Extensions

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1 Convolution

For the model problem with two independent drawings:

$$I[f] = \int_0^1 r(x) \ln (r(x)) \, dx$$

$$I(X,Y;\Theta) = \int_0^1 dx \int_0^1 dy r(x) r(y) \ln \left(\frac{r(x)r(y)}{\int_0^1 d\theta r(x-\theta) r(y-\theta)} \right) \tag{1}$$

where $r(x) = \int_0^1 f(x-\theta) s(\theta) d\theta = \int_0^1 f(\theta) s(x-\theta) d\theta$, $\int_0^1 s(\theta) = 1$, $\int_0^1 f(\theta) d\theta = \bar{f}$, $f_+ \leq f \leq f_-$. For this model we can also calculate the auto-correlation:

$$\begin{aligned} A_r(x-y) &= \int d\theta r(x-\theta) r(y-\theta) \\ &= \int d\theta \int s(u) du f(x-\theta-u) \int s(v) dv f(y-\theta-v) \\ &= \int du \int dv s(u) s(v) \int d\theta f(x-\theta-u) f(y-\theta-v) \\ &= \int du \int dv s(u) s(v) A_f\left(x-y-(u-v)\right) \end{aligned}$$

Then $I[f] = -2H[r] + H[A_r] \leq -2H[r]$, which also maximizes when $r = f_+$ and f_-, which encourages f to saturate. However, it is unclear about the upper bound of $-H[r]$ or $H[A_r]$ with the constraints of f.

Consider the simplest case:

$$I[f] = -H[r] = \int_0^1 r(x) \ln (r(x)) \, dx \tag{2}$$

Compute the gradient and write the KKT condition:

$$\begin{aligned} \nabla_f [r(x)](t) &= s(x-t) \\ \nabla_f [I](x) &= \int \nabla_r [I](y) \nabla_f [r(y)](x) dy \\ &= \int \left(\log r(y) + 1\right) s(y-x) dy \\ &= \int \left(\log r(y)\right) s(y-x) dy + 1 && (3) \\ \nabla_f [I](x) &= \alpha^+(x) - \alpha^-(x) + \mu && (4) \end{aligned}$$

Thus if $f \neq f_+$ and $f \neq f_-$,

$$\int s(y-x)\log r(y)dy = \mu - 1 \text{ for } \forall \text{x} . \tag{5}$$

Differentiate the above equation w.r.t. x:

$$\begin{aligned}
-\int_0^1 s'(y-x)\log r(y)dy &= 0 \\
\left(\log r(y)\right) s(y-x)|_{y=0}^{y=1} - \int_0^1 \frac{r'(y)}{r(y)} s'(y-x)dy &= 0 \\
\int_0^1 \frac{r'(y)}{r(y)} s'(y-x)dy &= 0.
\end{aligned}$$

To further simplify, we consider the convolution kernel to be a rectangle: $s(\theta) = \frac{1}{c} 1_{[0,c]}(\theta)$. Differentiate the equation 5 w.r.t. x:

$$\begin{aligned}
\frac{1}{c}\int_x^{x+c} \log r(y)dy &= \mu - 1 \\
\log r(x+c) - \log r(x) &= 0 \\
r(x) &= r(x+c) \\
\int_0^c f(x+c-\theta)d\theta &= \int_0^c f(x-\theta)d\theta \qquad (6)
\end{aligned}$$

Let D be the set of discontinuities of f . Since $f'(x) = f'_p(x) + \sum_{x_i \in D} (\triangle_i f)\delta_{x_i}(x)$ in the sense of distributions (where $f'_p(x)$ is the point-wise derivative), we differentiate the above equation

$$\begin{aligned}
\frac{\partial}{\partial x}\int_0^c f(x-\theta)d\theta &= \int_0^c f'_p(x-\theta) + \sum_{x_i \in D} (\triangle_i f)\delta_{x_i}(x-\theta)d\theta \\
&= \int_0^c f'_p(x-\theta)d\theta + \sum_{x_i \in D} (\triangle_i f) 1_{\{x - x_i \in [0,c]\}} \\
&= \int_{-c}^0 f'_p(x+\theta)d\theta + \sum_{x_i \in D} (\triangle_i f) 1_{\{x_i \in [x-c,x]\}} \\
&= f\left(x^-\right) - f\left((x-c)^+\right) - \sum_{x_i \in D, x_i \in [x-c,x]} (\triangle_i f) + \sum_{x_i \in D} (\triangle_i f) 1_{\{x_i \in [x-c,x]\}} \\
\frac{\partial}{\partial x}\int_0^c f(x-\theta)d\theta &= f\left(x^-\right) - f\left((x-c)^+\right) \qquad (7)
\end{aligned}$$

Similarily $\frac{\partial}{\partial x}\int_0^c f(x+c-\theta)d\theta = f\left((x+c)^-\right) - f\left(x^+\right)$,

$$\begin{aligned}
f\left((x+c)^-\right) - f\left(x^+\right) &= f\left(x^-\right) - f\left((x-c)^+\right) \\
\frac{f\left(x^+\right) + f\left(x^-\right)}{2} &= \frac{f\left((x+c)^-\right) + f\left((x-c)^+\right)}{2} \qquad (8)
\end{aligned}$$

* If f is continuous on $[0,1)$ and does not reach f^+ or f_- then $f(x)$ satisfies $f(x) = f(x+c)$ for all x: f is periodic with period c. Also, $f(x) = f(x+1)$. If c is irrational, then f is a constant, i.e. $f \equiv 1$.

We also compute $\nabla_\delta I[f]$ using the change of variable $\eta = x + \epsilon\delta(x)$, $dx = \frac{1}{1+\epsilon\delta'(x)}d\eta = \left(1 - \epsilon\delta'(\eta) + O(\epsilon^2)\right)d\eta$,

$$\begin{aligned}
I[f(\cdot+\epsilon\delta(\cdot))] &= \int_0^1 dx r\left(x+\epsilon\delta(x)\right)\log r\left(x+\epsilon\delta(x)\right) \\
&= \int_0^1 d\eta r\left(\eta\right)\log r\left(\eta\right)\left(1-\epsilon\delta'(\eta)+O(\epsilon^2)\right) \\
&= I[f]-\epsilon\int_0^1 d\eta\delta'(\eta)r\left(\eta\right)\log r\left(\eta\right)+O(\epsilon^2) \\
&= I[f]+\epsilon\int_0^1 d\eta\delta(\eta)\frac{d}{d\eta}\left[r\left(\eta\right)\log r\left(\eta\right)\right] \\
&= I[f]+\epsilon\int_0^1 d\eta\delta(\eta)\left[\int_0^1 f'(\eta-\theta)s(\theta)d\theta\left(\log\left(r(\eta)\right)+1\right)\right] \\
\nabla_\delta I[f] &= \int_0^1 f'(\eta-\theta)s(\theta)d\theta\left(\log\left(r(\eta)\right)+1\right) \qquad (9)
\end{aligned}$$

When $s(\theta)=\frac{1}{c}1_{[0,c]}(\theta)$,

$$\begin{aligned}
\nabla_\delta I[f] &= \frac{1}{c}\int_0^c f'(\eta-\theta)d\theta\left(\log\left(r(\eta)\right)+1\right) \\
&= \frac{1}{c}\frac{\partial}{\partial\eta}\left[\int_0^c f'(\eta-\theta)d\theta\right]\left(\log\left(r(\eta)\right)+1\right) \\
&= \frac{1}{c}\left(f\left(\eta^-\right)-f\left(\left(\eta-c\right)^+\right)\right)\left(\log\left(r(\eta)\right)+1\right) \qquad (10)
\end{aligned}$$

