{k-1} f(y_l)\right) \ln\left(\frac{f_c'(x) + \sum_{l=1}^{k-1} f(y_l)}{k}\right) + \sum_{x_i \in D} \triangle_i \tilde{f} \cdot \delta_{x_i}(x) &= \nu\left(f_c'(x) + \sum_{x_i \in D} \triangle_i f \cdot \delta_{x_i}(x)\right)\end{aligned}$$

At discontinuity x_i

$$\begin{aligned}\triangle_i \tilde{f} &= \nu \triangle_i f\\ \frac{\triangle_i \tilde{f}}{\triangle_i f} &= \nu\end{aligned}$$

Since $\triangle_i \tilde{f} = \int \cdots \int \prod_{l=1}^{k-1} dy_l \left(f(x_i+) + \sum_{l=1}^{k-1} f(y_l)\right) dy \ln\left(\frac{f(x_i+)+\sum_{l=1}^{k-1} f(y_l)}{k}\right) - \int \cdots \int \prod_{l=1}^{k-1} dy_l \left(f(x_i-) + \sum_{l=1}^{k-1} f(y_l)\right) dy \ln \left(\right.$
$\triangle_i f = f(x_i+) - f(x_i-)$,
let $g(t) := \int \cdots \int \prod_{l=1}^{k-1} dy_l \left(t + \sum_{l=1}^{k-1} f(y_l)\right) \ln\left(\frac{t+\sum_{l=1}^{k-1} f(y_l)}{k}\right) = \int \cdots \int \prod_{l=1}^{k-1} dy_l \left(f(t) + \sum_{l=1}^{k-1} f(y_l)\right) \ln\left(t + \sum_{l=1}^{k-1} f(y_l)\right) -$ $(t+k-1)\ln k$ is a convex function on $t \in [f_-, f_+]$:
$g''(t) = \int \cdots \int \prod_{l=1}^{k-1} dy_l \frac{1}{t+\sum_{l=1}^{k-1} f(y_l)} > 0$
Therefore, when f only takes f_+ or f_-, $I[f]$ reaches a stationary point:
Assume $\int f(x)dx = 1$, $|\{f = f_+\}| = \triangle$, $|\{f = f_-\}| = 1 - \triangle$, and $\triangle = \frac{1-f_-}{f_+-f_-}$.

$$\begin{aligned}I_k[f] &= \int \cdots \int \prod_{l=1}^{k} ds_l \frac{\sum_{l=1}^{k} f(s_l)}{k} \ln\left(\frac{\sum_{l=1}^{k} f(s_l)}{k}\right)\\ B(k, f_+, f_-) &:= \sum_{m=0}^{k} \binom{k}{m} \triangle^m (1-\triangle)^{k-m} \frac{mf_+ + (k-m)f_-}{k} \ln\left(\frac{mf_+ + (k-m)f_-}{k}\right) \qquad (1)\end{aligned}$$

special case when $k = 1$:

$$\begin{aligned}\max I_k[f] &= \int ds f(s) \ln f(s)\\ B(1, f_+, f_-) &= \triangle f_+ \ln(f_+) + (1-\triangle) f_- \ln(f_-)\end{aligned}$$

General case:

The model (not assuming $\bar{f} = 1$):

Fix k and n:

stimulus $\theta_1, ..., \theta_k$, firing cell positions $x_1, ..., x_n$:
One population:

$$
\begin{aligned}
I[f;n,k] &= \int \prod_{l=1}^{k} d\theta_l \int \prod_{i=1}^{n} dx_i \prod{}_{i=1}^{n} \left(\sum_{l=1}^{k} f(\theta_l - x_i)\right) \ln\left(\frac{\prod{}_{i=1}^{n}\left(\sum_{l=1}^{k} f(\theta_l - x_i)\right)}{\int \prod_{l=1}^{k} d\theta'_l \prod{}_{i=1}^{n}\left(\sum_{l=1}^{k} f(\theta'_l - x_i)\right)}\right) \\
&= \bar{f}^n \int \prod_{l=1}^{k} d\theta_l \int \prod_{i=1}^{n} dx_i \prod{}_{i=1}^{n} \left(\sum_{l=1}^{k} \tilde{f}(\theta_l - x_i)\right) \ln\left(\frac{\prod{}_{i=1}^{n}\left(\sum_{l=1}^{k} \tilde{f}(\theta_l - x_i)\right)}{\int \prod_{l=1}^{k} d\theta'_l \prod{}_{i=1}^{n}\left(\sum_{l=1}^{k} \tilde{f}(\theta'_l - x_i)\right)}\right) \\
&= n\bar{f}^n k^n \int \cdots \int \prod d\theta_l \frac{\sum_{l=1}^{k} \tilde{f}(\theta_l)}{k} \ln\left(\frac{\sum_{l=1}^{k} \tilde{f}(\theta_l)}{k}\right) \\
&\quad -\bar{f}^n k^n \int \cdots \int \prod_{i=1}^{n} dx_i A_k(x_1,...,x_n) \ln A_k(x_1,...,x_n) \\
\max I[f;n,k] &\leq \bar{f}^n k^n n B\left(k, \frac{f_+}{\bar{f}}, \frac{f_-}{\bar{f}}\right) \qquad (2)
\end{aligned}
$$

where $A_k(x_1,...,x_n) = \int \prod_{l=1}^{k} d\theta_l \prod{}_{i=1}^{n} \left(\frac{\sum_{l=1}^{k} \tilde{f}(\theta_l - x_i)}{k}\right)$.

Two populations:

$$
\begin{aligned}
I[f_1,f_2;n_1,n_2,k] &= \int \cdots \int \prod_{l=1}^{k} ds_l \prod_{i=1}^{n_1} dx_i \prod_{j=1}^{n_2} dy_j \Bigg\{ \prod_{i,j} d_1\left(\sum_l f_1(s_l - x_i)\right) d_2\left(\sum_l f_2(s_l - y_j)\right) \\
&\quad \ln\left(\frac{\prod_{i,j}(d_1(\sum_l f_1(s_l - x_i))\, d_2(\sum_l f_2(s_l - y_j)))}{\int \cdots \int \prod_{l=1}^{k} ds'_l \prod_{i,j}(d_1(\sum_l f_1(s'_l - x_i))\, d_2(\sum_l f_2(s'_l - y_j)))}\right) \Bigg\} \\
&= (d_1\bar{f}_1)^{n_1} (d_2\bar{f}_2)^{n_2} \int \cdots \int \prod_{l=1}^{k} ds_l \prod_{i=1}^{n_1} dx_i \prod_{j=1}^{n_2} dy_j \Bigg\{ \prod_{i,j} \left(\sum_l \tilde{f}_1(s_l - x_i)\right) \left(\sum_l \tilde{f}_2(s_l - y_j)\right) \\
&\quad \ln\left(\frac{\prod_{i,j}\left(\sum_l \tilde{f}_1(s_l - x_i)\right)\left(\sum_l \tilde{f}_2(s_l - y_j)\right)}{\int \cdots \int \prod_{l=1}^{k} ds'_l \prod_{i,j}\left(\sum_l \tilde{f}_1(s'_l - x_i)\right)\left(\sum_l \tilde{f}_2(s'_l - y_j)\right)}\right) \Bigg\} \\
&= (d_1\bar{f}_1)^{n_1} (d_2\bar{f}_2)^{n_2} k^{n_1+n_2} n_1 \int \cdots \int \prod_{l=1}^{k} ds_l \prod_{i=1}^{n_1} dx_i \left(\frac{\sum_l \tilde{f}_1(s_l - x_i)}{k}\right) \ln\left(\frac{\sum_l \tilde{f}_1(s_l - x_i)}{k}\right) \\
&\quad + (d_1\bar{f}_1)^{n_1} (d_2\bar{f}_2)^{n_2} k^{n_1+n_2} n_2 \int \cdots \int \prod_{l=1}^{k} ds_l \prod_{j=1}^{n_2} dy_j \left(\frac{\sum_l \tilde{f}_2(s_l - y_j)}{k}\right) \ln\left(\frac{\sum_l \tilde{f}_2(s_l - y_j)}{k}\right) \\
&\quad - (d_1\bar{f}_1)^{n_1} (d_2\bar{f}_2)^{n_2} k^{n_1+n_2} \int \cdots \int \prod_{i=1}^{n_1} dx_i \prod_{j=1}^{n_2} dy_j A_k(x_1,...,x_{n_1},y_1,...,y_{n_2}) \ln\left(A_k(x_1,...,x_{n_1},y_1,...,y_{n_2})\right) \\
\max I[f_1,f_2;n_1,n_2,k] &\leq (d_1\bar{f}_1)^{n_1} (d_2\bar{f}_2)^{n_2} k^{n_1+n_2} \left[n_1 B\left(k, \frac{f_+}{\bar{f}_1}, \frac{f_-}{\bar{f}_1}\right) + n_2 B\left(k, \frac{f_+}{\bar{f}_2}, \frac{f_-}{\bar{f}_2}\right)\right] \qquad (3)
\end{aligned}
$$

where

$$
A_k(x_1,...,x_{n_1},y_1,...,y_{n_2}) \coloneqq \int \cdots \int \prod_{l=1}^{k} ds'_l \prod_{i,j} \left(\frac{\sum_l \tilde{f}_1(s'_l - x_i)}{k}\right) \left(\frac{\sum_l \tilde{f}_2(s'_l - y_j)}{k}\right)
$$

and maximum is achieved when $\tilde{A} \equiv 1$, i.e. f_1 and f_2 randomly takes f_+ and f_-.

