Several tuning curves:

Assume that different responses of tuning curves ($\mathbf{r} = (\mathbf{r}_1, ..., \mathbf{r}_N)$) are conditionally independent on θ:

$$
\begin{aligned}
\mathbb{P}(\mathbf{r}|\theta) &= \prod_{i=1}^{N}\prod_{j=1}^{M} \mathbb{P}(r_{i,j}|\theta) \\
&= \prod_{i=1}^{N}\prod_{j=1}^{M} \frac{(f_{i,j}(\theta))^{r_{i,j}}}{r_{i,j}!} e^{-f_{i,j}(\theta)}
\end{aligned}
$$

For example, two tuning curves:

$$
\begin{aligned}
\mathbb{P}(\mathbf{r}_1, \mathbf{r}_2|\theta) &= \mathbb{P}(\mathbf{r}_1|\theta)\mathbb{P}(\mathbf{r}_2|\theta) \\
&= \prod_{i_1=1}^{M} \mathbb{P}(r_{1,i_1}|\theta) \prod_{i_2=1}^{M} \mathbb{P}(r_{2,i_2}|\theta) \\
&= \prod_{i_1=1}^{M} \frac{(f_{1,i_1}(\theta))^{r_{1,i_1}}}{r_{1,i_1}!} e^{-f_{1,i_1}(\theta)} \prod_{i_2=1}^{M} \frac{(f_{2,i_2}(\theta))^{r_{2,i_2}}}{r_{2,i_2}!} e^{-f_{2,i_2}(\theta)}
\end{aligned}
$$

Here $f(\theta) = \int t(\theta - y)s(y)dy$ is the rate function, t is the tuning curve (periodic on 2π), s is the stimulus, $p(\theta) = \frac{1}{2\pi}$ is uniform, N is the number of populations ($N = numPop$), $M = numBin$.
Problem:

$$
\begin{gathered}
\max_{t_{k,1},\cdots,t_{k,M}} I(\mathbf{r};\theta) \\
0 < FM \leq t_k(\theta) \leq FP \\
\int t_k(\theta)p(\theta)d\theta = const
\end{gathered}
$$

Discretize: $\theta \in (0, 2\pi)$ to be $(\theta_1, \ldots, \theta_M)$. Also, assume each tuning curve t_k (also the rate curve f_k) is rotationally invariant, i.e.

$$f_{k,i}(\theta_j) = f_{k,i-j}$$

where$(i-j)$ stands for $(i-j)\%M$ here. Therefore from convolution,

$$f_{k,i} = stimWidth \sum_{j=0}^{M-1} t_{k,i-j}s_j$$

Denote $\mathbf{r} = (r_{j,k})$, $j = 1, \ldots, N$, $k = 1, \ldots, M$. ($N = numPop$)
As in the 1-population case,

$$I(\mathbf{r};\theta) = D_{KL}(p(\mathbf{r},\theta)||p(\mathbf{r})p(\theta)) = -E_{\mathbf{r}|\theta=0}\left[\ln\left(\frac{p(\mathbf{r})}{p(\mathbf{r}|\theta=0)}\right)\right]$$