Thus$\nabla_\delta I[f]=0$ implies that

$$\left(f\left(x^-\right)-f\left(\left(x-c\right)^+\right)\right)\left(\log\left(r(x)\right)+1\right)=0 \text{ for } \forall x\in[0,1) \qquad (11)$$

First, the set $\{x\in[0,1):\log r(x)+1\neq 0\}$ has positive measure (otherwise $r\equiv\frac{1}{e}$ contradicts with $\int r=1$). Since the function $r(x)$ is continuous, there exists an interval (α,β) such that $\log r+1\neq 0$ on that inverval. Thus

$$f\left(x^-\right)=f\left(\left(x-c\right)^+\right)$$

Assume f does not reach f^+,f_- on (α,β), then using the previous argument (7),

$$\frac{f\left(x^+\right)+f\left(x^-\right)}{2}=\frac{f\left(\left(x+c\right)^-\right)+f\left(\left(x-c\right)^+\right)}{2}$$

Combined with pervious result, we have $f\left(x^+\right)=f\left(\left(x+c\right)^-\right)$ for $x\in(\alpha,\beta)$. These two statements have extended our conclusion outside of (α,β). (However, it is still far from proving that f should saturate).

2 Multi populations

For two populations with tuning curves f and g, the discretized model

$$\begin{aligned}
I(\mathbf{r};\theta) &= E_{\mathbf{r}|\theta=0}\left[-\ln\left(\frac{p(\mathbf{r})}{p(\mathbf{r}|\theta=0)}\right)\right] \\
&= \sum_{r_{1,1},\ldots,r_{1,M}}\sum_{r_{2,1},r_{2,M}}\prod_{i=1}^M\frac{(f_i)^{r_{1,i}}}{r_{1,i}!}e^{-f_i}\prod_{j=1}^M\frac{(g_j)^{r_{2,j}}}{r_{2,j}!}e^{-g_j}\ln\left(\frac{\prod_{i=1}^M\prod_{j=1}^M(f_i)^{r_{1,i}}(g_j)^{r_{2,j}}}{\frac{1}{M}\sum_{k=1}^M\prod_{i=1}^M\prod_{j=1}^M(f_{i-k})^{r_{1,i}}(g_{j-k})^{r_{2,j}}}\right)
\end{aligned}$$

has the following Poissonian limit when $M\to\infty$:

$$I[f,g]=\sum_{n_1,n_2}\frac{1}{n_1!}\frac{1}{n_2!}\int\cdots\int\prod_{i=1}^{n_1}f(s_i)ds_i\int\cdots\int\prod_{j=1}^{n_2}g(t_j)dt_je^{-\int f(s)ds}e^{-\int g(t)dt}\ln\left(\frac{\prod f(s_i)\prod g(t_j)}{\int f(s_i-\theta)g(t_j-\theta)d\theta}\right)$$

let $\tilde{f}=\frac{f}{\bar{f}}$,$\tilde{g}=\frac{g}{\bar{g}}$, then

$$\begin{aligned}
I[f,g] &= \sum_{n_1,n_2}\frac{1}{n_1!}\frac{1}{n_2!}\int\cdots\int\prod_{i=1}^{n_1}f(s_i)ds_i\int\cdots\int\prod_{j=1}^{n_2}g(t_j)dt_je^{-\int f(s)ds}e^{-\int g(t)dt}\ln\left(\frac{\prod f(s_i)\prod g(t_j)}{\int f(s_i-\theta)g(t_j-\theta)d\theta}\right)\\
&\leq \sum_{n_1,n_2}\frac{1}{n_1!}\frac{1}{n_2!}e^{-\bar{f}}e^{-\bar{g}}\left(\bar{f}\right)^{n_1}\left(\bar{g}\right)^{n_2}\int\cdots\int\prod_{i=1}^{n_1}\tilde{f}(s_i)ds_i\int\cdots\int\prod_{j=1}^{n_2}\tilde{g}(t_j)dt_j\ln\left(\prod\tilde{f}(s_i)\prod\tilde{g}(t_j)\right)\\
&= \sum_{n_1,n_2}\frac{1}{n_1!}\frac{1}{n_2!}e^{-\bar{f}}e^{-\bar{g}}\left(\bar{f}\right)^{n_1}\left(\bar{g}\right)^{n_2}\left[\left(-n_1H[\tilde{f}]\right)+\left(-n_2H[\tilde{g}]\right)\right]\\
&= -\bar{f}H[\tilde{f}]-\bar{g}H[\tilde{g}]
\end{aligned}$$

where, if we define $H\left(\frac{f_+}{\bar{f}},\frac{f_-}{\bar{f}}\right):=\frac{f_+}{\bar{f}}\ln\left(\frac{f_+}{\bar{f}}\right)\triangle+\frac{f_-}{\bar{f}}\ln\left(\frac{f_-}{\bar{f}}\right)(1-\triangle)$ with $\triangle=\frac{\bar{f}-f_-}{f_+-f_-}$, then

$$I[f,g]\leq-\bar{f}H\left(\frac{f_+}{\bar{f}},\frac{f_-}{\bar{f}}\right)-\bar{g}H\left(\frac{g_+}{\bar{g}},\frac{g_-}{\bar{g}}\right).$$

3 Different Limit

For the 'red' model, i.e. neurons centered on a subset of positions,

$$\begin{aligned}
I(\mathbf{r};\theta) &= \frac{1}{\delta}\sum_{m=1}^{\delta}E_{\mathbf{r}|\theta_m}\left[-\ln\left(\frac{p(\mathbf{r})}{p(\mathbf{r}|\theta_m)}\right)\right] &(12)\\
&= \frac{1}{\delta}\sum_{m=1}^{\delta}E_{\mathbf{r}|\theta_m}\left[-\ln\left(\frac{1}{M}\sum_{i=1}^{M}\prod_{k=1}^{K}\left(\frac{f_{i-c_k}}{f_{m-c_k}}\right)^{r_k}e^{-(f_{i-c_k}-f_{m-c_k})}\right)\right] &(13)
\end{aligned}$$

Consider the limit when the number of centers (K) remain finite, as the number of discretizations $M\to\infty$:

$$I=M\int_0^{\frac{1}{M}}d\theta\sum_{r_1,\ldots,r_k}\prod_{i=1}^{K}\frac{f(\theta-\frac{i}{M})^{r_i}}{r_i!}e^{-f(\theta-\frac{i}{K})}\ln\left(\frac{\prod_{i=1}^{K}f(\theta-\frac{i}{K})^{r_i}e^{-f(\theta-\frac{i}{K})}}{M\int_0^{\frac{1}{M}}d\theta'\prod_{i=1}^{K}f(\theta'-\frac{i}{K})^{r_i}e^{-f(\theta'-\frac{i}{K})}}\right)$$

or

$$I=\int_0^{1}d\theta\sum_{r_1,\ldots,r_k}\prod_{i=1}^{K}\frac{f(\theta-\frac{i}{M})^{r_i}}{r_i!}e^{-f(\theta-\frac{i}{K})}\ln\left(\frac{\prod_{i=1}^{K}f(\theta-\frac{i}{K})^{r_i}e^{-f(\theta-\frac{i}{K})}}{\int_0^{1}d\theta'\prod_{i=1}^{K}f(\theta'-\frac{i}{K})^{r_i}e^{-f(\theta'-\frac{i}{K})}}\right)$$

4 Place cells

(Example: place cells)

A cell at x responses not only to stimulus at x, but also to stimulus at other places.