Fix k and sum over n:

One population:

$$\begin{aligned} I[f;k] &= e^{-k\bar{f}}\sum_n \frac{1}{n!} I[f;n,k] \\ \max I[f;k] &\leq e^{-k\bar{f}}\sum_n \frac{\bar{f}^n k^n}{n!} nB\left(k, \frac{f_+}{\bar{f}}, \frac{f_-}{\bar{f}}\right) \\ &= k\bar{f}B\left(k, \frac{f_+}{\bar{f}}, \frac{f_-}{\bar{f}}\right) \end{aligned} \tag{4}$$

Two populations:remove , $\sum_{n_1,n_2} \frac{1}{n_1!n_2!}$

$$\begin{aligned} I[f_1,f_2;k] &= e^{-k\left(d_1\bar{f}_1+d_2\bar{f}_2\right)}\sum_{n_1,n_2} \frac{1}{n_1!n_2!} I[f_1,f_2;n_1,n_2,k] \\ \max I[f;k] &\leq e^{-k\left(d_1\bar{f}_1+d_2\bar{f}_2\right)}\sum_{n_1,n_2} \frac{\left(d_1\bar{f}_1\right)^{n_1}\left(d_2\bar{f}_2\right)^{n_2} k^{n_1+n_2}}{n_1!n_2!}\left[n_1B\left(k, \frac{f_+}{\bar{f}_1}, \frac{f_-}{\bar{f}_1}\right) + n_2B\left(k, \frac{f_+}{\bar{f}_2}, \frac{f_-}{\bar{f}_2}\right)\right] \\ &= kd_1\bar{f}_1B\left(k, \frac{f_+}{\bar{f}_1}, \frac{f_-}{\bar{f}_1}\right) + kd_2\bar{f}_2B\left(k, \frac{f_+}{\bar{f}_2}, \frac{f_-}{\bar{f}_2}\right) \end{aligned} \tag{5}$$

Recall that $B(k,p,q)$ is defined as:

$$B(k,p,q) \quad \coloneqq \quad \sum_{m=0}^{k} \begin{pmatrix} k \\ m \end{pmatrix} \triangle^m (1-\triangle)^{k-m} \frac{mp+(k-m)q}{k} \ln\left(\frac{mp+(k-m)q}{k}\right) \tag{6}$$

where $p > 1 > q$.

As a function of $\bar{f}$

$\max I[f;k] = k\bar{f}B(k, \frac{f_+}{\bar{f}}, \frac{f_-}{\bar{f}}) = k\bar{f} \cdot \sum_{m=0}^{k} \begin{pmatrix} k \\ m \end{pmatrix} \triangle^m (1-\triangle)^{k-m} \frac{mf_+ + (k-m)f_-}{k\bar{f}} \ln\left(\frac{mf_+ + (k-m)f_-}{k\bar{f}}\right)$

where $\triangle = \frac{\bar{f}-f_-}{f_+-f_-}$.

If we look at $H(k,a,b,t) = t \cdot \sum_{m=0}^{k} \begin{pmatrix} k \\ m \end{pmatrix} \left(\frac{t-b}{a-b}\right)^m (1-\frac{t-b}{a-b})^{k-m} \frac{ma+(k-m)b}{kt} \ln\left(\frac{ma+(k-m)b}{kt}\right)$

where $b < t < a$.

$$
\begin{aligned}
H(k,a,b,t) &= t \cdot \sum_{m=0}^{k} \binom{k}{m} \left(\frac{t-b}{a-b}\right)^{m} \left(1-\frac{t-b}{a-b}\right)^{k-m} \frac{ma+(k-m)b}{kt} \ln\left(\frac{ma+(k-m)b}{kt}\right) \\
&= \sum_{m=0}^{k} \binom{k}{m} \frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k} \cdot \frac{ma+(k-m)b}{k} \ln\left(\frac{ma+(k-m)b}{kt}\right) \qquad (7) \\
&= \sum_{m=0}^{k} \binom{k}{m} \frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k} \cdot \frac{ma+(k-m)b}{k} \ln\left(\frac{ma+(k-m)b}{k}\right) \\
&\quad - \ln t \sum_{m=0}^{k} \binom{k}{m} \frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k} \cdot \frac{ma+(k-m)b}{k} \\
&= \sum_{m=0}^{k} \binom{k}{m} \frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k} \cdot \frac{ma+(k-m)b}{k} \ln\left(\frac{ma+(k-m)b}{k}\right) - t\ln t \qquad (8) \\
\frac{\partial H(k,a,b,t)}{\partial t} &= -(1+\ln t) + \sum_{m=0}^{k} \binom{k}{m} \frac{\partial}{\partial t}\left[\frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k}\right] \cdot \frac{ma+(k-m)b}{k} \ln\left(\frac{ma+(k-m)b}{k}\right) \\
&= -(1+\ln t) + \sum_{m=0}^{k} \binom{k}{m} \frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k} \cdot \left(\frac{m}{t-b} - \frac{k-m}{a-t}\right) \frac{ma+(k-m)b}{k} \ln\left(\frac{ma+(k-m)b}{k}\right) \qquad (9) \\
poly(t) &= 1+\ln t \qquad (10)
\end{aligned}
$$

$\frac{ma+(k-m)b-kt}{(t-b)(a-t)}$

* Show that $\sum_{m=0}^{k} \binom{k}{m} \frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k} \cdot \frac{ma+(k-m)b}{k} = t$ and $\sum_{m=0}^{k} \binom{k}{m} \frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k} \cdot \frac{ma+(k-m)b}{k} = kt^2 + (t-b)(a-t)$:

$$
\begin{aligned}
(a-b)^k &= [(t-b)+(a-t)]^k \\
&= \sum_{m=0}^{k} \binom{k}{m} (t-b)^m (a-t)^{k-m} \\
\frac{d}{dt}(a-b)^k &= \sum_{m=0}^{k} \binom{k}{m} (t-b)^m (a-t)^{k-m} \left[\frac{m}{t-b} - \frac{k-m}{a-t}\right] \\
0 &= \sum_{m=0}^{k} \binom{k}{m} (t-b)^m (a-t)^{k-m} \left[\frac{ma+(k-m)b-kt}{(t-b)(a-t)}\right] \\
\sum_{m=0}^{k} \binom{k}{m} (t-b)^m (a-t)^{k-m} \left(ma+(k-m)b\right) &= kt \sum_{m=0}^{k} \binom{k}{m} (t-b)^m (a-t)^{k-m} \\
\sum_{m=0}^{k} \binom{k}{m} (t-b)^m (a-t)^{k-m} \left(\frac{ma+(k-m)b}{k}\right) &= t(a-b)^k \\
\sum_{m=0}^{k} \binom{k}{m} \frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k} \left(\frac{ma+(k-m)b}{k}\right) &= t
\end{aligned}
$$

$$\begin{aligned}
\sum_{m=0}^{k}\binom{k}{m}(t-b)^m(a-t)^{k-m}\,(ma+(k-m)b) &= kt\sum_{m=0}^{k}\binom{k}{m}(t-b)^m(a-t)^{k-m} \\
\sum_{m=0}^{k}\binom{k}{m}(t-b)^m(a-t)^{k-m}\left(\frac{ma+(k-m)b}{k}\right) &= t(a-b)^k \\
\sum_{m=0}^{k}\binom{k}{m}\frac{(t-b)^m(a-t)^{k-m}}{(a-b)^k}\left(\frac{ma+(k-m)b}{k}\right) &= t \\
\sum_{m=0}^{k}\binom{k}{m}\frac{(t-b)^m(a-t)^{k-m}}{(a-b)^k}\left[\frac{ma+(k-m)b-kt}{(t-b)(a-t)}\right]\left(\frac{ma+(k-m)b}{k}\right) &= 1 \\
\sum_{m=0}^{k}\binom{k}{m}\frac{(t-b)^m(a-t)^{k-m}}{(a-b)^k}\frac{(ma+(k-m)b)^2}{k} &= kt\sum_{m=0}^{k}\binom{k}{m}\frac{(t-b)^m(a-t)^{k-m}}{(a-b)^k}\left(\frac{ma+(k-m)}{k}\right. \\
&= kt^2+(t-b)(a-t)
\end{aligned}$$