Since

$$
\begin{aligned}
p(\mathbf{r}) &= \sum_{i=1}^{M} p(\theta_i)p(\mathbf{r}|\theta=\theta_i) \\
&= \sum_{i=1}^{M} \frac{1}{M} \prod_{j=1}^{N}\prod_{k=1}^{M} \frac{(f_{j,k}(\theta_i))^{r_{j,k}}}{r_{j,k}!} e^{-f_{j,k}(\theta_i)} \\
&= \frac{1}{M} \sum_{i=1}^{M}\prod_{j=1}^{N}\prod_{k=1}^{M} \frac{{f_{j,k-i}}^{r_{j,k}}}{r_{j,k}!} e^{-f_{j,k-i}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{p(\mathbf{r})}{p(\mathbf{r}|\theta=0)} &= \frac{\frac{1}{M}\sum_{i=1}^{M}\prod_{j=1}^{N}\prod_{k=1}^{M}\frac{{f_{j,k-i}}^{r_{j,k}}}{r_{j,k}!}e^{-f_{j,k-i}}}{\prod_{j=1}^{N}\prod_{k=1}^{M}\frac{{f_{j,k-0}}^{r_{j,k}}}{r_{j,k}!}e^{-f_{j,k-0}}} \\
&= \frac{1}{M}\sum_{i=1}^{M}\left(\prod_{j=1}^{N}\prod_{k=1}^{M}\left(\frac{f_{j,k-i}}{f_{j,k}}\right)^{r_{j,k}}\left(\frac{r_{j,k}!}{r_{j,k}!}\right)\right)\cdot\left(\frac{\prod_{j=1}^{N}\prod_{k=1}^{M}e^{-f_{j,k-i}}}{\prod_{j=1}^{N}\prod_{k=1}^{M}e^{-f_{j,k}}}\right) \\
&= \frac{1}{M}\sum_{i=1}^{M}\prod_{j=1}^{N}\prod_{k=1}^{M}\left(\frac{f_{j,k-i}}{f_{j,k}}\right)^{r_{j,k}}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
I(\mathbf{r};\theta) &= -E_{\mathbf{r}|\theta=0}\left[\ln\left(\frac{p(\mathbf{r})}{p(\mathbf{r}|\theta=0)}\right)\right] \\
&= -E_{\mathbf{r}|\theta=0}\ln\left[\frac{1}{M}\sum_{i=1}^{M}\prod_{j=1}^{N}\prod_{k=1}^{M}\left(\frac{f_{j,k-i}}{f_{j,k}}\right)^{r_{j,k}}\right] \\
&= -E_{\mathbf{r}|\theta=0}\ln\left(S(\mathbf{r})\right) \\
&= -\sum_{\mathbf{r}}P(\mathbf{r}|\theta=0)\ln\left(S(\mathbf{r})\right)
\end{aligned}
$$

Where $(r_{j,k}|\theta=0)\sim Poisson(f_{j,k})$ are Poisson random variables,

$$
S(\mathbf{r})=\frac{1}{M}\sum_{i=1}^{M}Q^{i}(\mathbf{r})
$$

$$
Q^{i}(\mathbf{r})=\prod_{j=1}^{N}\prod_{k=1}^{M}\left(\frac{f_{j,k-i}}{f_{j,k}}\right)^{r_{j,k}}=\prod_{j=1}^{N}\prod_{k=1}^{M}f_{j,k}^{r_{j,k+i}-r_{j,k}}
$$

1st Order Derivatives

For simplification denote

$$
P=P(\mathbf{r}|\theta=0)=\prod_{j=1}^{N}\prod_{k=1}^{M}\frac{{f_{j,k}}^{r_{j,k}}}{r_{j,k}!}e^{-f_{j,k}}
$$

Partial derivatives of $P=\prod_{j=1}^{N}\prod_{k=1}^{M}\frac{{f_{j,k}}^{r_{j,k}}}{r_{j,k}!}e^{-f_{j,k}}$ are:

$$
\begin{aligned}
\frac{\partial P}{\partial f_{j,k}} &= \left(\frac{r_{j,k}}{f_{j,k}}-1\right)P \\
\frac{\partial^2 P}{\partial f_{j,k}\partial f_{l,m}} &= \left(\frac{r_{j,k}}{f_{j,k}}-1\right)\left(\frac{r_{l,m}}{f_{l,m}}-1\right)P,\text{ for }(j,k)\neq(l,m) \\
\frac{\partial^2 P}{\partial^2 f_{j,k}} &= \left(\frac{r_{j,k}}{f_{j,k}}-1\right)^2 P+\left(-\frac{r_{j,k}}{f_{j,k}^2}\right)P
\end{aligned}
$$

Denote

$$
L=-I=\sum_{\mathbf{r}}P(\mathbf{r})\ln\left(S(\mathbf{r})\right)
$$

$$
\frac{\partial L}{\partial f_{j,k}}=\sum_{r}P\frac{1}{S}\frac{\partial S}{\partial f_{j,k}}+\sum_{r}\frac{\partial P}{\partial f_{j,k}}\ln(S)
$$