Cell position x_i, $i=1,2,...,n$

Stimulus positions θ_l, $l=1,2,...,k$. (Poisson point process on $(0,1)$ with intensity λ: $p(\theta_1,...,\theta_k)=\frac{\lambda^k}{k!}e^{-\lambda}$)

Firing rate at x_i: $\sum_{l=1}^{k}f(\theta_l-x_i)=\int f(\theta-x_i)\sum_l\delta_{\theta_l}(\theta)d\theta$

$p(c_1,...,c_n|\theta_1,...,\theta_k)=\prod_{i=1}^{n}\frac{\left(\sum_{l=1}^{k}f(\theta_l-x_i)\right)^{c_i}}{c_i!}e^{-\sum_l f(\theta_l-x_i)}$

$$
\begin{aligned}
I &= \int_\theta \int_{\mathbf{c}} p(\mathbf{c}|\theta)p(\theta)\ln\left(\frac{p(\mathbf{c}|\theta)}{\int p(\mathbf{c}|\theta)p(\theta)d\theta}\right) \\
&= \sum_k \frac{\lambda^k}{k!}\int d\theta_1\cdots\int d\theta_k \sum_{\mathbf{c}} \prod_{i=1}^n \frac{\left(\sum_l^k f(\theta_l - x_i)\right)^{c_i}}{c_i!} e^{-\sum_l^k f(\theta_l - x_i)} \ln\left(\frac{\prod_{i=1}^n \frac{\left(\sum_l^k f(\theta_l - x_i)\right)^{c_i}}{c_i!} e^{-\sum_l^k f(\theta_l - x_i)}}{E_{\theta'}\left[\prod_{i=1}^n \frac{\left(\sum_l^{k'} f(\theta'_l - x_i)\right)^{c_i}}{c_i!} e^{-\sum_l^{k'} f(\theta'_l - x_i)}\right]}\right)
\end{aligned}
$$

Take the limit: $nf_n(x) \to f(x)$,

$$
I = \sum_k \frac{\lambda^k}{k!}\int d\theta_1\cdots\int d\theta_k \sum_n \frac{1}{n!}\int dx_1\cdots\int dx_n \prod_{i=1}^n \left(\sum_{l=1}^k f(\theta_l - x_i)\right) e^{-k\bar{f}} \ln\left(\frac{\prod_{i=1}^n \left(\sum_{l=1}^k f(\theta_l - x_i)\right) e^{-k\bar{f}}}{E_{\theta'}\left[\prod_{i=1}^n \left(\sum_{l=1}^{k'} f(\theta'_l - x_i)\right) e^{-k'\bar{f}}\right]}\right)
$$

Only one θ and one x (all with uniform distribution on $(0,1)$):

$$
\begin{aligned}
I &= \int d\theta \int dx f(\theta - x)\ln\left(\frac{f(\theta - x)e^{-\bar{f}}}{E_{\theta'}\left[f(\theta' - x)e^{-\bar{f}}\right]}\right) \\
&= \int d\theta \int dx f(\theta - x)\ln\left(\frac{f(\theta - x)}{\int f(\theta' - x)d\theta'}\right) \\
&= -H[f]
\end{aligned}
$$

Two θ and one x:

$$
\begin{aligned}
I &= \int d\theta_1 \int d\theta_2 \int dx\,(f(\theta_1 - x) + f(\theta_2 - x))\ln\left(\frac{f(\theta_1 - x) + f(\theta_2 - x)}{\int d\theta'_1 \int d\theta'_2 f(\theta'_1 - x) + f(\theta'_2 - x)}\right) \\
&= \int d\theta_1 \int d\theta_2 \int dx\,(f(\theta_1 - x) + f(\theta_2 - x))\ln(f(\theta_1 - x) + f(\theta_2 - x)) - \int d\theta_1 \int d\theta_2 \int dx\,(f(\theta_1 - x) + f(\theta_2 - x))\ln(2) \\
&= \int d\theta_1 \int d\theta_2 \int dx\,(f(\theta_1 - x) + f(\theta_2 - x))\ln(f(\theta_1 - x) + f(\theta_2 - x)) - 2\ln(2) \\
&= \int d\theta_1 \int d\theta_2 \int dx'\,(f(\theta_1 - \theta_2 + x') + f(x'))\ln(f(\theta_1 - \theta_2 + x') + f(x')) - 2\ln(2) \\
&= \int d\theta_1 \int dx' \int d\theta'\,(f(\theta') + f(x'))\ln(f(\theta') + f(x')) - 2\ln(2) \\
&= \int dx \int d\theta\,(f(\theta) + f(x))\ln(f(\theta) + f(x)) - 2\ln(2) \\
&= 2\int dx \int d\theta f(\theta)\ln(f(\theta) + f(x)) - 2\ln(2)
\end{aligned}
\tag{14}
$$

Gradient:

$$
\begin{aligned}
I &= \int d\theta_1 \int d\theta_2 \int dx\,(f(\theta_1 - x) + f(\theta_2 - x))\ln(f(\theta_1 - x) + f(\theta_2 - x)) - 2\ln 2 \\
&= -H[F] - 2\ln 2
\end{aligned}
$$

where $F(x,\theta_1,\theta_2) = f(\theta_1 - x) + f(\theta_2 - x) = \int f(t)\delta_{\theta_1 - x}(t)dt + \int f(t)\delta_{\theta_2 - x}(t)dt$
$[\nabla_f F(x,\theta_1,\theta_2)](t) = \delta_{\theta_1 - x}(t) + \delta_{\theta_2 - x}(t)$
$\nabla_F H[F](x,\theta_1,\theta_2) = -(\ln F(x,\theta_1,\theta_2) + 1)$

$$
\begin{aligned}
\nabla_f[I](t) &= \int d\theta_1 \int d\theta_2 \int dx\,(\ln F(x,\theta_1,\theta_2)+1)\,(\delta_{\theta_1-x}(t)+\delta_{\theta_2-x}(t)) \\
&= \int d\theta_1 \int d\theta_2\,(\ln(f(t)+f(\theta_2-\theta_1+t))+1) + \int d\theta_1 \int d\theta_2\,(\ln(f(\theta_1-\theta_2+t)+f(t))+1) \\
&= 2+2\int d\theta_1 \int d\theta_2' \ln(f(t)+f(\theta_2')) \\
&= 2+2\int d\eta \ln(f(t)+f(\eta))
\end{aligned}
$$

OR use: $(x+y)\ln(x+y) = x\ln x + (\ln x + 1)y + O(y^2)$

$$
\begin{aligned}
I[f+h] &= \int d\theta \int dx\,(f(\theta-x)+f(x)+h(\theta-x)+h(x))\ln(f(\theta-x)+f(x)+h(\theta-x)+h(x)) \\
&= I[f] + \int d\theta \int dx\,(\ln(f(\theta-x)+f(x))+1)\,(h(\theta-x)+h(x)) + O(h^2) \\
&= I[f] + 2\int dx h(x)\left(\int d\theta \ln(f(\theta-x)+f(x))+1\right) + O(h^2) \\
&= I[f] + 2\int dx h(x)\left(\int d\theta' \ln(f(\theta')+f(x))+1\right) + O(h^2)
\end{aligned}
$$