(weighted sum = 0 and weights depends on t, not solvable.)
$k=1$:

$$\begin{aligned}
\frac{\partial H(1,a,b,t)}{\partial t} &= -1-\ln t+\frac{a\ln a-b\ln b}{a-b} \\
&= 0 \\
1+\ln t &= \frac{a\ln a-b\ln b}{a-b} \\
t &= \exp\left(\frac{a\ln a-b\ln b}{a-b}-1\right) \\
\max H(1,a,b,t) &= \frac{(t-b)a\ln a+(a-t)b\ln b}{a-b}-t\ln t \\
&= \frac{ab(\ln a-\ln b)}{a-b}+t\frac{a\ln a-b\ln b}{a-b}-t\ln t \\
&= \exp\left(\frac{a\ln a-b\ln b}{a-b}-1\right)-\frac{ab(\ln a-\ln b)}{a-b}
\end{aligned}$$

$\sum_{m=0}^{k}\binom{k}{m}\frac{(t-b)^m(a-t)^{k-m}}{(a-b)^k}\cdot\frac{ma+(k-m)b}{k}\ln\left(\frac{ma+(k-m)b}{kt}\right)$

$$\begin{aligned}
\max H(1,f_+,f_-,\bar{f}) &= \bar{f}\cdot\left[\triangle\frac{f_+}{\bar{f}}\ln\left(\frac{f_+}{\bar{f}}\right)+(1-\triangle)\frac{f_-}{\bar{f}}\ln\left(\frac{f_-}{\bar{f}}\right)\right] \\
&= \bar{f}\cdot\left[\frac{\bar{f}-f_-}{f_+-f_-}\frac{f_+}{\bar{f}}\ln\left(\frac{f_+}{\bar{f}}\right)+\left(\frac{\bar{f}-f_-}{f_+-f_-}\right)\frac{f_-}{\bar{f}}\ln\left(\frac{f_-}{\bar{f}}\right)\right] \\
&\leq \exp\left(\frac{f_+\ln f_+-f_-\ln f_-}{f_+-f_-}-1\right)-\frac{f_+f_-(\ln f_+-\ln f_-)}{f_+-f_-}
\end{aligned}$$

$k=2$:

$$0=\frac{\partial H(2,a,b,t)}{\partial t} = -1-\ln t+\sum_{m=0}^{2}\binom{2}{m}\frac{(t-b)^m(a-t)^{2-m}}{(a-b)^2}\cdot\left(\frac{m}{t-b}-\frac{2-m}{a-t}\right)\frac{ma+(2-m)b}{2}\ln\left(\frac{ma+(2-m)b}{2}\right)$$

$$
\begin{aligned}
1+\ln t &= \frac{1}{(a-b)^2}\left[(a-t)^2\frac{-2}{a-t}a\ln a+(t-b)(a-t)\left(\frac{1}{t-b}-\frac{1}{a-t}\right)\frac{a+b}{2}\ln\left(\frac{a+b}{2}\right)+(t-b)^2\frac{2}{t-b}b\ln b\right]\\
&= \frac{-2(a-t)a\ln a+(a+b-2t)\frac{a+b}{2}\ln\left(\frac{a+b}{2}\right)+2(t-b)b\ln b}{(a-b)^2}\\
1+\ln t &= \frac{2}{(a-b)^2}\left[(t-a)a\ln a+(t-b)b\ln b-\left(t-\frac{a+b}{2}\right)\frac{a+b}{2}\ln\left(\frac{a+b}{2}\right)\right]\\
&= \frac{2}{(a-b)^2}\left[t\left(a\ln a+b\ln b-\frac{a+b}{2}\ln\left(\frac{a+b}{2}\right)\right)-\left(a^2\ln a+b^2\ln b-\left(\frac{a+b}{2}\right)^2\ln\left(\frac{a+b}{2}\right)\right)\right]
\end{aligned}
$$

can solve solution by Lambert W function.
higher k?

Random Stimulus positions (sum over k):

$$
I=\sum_k\frac{\lambda^k e^{-\lambda}}{k!}\int ds_1\cdots\int ds_k\sum_n\frac{1}{n!}\int dx_1\cdots\int dx_n\prod_{i=1}^n\left(\sum_{l=1}^k f(s_l-x_i)\right)e^{-k\bar f}\ln\left(\frac{\prod_{i=1}^n\left(\sum_{l=1}^k f(s_l-x_i)\right)e^{-k\bar f}}{E_{s'}\left[\prod_{i=1}^n\left(\sum_{l=1}^{k'} f(s'_l-x_i)\right)e^{-k'\bar f}\right]}\right)
$$

$$
\begin{aligned}
E_s\left[\prod_{i=1}^n\left(\sum_{l=1}^k f(s'_l-x_i)\right)e^{-k\bar f}\right] &= \sum_k\frac{\lambda^k e^{-\lambda}}{k!}\int ds'_1\cdots\int ds'_k\prod_{i=1}^n\left(\sum_{l=1}^k f(s'_l-x_i)\right)e^{-k\bar f}\\
&= \sum_k\frac{\lambda^k e^{-\lambda}}{k!}\left(\bar f k\right)^n\int ds'_1\cdots\int ds'_k\prod_{i=1}^n\left(\frac{\sum_{l=1}^k\tilde f(s'_l-x_i)}{k}\right)e^{-k\bar f}\\
&= \sum_k\frac{\lambda^k e^{-\lambda}}{k!}\left(\bar f k\right)^n e^{-k\bar f}\tilde A_k(x_1,...,x_n)
\end{aligned}
$$

where $\tilde A_k(x_1,...,x_n):=\int ds_1\cdots\int ds_k\prod_{i=1}^n\left(\frac{\sum_{l=1}^k\tilde f(s_l-x_i)}{k}\right)$ satisfies $\int\cdots\int\prod_{i=1}^n dx_i\tilde A_k(x_1,...,x_n)=1$. The 2nd term produces:

$$
\begin{aligned}
&\sum_k\frac{\lambda^k e^{-\lambda}}{k!}\int ds_1\cdots\int ds_k\sum_n\frac{1}{n!}\int dx_1\cdots\int dx_n\prod_{i=1}^n\left(\sum_{l=1}^k f(s_l-x_i)\right)e^{-k\bar f}\ln\left(\sum_{k'}\frac{\lambda^{k'}e^{-\lambda}}{k'!}\left(k'\bar f\right)^n e^{-k'\bar f}\tilde A_{k'}(x_1,...,x_n)\right)\\
=&\sum_n\frac{1}{n!}\sum_k\frac{\lambda^k e^{-\lambda}}{k!}e^{-k\bar f}\int dx_1\cdots\int dx_n\int ds_1\cdots\int ds_k\prod_{i=1}^n\left(\frac{\sum_{l=1}^k\tilde f(s_l-x_i)}{k}\right)\ln\left(\sum_{k'}\frac{\lambda^{k'}e^{-\lambda}}{k'!}\left(k'\bar f\right)^n e^{-k'\bar f}\tilde A_{k'}(x_1,...,x_n)\right)\\
=&\sum_n\frac{1}{n!}\int dx_1\cdots\int dx_n\sum_k\frac{\lambda^k e^{-\lambda}}{k!}e^{-k\bar f}\left(k\bar f\right)^n\tilde A_k(x_1,...,x_n)\ln\left(\sum_{k'}\frac{\lambda^{k'}e^{-\lambda}}{k'!}\left(k'\bar f\right)^n e^{-k'\bar f}\tilde A_{k'}(x_1,...,x_n)\right)
\end{aligned}
$$

