Continuous limit

For M possible positions of the stimulus and M tuning curves centered on every position, the discrete version of the mutual information is given by

$$I[f] = \mathbb{E}\left[- \ln \left(\sum_m \prod_k \frac{f(k-m)}{f(k)} \right) \right] \tag{1}$$

where the expectation is with respect to the product of independent Poisson distributions

$$p(r_1, \dots r_M) = \prod_k \frac{f(k)^{r_k}}{r_k!} e^{-\sum_l f(l)} \,. \tag{2}$$

We want to consider our model in the continuous limit, i.e. for $M \to \infty$. For the mutual information to be finite, we need the expected number of spikes to remain finite when $M \to \infty$. A natural limit to consider is when the expected number of spikes per neuron scales as $1/M$ so that the total average number of spikes remains constant while $M \to \infty$. This corresponds to consider a sequence of tuning curves f_M satisfying the constraints:

$$\sum_{m=1}^{M} f_M(m) = \bar{f}/M \quad \text{and} \quad f_-/M \leq f_M(m) \leq f_+/M \,. \tag{3}$$

For all $0 \leq s < 1$, we assume that the limit

$$\lim_{M \to \infty} M f(\lfloor sM \rfloor) = f(s) \tag{4}$$

exists, thus defining a limit rate function f. In the limit $M \to \infty$, neurons become silent except for a finite number of cells, which spike at most once almost surely. These finite collection of neurons can be labelled by a continuous coordinate s indicating the position of their center on the circle. Moreover, by independence of our Poissonian model, the coordinates of the spiking neurons follows a Poisson process on the circle with rate f such that:

$$\int_0^1 f(s)\, ds = \bar{f} \quad \text{and} \quad f_- \leq f(s) \leq f_+ \,. \tag{5}$$

From there, in this continuous limit, the mutual information becomes an expectation with respect to that Poisson process. Specifically, we have

$$\begin{aligned} I[f] &= -\sum_n \frac{1}{n!} \int_0^1 \dots \int_0^1 \prod_i ds_i f(s_i) e^{-\int_0^1 d\theta f(s)\, ds} \ln \left(\frac{\int_0^1 d\theta \prod_i f(s_i - \theta)}{\prod_i f(s_i)} \right) , & (6) \\ &= -\mathbb{E}_N \left[\ln \left(\int_0^1 d\theta e^{\int_0^1 \ln\left(\frac{f(s-\theta)}{s}\right) dN(s)} \right) \right] , & (7) \end{aligned}$$

where N is a Poisson process over the circle with rate f. Computing the gradient of I with respect to f via functional calculus yields two terms, one of which should be zero by cyclic invariance. The remaining term is

$$\nabla_f I(t) = -\sum_n \frac{1}{n!} \sum_{j=1}^{n} \int_0^1 \ldots \int_0^1 \prod_{i=1}^{n} ds_i f(s_i) e^{-\int_0^1 d\theta f(s)\, ds} \left(\frac{\delta_t(s_j)}{f(t)} - 1 \right) \ln \left(\int_0^1 d\theta \prod_{i=1}^{n} \frac{f(s_i - \theta)}{f(s_i)} \right). \quad (8)$$

Integrating the Dirac delta functions shows that the gradient can be expressed as an expectation with respect to N:

$$\begin{aligned} \nabla_f I(t) - I[f] &= -\sum_n \frac{1}{(n-1)!} \int_0^1 \ldots \int_0^1 \prod_{i=1}^{n-1} ds_i f(s_i) e^{-\int_0^1 d\theta f(s)\, ds} \ln \left(\int_0^1 d\theta \frac{f(t-\theta)}{f(t)} \prod_{i=1}^{n-1} \frac{f(s_i - \theta)}{f(s_i)} \right) \quad (9) \\ &= -\mathbb{E}_N \left[\ln \left(\int_0^1 d\theta \frac{f(t-\theta)}{f(t)} e^{\int_0^1 \ln\left(\frac{f(s-\theta)}{s}\right) dN(s)} \right) \right]. \quad (10) \end{aligned}$$

Introducing the positive random variables $X_{\theta,N} = \exp\left(\int_0^1 \ln f(s-\theta)\, dN(s) \right)$, we finally have the expression

$$\nabla_f I(t) = -\mathbb{E}_N \left[\ln \left(\frac{\int_0^1 d\theta \frac{f(t-\theta)}{f(t)} X_{\theta,N}}{\int_0^1 d\theta X_{\theta,N}} \right) \right], \quad (11)$$

which readily shows that $\nabla_f I(t) \geq 0$ if $f(t) = f_+$ and $\nabla_f I(t) \leq 0$ if $f(t) = f_-$. This property might be all we need!