—-

Two θ and two x:

$$
I = \int d\theta_1 \int d\theta_2 \int dx_1 \int dx_2 \prod{}_{i=1}^{2}(f(\theta_1-x_i)+f(\theta_2-x_i))\ln\left(\frac{\prod_{i=1}^{2}(f(\theta_1-x_i)+f(\theta_2-x_i))}{\int d\theta_1' \int d\theta_2' \prod_{i=1}^{2}(f(\theta_1'-x_i)+f(\theta_2'-x_i))}\right)
$$

The first term:

$$
\begin{aligned}
& \int d\theta_1 \int d\theta_2 \int dx_1 \int dx_2\,(f(\theta_1-x_1)+f(\theta_2-x_1))\,(f(\theta_1-x_2)+f(\theta_2-x_2))\ln((f(\theta_1-x_1)+f(\theta_2-x_1))\,(f(\theta_1-x_2)+f(\theta_2 \\
=& \int d\theta_1 \int d\theta_2 \int dx_1' \int dx_2'\,(f(\theta_1-\theta_2+x_1')+f(x_1'))\,(f(\theta_1-\theta_2+x_2')+f(x_2'))\ln((f(\theta_1-\theta_2+x_1')+f(x_1'))\,(f(\theta_1-\theta_2+x_2')+ \\
=& \int dx_1' \int dx_2' \int d\theta_1' d\theta_2'\,(f(\theta_1')+f(x_1'))\,(f(\theta_2')+f(x_2'))\ln((f(\theta_1')+f(x_1'))\,(f(\theta_2')+f(x_2'))) \\
=& 2\int dx_2 d\theta_2\,(f(\theta_2)+f(x_2)) \cdot \int dx_1 \int d\theta_1\,(f(\theta_1)+f(x_1))\ln(f(\theta_1)+f(x_1)) \\
=& 4\bar{f}\int dx \int d\theta\,(f(\theta)+f(x))\ln(f(\theta)+f(x)) \\
=& 8\bar{f}\int dx \int d\theta f(\theta)\ln(f(\theta)+f(x))
\end{aligned}
$$

The 2nd term:

$$
\begin{aligned}
A(x_1,x_2) &= \int d\theta_1 \int d\theta_2\,(f(\theta_1-x_1)+f(\theta_2-x_1))\,(f(\theta_1-x_2)+f(\theta_2-x_2)) \\
&= \int d\theta'{}_1 \int d\theta_2'\,(f(\theta_1')+f(\theta_2'))\,(f(\theta_1'+x_1-x_2)+f(\theta_2'+x_1-x_2)) \\
A(u) &= \int d\theta_1 \int d\theta_2\,(f(\theta_1)+f(\theta_2))\,(f(\theta_1+u)+f(\theta_2+u)) \\
&= \int d\theta_1 \int d\theta_2\,(f(\theta_1)f(\theta_1+u)+f(\theta_1)f(\theta_2+u)+f(\theta_2)f(\theta_1+u)+f(\theta_2)f(\theta_2+u)) \\
&= 2A_f(u)+2 \qquad (15)
\end{aligned}
$$

Thus

$$\begin{aligned}
& \int d\theta_1 \int d\theta_2 \int dx_1 \int dx_2 \left(f(\theta_1 - x_1) + f(\theta_2 - x_1)\right)\left(f(\theta_1 - x_2) + f(\theta_2 - x_2)\right)\ln\left(A(x_1 - x_2)\right) \\
= & \int dx_1 \int dx_2 A(x_1 - x_2)\ln\left(A(x_1 - x_2)\right) \\
= & \int dx_1 \int du A(u)\ln A(u) \\
= & \int du A(u)\ln A(u) \\
= & \ 2\int du(A_f(u) + 1)\ln(2A_f(u) + 2) \\
= & \ 2\int du(A_f(u) + 1)\ln(A_f(u) + 1) + 2\ln 2((\bar{f})^2 + 1)
\end{aligned}$$

$$I \ = \ 8\bar{f}\int dx \int d\theta f(\theta)\ln\left(f(\theta) + f(x)\right) + 2\int du(A_f(u) + 1)\ln(A_f(u) + 1) - 2\ln 2((\bar{f})^2 + 1)$$

If $\bar{f} = 1$:

$$I \ = \ 8\int dx \int d\theta f(\theta)\ln\left(f(\theta) + f(x)\right) + \int du A(u)\ln A(u) \tag{16}$$

where $A(u) = 2A_f(u) + 2$.
Gradient:
First term: $8 + 8\int dx \ln\left(f(t) + f(x)\right)$.
2nd term: $A_f(u) = \int f(x)f(u + x)dx$
$\int (f(x) + h(x))(f(u + x) + h(u + x))dx = A_f(u) + \int h(x)f(u + x)dx + \int h(y)f(y - u)dy + O(h^2)$
$\nabla_f[A_f(u)](t) = f(t + u) + f(t - u)$
$\nabla_f[A(u)](t) = 2\left(f(t + u) + f(t - u)\right)$

$$\begin{aligned}
\nabla_f[\int du A(u)\ln A(u)](t) \ &= \ 2\int du\left(1 + \ln(A(u))\right)\left(f(t + u) + f(t - u)\right) \\
&= \ 2\left[2 + 2\int du f(t + u)\ln(A(u))\right] \\
&= \ 4 + 4\int du f(t + u)\ln(A(u))
\end{aligned}$$

Thus

$$\begin{aligned}
\nabla_f[I](t) \ &= \ 8 + 8\int dx \ln\left(f(t) + f(x)\right) - 4 - 4\int du f(t + u)\ln(A(u)) \\
&= \ 8\int dx \ln\left(f(t) + f(x)\right) - 4\int du f(t + u)\ln(A(u)) + 4 \qquad (17)
\end{aligned}$$

where $A(u) = 2A_f(u) + 2 = 2\int f(x)f(x + u)du + 2$.
—-
*
Gradient:

$$
\begin{aligned}
I &= \int d\theta_1 \int d\theta_2 \int dx_1 \int dx_2 F(\theta_1,\theta_2,x_1,x_2)\ln\left(F(\theta_1,\theta_2,x_1,x_2)\right) - \int d\theta_1 \int d\theta_2 \int dx_1 \int dx_2 F(\theta_1,\theta_2,x_1,x_2)\ln\left(A(x_1,x_2)\right) \\
&= \int d\theta_1 \int d\theta_2 \int dx_1 \int dx_2 F(\theta_1,\theta_2,x_1,x_2)\ln\left(F(\theta_1,\theta_2,x_1,x_2)\right) - \int dx_1 \int dx_2 A(x_1,x_2)\ln\left(A(x_1,x_2)\right) \\
&= -H[F] + H[A]
\end{aligned}
$$

where $F(\theta_1,\theta_2,x_1,x_2) = \left(f(\theta_1 - x_1) + f(\theta_2 - x_1)\right)\left(f(\theta_1 - x_2) + f(\theta_2 - x_2)\right),\ A(x_1,x_2) = \int d\theta_1 \int d\theta_2 F(\theta_1,\theta_2,x_1,x_2)$.