maximized when $A_n(x_1,...,x_n)=\sum_k\frac{\lambda^k e^{-\lambda}}{k!}e^{-k\bar f}\left(k\bar f\right)^n\tilde A_k(x_1,...,x_n)=constant$, i.e. when $\tilde A(x_1,...,x_n)\equiv 1$:

$$
\sum_k\frac{\lambda^k e^{-\lambda}}{k!}\left(k\bar f\right)^n e^{-k\bar f}=\bar f^n e^{-\lambda}\sum_k\frac{\left(\lambda e^{-\bar f}\right)^k}{k!}k^n=\bar f^n e^{-\lambda+\lambda e^{-\bar f}}p_n\left(\lambda e^{-\bar f}\right) \tag{11}
$$

where $p_n(x)=\frac{1}{e^x}\sum_k\frac{x^k}{k!}k^n=x^n+\sum_{j=1}^{n-1}b_jx^j$, (poisson distribution moments: Touchard polynomials)

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	0	1	1								
3	0	1	3	1							
4	0	1	7	6	1						
5	0	1	15	25	10	1					
6	0	1	31	90	65	15	1				
7	0	1	63	301	350	140	21	1			
8	0	1	127	966	1701	1050	266	28	1		
9	0	1	255	3025	7770	6951	2646	462	36	1	
10	0	1	511	9330	34105	42525	22827	5880	750	45	1

Table 1: Touchard polynomial Coefficients

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{a^k}{k!}k^n &= \sum_{k=1}^{\infty}\frac{a^k}{k!}\left[k(k-1)\cdots(k-n+1)+b_{n-1}k(k-1)\cdots(k-n+2)+\cdots b_3k(k-1)(k-2)+b_2k(k-1)+b_1k\right] \\
&= a^ne^a+b_{n-1}a^{n-1}e^a+\cdots+b_3a^3e^a+b_2a^2e^a+b_1ase^a \\
&= e^a\left(b_1a+b_2a^2+\cdots+b_{n-1}a^{n-1}+a^n\right) \\
&= e^ap_n(a)
\end{aligned}$$

b_j satisfies a system of equations followed from

$$\begin{aligned}
k^n &= k(k-1)\cdots(k-n+1)+b_{n-1}k(k-1)\cdots(k-n+2)+\cdots b_3k(k-1)(k-2)+b_2k(k-1)+b_1k \\
k^{n-1} &= (k-1)\cdots(k-n+1)+b_{n-1}(k-1)\cdots(k-n+2)+\cdots b_3(k-1)(k-2)+b_2(k-1)+b_1 \\
k^{n-1} &= \frac{(k-1)!}{(k-n)!}+\sum_{j=1}^{n-1}b_j\frac{(k-1)!}{(k-j)!}
\end{aligned}$$

plugging in $k=1,2,...,n-1$:

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 2\cdot 1 & & & \\ 1 & 3 & 3\cdot 2 & 3\cdot 2\cdot 1 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ 1 & (n-2) & \frac{(n-2)!}{(n-4)!} & \frac{(n-2)!}{(n-5)!} & \cdots & (n-2)! \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} 1^{n-1} \\ 2^{n-1} \\ 3^{n-1} \\ 4^{n-1} \\ \vdots \\ (n-1)^{n-1} \end{bmatrix}$$

The Touchard polynomials satisfy:
$p_n(x)=\sum_{k=0}^{n}S(n,k)x^k$
where $S(n,k)=\left\{ {n \atop k} \right\}$ is a Stirling number of the second kind, i.e., the number of partitions of a set of size n into k disjoint non-empty subsets.
Below is a triangular array of values for the Stirling numbers of the second kind (sequence A008277 in the OEIS):
Representation by Touchard polynomials:

$$\ln\left(\sum_k\frac{\lambda^ke^{-\lambda}}{k!}\left(\bar{f}k\right)^n e^{-k\bar{f}}\right) = \ln\left(\bar{f}^ne^{-\lambda}p_n\left(\lambda e^{-\bar{f}}\right)\cdot e^{\lambda e^{-\bar{f}}}\right)$$

$$I=\sum_k \frac{\lambda^k e^{-\lambda}}{k!}\int ds_1\cdots\int ds_k \sum_n \frac{1}{n!}\int dx_1\cdots\int dx_n \prod\nolimits_{i=1}^n \left(\sum_{l=1}^k f(s_l-x_i)\right) e^{-k\bar{f}} \ln\left(\frac{\prod\nolimits_{i=1}^n\left(\sum_{l=1}^k f(s_l-x_i)\right)e^{-k\bar{f}}}{E_{s'}\left[\prod\nolimits_{i=1}^n\left(\sum_{l=1}^{k'} f(s'_l-x_i)\right)e^{-k'\bar{f}}\right]}\right)$$

$$E_{s'}\left[\prod\nolimits_{i=1}^n\left(\sum_{l=1}^{k'} f(s'_l-x_i)\right)e^{-k'\bar{f}}\right] \quad = \quad \sum_{k'}\frac{\lambda^{k'}e^{-\lambda}}{k'!}\left(\bar{f}k'\right)^n e^{-k'\bar{f}}\tilde{A}(x_1,...,x_n)$$

