Calculations of gradient and Hessian for the continuous limit

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In Poissonian Limit,

$$\begin{aligned}
I[f] &= -E_N\left[\int_0^1 \prod_{i=1}^n \frac{f(s_i-\theta)}{f(s_i)}d\theta\right] && (1)\\
&= -\sum_n \frac{1}{n!}\int_0^1\cdots\int_0^1 ds_1\cdots ds_n \prod_{i=1}^n f(s_i)e^{-\int_0^1 f(s)ds}\ln\left(\int_0^1\prod_{i=1}^n \frac{f(s_i-\theta)}{f(s_i)}d\theta\right) && (2)\\
&= -\sum_n \frac{1}{n!}\int\cdots\int\prod ds_i P[f](s)\ln(W[f](s)) && (3)
\end{aligned}$$

where

$$\begin{aligned}
W[f](s) &:= \int_0^1 \prod_{i=1}^n \frac{f(s_i-\theta)}{f(s_i)}d\theta && (4)\\
P[f](s) &:= \prod_{i=1}^n f(s_i)e^{-\int_0^1 f(s)ds} && (5)
\end{aligned}$$

N is a Poisson Point Process over the circle with intensity f.

Gradient

We aim to show that

$$\frac{\delta I[f]}{\delta f}(t) = -E_N\left[\ln\left(\frac{\int_0^1 \frac{f(t-\theta)}{f(t)}X_{\theta,N}d\theta}{\int_0^1 X_{\theta,N}d\theta}\right)\right] \quad (6)$$

where $X_{\theta,N} = \exp\left[\int_0^1 \ln\left(\frac{f(s-\theta)}{f(s)}\right)dN(s)\right] = \prod_{i=1}^n \frac{f(s_i-\theta)}{f(s_i)}$.

By applying product rule, we can take derivatives of $P[f](s)$ and $W[f](s)$ separately:

$$\begin{aligned}
\frac{\delta I[f]}{\delta f}(t) &= -\sum_n \frac{1}{n!}\int\cdots\int\prod ds_i P[f](s)\frac{\delta(\ln W[f](s))}{\delta f}(t) - \sum_n \frac{1}{n!}\int\cdots\int\prod ds_i \frac{\delta(P[f](s))}{\delta f}(t)\ln W[f](s) && (7)\\
&:= D_1[f](t) + D_2[f](t)
\end{aligned}$$

We begin by showing that $D_1[f](t) = 0$:

Lemma 1.

$$D_1[f](t) = -\sum_n \frac{1}{n!}\int_0^1\cdots\int_0^1 ds_1\cdots ds_n P[f](s)\frac{\delta(\ln W[f](s))}{\delta f}(t) = 0 \quad (8)$$

Proof. In order to derive the derivative $\frac{\delta(\ln W[f](s))}{\delta f}$, we compute

