The Model:

P neurons $\{f_1, ..., f_P\}$, each discretized with M parameters $(\theta_1, \cdots, \theta_M)$.

$r_k|\theta \sim Poisson(f_{k,j})$, $k = 1, \ldots, P$, $j = 1, \ldots, M$, if $\theta \in I_j$.

Can think of f_k as a piecewise constant tuning curve with each piece length $= w_j$:

$$f_k(\theta) = \sum_{j=1}^{M} f_{k,j} 1_{\{\theta \in I_j\}}$$

where $|I_j| = w_i$, $I_j = [\sum_{l=1}^{j-1} w_l, \sum_{l=1}^{j} w_l)$, $\sum_{l=1}^{M} w_l = 1$.

The stimulus distribution is uniform, so:

$$p(\theta) = \sum_{j=1}^{M} w_j 1_{\{\theta \in I_j\}}$$

Thus

$$P(r_k|\theta) = \sum_j 1_{\{\theta \in I_j\}} \frac{f_{k,j}^{r_k}}{r_k!} e^{-f_{k,j}}$$

$$P(\mathbf{r}|\theta) = \sum_j 1_{\{\theta \in I_j\}} \prod_{k=1}^{P} \frac{f_{k,j}^{r_k}}{r_k!} e^{-f_{k,j}}$$

$$P(\mathbf{r}) = \int p(\theta) d\theta P(\mathbf{r}|\theta) = \sum_{j=1}^{M} w_j P(\mathbf{r}|\theta_j)$$

The Mutual Information:

$$\begin{aligned} I(\mathbf{r};\theta) &= \int d\theta p(\theta) D_{KL}\left(p(\mathbf{r}|\theta) \| p(\mathbf{r})\right) \\ &= \sum_{j=1}^{M} w_j D_{KL}\left(p(\mathbf{r}|\theta_j) \| p(\mathbf{r})\right) \\ &= \sum_{j=1}^{M} w_j \left[-\sum_{\mathbf{r}} P_j(\mathbf{r}) \ln S_j(\mathbf{r}) \right] \\ &= \sum_{j=1}^{M} w_j E_{\mathbf{r}|\theta_j}\left[-\ln S_j(\mathbf{r}) \right] \end{aligned} \tag{1}$$

where

$$\begin{aligned} P_j(\mathbf{r}) &:= P(\mathbf{r}|\theta_j) = \prod_{k=1}^{P} \frac{f_{k,j}^{r_k}}{r_k!} e^{-f_{k,j}} \\ S_j(\mathbf{r}) &:= \frac{p(\mathbf{r})}{P(\mathbf{r}|\theta_j)} = \sum_{l=1}^{M} w_l \prod_{k=1}^{P} \frac{f_{k,l}^{r_k} e^{-f_{k,l}}}{f_{k,j}^{r_k} e^{-f_{k,j}}} \end{aligned}$$

Derivatives:

$(x^r e^{-x})' = (\frac{r}{x} - 1) x^r e^{-x}$, $(x^{-r} e^{x})' = -(\frac{r}{x} - 1) x^{-r} e^{-x}$