Thus when $\tilde{A}=1$, the second term is

$$\begin{aligned}
&\sum_k \frac{\lambda^k e^{-\lambda}}{k!}\int ds_1\cdots\int ds_k \sum_n \frac{1}{n!}\int dx_1\cdots\int dx_n \prod\nolimits_{i=1}^n \left(\sum_{l=1}^k f(s_l-x_i)\right) e^{-k\bar{f}} \ln\left(\sum_{k'}\frac{\lambda^{k'}e^{-\lambda}}{k'!}\left(k'\bar{f}\right)^n e^{-k'\bar{f}}\right)\\
=\ &\sum_k \frac{\lambda^k e^{-\lambda}e^{-k\bar{f}}}{k!}\sum_n \frac{\left(k\bar{f}\right)^n}{n!}\ln\left(\bar{f}^n e^{-\lambda+\lambda e^{-\bar{f}}}p_n\left(\lambda e^{-\bar{f}}\right)\right)\\
=\ &\sum_k \frac{\lambda^k e^{-\lambda}e^{-k\bar{f}}}{k!}\sum_n \frac{\left(k\bar{f}\right)^n}{n!} n\ln\bar{f} + \sum_n \frac{1}{n!}\sum_k \frac{\lambda^k e^{-\lambda}e^{-k\bar{f}}}{k!}\left(k\bar{f}\right)^n \ln\left(e^{-\lambda+\lambda e^{-\bar{f}}}p_n\left(\lambda e^{-\bar{f}}\right)\right)\\
=\ &\sum_k \frac{\lambda^k e^{-\lambda}}{k!} k\bar{f}\ln\bar{f} + \sum_n \frac{\bar{f}^n}{n!}e^{-\lambda+\lambda e^{-\bar{f}}}p_n\left(\lambda e^{-\bar{f}}\right)\ln\left(e^{-\lambda+\lambda e^{-\bar{f}}}p_n\left(\lambda e^{-\bar{f}}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
I \quad \leq \quad & \sum_k \frac{\lambda^k e^{-\lambda}}{k!} \int ds_1 \cdots \int ds_k \sum_n \frac{1}{n!} \int dx_1 \cdots \int dx_n \prod\nolimits_{i=1}^n \left(\sum_{l=1}^k f(s_l - x_i) \right) e^{-k\bar{f}} \ln \left(\frac{\prod_{i=1}^n \left(\sum_{l=1}^k f(s_l - x_i) \right) e^{-k\bar{f}}}{\sum_{k'} \frac{\lambda^{k'} e^{-\lambda}}{k'!} \left(\bar{f} k'\right)^n e^{-k'\bar{f}}} \right) \\
= \quad & \sum_k \frac{\lambda^k e^{-\lambda}}{k!} e^{-k\bar{f}} \int ds_1 \cdots \int ds_k \sum_n \frac{\left(k\bar{f}\right)^n}{n!} \int dx_1 \cdots \int dx_n \prod\nolimits_{i=1}^n \left(\frac{\sum_{l=1}^k \tilde{f}(s_l - x_i)}{k} \right) \left[\ln \left[\left(k\bar{f}\right)^n \prod\nolimits_{i=1}^n \left(\frac{\sum_{l=1}^k \tilde{f}(s_l - x_i)}{k} \right) \right. \right. \\
& - \sum_k \frac{\lambda^k e^{-\lambda}}{k!} \sum_n \frac{\left(k\bar{f}\right)^n}{n!} e^{-k\bar{f}} \ln \left(\sum_{k'} \frac{\lambda^{k'} e^{-\lambda}}{k'!} \left(\bar{f} k'\right)^n e^{-k'\bar{f}} \right) \\
= \quad & \sum_k \frac{\lambda^k e^{-\lambda}}{k!} e^{-k\bar{f}} \sum_n \frac{\left(k\bar{f}\right)^n}{n!} n \ln\left(k\bar{f}\right) - \sum_k \frac{\lambda^k e^{-\lambda}}{k!} e^{-k\bar{f}} \sum_n \frac{\left(k\bar{f}\right)^n}{n!} \left(k\bar{f}\right) \\
& + \sum_k \frac{\lambda^k e^{-\lambda}}{k!} e^{-k\bar{f}} \int ds_1 \cdots \int ds_k \sum_n \frac{\left(k\bar{f}\right)^n}{n!} n \int dx_1 \cdots \int dx_n \prod\nolimits_{i=1}^n \left(\frac{\sum_{l=1}^k \tilde{f}(s_l - x_i)}{k} \right) \ln \left(\frac{\sum_{l=1}^k \tilde{f}(s_l - x_i)}{k} \right) \\
& - \sum_k \frac{\lambda^k e^{-\lambda}}{k!} \sum_n \frac{\left(k\bar{f}\right)^n}{n!} e^{-k\bar{f}} \ln \left(\bar{f}^n e^{-\lambda} p_n \left(\lambda e^{-\bar{f}}\right) \cdot e^{\lambda e^{-\bar{f}}} \right) \\
= \quad & \sum_k \frac{\lambda^k e^{-\lambda}}{k!} k\bar{f} \ln\left(k\bar{f}\right) - \sum_k \frac{\lambda^k e^{-\lambda}}{k!} e^{-k\bar{f}} \sum_n \frac{\left(k\bar{f}\right)^n}{n!} \left(k\bar{f}\right) \\
& + \sum_k \frac{\lambda^k e^{-\lambda}}{k!} e^{-k\bar{f}} \int ds_1 \cdots \int ds_k \sum_n \frac{\left(k\bar{f}\right)^n}{n!} n B \left(k, \frac{f_+}{\bar{f}}, \frac{f_-}{\bar{f}} \right) \\
& - \sum_k \frac{\lambda^k e^{-\lambda}}{k!} k\bar{f} \ln \bar{f} - \sum_n \frac{\bar{f}^n}{n!} e^{-\lambda + \lambda e^{-\bar{f}}} p_n \left(\lambda e^{-\bar{f}}\right) \ln \left(e^{-\lambda + \lambda e^{-\bar{f}}} p_n \left(\lambda e^{-\bar{f}}\right) \right) \\
\leq \quad & \sum_k \frac{\lambda^k e^{-\lambda}}{k!} k\bar{f} B \left(k, \frac{f_+}{\bar{f}}, \frac{f_-}{\bar{f}} \right) - \sum_n \frac{\bar{f}^n}{n!} e^{-\lambda + \lambda e^{-\bar{f}}} p_n \left(\lambda e^{-\bar{f}}\right) \ln \left(e^{-\lambda + \lambda e^{-\bar{f}}} p_n \left(\lambda e^{-\bar{f}}\right) \right) \\
& + \sum_k \frac{\lambda^k e^{-\lambda}}{k!} k\bar{f} \ln(k) - \sum_k \frac{\lambda^k e^{-\lambda}}{k!} e^{-k\bar{f}} \sum_n \frac{\left(k\bar{f}\right)^n}{n!} \left(k\bar{f}\right) \\
= \quad & \sum_k \frac{\lambda^k e^{-\lambda}}{k!} k\bar{f} B \left(k, \frac{f_+}{\bar{f}}, \frac{f_-}{\bar{f}} \right) - \sum_n \frac{\bar{f}^n}{n!} e^{-\lambda + \lambda e^{-\bar{f}}} p_n \left(\lambda e^{-\bar{f}}\right) \ln \left(e^{-\lambda + \lambda e^{-\bar{f}}} p_n \left(\lambda e^{-\bar{f}}\right) \right) \\
& + \bar{f} \sum_k \frac{\lambda^k e^{-\lambda}}{k!} k \ln(k) - \bar{f} + \bar{f} \lambda e^{-\lambda - \bar{f}}
\end{aligned}
$$

$$
\begin{aligned}
\max I \quad = \quad & \sum_k \frac{\lambda^k e^{-\lambda}}{k!} k\bar{f} B \left(k, \frac{f_+}{\bar{f}}, \frac{f_-}{\bar{f}} \right) - \sum_n \frac{\bar{f}^n}{n!} e^{-\lambda + \lambda e^{-\bar{f}}} p_n \left(\lambda e^{-\bar{f}}\right) \ln \left(e^{-\lambda + \lambda e^{-\bar{f}}} p_n \left(\lambda e^{-\bar{f}}\right) \right) \qquad (12) \\
& + \sum_k \frac{\lambda^k e^{-\lambda}}{k!} k\bar{f} \ln(k) - \sum_k \frac{\lambda^k e^{-\lambda}}{k!} e^{-k\bar{f}} \sum_n \frac{\left(k\bar{f}\right)^n}{n!} \left(k\bar{f}\right) \\
= \quad & \sum_k \frac{\lambda^k e^{-\lambda}}{k!} k\bar{f} B \left(k, \frac{f_+}{\bar{f}}, \frac{f_-}{\bar{f}} \right) - \sum_n \frac{\bar{f}^n}{n!} e^{-\lambda + \lambda e^{-\bar{f}}} p_n \left(\lambda e^{-\bar{f}}\right) \left(-\lambda + \lambda e^{-\bar{f}} \right) \ln \left(p_n \left(\lambda e^{-\bar{f}}\right) \right) \\
& + \bar{f} \sum_k \frac{\lambda^k e^{-\lambda}}{k!} k \ln(k) - \bar{f} + \bar{f} \lambda e^{-\lambda - \bar{f}}
\end{aligned}
$$

$$
\begin{aligned}
& \bar{f}\sum_{k=1}^{\infty}\frac{k\lambda^k e^{-\lambda}}{k!}e^{-k\bar{f}}\sum_{n=1}^{\infty}\frac{\left(k\bar{f}\right)^n}{n!} \\
= & \bar{f}\sum_{k=1}^{\infty}\frac{k\lambda^k e^{-\lambda}}{k!}e^{-k\bar{f}}\left(e^{k\bar{f}}-1\right) \\
= & \bar{f}\sum_{k}\frac{k\lambda^k e^{-\lambda}}{k!}-\bar{f}\sum_{k}\frac{k\lambda^k e^{-\lambda}}{k!}e^{-k\bar{f}} \\
= & \bar{f}-\bar{f}e^{-\lambda}\lambda e^{-\bar{f}}
\end{aligned}
$$

$\sum_k \frac{\lambda^k e^{-\lambda}}{k!}\sum_n \frac{\left(k\bar{f}\right)^n}{n!}e^{-k\bar{f}}\ln\left(\sum_{k'}\frac{\lambda^{k'}e^{-\lambda}}{k'!}\left(\bar{f}k'\right)^n e^{-k'\bar{f}}\right)$
Special case when $f_- = 0$:
$H(k,a,0,t) = t\cdot\sum_{m=0}^{k}\left(\begin{array}{c}k\\m\end{array}\right)\left(\frac{t}{a}\right)^m(1-\frac{t}{a})^{k-m}\frac{ma}{kt}\ln\left(\frac{ma}{kt}\right)$
where $b < t < a$.