$$\begin{aligned}
& \ln W[f+h](s) - \ln W[f](s) \\
= \; & \ln\left(\int_0^1 d\theta \frac{\prod_{i=1}^n (f(s_i-\theta)+h(s_i-\theta))}{\prod_{i=1}^n (f(s_i)+h(s_i))}\right) - \ln\left(\int_0^1 d\theta \frac{\prod_{i=1}^n f(s_i-\theta)}{\prod_{i=1}^n f(s_i)}\right) \\
= \; & \ln\left(\int d\theta \frac{\prod_i f(s_i-\theta) + \sum_{i=1}^n h(s_i-\theta)\prod_{j\neq i} f(s_j-\theta) + O(h^2)}{\prod_i f(s_i) + \sum_{i=1}^n h(s_i)\prod_{j\neq i} f(s_j) + O(h^2)}\right) - \ln\left(\int d\theta \frac{\prod_i f(s_i-\theta)}{\prod_i f(s_i)}\right) \\
= \; & \ln\left(\int d\theta \left(\frac{\prod_i f(s_i-\theta)}{\prod_i f(s_i)}\right) \frac{1+\sum_{i=1}^n \frac{h(s_i-\theta)}{f(s_i-\theta)} + O(h^2)}{1+\sum_{i=1}^n \frac{h(s_i)}{f(s_i)} + O(h^2)}\right) - \ln\left(\int d\theta \frac{\prod_i f(s_i-\theta)}{\prod_i f(s_i)}\right) \\
= \; & \ln\left(\int d\theta \left(\frac{\prod_i f(s_i-\theta)}{\prod_i f(s_i)}\right)\left(1+\frac{\sum_i \frac{h(s_i-\theta)}{f(s_i-\theta)} - \sum_i \frac{h(s_i)}{f(s_i)}}{1+\sum_{i=1}^n \frac{h(s_i)}{f(s_i)}} + O(h^2)\right)\right) - \ln\left(\int d\theta \frac{\prod_i f(s_i-\theta)}{\prod_i f(s_i)}\right) \\
= \; & \ln\left(\int d\theta \left(\frac{\prod_i f(s_i-\theta)}{\prod_i f(s_i)}\right)\left(1+\sum_{i=1}^n \left(\frac{h(s_i-\theta)}{f(s_i-\theta)} - \frac{h(s_i)}{f(s_i)}\right) + O(h^2)\right)\right) - \ln\left(\int d\theta \frac{\prod_i f(s_i-\theta)}{\prod_i f(s_i)}\right) \\
= \; & \frac{1}{\int_0^1 d\theta \frac{\prod_i f(s_i-\theta)}{\prod_i f(s_i)}} \cdot \int_0^1 \left(\frac{\prod_i f(s_i-\theta)}{\prod_i f(s_i)}\right) \sum_{j=1}^n \left(\frac{h(s_j-\theta)}{f(s_j-\theta)} - \frac{h(s_j)}{f(s_j)}\right) d\theta + O(h^2) \qquad (9) \\
= \; & \frac{\sum_{j=1}^n \int_0^1 \prod_i f(s_i-\theta)\frac{h(s_j-\theta)}{f(s_j-\theta)} d\theta}{\int_0^1 \prod_i f(s_i-\theta) d\theta} - \sum_{j=1}^n \frac{h(s_j)}{f(s_j)} + O(h^2) \\
= \; & \frac{\sum_{j=1}^n \int_0^1 \prod_i f(s_i-s_j+\theta')\frac{h(\theta')}{f(\theta')} d\theta'}{\int_0^1 \prod_i f(s_i-\theta) d\theta} - \sum_{j=1}^n \frac{h(s_j)}{f(s_j)} + O(h^2) \\
= \; & \sum_{j=1}^n \left[\frac{\int_0^1 \prod_i f(s_i-s_j+\theta')\frac{h(\theta')}{f(\theta')} d\theta'}{\int_0^1 \prod_i f(s_i-\theta) d\theta} - \int_0^1 \delta_{s_j}(\theta)\frac{h(\theta)}{f(\theta)} d\theta\right] + O(h^2) \qquad (10)
\end{aligned}$$

where δ is the Dirac Delta function, Equation (9) is obtained by the Taylor expansion $\ln(x+y) = \ln(x) + \frac{1}{x}y + O(y^2)$, and the last step follows from the change of variable: $\theta' = s_j - \theta$. Therefore,

$$\frac{\delta(\ln W[f](s))}{\delta f}(t) = \sum_{j=1}^n \left[\frac{\prod_i f(s_i-s_j+t)\frac{1}{f(t)}}{\int_0^1 \prod_i f(s_i-\theta)d\theta} - \frac{\delta_{s_j}(t)}{f(t)}\right] \qquad (11)$$