$$
\begin{aligned}
\frac{\partial P_j(\mathbf{r})}{\partial f_{k,l}} &= 1_{\{l=j\}}\left(\frac{r_k}{f_{k,l}}-1\right)P_j(\mathbf{r}) \\
\frac{\partial S_j(\mathbf{r})}{\partial f_{k,l}} &= \sum_{l'=1}^{M} w_{l'}\frac{\partial}{\partial f_{k,l}}\left(\prod_{k=1}^{P}\frac{f_{k,l'}^{r_k}e^{-f_{k,l'}}}{f_{k,j}^{r_k}e^{-f_{k,j}}}\right) \\
&= \sum_{l'=1}^{M} w_{l'}\prod_{k'\neq k}\frac{f_{k',l'}^{r_{k'}}e^{-f_{k',l'}}}{f_{k',j}^{r_{k'}}e^{-f_{k',j}}}\frac{\partial}{\partial f_{k,l}}\left(\frac{f_{k,l'}^{r_k}e^{-f_{k,l'}}}{f_{k,j}^{r_k}e^{-f_{k,j}}}\right) \\
&= \sum_{l'=1}^{M} w_{l'}\prod_{k'\neq k}\frac{f_{k',l'}^{r_{k'}}e^{-f_{k',l'}}}{f_{k',j}^{r_{k'}}e^{-f_{k',j}}}\left(1_{\{l=l'\}}\left(\frac{r_k}{f_{k,l'}}-1\right)\frac{f_{k,l'}^{r_k}e^{-f_{k,l'}}}{f_{k,j}^{r_k}e^{-f_{k,j}}}+1_{\{l=j\}}\left(-\frac{r_k}{f_{k,j}}+1\right)\frac{f_{k,l'}^{r_k}e^{-f_{k,l'}}}{f_{k,j}^{r_k}e^{-f_{k,j}}}+1_{\{l'=j\}}0\right) \\
&= w_l\left(\frac{r_k}{f_{k,l}}-1\right)\prod_{k'=1}^{P}\frac{f_{k',l}^{r_{k'}}e^{-f_{k',l}}}{f_{k',j}^{r_{k'}}e^{-f_{k',j}}}+1_{\{l=j\}}\left(-\frac{r_k}{f_{k,j}}+1\right)\sum_{l'=1}^{M} w_{l'}\prod_{k'=1}^{P}\frac{f_{k',l'}^{r_{k'}}e^{-f_{k',l'}}}{f_{k',j}^{r_{k'}}e^{-f_{k',j}}} \\
&= w_l\left(\frac{r_k}{f_{k,l}}-1\right)\prod_{k'=1}^{P}\frac{f_{k',l}^{r_{k'}}e^{-f_{k',l}}}{f_{k',j}^{r_{k'}}e^{-f_{k',j}}}+1_{\{l=j\}}\left(-\frac{r_k}{f_{k,j}}+1\right)S_j(\mathbf{r})
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial f_{k,l}}\left(\sum_{\mathbf{r}}P_j(\mathbf{r})\ln S_j(\mathbf{r})\right) &= \sum_{\mathbf{r}}\frac{\partial P_j(\mathbf{r})}{\partial f_{k,l}}\ln S_j(\mathbf{r})+\sum_{\mathbf{r}}P_j(\mathbf{r})\frac{1}{S_j(\mathbf{r})}\frac{\partial S_j(\mathbf{r})}{\partial f_{k,l}} \\
&= 1_{\{l=j\}}\sum_{\mathbf{r}}\left(\frac{r_k}{f_{k,l}}-1\right)P_j(\mathbf{r})\ln S_j(\mathbf{r})+\sum_{\mathbf{r}}P_j(\mathbf{r})\frac{1}{S_j(\mathbf{r})}\frac{\partial S_j(\mathbf{r})}{\partial f_{k,l}} \\
&= 1_{\{l=j\}}E_{\mathbf{r}|\theta_j}\left[\left(\frac{r_k}{f_{k,l}}-1\right)\ln S_j(\mathbf{r})\right]+E_{\mathbf{r}|\theta_j}\left[\frac{1}{S_j(\mathbf{r})}\frac{\partial S_j(\mathbf{r})}{\partial f_{k,l}}\right] \\
&= 1_{\{l=j\}}E_{\mathbf{r}|\theta_j}\left[\left(\frac{r_k}{f_{k,l}}-1\right)\ln S_j(\mathbf{r})\right] \\
&\quad +E_{\mathbf{r}|\theta_j}\left[w_l\left(\frac{r_k}{f_{k,l}}-1\right)\frac{\prod_{k'=1}^{P}\frac{f_{k',l'}^{r_{k'}}e^{-f_{k',l'}}}{f_{k',j}^{r_{k'}}e^{-f_{k',j}}}}{\sum_{l=1}^{M}w_l\prod_{k=1}^{P}\frac{f_{k,l}^{r_k}e^{-f_{k,l}}}{f_{k,j}^{r_k}e^{-f_{k,j}}}}\right] \\
&\quad +1_{\{l=j\}}E_{\mathbf{r}|\theta_j}\left[\left(-\frac{r_k}{f_{k,j}}+1\right)\frac{S_j(\mathbf{r})}{S_j(\mathbf{r})}\right] \\
&= 1_{\{l=j\}}E_{\mathbf{r}|\theta_j}\left[\left(\frac{r_k}{f_{k,l}}-1\right)\ln S_j(\mathbf{r})\right]+1_{\{l=j\}}E_{\mathbf{r}|\theta_j}\left(-\frac{r_k}{f_{k,j}}\right)+1 \\
&\quad +E_{\mathbf{r}|\theta_j}\left[\left(\frac{r_k}{f_{k,l}}-1\right)\frac{w_l\prod_{k'=1}^{P}f_{k',l}^{r_{k'}}e^{-f_{k',l}}}{\sum_{l'=1}^{M}w_{l'}\prod_{k'=1}^{P}f_{k',l'}^{r_{k'}}e^{-f_{k',l'}}}\right] \\
&= 1_{\{l=j\}}E_{\mathbf{r}|\theta_j}\left[\left(\frac{r_k}{f_{k,l}}-1\right)\ln S_j(\mathbf{r})\right]+E_{\mathbf{r}|\theta_j}\left[\left(\frac{r_k}{f_{k,l}}-1\right)\frac{w_l\prod_{k'=1}^{P}f_{k',l}^{r_{k'}}e^{-f_{k',l}}}{\sum_{l'=1}^{M}w_{l'}\prod_{k'=1}^{P}f_{k',l'}^{r_{k'}}e^{-f_{k',l'}}}\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial I(\mathbf{r};\theta)}{\partial f_{k,l}} &= -\sum_j w_j \sum_{\mathbf{r}} \frac{\partial}{\partial f_{k,l}} \left(\sum_{\mathbf{r}} P_j(\mathbf{r}) \ln S_j(\mathbf{r}) \right) \\
&= -\sum_{j=1}^{M} w_j 1_{\{l=j\}} E_{\mathbf{r}|\theta_j} \left[\left(\frac{r_k}{f_{k,l}} - 1 \right) \ln S_j(\mathbf{r}) \right] \\
&\quad -\sum_{j=1}^{M} w_j E_{\mathbf{r}|\theta_j} \left[\left(\frac{r_k}{f_{k,l}} - 1 \right) \frac{w_l \prod_{k'=1}^{P} f_{k',l}^{r_{k'}} e^{-f_{k',l}}}{\sum_{l'=1}^{M} w_{l'} \prod_{k'=1}^{P} f_{k',l'}^{r_{k'}} e^{-f_{k',l'}}} \right] \\
&= -w_l E_{\mathbf{r}|\theta_l} \left[\left(\frac{r_k}{f_{k,l}} - 1 \right) \ln S_l(\mathbf{r}) \right] - \sum_{j=1}^{M} w_j E_{\mathbf{r}|\theta_j} \left[\left(\frac{r_k}{f_{k,l}} - 1 \right) \frac{w_l \prod_{k'=1}^{P} f_{k',l}^{r_{k'}} e^{-f_{k',l}}}{\sum_{l'=1}^{M} w_{l'} \prod_{k'=1}^{P} f_{k',l'}^{r_{k'}} e^{-f_{k',l'}}} \right] \qquad (2)
\end{aligned}
$$