$$
\begin{aligned}
H(k,a,0,t) &= t\cdot\sum_{m=0}^{k}\left(\begin{array}{c}k\\m\end{array}\right)\left(\frac{t}{a}\right)^m(1-\frac{t}{a})^{k-m}\frac{ma}{kt}\ln\left(\frac{ma}{kt}\right) \\
&= \sum_{m=0}^{k}\left(\begin{array}{c}k\\m\end{array}\right)\frac{t^m(a-t)^{k-m}}{a^k}\cdot\frac{ma}{k}\ln\left(\frac{ma}{k}\right) \\
&\quad -\ln t\sum_{m=0}^{k}\left(\begin{array}{c}k\\m\end{array}\right)\frac{t^m(a-t)^{k-m}}{a^k}\cdot\frac{ma}{k} \\
&= \sum_{m=0}^{k}\left(\begin{array}{c}k\\m\end{array}\right)\frac{t^m(a-t)^{k-m}}{a^k}\cdot\frac{ma}{k}\ln\left(\frac{ma}{k}\right)-t\ln t \qquad (13) \\
\frac{\partial H(k,a,b,t)}{\partial t} &= -(1+\ln t)+\sum_{m=0}^{k}\left(\begin{array}{c}k\\m\end{array}\right)\frac{\partial}{\partial t}\left[\frac{t^m(a-t)^{k-m}}{a^k}\right]\cdot\frac{ma}{k}\ln\left(\frac{ma}{k}\right) \\
&= -(1+\ln t)+\sum_{m=0}^{k}\left(\begin{array}{c}k\\m\end{array}\right)\frac{t^m(a-t)^{k-m}}{a^k}\cdot\left(\frac{m}{t}-\frac{k-m}{a-t}\right)\frac{ma}{k}\ln\left(\frac{ma}{k}\right) \qquad (14) \\
\mathit{poly}(t) &= 1+\ln t \qquad (15)
\end{aligned}
$$

Normailizations

- Stimulus positions θ_l, $l = 1, 2, ..., k$
- Firing Cell positions x_i, $i = 1, 2, ..., n$, random Poisson
- Firing rate at x_i: $\frac{\sum_{l=1}^{k} f(\theta_l - x_i)}{k} = \frac{1}{k}\int f(\theta - x_i)\sum_l \delta_{\theta_l}(\theta)d\theta$

$$
p(c_1,...,c_n|\theta_1,...,\theta_k) = \prod_{i=1}^{n}\frac{\left(\frac{\sum_{l=1}^{k} f(\theta_l-x_i)}{k}\right)^{c_i}}{c_i!}e^{-\frac{1}{k}\sum_l f(\theta_l-x_i)}
$$

Uniform stimulus positions:

$$\begin{aligned}
\tilde{I} &= \int_\theta \int_{\mathbf{c}} p(\mathbf{c}|\theta)p(\theta)\ln\left(\frac{p(\mathbf{c}|\theta)}{\int p(\mathbf{c}|\theta)p(\theta)d\theta}\right) \\
&= \int d\theta_1 \cdots \int d\theta_k \sum_{\mathbf{c}} \prod{}_{i=1}^{n} \frac{\left(\frac{\sum_l^k f(\theta_l - x_i)}{k}\right)^{c_i}}{c_i!} e^{-\frac{\sum_l^k f(\theta_l - x_i)}{k}} \ln\left(\frac{\prod{}_{i=1}^{n} \frac{\left(\frac{\sum_l^k f(\theta_l - x_i)}{k}\right)^{c_i}}{c_i!} e^{-\frac{\sum_l^k f(\theta_l - x_i)}{k}}}{\int d\theta_1' \cdots \int d\theta_k' \prod{}_{i=1}^{n} \frac{\left(\frac{\sum_l^k f(\theta_l' - x_i)}{k}\right)^{c_i}}{c_i!} e^{-\frac{\sum_l^k f(\theta_l' - x_i)}{k}}}\right)
\end{aligned}$$

Take the limit: $nf_n(x) \to f(x)$

$$\tilde{I} = \int d\theta_1 \cdots \int d\theta_k \sum_n \frac{1}{n!} \int dx_1 \cdots \int dx_n e^{-\bar{f}} \prod{}_{i=1}^{n} \left(\frac{\sum_{l=1}^k f(\theta_l - x_i)}{k}\right) \ln\left(\frac{\prod{}_{i=1}^{n} \left(\frac{\sum_{l=1}^k f(\theta_l - x_i)}{k}\right)}{\int d\theta_1' \cdots \int d\theta_k' \prod{}_{i=1}^{n} \left(\frac{\sum_{l=1}^k f(\theta_l' - x_i)}{k}\right)}\right)$$

Fix k:

$$\begin{aligned}
\tilde{I}[f;n,k] &= \int \prod_{l=1}^{k} d\theta_l \int \prod_{i=1}^{n} dx_i \cdot \prod{}_{i=1}^{n} \left(\frac{\sum_{l=1}^k f(\theta_l - x_i)}{k}\right) \ln\left(\frac{\prod{}_{i=1}^{n} \left(\frac{\sum_{l=1}^k f(\theta_l - x_i)}{k}\right)}{\int \prod_{l=1}^k d\theta_l' \cdot \prod{}_{i=1}^{n} \left(\frac{\sum_{l=1}^k f(\theta_l' - x_i)}{k}\right)}\right) \\
&= \bar{f}^n \int \prod_{l=1}^{k} d\theta_l \int \prod_{i=1}^{n} dx_i \cdot \prod{}_{i=1}^{n} \left(\frac{\sum_{l=1}^k \tilde{f}(\theta_l - x_i)}{k}\right) \ln\left(\frac{\prod{}_{i=1}^{n} \left(\sum_{l=1}^k \tilde{f}(\theta_l - x_i)\right)}{\int \prod_{l=1}^k d\theta_l' \cdot \prod{}_{i=1}^{n} \left(\sum_{l=1}^k \tilde{f}(\theta_l' - x_i)\right)}\right) \\
&= n\bar{f}^n \int \cdots \int \prod d\theta_l \frac{\sum_{l=1}^k \tilde{f}(\theta_l)}{k} \ln\left(\frac{\sum_{l=1}^k \tilde{f}(\theta_l)}{k}\right) - \bar{f}^n \int \cdots \int \prod_{i=1}^{n} dx_i A_k(x_1, ..., x_n) \ln A_k(x_1, ..., x_n) \\
&\leq \bar{f}^n nB\left(k, \frac{f_+}{\bar{f}}, \frac{f_-}{\bar{f}}\right) \qquad (16)
\end{aligned}$$

$$\begin{aligned}
\tilde{I}[f_1, f_2; n_1, n_2, k] &= \int \cdots \int \prod_{l=1}^{k} ds_l \prod_{i=1}^{n_1} dx_i \prod_{j=1}^{n_2} dy_j \Bigg\{ \prod_{i,j} d_1 \left(\frac{\sum_l f_1(s_l - x_i)}{k}\right) d_2 \left(\frac{\sum_l f_2(s_l - y_j)}{k}\right) \\
&\quad \ln\left(\frac{\prod_{i,j}(d_1 \left(\frac{\sum_l f_1(s_l - x_i)}{k}\right) d_2 \left(\frac{\sum_l f_2(s_l - y_j)}{k}\right)}{\int \cdots \int \prod_{l=1}^k ds_l' \prod_{i,j}(d_1 \left(\frac{\sum_l f_1(s_l' - x_i)}{k}\right) d_2 \left(\frac{\sum_l f_2(s_l' - y_j)}{k}\right)}\right) \Bigg\} \\
&\leq \left(d_1 \bar{f}_1\right)^{n_1} \left(d_2 \bar{f}_2\right)^{n_2} \left[n_1 B\left(k, \frac{f_+}{\bar{f}_1}, \frac{f_-}{\bar{f}_1}\right) + n_2 B\left(k, \frac{f_+}{\bar{f}_2}, \frac{f_-}{\bar{f}_2}\right)\right] \qquad (17)
\end{aligned}$$

sum over n:

$$\begin{aligned}
\tilde{I}[f;k] &= e^{-\bar{f}} \sum_n \frac{1}{n!} I[f;n,k] \\
&\leq e^{-\bar{f}} \sum_n \frac{\bar{f}^n}{n!} nB\left(k, \frac{f_+}{\bar{f}}, \frac{f_-}{\bar{f}}\right) \\
&= \bar{f} B\left(k, \frac{f_+}{\bar{f}}, \frac{f_-}{\bar{f}}\right) \qquad (18)
\end{aligned}$$

$$
\begin{aligned}
\tilde{I}[f_1,f_2;k] &= e^{-\left(d_1\bar{f}_1+d_2\bar{f}_2\right)}\sum_{n_1,n_2}\frac{1}{n_1!n_2!}I[f_1,f_2;n_1,n_2,k] \\
&\leq e^{-\left(d_1\bar{f}_1+d_2\bar{f}_2\right)}\sum_{n_1,n_2}\frac{\left(d_1\bar{f}_1\right)^{n_1}\left(d_2\bar{f}_2\right)^{n_2}}{n_1!n_2!}\left[n_1 B\left(k,\frac{f_+}{\bar{f}_1},\frac{f_-}{\bar{f}_1}\right)+n_2 B\left(k,\frac{f_+}{\bar{f}_2},\frac{f_-}{\bar{f}_2}\right)\right] \\
&= d_1\bar{f}_1 B\left(k,\frac{f_+}{\bar{f}_1},\frac{f_-}{\bar{f}_1}\right)+d_2\bar{f}_2 B\left(k,\frac{f_+}{\bar{f}_2},\frac{f_-}{\bar{f}_2}\right) \qquad (19)
\end{aligned}
$$