$$\begin{aligned}
D_1[f](t) &= -\sum_n \frac{1}{n!} \int \cdots \int \prod_{i=1}^n ds_i f(s_i) e^{-\int_0^1 f(s)ds} \sum_{j=1}^n \left[\frac{\prod_i f(s_i-s_j+t)\frac{1}{f(t)}}{\int_0^1 \prod_i f(s_i-\theta)d\theta} - \frac{\delta_{s_j}(t)}{f(t)}\right] \\
e^{-\int_0^1 f(s)ds} f(t)\cdot D_1[f](t) &= -\sum_n \frac{1}{n!} \int \cdots \int \prod_{i=1}^n ds_i f(s_i) \sum_{j=1}^n \left[\frac{\prod_i f(s_i-s_j+t)}{\int_0^1 \prod_i f(s_i-\theta)d\theta} - \delta_{s_j}(t)\right] \\
&= -\sum_n \frac{1}{n!} \sum_{j=1}^n \int \cdots \int \prod_{i=1}^n ds_i \frac{\left(\prod_i f(s_i-s_j+t) - \delta_{s_j}(t)\int_0^1 \prod_{i=1}^n f(s_i-\theta)d\theta\right)\prod_i f(s_i)}{\int_0^1 \prod_{i=1}^n f(s_i-\theta)d\theta}
\end{aligned}$$

Now we take a closer look at the n-dimensional integral in the above formula. By introducing a dirac delta function, we can write $\prod_i f(s_i - s_j + t)$ as:

$$\prod_i f(s_i - s_j + t) = \int_0^1 \prod_i f(s_i - \theta)\delta_{s_j - t}(\theta) d\theta,$$

Therefore we are able to express the numerator as the difference of integrals, and the conclusion follows from symmetry after integrating over $\prod s_i$:

$$
\begin{aligned}
& \int_0^1 \cdots \int_0^1 ds_1 \cdots ds_n \frac{\left(\prod_i f(s_i - s_j + t) - \delta_{s_j}(t) \int_0^1 \prod_{i=1}^n f(s_i - \theta) d\theta\right) \prod_i f(s_i)}{\int_0^1 \prod_{i=1}^n f(s_i - \theta) d\theta} \\
= & \int_0^1 \cdots \int_0^1 ds_1 \cdots ds_n \frac{\int_0^1 d\theta \left[\delta_{s_j - t}(\theta) - \delta_{s_j}(t)\right] \prod_i f(s_i - \theta) \prod_i f(s_i)}{\int_0^1 \prod_{i=1}^n f(s_i - \theta) d\theta} \\
= & \int_0^1 \cdots \int_0^1 ds_1 \cdots ds_n \frac{\int_0^1 d\theta \left[\delta_0(s_j - t - \theta) - \delta_0(s_j - t)\right] \prod_i f(s_i - \theta) \prod_i f(s_i)}{\int_0^1 \prod_{i=1}^n f(s_i - \theta) d\theta} \\
= & \int_0^1 \cdots \int_0^1 ds_1 \cdots ds_n \frac{\int_0^1 d\theta \left[\delta_0(s_j + u - t - \theta) - \delta_0(s_j + u - t)\right] \prod_i f(s_i + u - \theta) \prod_i f(s_i + u)}{\int_0^1 \prod_{i=1}^n f(s_i + u - \theta) d\theta} \qquad (12) \\
= & \int_0^1 du \int_0^1 \cdots \int_0^1 ds_1 \cdots ds_n \frac{\int_0^1 d\theta \left[\delta_0(s_j + u - t - \theta) - \delta_0(s_j + u - t)\right] \prod_i f(s_i + u - \theta) \prod_i f(s_i + u)}{\int_0^1 \prod_{i=1}^n f(s_i + u - \theta) d\theta} \\
= & \int_0^1 \cdots \int_0^1 ds_1 \cdots ds_n \frac{\int_0^1 du \int_0^1 d\theta \left[\delta_0(s_j + u - t - \theta) - \delta_0(s_j + u - t)\right] \prod_i f(s_i + u - \theta) \prod_i f(s_i + u)}{\int_0^1 \prod_{i=1}^n f(s_i - \theta) d\theta} \\
= & \int_0^1 \cdots \int_0^1 ds_1 \cdots ds_n \frac{\int_0^1 du \int_0^1 d\theta' \left[\delta_0(s_j + \theta' - t) - \delta_0(s_j + u - t)\right] \prod_i f(s_i + \theta') \prod_i f(s_i + u)}{\int_0^1 \prod_{i=1}^n f(s_i - \theta) d\theta} \\
= & \; 0
\end{aligned}
$$