In conclusion:

$$
I(\mathbf{r};\theta) = \sum_{j=1}^{M} w_j E_{\mathbf{r}|\theta_j} \left[-\ln S_j(\mathbf{r}) \right] \qquad (3)
$$

$$
\frac{\partial I(\mathbf{r};\theta)}{\partial f_{k,l}} = -w_l E_{\mathbf{r}|\theta_l} \left[\left(\frac{r_k}{f_{k,l}} - 1 \right) \ln S_l(\mathbf{r}) \right] - \sum_{j=1}^{M} w_j E_{\mathbf{r}|\theta_j} \left[\left(\frac{r_k}{f_{k,l}} - 1 \right) \frac{w_l \prod_{k'=1}^{P} f_{k',l}^{r_{k'}} e^{-f_{k',l}}}{\sum_{l'=1}^{M} w_{l'} \prod_{k'=1}^{P} f_{k',l'}^{r_{k'}} e^{-f_{k',l'}}} \right] \qquad (4)
$$

where

$$
\begin{aligned}
S_j(\mathbf{r}) &= \sum_{l=1}^{M} w_l \prod_{k=1}^{P} \frac{f_{k,l}^{r_k} e^{-f_{k,l}}}{f_{k,j}^{r_k} e^{-f_{k,j}}} \\
&= \sum_{l=1}^{M} w_l \exp \sum_{k=1}^{P} \log \left(\frac{f_{k,l}^{r_k} e^{-f_{k,l}}}{f_{k,j}^{r_k} e^{-f_{k,j}}} \right) \\
&= \sum_{l=1}^{M} w_l \exp \sum_{p=1}^{P} \left[r_p \log \left(\frac{f_{p,l}}{f_{p,j}} \right) + (f_{p,j} - f_{p,l}) \right]
\end{aligned}
$$