Recall that $B(k,p,q)$ is defined as:

$$
B(k,p,q) \coloneqq \sum_{m=0}^{k}\left(\begin{array}{c}k\\m\end{array}\right)\triangle^m(1-\triangle)^{k-m}\frac{mp+(k-m)q}{k}\ln\left(\frac{mp+(k-m)q}{k}\right) \qquad (20)
$$

Poisson stimulus positions:

Stimulus $\theta_1,...,\theta_k$ follows a Poisson point process on $[0,1)$ with intensity λ: $p(\theta_1,...,\theta_k)=\frac{\lambda^k}{k!}e^{-\lambda}$, $P\left(N([0,1))=k\right)=\frac{\lambda^k}{k!}e^{-\lambda}$

$$
\begin{aligned}
\tilde{I} &= \int_\theta\int_{\mathbf{c}}p(\mathbf{c}|\theta)p(\theta)\ln\left(\frac{p(\mathbf{c}|\theta)}{\int p(\mathbf{c}|\theta)p(\theta)d\theta}\right) \\
&= \sum_k\frac{\lambda^k e^{-\lambda}}{k!}\int d\theta_1\cdots\int d\theta_k\sum_{\mathbf{c}}\prod\nolimits_{i=1}^{n}\frac{\left(\frac{\sum_l^k f(\theta_l-x_i)}{k}\right)^{c_i}}{c_i!}e^{-\frac{\sum_l^k f(\theta_l-x_i)}{k}}\ln\left(\frac{\prod_{i=1}^{n}\frac{\left(\frac{\sum_l^k f(\theta_l-x_i)}{k}\right)^{c_i}}{c_i!}e^{-\frac{\sum_l^k f(\theta_l-x_i)}{k}}}{E_{\theta'}\left[\prod_{i=1}^{n}\frac{\left(\frac{\sum_l^{k'} f(\theta'_l-x_i)}{k'}\right)^{c_i}}{c_i!}e^{-\frac{\sum_l^{k'} f(\theta'_l-x_i)}{k'}}\right]}\right)
\end{aligned}
$$

Take the limit: $nf_n(x)\to f(x)$,

$$
\tilde{I}=\sum_k\frac{\lambda^k e^{-\lambda}}{k!}\int d\theta_1\cdots\int d\theta_k\sum_n\frac{1}{n!}\int dx_1\cdots\int dx_n\prod\nolimits_{i=1}^{n}\left(\frac{\sum_{l=1}^k f(\theta_l-x_i)}{k}\right)e^{-\bar{f}}\ln\left(\frac{\prod_{i=1}^{n}\left(\frac{\sum_{l=1}^k f(\theta_l-x_i)}{k}\right)e^{-\bar{f}}}{E_{\theta'}\left[\prod_{i=1}^{n}\left(\frac{\sum_{l=1}^{k'} f(\theta'_l-x_i)}{k'}\right)e^{-\bar{f}}\right]}\right)
$$

Random Stimulus positions (sum over k):

$$
I=\sum_k\frac{\lambda^k e^{-\lambda}}{k!}\int ds_1\cdots\int ds_k\sum_n\frac{1}{n!}\int dx_1\cdots\int dx_n\prod\nolimits_{i=1}^{n}\left(\frac{\sum_{l=1}^k f(s_l-x_i)}{k}\right)e^{-\bar{f}}\ln\left(\frac{\prod_{i=1}^{n}\left(\frac{\sum_{l=1}^k f(s_l-x_i)}{k}\right)e^{-\bar{f}}}{E_{s'}\left[\prod_{i=1}^{n}\left(\frac{\sum_{l=1}^{k'} f(s'_l-x_i)}{k'}\right)e^{-\bar{f}}\right]}\right)
$$

$$
\begin{aligned}
E_s\left[\prod\nolimits_{i=1}^{n}\left(\frac{\sum_{l=1}^{k'} f(s'_l-x_i)}{k'}\right)\right] &= \sum_{k'}\frac{\lambda^{k'}e^{-\lambda}}{k'!}\int ds'_1\cdots\int ds'_{k'}\prod\nolimits_{i=1}^{n}\left(\frac{\sum_{l=1}^{k'} f(s'_l-x_i)}{k'}\right) \\
&= \sum_{k'}\frac{\lambda^{k'}e^{-\lambda}}{k'!}\left(\bar{f}\right)^n\int ds'_1\cdots\int ds'_{k'}\prod\nolimits_{i=1}^{n}\left(\frac{\sum_{l=1}^{k'} f(s'_l-x_i)}{k'}\right) \\
&= \sum_{k'}\frac{\lambda^{k'}e^{-\lambda}}{k'!}\left(\bar{f}\right)^n\tilde{A}(x_1,...,x_n) \\
&= \left(\bar{f}\right)^n\tilde{A}(x_1,...,x_n)e^{-\lambda}(e^\lambda-1)
\end{aligned}
$$

where $\tilde{A}(x_1,...,x_n) \coloneqq \int ds_1 \cdots \int ds_k \prod_{i=1}^{n} \left(\frac{\sum_{l=1}^{k} \tilde{f}(s_l - x_i)}{k}\right)$ satisfies $\int \cdots \int \prod_{i=1}^{n} dx_i \tilde{A}(x_1,...,x_n) = 1$.
When $\tilde{A}(x_1,...,x_n) \equiv 1$:

$$
\begin{aligned}
I \quad &\leq \quad \sum_k \frac{\lambda^k e^{-\lambda}}{k!} \int ds_1 \cdots \int ds_k \sum_n \frac{1}{n!} \int dx_1 \cdots \int dx_n \prod_{i=1}^{n} \left(\frac{\sum_{l=1}^{k} f(s_l - x_i)}{k}\right) e^{-\bar{f}} \ln\left(\frac{\prod_{i=1}^{n}\left(\frac{\sum_{l=1}^{k} f(s_l-x_i)}{k}\right) e^{-\bar{f}}}{\left(\bar{f}\right)^n e^{-\lambda}(e^{\lambda}-1)e^{-\bar{f}}}\right) \\
&= \quad \sum_k \frac{\lambda^k e^{-\lambda}}{k!} \int ds_1 \cdots \int ds_k \sum_n \frac{\left(\bar{f}\right)^n}{n!} \int dx_1 \cdots \int dx_n \prod_{i=1}^{n} \left(\frac{\sum_{l=1}^{k} \tilde{f}(s_l - x_i)}{k}\right) e^{-\bar{f}} \ln\left(\frac{\prod_{i=1}^{n}\left(\frac{\sum_{l=1}^{k} \tilde{f}(s_l-x_i)}{k}\right)}{e^{-\lambda}(e^{\lambda}-1)}\right) \\
&= \quad \sum_k \frac{\lambda^k e^{-\lambda}}{k!} \int ds_1 \cdots \int ds_k \sum_n \frac{\left(\bar{f}\right)^n}{n!} \int dx_1 \cdots \int dx_n \prod_{i=1}^{n} \left(\frac{\sum_{l=1}^{k} \tilde{f}(s_l - x_i)}{k}\right) e^{-\bar{f}} \ln\left(\prod_{i=1}^{n}\left(\frac{\sum_{l=1}^{k} \tilde{f}(s_l-x_i)}{k}\right)\right) \\
&\quad\quad - \ln\left(e^{-\lambda}(e^{\lambda}-1)\right) \sum_k \frac{\lambda^k e^{-\lambda}}{k!} \int ds_1 \cdots \int ds_k \sum_n \frac{\left(\bar{f}\right)^n}{n!} \int dx_1 \cdots \int dx_n \prod_{i=1}^{n} \left(\frac{\sum_{l=1}^{k} \tilde{f}(s_l - x_i)}{k}\right) e^{-\bar{f}} \\
&\leq \quad \sum_k \frac{\lambda^k e^{-\lambda}}{k!} e^{-\bar{f}} \sum_n \frac{\left(\bar{f}\right)^n}{n!} nB\left(k, \frac{f_+}{\bar{f}}, \frac{f_-}{\bar{f}}\right) - \ln\left(e^{-\lambda}(e^{\lambda}-1)\right) \sum_k \frac{\lambda^k e^{-\lambda}}{k!} \sum_n \frac{(\bar{f})^n}{n!} e^{-\bar{f}} \\
&= \quad \sum_k \frac{\lambda^k e^{-\lambda}}{k!} \bar{f} B\left(k, \frac{f_+}{\bar{f}}, \frac{f_-}{\bar{f}}\right) - \ln\left(1-e^{-\lambda}\right)(1-e^{-\lambda})(1-e^{-\bar{f}})
\end{aligned}
$$