Note that the equation (12) holds for any fixed u by applying the change of variables $s_i = s_i' + u$ for all i.
Since $e^{-\int_0^1 f(s) ds} f(t)$ is a constant, $D_1[f](t) = 0$. $\square$

Now take derivative of P in equation (7),

$$
\begin{aligned}
P[f+h](s) - P[f](s) &= \prod_{i=1}^n \left(f(s_i) + h(s_i)\right) e^{-\int_0^1 f(s) + h(s) ds} - \prod_{i=1}^n f(s_i) e^{-\int_0^1 f(s) ds} \\
&= \left(\prod_{i=1}^n f(s_i) + \sum_{i=1}^n h(s_i) \prod_{j \neq i} f(s_j)\right) e^{-\int_0^1 f(s) ds} \left(1 - \int_0^1 h(s) ds + O(h^2)\right) - \prod_{i=1}^n f(s_i) e^{-\int_0^1 f(s) ds} \\
&= \left(-\prod_{i=1}^n f(s_i) \int_0^1 h(s) ds + \prod_j f(s_j) \sum_{i=1}^n \frac{h(s_i)}{f(s_i)}\right) e^{-\int_0^1 f(s) ds} + O(h^2)
\end{aligned}
$$

Thus

$$
\frac{\delta(P[f](s))}{\delta f}(t) = \prod_{i=1}^n f(s_i) e^{-\int_0^1 f(s) ds} \left(\sum_{j=1}^n \frac{\delta_{s_j}(t)}{f(t)} - 1\right) \qquad (13)
$$

$$
\begin{aligned}
\frac{\delta I[f]}{\delta f}(t) = D_2[f](t) &= -\sum_n \frac{1}{n!} \int \cdots \int \prod_{i=1}^n ds_i \frac{\delta(P[f](s))}{\delta f}(t) \ln W[f](s) \\
&= -\sum_n \frac{1}{n!} \int \cdots \int \prod_{i=1}^n ds_i f(s_i) e^{-\int_0^1 f(s) ds} \left(\sum_{j=1}^n \frac{\delta_{s_j}(t)}{f(t)} - 1\right) \ln W[f](s) \\
&= -E_N \left[\left(\sum_{j=1}^n \frac{\delta_{s_j}(t)}{f(t)} - 1\right) \ln \left(\int_0^1 \prod_{i=1}^n \frac{f(s_i - \theta)}{f(s_i)} d\theta\right)\right] \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta I[f]}{\delta f}(t)+I[f] &= -E_N\left[\left(\sum_{j=1}^{n}\frac{\delta_{s_j}(t)}{f(t)}-1\right)\ln\left(\int_0^1\prod_{i=1}^{n}\frac{f(s_i-\theta)}{f(s_i)}d\theta\right)\right]+\left(-E_N\left[\ln\left(\int_0^1\prod_{i=1}^{n}\frac{f(s_i-\theta)}{f(s_i)}d\theta\right)\right]\right)\\
&= -E_N\left[\sum_{j=1}^{n}\frac{\delta_{s_j}(t)}{f(t)}\ln\left(\int_0^1\prod_{i=1}^{n}\frac{f(s_i-\theta)}{f(s_i)}d\theta\right)\right]\\
&= -\sum_n\frac{1}{n!}\sum_{j=1}^{n}\int\cdots\int\prod ds_i f(s_i)e^{-\int_0^1 f(s)ds}\frac{\delta_t(s_j)}{f(t)}\ln\left(\int_0^1\prod_{i=1}^{n}\frac{f(s_i-\theta)}{f(s_i)}d\theta\right)\\
&= -\sum_n\frac{1}{n!}\sum_{j=1}^{n}\int\cdots\int\prod_{i\neq j} ds_i f(s_i)e^{-\int_0^1 f(s)ds}\ln\left(\int_0^1\prod_{i\neq j}\frac{f(s_i-\theta)}{f(s_i)}\frac{f(t-\theta)}{f(t)}d\theta\right)\\
&= -\sum_n\frac{1}{(n-1)!}\int\cdots\int\prod_{i=1}^{n-1} ds_i f(s_i)e^{-\int_0^1 f(s)ds}\ln\left(\int_0^1\prod_{i=1}^{n-1}\frac{f(s_i-\theta)}{f(s_i)}\frac{f(t-\theta)}{f(t)}d\theta\right)\\
&= -E_N\left[\ln\left(\int_0^1\frac{f(t-\theta)}{f(t)}\prod_{i=1}^{n}\frac{f(s_i-\theta)}{f(s_i)}d\theta\right)\right]\\
&= -E_N\left[\ln\left(\int_0^1\frac{f(t-\theta)}{f(t)}e^{\sum\ln\frac{f(s_i-\theta)}{f(s_i)}}d\theta\right)\right]\\
&= -E_N\left[\ln\left(\int_0^1\frac{f(t-\theta)}{f(t)}e^{\int_0^1\ln\left(\frac{f(s-\theta)}{f(s)}\right)dN(s)}d\theta\right)\right]
\end{aligned}
$$