$$
\begin{aligned}
F[f+h](\theta_1,\theta_2,x_1,x_2) &= \left(f(\theta_1 - x_1) + f(\theta_2 - x_1) + h(\theta_1 - x_1) + h(\theta_2 - x_1)\right)\left(f(\theta_1 - x_2) + f(\theta_2 - x_2) + h(\theta_1 - x_2) + h(\theta_2 - x_2)\right) \\
&= F + \left(h(\theta_1 - x_1) + h(\theta_2 - x_1)\right)\left(f(\theta_1 - x_2) + f(\theta_2 - x_2)\right) \\
&\quad + \left(f(\theta_1 - x_1) + f(\theta_2 - x_1)\right)\left(h(\theta_1 - x_2) + h(\theta_2 - x_2)\right) + O(h^2) \\
&= F + \int dt h(t)\left[\left(\delta_{\theta_1 - x_1}(t) + \delta_{\theta_2 - x_1}(t)\right)\left(f(\theta_1 - x_2) + f(\theta_2 - x_2)\right)\right] \\
&\quad + \int dt h(t)\left[\left(\delta_{\theta_1 - x_2}(t) + \delta_{\theta_2 - x_2}(t)\right)\left(f(\theta_1 - x_1) + f(\theta_2 - x_1)\right)\right]
\end{aligned}
$$

$[\nabla_f F(\theta_1,\theta_2,x_1,x_2)](t) = \left(\delta_{\theta_1 - x_1}(t) + \delta_{\theta_2 - x_1}(t)\right)\left(f(\theta_1 - x_2) + f(\theta_2 - x_2)\right) + \left(\delta_{\theta_1 - x_2}(t) + \delta_{\theta_2 - x_2}(t)\right)\left(f(\theta_1 - x_1) + f(\theta_2 - x_1)\right)$
$\nabla_F H[F](\theta_1,\theta_2,x_1,x_2) = -(\ln F(\theta_1,\theta_2,x_1,x_2) + 1)$

$$
\begin{aligned}
\nabla_f[-H[F]](t) &= \int d\theta_1 \int d\theta_2 \int dx_1 \int dx_2 \left(\ln F(\theta_1,\theta_2,x_1,x_2) + 1\right)\left(\delta_{\theta_1 - x_1}(t) + \delta_{\theta_2 - x_1}(t)\right)\left(f(\theta_1 - x_2) + f(\theta_2 - x_2)\right) \\
&\quad + \int d\theta_1 \int d\theta_2 \int dx_1 \int dx_2 \left(\ln F(\theta_1,\theta_2,x_1,x_2) + 1\right)\left(\delta_{\theta_1 - x_2}(t) + \delta_{\theta_2 - x_2}(t)\right)\left(f(\theta_1 - x_1) + f(\theta_2 - x_1)\right) \\
&= 4\int d\theta_1 \int d\theta_2 \int dx_2 \ln\left[\left(f(t) + f(\theta_2 - \theta_1 + t)\right)\left(f(\theta_1 - x_2) + f(\theta_2 - x_2)\right)\right]\left(f(\theta_1 - x_2) + f(\theta_2 - x_2)\right) \\
&\quad +8 \\
&= 4\int d\theta_1' \int d\theta_2' \int dx_2 \ln\left[\left(f(t) + f(\theta_2' - \theta_1' + t)\right)\left(f(\theta_1') + f(\theta_2')\right)\right]\left(f(\theta_1') + f(\theta_2')\right) + 8 \\
&= 4\int d\theta_1 \int d\theta_2 \ln\left[\left(f(t) + f(\theta_2 - \theta_1 + t)\right)\left(f(\theta_1) + f(\theta_2)\right)\right]\left(f(\theta_1) + f(\theta_2)\right) + 8
\end{aligned}
$$

$A(x_1,x_2) = \int d\theta_1 \int d\theta_2 F(\theta_1,\theta_2,x_1,x_2)$
$\nabla_F[A(x_1,x_2)](\theta_1,\theta_2,y_1,y_2) = \delta_{x_1}(y_1)\delta_{x_2}(y_2)$

$$
\begin{aligned}
\nabla_f[A(x_1,x_2)](t) &= \int dy_1 dy_2 \int d\theta_1 \int d\theta_2 \nabla_F[A(x_1,x_2)](\theta_1,\theta_2,y_1,y_2)\cdot[\nabla_f F(\theta_1,\theta_2,y_1,y_2)](t) \\
&= \int d\theta_1 \int d\theta_2 \left(\delta_{\theta_1 - x_1}(t) + \delta_{\theta_2 - x_1}(t)\right)\left(f(\theta_1 - x_2) + f(\theta_2 - x_2)\right) \\
&\quad + \int d\theta_1 \int d\theta_2 \left(\delta_{\theta_1 - x_2}(t) + \delta_{\theta_2 - x_2}(t)\right)\left(f(\theta_1 - x_1) + f(\theta_2 - x_1)\right) \\
&= 2\left[\int d\theta_2 \left(f(x_1 - x_2 + t) + f(\theta_2 - x_2)\right) + \int d\theta_2 \left(f(x_2 - x_1 + t) + f(\theta_2 - x_1)\right)\right]
\end{aligned}
$$

$$\begin{aligned}
\nabla_f[-H[A]](t) &= \int dx_1 \int dx_2 \left(\ln A(x_1,x_2)+1\right) \nabla_f[A(x_1,x_2)](t) \\
&= 2\int dx_1 \int dx_2 \int d\theta_2 \left(f(x_1-x_2+t)+f(\theta_2-x_2)\right)\left(\ln A(x_1,x_2)+1\right) \\
&\quad +2\int dx_1 \int dx_2 \int d\theta_2 \left(f(x_2-x_1+t)+f(\theta_2-x_1)\right)\left(\ln A(x_1,x_2)+1\right) \\
&= 4\int dx_1 \int dx_2 f(x_1-x_2+t)\ln A(x_1,x_2) + 4\int dx_1 \int dx_2 \int d\theta_2' f(\theta_2')\ln A(x_1,x_2) \\
&\quad +4 \\
&= 4\int dx_1 \int dx_2 \left[f(x_1-x_2+t)+1\right]\ln A(x_1,x_2)+4
\end{aligned}$$

$$\begin{aligned}
\nabla_f[I](t) &= 4\int d\theta_1 \int d\theta_2 \ln\left[\left(f(t)+f(\theta_2-\theta_1+t)\right)\left(f(\theta_1)+f(\theta_2)\right)\right]\left(f(\theta_1)+f(\theta_2)\right) \\
&\quad -4\int dx_1 \int dx_2 \left[f(x_1-x_2+t)+1\right]\ln A(x_1,x_2)+4
\end{aligned}$$