Similar to the arguments in Lorenzo's notes, we deduce that the 1st term is zero:

$$\begin{aligned}
\frac{\partial S}{\partial f_{j,k}} &= \frac{1}{M}\sum_{i=1}^{M}(r_{j,k+i}-r_{j,k})/f_{j,k}\prod_{p=1}^{N}\prod_{q=1}^{M} f_{p,q}^{r_{p,q+i}-r_{p,q}} \\
f_{j,k}\frac{\partial S}{\partial f_{j,k}}\frac{1}{S} &= \frac{\frac{1}{M}\sum_{i=1}^{M}(r_{j,k+i}-r_{j,k})\prod_{p=1}^{N}\prod_{q=1}^{M} f_{p,q}^{r_{p,q+i}-r_{p,q}}}{\frac{1}{M}\sum_{i=1}^{M}\prod_{p=1}^{N}\prod_{q=1}^{M} f_{p,q}^{r_{p,q+i}-r_{p,q}}} \\
&= \frac{\sum_{i=1}^{M}(r_{j,k+i}-r_{j,k})\prod_{p=1}^{N}\prod_{q=1}^{M} f_{p,q}^{r_{p,q+i}}}{\sum_{i=1}^{M}\prod_{p=1}^{N}\prod_{q=1}^{M} f_{p,q}^{r_{p,q+i}}}
\end{aligned}$$

Taking expectation:

$$\begin{aligned}
\sum_{\mathbf{r}} P\frac{1}{S}\frac{\partial S}{\partial f_{j,k}} &= \frac{1}{f_{j,k}}\sum_{\mathbf{r}}\prod_{p,q}\frac{f_{p,q}^{r_{p,q}}}{r_{p,q}!}e^{-f_{p,q}}\frac{\sum_{i=1}^{M}(r_{j,k+i}-r_{j,k})\prod_{p,q} f_{p,q}^{r_{p,q+i}}}{\sum_{i=1}^{M}\prod_{p,q} f_{p,q}^{r_{p,q+i}}} \\
&= \left(\frac{1}{f_{j,k}}\prod_{p,q} e^{-f_{p,q}}\right)\sum_{\mathbf{r}}\frac{\sum_{i=1}^{M}(r_{j,k+i}-r_{j,k})\prod_{p,q} f_{p,q}^{r_{p,q+i}+r_{p,q}}}{\sum_{i=1}^{M}\prod_{p,q} f_{p,q}^{r_{p,q+i}}r_{p,q}!}
\end{aligned}$$

Observe that the denominator $\sum_{i=1}^{M}\prod_{p,q} f_{p,q}^{r_{p,q+i}}r_{p,q}!$ is invariant under cyclic permutations of the entries of r **on the 2nd coordinate:**

$$\sum_{i=1}^{M}\prod_{p,q} f_{p,q}^{r_{p,q+i}}r_{p,q}! = \left(\prod_{p,q} r_{p,q}!\right)\sum_{i=1}^{M}\prod_{p,q} f_{p,q}^{r_{p,q+i+l}} \text{ (for any } l\text{)}$$

So when summing up over r , from $\sum_{k=1}^{M} N(r_{j,k}) = \sum_{l=1}^{M} N(r_{j,k+l})$ and invariance of the denominator, we can average the numerator over those cyclic permutations (why?):

$$\begin{aligned}
\sum_{\mathbf{r}}\frac{N(r)}{D(r)} &= \sum_{j=1}^{N}\sum_{k=1}^{M}\frac{N(r_{j,k})}{D(r_{j,k})} = \sum_{j=1}^{N}\sum_{k=1}^{M}\frac{N(r_{j,k})}{D(r_{j,w})} \text{ (for any cyclic permuation } w \text{ of } 1,...,M) \\
&= \sum_{j=1}^{N}\frac{\sum_{l=1}^{M} N(r_{j,k+l})}{D(r_{j,w})} \text{ (for any } k,w)
\end{aligned}$$

getting:

$$\sum_{l=1}^{M}\sum_{i=1}^{M}(r_{j,k+l+i}-r_{j,k+l})\prod_{p,q} f_{p,q}^{r_{p,q+l+i}+r_{p,q+l}}.$$

