The proof of Poissonian Limit

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n cell positions, discrete model:

$$I(\mathbf{r};\theta) = E_{\mathbf{r}|\theta=0}\left[-\ln\left(\frac{p(\mathbf{r})}{p(\mathbf{r}|\theta=0)}\right)\right] \tag{1}$$

$$= \sum_{\mathbf{c}\in\mathbb{N}^n}\prod_{i=1}^{n}\frac{f(\frac{i}{n})^{c_i}}{c_i!}e^{-\sum_{i=1}^{n} f(\frac{i}{n})}\left[\ln\left(\frac{\frac{1}{n}\sum_{l=1}^{n}\prod_{i=1}^{n} f(\frac{i}{n}-\frac{l}{n})^{c_i}}{\prod_{i=1}^{n} f(\frac{i}{n})^{c_i}}\right)\right] \tag{2}$$

We want to consider our model in the continuous limit, i.e. for $n \to \infty$. For the mutual information to be finite, we need the expected number of spikes to remain finite when $n \to \infty$. A natural limit to consider is when the expected number of spikes per neuron scales as $1/n$ so that the total average number of spikes remains constant while $n \to \infty$. Consider a sequence of tuning curves f_n satisfying

$$\bar{f_n} = \frac{1}{n}\sum_{i=1}^{n} f_n(\frac{i}{n}) = \frac{1}{n}\bar{f}$$

$$\frac{1}{n}f_- \leq f_n \leq \frac{1}{n}f_+$$

For all $s\in[0,1)$, assume the limit

$$\lim_{n\to\infty} n f_n(\frac{\lfloor sn\rfloor}{n}) = f(s)$$

exists.

First show that

$$\sum_{\mathbf{c}\in\mathbb{N}^n}\prod_{i=1}^{n}\frac{f_n(\frac{i}{n})^{c_i}}{c_i!} \sim \sum_{\mathbf{c}\in\{0,1\}^n}\prod_{i=1}^{n}\frac{f_n(\frac{i}{n})^{c_i}}{c_i!}$$

i.e. if a vector $\mathbf{c}\in\mathbb{N}^n$ has at least one coordinate c_i with $c_i \geq 2$, then it makes no contribution to the sum.

Suppose $\|\mathbf{a}\|_0 = \sum_{i=1}^{n} 1_{\{a_i\neq 0\}} = k$, then the permutations of $\mathbf{a}$ contains $\leq n^k$ vectors. The other terms of order 1,

$$\begin{aligned}
\sum_{\mathbf{c}\in P(\mathbf{a})}\prod_{i=1}^{n}\frac{f_n(\frac{i}{n})^{c_i}}{c_i!} &= \sum_{\mathbf{c}\in P(\mathbf{a})}\prod_{j=1}^{k}\frac{f_n(\frac{i_j}{n})^{c_{i_j}}}{c_{i_j}!} \\
&\leq n^k\prod_{j=1}^{k}\frac{\left(\frac{1}{n}f_+\right)^{c_{i_j}}}{c_{i_j}!} \\
&\leq n^k O\left(n^{-\sum_{j=1}^{k} a_{i_j}}\right) \\
&= O(n^{k-\sum_{j=1}^{k} a_{i_j}})
\end{aligned}$$

So if there exists $a_{i_j} \geq 2$ then the above term goes to zero as $n \to \infty$.

Next,

$$
\begin{aligned}
I_n &= \sum_{\mathbf{c}\in\mathbb{N}^n}\prod_{i=1}^n \frac{f_n(\frac{i}{n})^{c_i}}{c_i!}e^{-\sum_{i=1}^n f_n(\frac{i}{n})}\left[\ln\left(\frac{\frac{1}{n}\sum_{l=1}^n\prod_{i=1}^n f_n(\frac{i}{n}-\frac{l}{n})^{c_i}}{\prod_{i=1}^n f_n(\frac{i}{n})^{c_i}}\right)\right] \\
&= \sum_{\mathbf{c}\in\{0,1\}^n}\prod_{i=1}^n \frac{f_n(\frac{i}{n})^{c_i}}{c_i!}e^{-\sum_{i=1}^n f_n(\frac{i}{n})}\left[\ln\left(\frac{\frac{1}{n}\sum_{l=1}^n\prod_{i=1}^n f_n(\frac{i}{n}-\frac{l}{n})^{c_i}}{\prod_{i=1}^n f_n(\frac{i}{n})^{c_i}}\right)\right] \\
&= \sum_{k=0}^n \sum_{c_{i_j}=1 \text{ at } i_1,\dots,i_k}\prod_{j=1}^k \frac{f_n(\frac{i_j}{n})^1}{1!}e^{-\sum_{i=1}^n f_n(\frac{i}{n})}\left[\ln\left(\frac{\frac{1}{n}\sum_{l=1}^n\prod_{j=1}^k f_n(\frac{i_j}{n}-\frac{l}{n})^1}{\prod_{j=1}^k f_n(\frac{i_j}{n})^1}\right)\right] \\
&= \sum_{k=0}^n \sum_{1\le i_1\le\cdots\le i_k\le n}\frac{1}{n^k}\prod_{j=1}^k \left(nf_n\left(\frac{i_j}{n}\right)\right)e^{-\frac{1}{n}\sum_{i=1}^n nf_n(\frac{i}{n})}\left[\ln\left(\frac{\frac{1}{n}\sum_{l=1}^n\prod_{j=1}^k\left(nf_n(\frac{i_j}{n}-\frac{l}{n})\right)}{\prod_{j=1}^k\left(nf_n(\frac{i_j}{n})\right)}\right)\right]
\end{aligned}
$$

As $n\to\infty$, let $nf_n\left(\frac{i_j}{n}\right)\to f(\theta_j)$, then take the limit of the mutual information by writing the sum as an integral:

$$
\begin{aligned}
I_n &\to \sum_{k=0}^\infty \int\cdots\int_{\{0\le\theta_1\le\cdots\le\theta_k\le 1\}} d\theta_1\cdots d\theta_k \prod_{j=1}^k f(\theta_j)e^{-\int_0^1 f(s)ds}\left[\ln\left(\frac{\int_0^1\prod_{j=1}^k f(\theta_j-s)ds}{\prod_{j=1}^k f(\theta_j)}\right)\right] \\
&= \sum_{k=0}^\infty \frac{1}{k!}\int_0^1 d\theta_1\cdots\int_0^1 d\theta_k \prod_{j=1}^k f(\theta_j)e^{-\int_0^1 f(s)ds}\left[\ln\left(\frac{\int_0^1\prod_{j=1}^k f(\theta_j-s)ds}{\prod_{j=1}^k f(\theta_j)}\right)\right] \qquad (3)
\end{aligned}
$$

where the last step comes from the number of permutations of $(\theta_1,\dots,\theta_k)$.
Note: $k=0$ makes no contribution to the above sum.