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*

$$\begin{aligned}
&\int d\theta_1 \int d\theta_2 \int dx_1 \int dx_2 \left(f(\theta_1-x_1)+f(\theta_2-x_1)\right)\left(f(\theta_1-x_2)+f(\theta_2-x_2)\right)\ln\left(A(x_1-x_2)\right) \\
=& \int d\theta_1 \int d\theta_2 \int dx_1' \int dx_2' \left(f(\theta_1-\theta_2+x_1')+f(x_1')\right)\left(f(\theta_1-\theta_2+x_2')+f(x_2')\right)\ln\left(A(x_1'-x_2')\right) \\
=& \int d\theta_1' \int d\theta_2' \int dx_1' \int dx_2' \left(f(\theta_1')+f(x_1')\right)\left(f(\theta_2')+f(x_2')\right)\ln\left(A(x_1'-x_2')\right) \\
=& \int d\theta_1 \int d\theta_2 \int dx_1 \int dx_2 \left(f(\theta_1)f(\theta_2)+f(\theta_1)f(x_2)+f(x_1)f(\theta_2)+f(x_1)f(x_2)\right)\ln\left(A(x_1-x_2)\right) \\
=& \int dx_1 \int dx_2 \ln A(x_1-x_2) + 2\int dx_1 \int dx_2 f(x_1)\ln A(x_1-x_2) + \int dx_1 \int dx_2 f(x_1)f(x_2)\ln A(x_1-x_2) \\
=& \int du \ln A(u) + 2\int dx_1 f(x_1)\int du \ln A(u) + \int dx_1 \int du f(x_1)f(x_1-u)\ln A(u)
\end{aligned}$$

5 Fourier

$$f(x)=\bar{f}+\sum_{n=1}^{N} a_n\cos(2\pi nx)+\sum_{n=1}^{N} b_n\sin(2\pi nx)\ \ ,x\in[0,1) \tag{18}$$

Fourier coefficients of f:

$$a_n = 2\int_0^1 f(x)\cos(2\pi nx)dx \tag{19}$$

$$b_n = 2\int_0^1 f(x)\sin(2\pi nx)dx \tag{20}$$

Note:

$$\begin{aligned}
\int_0^1 \cos(2\pi nx)\cos(2\pi mx)dx &= \frac{1}{2}\delta_{m,n} \\
\int_0^1 \sin(2\pi nx)\sin(2\pi mx)dx &= \frac{1}{2}\delta_{m,n} \\
\int_0^1 \cos(2\pi n\theta)\cos(2\pi m(x+\theta))d\theta &= \frac{1}{2}\delta_{m,n}\cos(2\pi nx) \\
\int_0^1 \cos(2\pi n\theta)\sin(2\pi m(x+\theta))d\theta &= \frac{1}{2}\delta_{m,n}\sin(2\pi nx) \\
\int_0^1 \sin(2\pi n\theta)\cos(2\pi m(x+\theta))d\theta &= -\frac{1}{2}\delta_{m,n}\sin(2\pi nx) \\
\int_0^1 \sin(2\pi n\theta)\sin(2\pi m(x+\theta))d\theta &= \frac{1}{2}\delta_{m,n}\cos(2\pi nx)
\end{aligned}$$

Auto-correlation:

$$\begin{aligned}
A_{f_1,f_2}(x) &= \int_0^1 f_1(\theta)f_2(x+\theta)d\theta \\
&= \int_0^1 \left(\bar{f}_1 + \sum_{n=1}^N a_n^{(1)}\cos(2\pi n\theta) + \sum_{n=1}^N b_n^{(1)}\sin(2\pi n\theta)\right)\left(\bar{f}_2 + \sum_{m=1}^N a_m^{(2)}\cos(2\pi m(x+\theta)) + \sum_{m=1}^N b_n^{(2)}\sin(2\pi m(x+\theta))\right)d\theta \\
&= \bar{f}_1\bar{f}_2 + \sum_{n=1}^N\sum_{m=1}^N a_n^{(1)}a_m^{(2)}\cos(2\pi nx)\frac{1}{2}\delta_{m,n} + \sum_{n=1}^N\sum_{m=1}^N a_n^{(1)}b_m^{(2)}\sin(2\pi nx)\frac{1}{2}\delta_{m,n} \\
&\quad + \sum_{n=1}^N\sum_{m=1}^N b_n^{(1)}a_m^{(2)}(-\sin(2\pi nx))\frac{1}{2}\delta_{m,n} + \sum_{n=1}^N\sum_{m=1}^N b_n^{(1)}b_m^{(2)}\cos(2\pi nx)\frac{1}{2}\delta_{m,n} \\
&= \bar{f}_1\bar{f}_2 + \sum_{n=1}^N \frac{a_n^{(1)}a_n^{(2)} + b_n^{(1)}b_n^{(2)}}{2}\cos(2\pi nx) + \sum_{n=1}^N \frac{a_n^{(1)}b_n^{(2)} - b_n^{(1)}a_n^{(2)}}{2}\sin(2\pi nx)
\end{aligned}$$

$$A_{f_1,f_2}(x) = \bar{f}_1\bar{f}_2 + \sum_{n=1}^N \frac{a_n^{(1)}a_n^{(2)} + b_n^{(1)}b_n^{(2)}}{2}\cos(2\pi nx) + \sum_{n=1}^N \frac{a_n^{(1)}b_n^{(2)} - b_n^{(1)}a_n^{(2)}}{2}\sin(2\pi nx) \tag{21}$$

For $f_1 = f_2 = f$:

$$A_f(x) = \int_0^1 f(\theta)f(x+\theta)d\theta = \left(\bar{f}\right)^2 + \sum_{n=1}^N \frac{a_n^2 + b_n^2}{2}\cos(2\pi nx) \tag{22}$$

model problem:

$$
\begin{aligned}
I[f] &= -2H[f] + H[A_f] && (23)\\
&= 2\int f(x)\ln f(x)dx - \int A_f(x)\ln(A_f(x))dx \\
&= 2\int\left[\bar{f} + \sum_{n=1}^{N} a_n\cos(2\pi nx) + \sum_{n=1}^{N} b_n\sin(2\pi nx)\right]\ln\left(f(x)\right)dx \\
&\quad -\int\left[\left(\bar{f}\right)^2 + \sum_{n=1}^{N}\frac{a_n^2+b_n^2}{2}\cos(2\pi nx)\right]\ln\left(A_f(x)\right)dx \\
&= 2\left[\bar{f}\int\ln f(x)dx + \sum_{n=1}^{N} a_n\int\cos(2\pi nx)\ln f(x)dx + \sum_{n=1}^{N} b_n\int\sin(2\pi nx)\ln f(x)dx\right] \\
&\quad -\left[\left(\bar{f}\right)^2\int\ln\left(A_f(x)\right)dx + \sum_{n=1}^{N}\frac{a_n^2+b_n^2}{2}\int\cos(2\pi nx)\ln\left(A_f(x)\right)dx\right] \\
&= 2\left[\overline{f\ln f} + \sum_{n=1}^{N}\frac{a_n^{(f)}a_n^{(\ln f)}}{2} + \sum_{n=1}^{N}\frac{b_n^{(f)}b_n^{(\ln f)}}{2}\right] - \left[\left(\bar{f}\right)^2\overline{(\ln A_f)} + \sum_{n=1}^{N}\frac{\left(a_n^{(f)}\right)^2+\left(b_n^{(f)}\right)^2}{2}\frac{a_n^{(\ln A_f)}}{2}\right] && (24)
\end{aligned}
$$