The above quantity is zero, since every term in this sum appears exactly twice with opposite signs. For instance, the term that contains a factor of $f_{p,q}^{r_{p,q+3}+r_{p,q}}$ comes once from $l=0$ and $i=3$, with coefficient $r_{j,k+3}-r_{j,k}$, and once from $l=3$ and $i=-3$, with coefficient $r_{j,k}-r_{j,k+3}$.
Therefore we obtain the first order partial derivatives:

$$\frac{\partial L}{\partial f_{j,k}} = \sum_{\mathbf{r}}\frac{\partial P}{\partial f_{j,k}}\ln(S) = \sum_{\mathbf{r}} P\left(\frac{r_{j,k}}{f_{j,k}}-1\right)\ln(S) = E_{\mathbf{r}|\theta=0}\left[(r_{j,k}/f_{j,k}-1)\ln(S)\right]$$

$$\frac{\partial I}{\partial f_{j,k}} = E_{\mathbf{r}|\theta=0}\left[(1-r_{j,k}/f_{j,k})\ln(S)\right] \tag{1}$$

2nd Order Derivatives

$$\frac{\partial^2 L}{\partial f_{j,k}\partial f_{l,m}} = \sum_{\mathbf{r}} \frac{\partial^2 P}{\partial f_{j,k}\partial f_{l,m}} \ln(S) + \sum_{\mathbf{r}} \frac{\partial P}{\partial f_{j,k}} \frac{\partial S}{\partial f_{l,m}} \frac{1}{S}$$

The 1st term is

$$\sum_{\mathbf{r}} \frac{\partial^2 P}{\partial f_{j,k}\partial f_{l,m}} \ln(S) = \sum_{\mathbf{r}} \left(\left(\frac{r_{j,k}}{f_{j,k}} - 1 \right) \left(\frac{r_{l,m}}{f_{l,m}} - 1 \right) + 1_{\{(j,k)=(l,m)\}} \left(-\frac{r_{j,k}}{f_{j,k}^2} \right) \right) P \ln(S)$$

To simplify the second term:

$$\frac{\partial S}{\partial f_{l,m}} = \frac{1}{M} \sum_{i=1}^{M} (r_{l,m+i} - r_{l,m})/f_{l,m} \prod_{p,q} f_{p,q}^{r_{p,q+i}-r_{p,q}}$$

$$\frac{\partial S}{\partial f_{l,m}} \frac{1}{S} = \frac{1}{f_{l,m}} \frac{\frac{1}{M}\sum_{i=1}^{M} (r_{l,m+i} - r_{l,m}) \prod_{p,q} f_{p,q}^{r_{p,q+i}-r_{p,q}}}{\frac{1}{M}\sum_{i=1}^{M} \prod_{p,q} f_{p,q}^{r_{p,q+i}-r_{p,q}}}$$