Constraints:

$$
f_{+} \leq f_{k,l} \leq f_{-}
$$

$$
\sum_{i=1}^{M} w_i f_{k,i} = \bar{f}_k
$$

If there is convolution:

$$
g_{k,i} = \sum_{l=1}^{M} f_{k,l} s_{i-l}
$$

$$
\sum_{i=1}^{M} w_i g_{k,i} = \sum_{i=1}^{M} \sum_{l=1}^{M} f_{k,l} s_{i-l} = \sum_{i=1}^{M} w_i \sum_{l=1}^{M} f_{k,l} s_{i-l} = \sum_{l=1}^{M} w_i s_{i-l} \sum_{i} f_{k,l}
$$

?, where

$$
\int s(\theta) d\theta = \sum_{i=1}^{M} w_i s_i = 1
$$

.

$$\frac{\partial I(\mathbf{r};\theta)}{\partial f_{k,l}} = \sum_{j=1}^{M} \frac{\partial I(\mathbf{r};\theta)}{\partial g_{k,j}} s_{j-l}$$

Notes for the code
$k, p = 1, ..., P,\ i, j, l, m = 1, ..., M$

$$\begin{aligned}
lrate_m(p,i) &:= \ln\left(\frac{f_{p,i}}{f_{p,m}}\right) \\
dexp_m(i) &:= \sum_{p=1}^{P}(f_{p,m} - f_{p,i}) \\
mexp_m(i) &:= \sum_{p=1}^{P} r_p lrate_m(p,i) + dexp_m(i) \\
Max_m &:= \max_i mexp_m(i) \\
prodq_m(i) = Q_m(i) &:= \exp\left[mexp_m(i) - Max\right] = \prod_{k=1}^{P} \frac{f_{k,i}^{r_k} e^{-f_{k,i}}}{f_{k,m}^{r_k} e^{-f_{k,m}}} e^{-Max} \\
S_m(\mathbf{r}) &= \sum_{i=1}^{M} w_i \exp\left[mexp(i)\right] \\
&= \sum_{i=1}^{M} w_i \exp\left[mexp(i) - Max\right] \cdot e^{Max} \\
mean_m &:= \ln\left(S_m(\mathbf{r})\right) \text{ where } \mathbf{r} \sim P(\mathbf{r}|\theta_m)
\end{aligned}$$

$$\begin{aligned}
grad1(p,i) &= E_{\mathbf{r}|\theta_i}\left[\left(\frac{r_k}{f_{k,i}} - 1\right) \ln S_i(\mathbf{r})\right], \mathbf{r} \sim P(\mathbf{r}|\theta_i) \\
tmpgrad2_m(p,i) &= E_{\mathbf{r}|\theta_m}\left[\left(\frac{r_p}{f_{p,i}} - 1\right) \frac{w_i Q_m(i)}{\sum_{l'=1}^{M} w_{l'} Q_m(l')}\right], \mathbf{r} \sim P(\mathbf{r}|\theta_m) \\
&= E_{\mathbf{r}|\theta_m}\left[\left(\frac{r_p}{f_{p,i}} - 1\right) \frac{w_i \exp\left[mexp_m(i) - Max\right]}{\sum_{l'=1}^{M} w_{l'} \exp\left[mexp_m(l') - Max\right]}\right] \\
grad(p,i) &= -w_i grad1(p,i) - \sum_{m=1}^{M} w_m tmpgrad2_m(p,i) \\
I &= -\sum_{m=1}^{M} w_m mean_m
\end{aligned}$$

$$I(\mathbf{r};\theta) = \sum_{j=1}^{M} w_j E_{\mathbf{r}|\theta_j}\left[-\ln S_j(\mathbf{r})\right]$$

$$\frac{\partial I(\mathbf{r};\theta)}{\partial f_{k,l}} = -w_l E_{\mathbf{r}|\theta_l}\left[\left(\frac{r_k}{f_{k,l}} - 1\right) \ln S_l(\mathbf{r})\right] - \sum_{j=1}^{M} w_j E_{\mathbf{r}|\theta_j}\left[\left(\frac{r_k}{f_{k,l}} - 1\right) \frac{w_l \prod_{k'=1}^{P} f_{k',l}^{r_{k'}} e^{-f_{k',l}}}{\sum_{l'=1}^{M} w_{l'} \prod_{k'=1}^{P} f_{k',l'}^{r_{k'}} e^{-f_{k',l'}}}\right]$$

Constraints:?