??
If $\lambda \approx 0$ and $\bar{f} \approx 0$,

$$1 - e^{-\lambda} \approx \lambda + O(\lambda^2)$$

$$\ln(1-e^{-\lambda}) \approx \ln(\lambda) - \frac{\lambda}{2} + O(\lambda^2)$$

$$\max I \approx \lambda \bar{f} B\left(k, \frac{f_+}{\bar{f}}, \frac{f_-}{\bar{f}}\right) - \ln(\lambda)\, \lambda \bar{f}$$

If λ is large, $1 - e^{-\lambda} \approx 0$,

$$\max I \approx \bar{f} B\left(k, \frac{f_+}{\bar{f}}, \frac{f_-}{\bar{f}}\right)$$

which is the same as before.

$$B(k, b = f_+, a = f_-) \quad = \quad \sum_{m=0}^{k} \begin{pmatrix} k \\ m \end{pmatrix} \left(\frac{1-a}{b-a}\right)^m \left(\frac{b-1}{b-a}\right)^{k-m} \frac{ma + (k-m)b}{k} \ln\left(\frac{ma+(k-m)b}{k}\right) \tag{21}$$

$\triangle = \frac{1-a}{b-a}$
when $a = f_- = 0$:

$$
\begin{aligned}
B(k, b = f_+, f_- = 0) \quad &= \quad \frac{1}{b^k} \sum_{m=0}^{k} \begin{pmatrix} k \\ m \end{pmatrix} (b-1)^{k-m} \frac{(k-m)b}{k} \ln\left(\frac{(k-m)b}{k}\right) \\
&= \quad \frac{1}{b^k} \sum_{m=0}^{k} \left(\frac{k}{m}\right) (b-1)^m \frac{mb}{k} \ln\left(\frac{mb}{k}\right) \\
\frac{\partial B}{\partial b} \quad &= \quad \sum_{m=0}^{k} \left(\frac{k}{m}\right) \frac{(b-1)^m}{b^k} ()
\end{aligned}
$$

If divide by t:

$$\sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array} \right) \frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k} \cdot \frac{ma+(k-m)b}{k} = t$$

$$\begin{aligned}
\frac{1}{t} H(k,a,b,t) &= \frac{1}{t} \sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array} \right) \frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k} \cdot \frac{ma+(k-m)b}{k} \ln\left(\frac{ma+(k-m)b}{k}\right) - \ln t \\
\frac{\partial \frac{1}{t} H(k,a,b,t)}{\partial t} &= -\frac{1}{t} + \sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array} \right) \frac{ma+(k-m)b}{k} \ln\left(\frac{ma+(k-m)b}{k}\right) \frac{\partial}{\partial t}\left[\frac{1}{t} \cdot \frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k}\right] \\
&= -\frac{1}{t} + \sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array} \right) \frac{ma+(k-m)b}{k} \ln\left(\frac{ma+(k-m)b}{k}\right) \left[\frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k} \left(\frac{m}{t-b} - \frac{k-m}{a-t}\right) \cdot \frac{1}{t} - \frac{1}{t^2} \frac{(t-b)}{(} \right. \\
&= \sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array} \right) \frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k} \frac{ma+(k-m)b}{k} \ln\left(\frac{ma+(k-m)b}{k}\right) \cdot \left[\left(\frac{m}{t-b} - \frac{k-m}{a-t}\right) \cdot \frac{1}{t} - \frac{1}{t^2}\right] - \frac{t}{t^2} \\
&= \frac{1}{t} \sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array} \right) \frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k} \cdot \frac{ma+(k-m)b}{k} \ln\left(\frac{ma+(k-m)b}{k}\right) \left[\left(\frac{m}{t-b} - \frac{k-m}{a-t}\right) - \frac{1}{t}\right] \\
&\quad -\frac{1}{t^2} \sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array} \right) \frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k} \frac{ma+(k-m)b}{k} \\
&= \frac{1}{t} \sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array} \right) \frac{(t-b)^m (a-t)^{k-m}}{(a-b)^k} \frac{ma+(k-m)b}{k} \left[\ln\left(\frac{ma+(k-m)b}{k}\right) \left(\left(\frac{m}{t-b} - \frac{k-m}{a-t}\right) - \frac{1}{t}\right) - \frac{1}{t}\right] \\
&=
\end{aligned}$$

Approximation using only the first term in polynomial:

$$\begin{aligned}
H(k,a,b,t) &= t \cdot \sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array} \right) \left(\frac{t-b}{a-b}\right)^m \left(1 - \frac{t-b}{a-b}\right)^{k-m} \frac{ma+(k-m)b}{kt} \ln\left(\frac{ma+(k-m)b}{kt}\right) \\
&\approx t \cdot \left(\frac{t-b}{a-b}\right)^0 \left(\frac{a-t}{a-b}\right)^k \frac{kb}{kt} \ln\left(\frac{kb}{kt}\right) + t \cdot \frac{(t-b)^1 (a-t)^{k-1}}{(a-b)^k} \cdot \frac{a+(k-1)b}{kt} \ln\left(\frac{a+(k-}{kt}\right. \\
&= \left(\frac{a-t}{a-b}\right)^k b \ln\left(\frac{b}{t}\right) + \frac{(t-b)(a-t)^{k-1}}{(a-b)^k} \cdot \frac{a+(k-1)b}{k} \ln\left(\frac{a+(k-1)b}{kt}\right) \\
deriv &= \left(\frac{a-t}{a-b}\right)^k b \left[\left(-\frac{k}{a-t}\right) \ln\left(\frac{b}{t}\right) - \frac{1}{t}\right] \\
&\quad + \frac{(t-b)(a-t)^{k-1}}{(a-b)^k} \frac{a+(k-1)b}{k} \left[\left(\frac{1}{t-b} - \frac{k-1}{a-t}\right) \ln\left(\frac{a+(k-1)b}{kt}\right) - \frac{1}{t}\right] \\
when b = 0 : & \\
0 &= \frac{t(a-t)^{k-1}}{(a-b)^k} \frac{a}{k} \left[\left(\frac{1}{t} - \frac{k-1}{a-t}\right) \ln\left(\frac{a}{kt}\right) - \frac{1}{t}\right] \\
0 &= \left(\frac{1}{t} - \frac{k-1}{a-t}\right) \ln\left(\frac{a}{kt}\right) - \frac{1}{t} \\
\left(1 + (k-1)\left(1 - \frac{a}{a-t}\right)\right) \left(\ln\left(\frac{a}{k}\right) - \ln t\right) &= 1 \\
(a-kt) \left(\ln\left(\frac{a}{k}\right) - \ln t\right) &= a-t \\
-a \ln t + kt \ln t &= a - a \ln\left(\frac{a}{k}\right) - t + kt \ln\left(\frac{a}{k}\right) \\
kt \ln t - a \ln t &= a\left(1 - \ln\left(\frac{a}{k}\right)\right) + t\left(k \ln\left(\frac{a}{k}\right) - 1\right) \\
&?
\end{aligned}$$