$$\begin{aligned}
f_{j,k} f_{l,m} \frac{\partial P}{\partial f_{j,k}} \frac{\partial S}{\partial f_{l,m}} \frac{1}{S} &= \frac{P(\mathbf{r})(r_{j,k} - f_{j,k}) \sum_{i=1}^{M} (r_{l,m+i} - r_{l,m}) \prod_{p,q} f_{p,q}^{r_{p,q+i}-r_{p,q}}}{\sum_{i=1}^{M} \prod_{p,q} f_{p,q}^{r_{p,q+i}-r_{p,q}}} \\
&= \frac{P(\mathbf{r}) r_{j,k} \sum_{i=1}^{M} (r_{l,m+i} - r_{l,m}) \prod_{p,q} f_{p,q}^{r_{p,q+i}}}{\sum_{i=1}^{M} \prod_{p,q} f_{p,q}^{r_{p,q+i}}} - \frac{P(\mathbf{r}) f_{j,k} \sum_{i=1}^{M} (r_{l,m+i} - r_{l,m}) \prod_{p,q} f_{p,q}^{r_{p,q+i}}}{\sum_{i=1}^{M} \prod_{p,q} f_{p,q}^{r_{p,q+i}}}
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{\mathbf{r}} \frac{\partial P}{\partial f_{j,k}} \frac{\partial S}{\partial f_{l,m}} \frac{1}{S} &= \frac{1}{f_{j,k} f_{l,m}} \sum_{\mathbf{r}} \frac{P(\mathbf{r}) r_{j,k} \sum_{i=1}^{M} (r_{l,m+i} - r_{l,m}) \prod_{p,q} f_{p,q}^{r_{p,q+i}}}{\sum_{i=1}^{M} \prod_{p,q} f_{p,q}^{r_{p,q+i}}} \\
&= \frac{1}{f_{j,k} f_{l,m}} \frac{P(\mathbf{r}) r_{j,k} \sum_{i=1}^{M} (r_{l,m+i} - r_{l,m}) \prod_{p,q} f_{p,q}^{r_{p,q+i}}}{\sum_{i=1}^{M} \prod_{p,q} f_{p,q}^{r_{p,q+i}}} \\
&= E_{\mathbf{r}|\theta=0} \left[\frac{r_{j,k}}{f_{j,k} f_{l,m}} \frac{\sum_{i=1}^{M} (r_{l,m+i} - r_{l,m}) \prod_{p,q} f_{p,q}^{r_{p,q+i}}}{\sum_{i=1}^{M} \prod_{p,q} f_{p,q}^{r_{p,q+i}}} \right] \\
&= E_{\mathbf{r}|\theta=0} \left[\frac{r_{j,k}}{f_{j,k} f_{l,m}} \frac{\sum_{i=1}^{M} r_{l,m+i} \prod_{p,q} f_{p,q}^{r_{p,q+i}}}{\sum_{i=1}^{M} \prod_{p,q} f_{p,q}^{r_{p,q+i}}} - \frac{r_{j,k} r_{l,m}}{f_{j,k} f_{l,m}} \right]
\end{aligned}$$

where the second term $\sum_{i=1}^{M} (r_{l,m+i} - r_{l,m}) \prod_{p,q} f_{p,q}^{r_{p,q+i}}$ produces zero after summing up over r, as we proved in the previous section.

If we wind up averaging $r_{j,k}(r_{l,m+i} - r_{l,m})$ and $r_{j,k+i}(r_{l,m} - r_{l,m+i})$, we get an expression that is manifestly symmetric in k and m, namely

$$2 f_{j,k} f_{l,m} \sum_{\mathbf{r}} \frac{\partial P}{\partial f_{j,k}} \frac{\partial S}{\partial f_{l,m}} \frac{1}{S} = \sum_{\mathbf{r}} P(\mathbf{r}) \frac{-\sum_{i=1}^{M} (r_{j,k+i} - r_{j,k})(r_{l,m+i} - r_{l,m}) \prod_{p,q} f_{p,q}^{r_{p,q+i}}}{\sum_{i=1}^{M} \prod_{p,q} f_{p,q}^{r_{p,q+i}}}$$

Therefore

$$\begin{aligned}
\frac{\partial^2 L}{\partial f_{j,k}\partial f_{l,m}} &= \sum_{\mathbf{r}} \frac{\partial^2 P}{\partial f_{j,k}\partial f_{l,m}} \ln(S) + \sum_{\mathbf{r}} \frac{\partial P}{\partial f_{j,k}} \frac{\partial S}{\partial f_{l,m}} \frac{1}{S} \\
&= E_{\mathbf{r}|\theta} \left[\left(\frac{r_{j,k}}{f_{j,k}} - 1 \right) \left(\frac{r_{l,m}}{f_{l,m}} - 1 \right) \ln(S) - \frac{\sum_{i=1}^{M} (r_{j,k+i} - r_{j,k})(r_{l,m+i} - r_{l,m}) \prod_{p,q} f_{p,q}^{r_{p,q+i}}}{2 f_{j,k} f_{l,m} \sum_{i=1}^{M} \prod_{p,q} f_{p,q}^{r_{p,q+i}}} + 1_{(j,k)}^{(l,m)} \left(-\frac{r_{j,k}}{f_{j,k}^2} \right) \ln(S) \right]
\end{aligned}$$