The Mutual Information:

$$
\begin{aligned}
I(\mathbf{r};\theta) &= \sum_{j=1}^{M} w_j \left[-\sum_{\mathbf{r}} P_j(\mathbf{r}) \ln S_j(\mathbf{r}) \right] \\
&\approx \sum_{r_1=0}^{T} \cdots \sum_{r_P=0}^{T} \left[-\sum_{j=1}^{M} w_j P_j(\mathbf{r}) \ln S_j(\mathbf{r}) \right]
\end{aligned}
$$

where

$$
\begin{aligned}
P_j(\mathbf{r}) &:= P(\mathbf{r}|\theta_j) = \prod_{k=1}^{P} \frac{f_{k,j}^{r_k}}{r_k!} e^{-f_{k,j}} \\
S_j(\mathbf{r}) &:= \frac{p(\mathbf{r})}{P(\mathbf{r}|\theta_j)} = \sum_{l=1}^{M} w_l \prod_{k=1}^{P} \frac{f_{k,l}^{r_k} e^{-f_{k,l}}}{f_{k,j}^{r_k} e^{-f_{k,j}}}
\end{aligned}
$$

$$
\begin{aligned}
P_j(\mathbf{r}) \ln S_j(\mathbf{r}) &= \prod_{k=1}^{P} \frac{f_{k,j}^{r_k}}{r_k!} e^{-f_{k,j}} \ln \left[\sum_{l=1}^{M} w_l \prod_{k=1}^{P} \frac{f_{k,l}^{r_k} e^{-f_{k,l}}}{f_{k,j}^{r_k} e^{-f_{k,j}}} \right] \\
&= \exp\left(\sum_{k=1}^{P} \left(r_k \log(f_{k,j}) - f_{k,j} - \log(r_k!) \right) \right) \cdot \ln \left[\sum_{l=1}^{M} w_l \exp \sum_{k=1}^{P} \left(r_k \log(f_{k,l}) - f_{k,l} - \left(r_k \log(f_{k,j}) - f_{k,j} \right) \right) \right] \\
&= \exp\left(\sum_{k=1}^{P} \left(texp_{k,j} - \log(r_k!) \right) \right) \ln \left[\sum_{l=1}^{M} w_l \exp \sum_{k=1}^{P} \left(texp_{k,l} - texp_{k,j} \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial I(\mathbf{r};\theta)}{\partial f_{k,l}} &= -w_l E_{\mathbf{r}|\theta_l} \left[\left(\frac{r_k}{f_{k,l}} - 1 \right) \ln S_l(\mathbf{r}) \right] - \sum_{j=1}^{M} w_j E_{\mathbf{r}|\theta_j} \left[\left(\frac{r_k}{f_{k,l}} - 1 \right) \frac{w_l \prod_{k'=1}^{P} f_{k',l}^{r_{k'}} e^{-f_{k',l}}}{\sum_{l'=1}^{M} w_{l'} \prod_{k'=1}^{P} f_{k',l'}^{r_{k'}} e^{-f_{k',l'}}} \right] \\
&= -\sum_{\mathbf{r}} \left[\left(\frac{r_k}{f_{k,l}} - 1 \right) w_l P_l(\mathbf{r}) \ln S_l(\mathbf{r}) \right] - \sum_{\mathbf{r}} \left[\left(\frac{r_k}{f_{k,l}} - 1 \right) \sum_{j=1}^{M} w_j P_j(\mathbf{r}) \frac{w_l \prod_{k'=1}^{P} f_{k',l}^{r_{k'}} e^{-f_{k',l}}}{\sum_{l'=1}^{M} w_{l'} \prod_{k'=1}^{P} f_{k',l'}^{r_{k'}} e^{-f_{k',l'}}} \right] \\
&= -\sum_{\mathbf{r}} \left[\left(\frac{r_k}{f_{k,l}} - 1 \right) w_l P_l(\mathbf{r}) \ln S_l(\mathbf{r}) \right] - \sum_{\mathbf{r}} \left[\left(\frac{r_k}{f_{k,l}} - 1 \right) \sum_{j=1}^{M} w_j P_j(\mathbf{r}) \frac{w_l}{S_l(\mathbf{r})} \right] \\
&= -\sum_{\mathbf{r}} \left[\left(\frac{r_k}{f_{k,l}} - 1 \right) \left(w_l P_l(\mathbf{r}) \ln S_l(\mathbf{r}) + \sum_{j=1}^{M} w_j P_j(\mathbf{r}) \frac{w_l}{S_l(\mathbf{r})} \right) \right]
\end{aligned}
$$