Gradients:

$$
\begin{aligned}
\partial_{a_n} f(x) &= \cos(2\pi nx)\\
\partial_{b_n} f(x) &= \sin(2\pi nx)\\
\partial_{a_n} A_f(x) &= a_n\cos(2\pi nx)\\
\partial_{b_n} A_f(x) &= b_n\cos(2\pi nx)
\end{aligned}
$$

$$
\begin{aligned}
I[f] &= 2\int f(x)\ln f(x)dx - \int A_f(x)\ln(A_f(x))dx\\
\partial_{a_n} I[f] &= 2\int\left(1+\ln f(x)\right)\cos(2\pi nx)dx - \int\left(1+\ln A_f(x)\right)a_n\cos(2\pi nx)dx\\
&= 2\int\left(\ln f(x)\right)\cos(2\pi nx)dx - a_n\int\left(\ln A_f(x)\right)\cos(2\pi nx)dx\\
&= a_n^{(\ln f)} - \frac{1}{2}a_n^{(f)}a_n^{(\ln A_f)}\\
\partial_{b_n} I[f] &= 2\int\left(\ln f(x)\right)\sin(2\pi nx)dx - b_n\int\left(\ln A_f(x)\right)\cos(2\pi nx)dx\\
&= b_n^{(\ln f)} - \frac{1}{2}b_n^{(f)}a_n^{(\ln A_f)}
\end{aligned}
$$

constraints:

$$
\begin{aligned}
\bar{f} + \sum|a_n| + \sum|b_n| &\leq f_+\\
\bar{f} - \sum|a_n| - \sum|b_n| &\geq f_-
\end{aligned}
$$

Too strong; change it to $f_- \leq f(x) \leq f_+$ at every x.

model problem with convolution:

$$I_2[f] = -2H[r] + H[A_r] \tag{25}$$

where $r(x) = \int_0^1 f(x-\theta)s(\theta)d\theta = \int_0^1 f(\theta)s(x-\theta)d\theta$.

$$\begin{aligned}
r(x) &= \bar{f} + \sum_{n=1}^{N} a_n \int_0^1 \cos(2\pi n(x-\theta))s(\theta)d\theta + \sum_{n=1}^{N} b_n \sin(2\pi n(x-\theta))s(\theta)d\theta \\
&= \bar{f} + \sum_{n=1}^{N} a_n \left[\cos(2\pi nx)\int \cos(2\pi n\theta)s(\theta)d\theta + \sin(2\pi nx)\int \sin(2\pi n\theta)s(\theta)d\theta\right] \\
&\quad + \sum_{n=1}^{N} b_n \left[\sin(2\pi nx)\int \cos(2\pi n\theta)s(\theta)d\theta - \cos(2\pi nx)\int \sin(2\pi n\theta)s(\theta)d\theta\right] \\
&= \bar{f} + \sum_{n=1}^{N} a_n \left[\cos(2\pi nx)\frac{a_n^{(s)}}{2} + \sin(2\pi nx)\frac{b_n^{(s)}}{2}\right] + \sum_{n=1}^{N} b_n \left[\sin(2\pi nx)\frac{a_n^{(s)}}{2} - \cos(2\pi nx)\frac{b_n^{(s)}}{2}\right] \\
&= \bar{f} + \sum_{n=1}^{N} \left(\frac{a_n^{(f)}a_n^{(s)} - b_n^{(f)}b_n^{(s)}}{2}\right)\cos(2\pi nx) + \sum_{n=1}^{N} \left(\frac{a_n^{(f)}b_n^{(s)} + b_n^{(f)}a_n^{(s)}}{2}\right)\sin(2\pi nx) \qquad (26) \\
A_r(x) &= \left(\bar{f}\right)^2 + \sum_{n=1}^{N} \frac{\left(a_n^{(r)}\right)^2 + \left(b_n^{(r)}\right)^2}{2}\cos(2\pi nx) \\
&= \left(\bar{f}\right)^2 + \sum_{n=1}^{N} \frac{\left(a_n^{(f)}a_n^{(s)} - b_n^{(f)}b_n^{(s)}\right)^2 + \left(a_n^{(f)}b_n^{(s)} + b_n^{(f)}a_n^{(s)}\right)^2}{8}\cos(2\pi nx) \\
&= \left(\bar{f}\right)^2 + \sum_{n=1}^{N} \frac{\left(a_n^{(f)}a_n^{(s)}\right)^2 + \left(b_n^{(f)}b_n^{(s)}\right)^2 + \left(a_n^{(f)}b_n^{(s)}\right)^2 + \left(b_n^{(f)}a_n^{(s)}\right)^2}{8}\cos(2\pi nx) \qquad (27) \\
&= \left(\bar{f}\right)^2 + \sum_{n=1}^{N} \frac{\left(\left(a_n^{(f)}\right)^2 + \left(b_n^{(f)}\right)^2\right)\left(\left(a_n^{(s)}\right)^2 + \left(b_n^{(s)}\right)^2\right)}{8}\cos(2\pi nx) \qquad (28) \\
\partial_{a_n^{(f)}} r(x) &= \frac{a_n^{(s)}\cos(2\pi nx) + b_n^{(s)}\sin(2\pi nx)}{2} \\
\partial_{b_n^{(f)}} r(x) &= \frac{b_n^{(s)}\cos(2\pi nx) + a_n^{(s)}\sin(2\pi nx)}{2} \\
\partial_{a_n^{(f)}} A_r(x) &= \frac{\left(a_n^{(f)}a_n^{(s)} + b_n^{(f)}b_n^{(s)}\right)a_n^{(s)} + \left(a_n^{(f)}b_n^{(s)} + b_n^{(f)}a_n^{(s)}\right)b_n^{(s)}}{4}\cos(2\pi nx) \\
&= \frac{a_n^{(r)}a_n^{(s)} + b_n^{(r)}b_n^{(s)}}{2}\cos(2\pi nx) \\
\partial_{b_n^{(f)}} A_r(x) &= \frac{a_n^{(r)}b_n^{(s)} + b_n^{(r)}a_n^{(s)}}{2}\cos(2\pi nx)
\end{aligned}$$

$$
\begin{aligned}
-2H[r] + H[A_r] &= 2\int r(x)\ln r(x)dx - \int A_r(x)\ln(A_r(x))dx \\
\partial_{a_n^{(f)}} I[f] &= 2\int (1+\ln r(x))\,\frac{a_n^{(s)}\cos(2\pi nx) + b_n^{(s)}\sin(2\pi nx)}{2}dx \\
&= -\int (1+\ln A_r(x))\,\frac{a_n^{(r)}a_n^{(s)} + b_n^{(r)}b_n^{(s)}}{2}\cos(2\pi nx)dx \\
\partial_{a_n^{(f)}} I[f] &= \int (\ln r(x))\left(a_n^{(s)}\cos(2\pi nx) + b_n^{(s)}\sin(2\pi nx)\right)dx \\
&\quad -\frac{1}{2}\int (\ln A_r(x))\left(a_n^{(r)}a_n^{(s)} + b_n^{(r)}b_n^{(s)}\right)\cos(2\pi nx)dx \\
\partial_{b_n^{(f)}} I[f] &= \int (\ln r(x))\left(b_n^{(s)}\cos(2\pi nx) + a_n^{(s)}\sin(2\pi nx)\right)dx \\
&\quad -\frac{1}{2}\int (\ln A_r(x))\left(a_n^{(r)}b_n^{(s)} + b_n^{(r)}a_n^{(s)}\right)\cos(2\pi nx)dx
\end{aligned}
$$