$$\frac{\partial^2 I}{\partial f_{j,k}\partial f_{l,m}} = E_{\mathbf{r}|\theta}\left[-\left(\frac{r_{j,k}}{f_{j,k}}-1\right)\left(\frac{r_{l,m}}{f_{l,m}}-1\right)\ln(S)+\frac{\sum_{i=1}^{M}(r_{j,k+i}-r_{j,k})(r_{l,m+i}-r_{l,m})\prod_{p,q}f_{p,q}^{r_{p,q+i}}}{2f_{j,k}f_{l,m}\sum_{i=1}^{M}\prod_{p,q}f_{p,q}^{r_{p,q+i}}}+1_{(j,k)}^{(l,m)}\left(\frac{r_{j,k}}{f_{j,k}^2}\right)\ln(S)\right] \quad (2)$$

$$= E_{\mathbf{r}|\theta}\left[-\left(\frac{r_{j,k}}{f_{j,k}}-1\right)\left(\frac{r_{l,m}}{f_{l,m}}-1\right)\ln(S)-\frac{r_{j,k}}{f_{j,k}f_{l,m}}\frac{\sum_{i=1}^{M}r_{l,m+i}\prod_{p,q}f_{p,q}^{r_{p,q+i}}}{\sum_{i=1}^{M}\prod_{p,q}f_{p,q}^{r_{p,q+i}}}+\frac{r_{j,k}r_{l,m}}{f_{j,k}f_{l,m}}+1_{(j,k)}^{(l,m)}\left(\frac{r_{j,k}}{f_{j,k}^2}\right)\ln(S)\right] \quad (3)$$

where $j,l \in \{1,...,N=numPop\}$, $k,m \in \{1,...,M=numBin\}$.
In other notations:

$$\frac{\partial^2 I}{\partial f_{p,i}\partial f_{q,j}} = E_{\mathbf{r}|\theta}\left[-\left(\frac{r_{p,i}}{f_{p,i}}-1\right)\left(\frac{r_{q,j}}{f_{q,j}}-1\right)\ln(S)+\frac{\sum_{k=1}^{M}(r_{p,i+k}-r_{p,i})(r_{q,j+k}-r_{q,j})\prod_{l,m}f_{l,m}^{r_{l,m+k}}}{2f_{p,i}f_{q,j}\sum_{k=1}^{M}\prod_{l,m}f_{l,m}^{r_{l,m+k}}}+1_{(p,i)}^{(q,j)}\left(\frac{r_{p,i}}{f_{p,i}^2}\right)\ln(S)\right] \quad (4)$$

$$= E_{\mathbf{r}|\theta}\left[-\left(\frac{r_{p,i}}{f_{p,i}}-1\right)\left(\frac{r_{q,j}}{f_{q,j}}-1\right)\ln(S)-\frac{r_{p,i}}{f_{p,i}f_{q,j}}\frac{\sum_{k=1}^{M}r_{q,j+k}\prod_{l,m}f_{l,m}^{r_{l,m+k}}}{\sum_{k=1}^{M}\prod_{l,m}f_{l,m}^{r_{l,m+k}}}+\frac{r_{p,i}r_{q,j}}{f_{p,i}f_{q,j}}+1_{(p,i)}^{(q,j)}\left(\frac{r_{p,i}}{f_{p,i}^2}\right)\ln(S)\right] \quad (5)$$