The Arimoto Algorithm:

$$\max_{p(\theta)} I(\mathbf{r};\theta) \Leftrightarrow \max_{p(\theta)} \max_{q(\theta|\mathbf{r})} \int d\mathbf{r} \int d\theta p(\theta) p(\mathbf{r}|\theta) \log \frac{q(\theta|\mathbf{r})}{p(\theta)}$$

At capacity, $D_{KL}(p(\mathbf{r}|\theta)\|p(\mathbf{r})) = const$, for all θ.
Iteration:

$$q^{(n+1)}(\theta_j|\mathbf{r}) \propto p^{(n)}(\theta_j) p(\mathbf{r}|\theta_j) \propto w_j^{(n)} \prod_{k=1}^{P} f_{k,j}^{r_k} e^{-f_{k,j}}$$

$$\begin{aligned}
p^{(n+1)}(\theta_j) &\propto \exp\left[\sum_{\mathbf{r}} p(\mathbf{r}|\theta_j) \log q^{(n+1)}(\theta_j|\mathbf{r})\right] \\
&\propto \exp\left[E_{\mathbf{r}|\theta_j} \log \frac{w_j^{(n)} \prod_{k=1}^{P} f_{k,j}^{r_k} e^{-f_{k,j}}}{\sum_l w_l^{(n)} \prod_{k=1}^{P} f_{k,l}^{r_k} e^{-f_{k,l}}}\right]
\end{aligned}$$

Codes notes: $\mathbf{r} \sim P(\mathbf{r}|\theta_m)$,

$$\frac{w_j^{(n)} \prod_{k=1}^{P} f_{k,j}^{r_k} e^{-f_{k,j}}}{\sum_l w_l^{(n)} \prod_{k=1}^{P} f_{k,l}^{r_k} e^{-f_{k,l}}} = \frac{w_j^{(n)} \exp \sum_{k=1}^{P} (r_k \log(f_{k,j}) - f_{k,j})}{\sum_l w_l^{(n)} \exp \sum_{k=1}^{P} (r_k \log(f_{k,l}) - f_{k,l})}$$

$$\begin{aligned}
rexp_m(i) &:= \sum_{p=1}^{P} (r_p \log f_{p,i} - f_{p,i}) \\
Max_m &:= \max_i rexp_m(i) \\
qexp_m &:= \frac{w_m \exp[rexp_m(m) - Max]}{\sum_{l=1}^{M} w_l \exp[rexp_m(l) - Max]} \\
mean_m &:= \ln(qexp_m)
\end{aligned}$$

$$\begin{aligned}
me\hat{a}n_m &= \frac{1}{N_{iter}} mean_m^{(N)} \\
p(\theta_m) &\propto \exp(m\hat{e}an_m)
\end{aligned}$$

$$\begin{aligned}
p^{(n+1)}(\theta_j) &\propto \exp\left[E_{\mathbf{r}|\theta_j} \log \frac{w_j^{(n)} \prod_{k=1}^{P} f_{k,j}^{r_k} e^{-f_{k,j}}}{\sum_l w_l^{(n)} \prod_{k=1}^{P} f_{k,l}^{r_k} e^{-f_{k,l}}}\right] \\
&\propto \exp\left[\sum_{\mathbf{r}} P_j(\mathbf{r}) \log\left(\frac{w_j^{(n)}}{S_j^{(n)}(\mathbf{r})}\right)\right]
\end{aligned}$$

$$\begin{aligned}
P_j(\mathbf{r}) &:= P(\mathbf{r}|\theta_j) = \prod_{k=1}^{P} \frac{f_{k,j}^{r_k}}{r_k!} e^{-f_{k,j}} \\
S_j^{(n)}(\mathbf{r}) &:= \frac{p(\mathbf{r})}{P(\mathbf{r}|\theta_j)} = \sum_{l=1}^{M} w_l^{(n)} \prod_{k=1}^{P} \frac{f_{k,l}^{r_k} e^{-f_{k,l}}}{f_{k,j}^{r_k} e^{-f_{k,j}}}
\end{aligned}$$