Gradients:

$$
\begin{aligned}
r(x) &= \bar{f} + \sum_{n=1}^{N} a_n \int_0^1 \cos(2\pi n(x-\theta))s(\theta)d\theta + \sum_{n=1}^{N} b_n \sin(2\pi n(x-\theta))s(\theta)d\theta \\
&= \bar{f} + \sum_{n=1}^{N} a_n \left[\cos(2\pi nx)\int \cos(2\pi n\theta)s(\theta)d\theta + \sin(2\pi nx)\int \sin(2\pi n\theta)s(\theta)d\theta\right] \\
&\quad + \sum_{n=1}^{N} b_n \left[\sin(2\pi nx)\int \cos(2\pi n\theta)s(\theta)d\theta - \cos(2\pi nx)\int \sin(2\pi n\theta)s(\theta)d\theta\right] \\
&= \bar{f} + \sum_{n=1}^{N} a_n \left[\cos(2\pi nx)\frac{a_n^{(s)}}{2} + \sin(2\pi nx)\frac{b_n^{(s)}}{2}\right] + \sum_{n=1}^{N} b_n \left[\sin(2\pi nx)\frac{a_n^{(s)}}{2} - \cos(2\pi nx)\frac{b_n^{(s)}}{2}\right] \\
&= \bar{f} + \sum_{n=1}^{N} \left(\frac{a_n^{(f)}a_n^{(s)} - b_n^{(f)}b_n^{(s)}}{2}\right)\cos(2\pi nx) + \sum_{n=1}^{N} \left(\frac{a_n^{(f)}b_n^{(s)} + b_n^{(f)}a_n^{(s)}}{2}\right)\sin(2\pi nx) \qquad (29) \\
A_r(x) &= \left(\bar{f}\right)^2 + \sum_{n=1}^{N} \frac{\left(a_n^{(r)}\right)^2 + \left(b_n^{(r)}\right)^2}{2}\cos(2\pi nx) \\
&= \left(\bar{f}\right)^2 + \sum_{n=1}^{N} \frac{\left(\frac{a_n^{(f)}a_n^{(s)} + b_n^{(f)}b_n^{(s)}}{2}\right)^2 + \left(\frac{a_n^{(f)}b_n^{(s)} + b_n^{(f)}a_n^{(s)}}{2}\right)^2}{2}\cos(2\pi nx) \\
&= \left(\bar{f}\right)^2 + \sum_{n=1}^{N} \frac{\left(a_n^{(f)}a_n^{(s)} - b_n^{(f)}b_n^{(s)}\right)^2 + \left(a_n^{(f)}b_n^{(s)} + b_n^{(f)}a_n^{(s)}\right)^2}{8}\cos(2\pi nx) \\
\partial_{a_n^{(f)}} r(x) &= \frac{a_n^{(s)}\cos(2\pi nx) + b_n^{(s)}\sin(2\pi nx)}{2} \\
\partial_{b_n^{(f)}} r(x) &= \frac{b_n^{(s)}\cos(2\pi nx) + a_n^{(s)}\sin(2\pi nx)}{2} \\
\partial_{a_n^{(f)}} A_r(x) &= \frac{\left(a_n^{(f)}a_n^{(s)} + b_n^{(f)}b_n^{(s)}\right)a_n^{(s)} + \left(a_n^{(f)}b_n^{(s)} + b_n^{(f)}a_n^{(s)}\right)b_n^{(s)}}{4}\cos(2\pi nx) \\
&= \frac{a_n^{(r)}a_n^{(s)} + b_n^{(r)}b_n^{(s)}}{2}\cos(2\pi nx) \\
\partial_{b_n^{(f)}} A_r(x) &= \frac{a_n^{(r)}b_n^{(s)} + b_n^{(r)}a_n^{(s)}}{2}\cos(2\pi nx)
\end{aligned}
$$

$$
\begin{aligned}
-2H[r] + H[A_r] &= 2\int r(x)\ln r(x)dx - \int A_r(x)\ln(A_r(x))dx \\
\partial_{a_n^{(f)}} I[f] &= 2\int (1+\ln r(x))\,\frac{a_n^{(s)}\cos(2\pi nx) + b_n^{(s)}\sin(2\pi nx)}{2}dx \\
&= -\int (1+\ln A_r(x))\,\frac{a_n^{(r)}a_n^{(s)} + b_n^{(r)}b_n^{(s)}}{2}\cos(2\pi nx)dx \\
\partial_{a_n^{(f)}} I[f] &= \int (\ln r(x))\left(a_n^{(s)}\cos(2\pi nx) + b_n^{(s)}\sin(2\pi nx)\right)dx \\
&\quad -\frac{1}{2}\int (\ln A_r(x))\left(a_n^{(r)}a_n^{(s)} + b_n^{(r)}b_n^{(s)}\right)\cos(2\pi nx)dx \\
\partial_{b_n^{(f)}} I[f] &= \int (\ln r(x))\left(b_n^{(s)}\cos(2\pi nx) + a_n^{(s)}\sin(2\pi nx)\right)dx \\
&\quad -\frac{1}{2}\int (\ln A_r(x))\left(a_n^{(r)}b_n^{(s)} + b_n^{(r)}a_n^{(s)}\right)\cos(2\pi nx)dx
\end{aligned}
$$

**

$Re[e^{2\pi inx}\int_0^1 e^{-2\pi in\theta}s(\theta)d\theta] = Re[(\cos(2\pi nx)+i\sin(2\pi nx))\int_0^1(\cos(2\pi n\theta)-i\sin(2\pi n\theta))s(\theta)d\theta] = \cos(2\pi nx)\int_0^1\cos(2\pi n\theta)s(\theta)d\theta + \sin(2\pi nx)\int\sin(2\pi n\theta)s(\theta)d\theta$

$\int_0^1\cos(2\pi n(x-\theta))s(\theta)d\theta = [\cos x\cos\theta + \sin x\sin\theta]\,s(\theta)$

$\int_0^1\sin(2\pi n(x-\theta))s(\theta)d\theta = [\sin x\cos\theta + \cos x\sin\theta]\,s(\theta)$

If $\bar{f}\neq 1$: $\tilde{r} = \frac{r}{\bar{f}}$,

$$\begin{aligned}
I(X,Y;\Theta) &= \int_0^1 dx\int_0^1 dy r(x)r(y)\ln\left(\frac{r(x)r(y)}{\int_0^1 d\theta r(x-\theta)r(y-\theta)}\right) \\
&= 2\bar{f}\int_0^1 dx r(x)\ln\left(r(x)\right) - \int_0^1 dx A_r(x)\ln\left(A_r(x)\right) \qquad (30) \\
&= 2(\bar{f})^2\int_0^1 dx\tilde{r}(x)\left(\ln\left(\tilde{r}(x)\right)+\ln\bar{f}\right) - \int_0^1 dx(\bar{f})^2A_{\tilde{r}}(x)\left[\ln\left(A_{\tilde{r}}(x)\right)+2\ln(\bar{f})\right] \\
&= (\bar{f})^2\left(-2H[\tilde{r}]+H[A_{\tilde{r}}]\right) \qquad (31)
\end{aligned}$$

$$\begin{aligned}
A_r(x) &= \int_0^1 d\theta r(\theta)r(x+\theta) \\
&= \int d\theta\int du\int dv f(\theta-u)s(u)f(x+\theta-v)s(v) \\
&= \int du\int dv s(u)s(v)\int d\theta f(\theta-u)f(x+\theta-v) \\
&= \int du\int dv s(u)s(v)A_f(x+u-v) \\
&= \text{'double convolution' of } A_f
\end{aligned}$$