Notations in the Code

To avoid overflow, compute S in the following way:

$$\begin{aligned}
S(\mathbf{r}) &= \frac{1}{M}\sum_{j=1}^{M}\prod_{p=1}^{N}\prod_{k=1}^{M}\left(\frac{f_{p,k-j}}{f_{p,k}}\right)^{r_{p,k}} \\
&= \frac{1}{M}\sum_{j=1}^{M}\exp\left[\sum_{p=1}^{N}\sum_{k=1}^{M}r_{p,k}\ln\left(\frac{f_{p,k-j}}{f_{p,k}}\right)\right] \\
&= \frac{1}{M}\sum_{j=1}^{M}\exp\left[\sum_{p,k}^{N,M}r_{p,k}\ln\left(\frac{f_{p,k-j}}{f_{p,k}}\right)-\max_j\left(\sum_{p,q}r_{p,k}\ln\left(\frac{f_{p,k-j}}{f_{p,k}}\right)\right)\right]\cdot\exp\left[\max_j\left(\sum_{p,q}r_{p,k}\ln\left(\frac{f_{p,k-j}}{f_{p,k}}\right)\right)\right]
\end{aligned}$$

For convenience of computation, define

$$\begin{aligned}
Lrate(p;k,j) &:= \ln\left(\frac{f_{p,k-j}}{f_{p,k}}\right) \\
Mexp(j) &:= \sum_{p=1}^{N}\sum_{k=1}^{M}r_{p,k}\cdot Lrate(p;k,j) \\
Max &:= \max_j Mexp(j) \\
S(\mathbf{r}) &= \sum_{j=1}^{M}\exp\left[Mexp(j)-Max\right]\cdot e^{Max} \\
\ln\left(S(\mathbf{r})\right) &= \ln\left(\frac{1}{M}\sum_{i=1}^{M}\exp\left[Mexp(i)-Max\right]\right)+Max, \\
Eexp(j) &:= \frac{\exp\left[Mexp(j)-Max\right]}{\sum_{j=1}^{M}\exp\left[Mexp(j)-Max\right]} \\
&= \frac{\prod_{p,k}^{N,M}\left(\frac{f_{p,k-j}}{f_{p,k}}\right)^{r_{p,k}}}{\sum_{j=1}^{M}\prod_{p,k}^{N,M}\left(\frac{f_{p,k-j}}{f_{p,k}}\right)^{r_{p,k}}} \\
&= \frac{\prod_{p,k}^{N,M}f_{p,k}^{r_{p,k+j}}}{\frac{1}{M}\sum_{j=1}^{M}\prod_{p,k}^{N,M}f_{p,k}^{r_{p,k+j}}} \in [0,1]
\end{aligned}$$

where the vector $Eexp$ is used in computation of Hessian (3), (5).

Tuning Derivatives

Now we compute the derivatives of I w.r.t. $t_{i,j}$'s.

Denote $w = stimWidth$, $a+b = (a+b)\%M$ for $a, b \in \{1, ..., M\}$.

Since

$$\begin{aligned}
f_{i,j} &= w\sum_{k=1}^{M} t_{i,j-k}s_k = w\sum_{k=1}^{M} t_{i,k}s_{j-k} \\
\frac{\partial f_{i,j}}{\partial t_{i,k}} &= ws_{j-k}
\end{aligned}$$

We have

$$\frac{\partial I}{\partial t_{i,j}} = \sum_{k=1}^{M} \frac{\partial I}{\partial f_{i,k}} \frac{\partial f_{i,k}}{\partial t_{i,j}} = w\sum_{k=1}^{M} \frac{\partial I}{\partial f_{i,k}} s_{k-j}$$

$$\begin{aligned}
\frac{\partial^2 I}{\partial t_{p,i}\partial t_{q,j}} &= \frac{\partial}{\partial t_{q,j}} \left(w\sum_{k=1}^{M} \frac{\partial I}{\partial f_{p,k}} s_{k-i} \right) \\
&= w\sum_{k=1}^{M}\sum_{l=1}^{M} s_{k-i} \frac{\partial^2 I}{\partial f_{p,k}\partial f_{q,l}} \frac{\partial f_{q,l}}{\partial t_{q,j}} \\
&= w^2\sum_{k=1}^{M}\sum_{l=1}^{M} s_{k-i}s_{l-j} \frac{\partial^2 I}{\partial f_{p,k}\partial f_{q,l}}
\end{aligned}$$