Introducing the random variable $X_{\theta,N}=\exp\left[\int_0^1\ln\left(\frac{f(s-\theta)}{f(s)}\right)dN(s)\right]=\prod_{i=1}^{n}\frac{f(s_i-\theta)}{f(s_i)}$,

$$
\begin{aligned}
\frac{\delta I[f]}{\delta f}(t) &= -E_N\left[\ln\left(\int_0^1\frac{f(t-\theta)}{f(t)}X_{\theta,N}d\theta\right)\right]-I[f]\\
&= -E_N\left[\ln\left(\int_0^1\frac{f(t-\theta)}{f(t)}X_{\theta,N}d\theta\right)\right]+E_N\left[\ln\left(\int_0^1X_{\theta,N}d\theta\right)\right]\\
&= -E_N\left[\ln\left(\frac{\int_0^1\frac{f(t-\theta)}{f(t)}X_{\theta,N}d\theta}{\int_0^1X_{\theta,N}d\theta}\right)\right] \qquad (15)
\end{aligned}
$$

Hessian

$$
\begin{aligned}
\frac{\delta I[f]}{\delta f}(t) &= -\sum_n\frac{1}{n!}\int_0^1\cdots\int_0^1\prod_{i=1}^{n}ds_i\frac{\delta(P[f](s))}{\delta f}(t)\ln\left(W[f](s)\right)\\
&= -\sum_n\frac{1}{n!}\int\cdots\int\prod ds_i P[f](s)\left(\frac{\sum_{j=1}^{n}\delta_{s_j}(t)}{f(t)}-1\right)\ln\left(W[f](s)\right)\\
\frac{\delta^2 I[f]}{\delta f^2}(u,v) &= -\sum_n\frac{1}{n!}\int\cdots\int\prod ds_i\frac{\delta}{\delta f}\left[P[f](s)\left(\frac{\sum_{j=1}^{n}\delta_{s_j}(u)}{f(u)}-1\right)\right](v)\ln\left(W[f](s)\right)\\
&\quad -\sum_n\frac{1}{n!}\int\cdots\int\prod ds_i P[f](s)\left(\frac{\sum_{j=1}^{n}\delta_{s_j}(u)}{f(u)}-1\right)\frac{\delta}{\delta f}\left(\ln\left(W[f](s)\right)\right)(v)
\end{aligned}
$$

The first term:

$$
\begin{aligned}
\frac{\delta}{\delta f}\left[P[f](s)\left(\frac{\sum_{j=1}^{n}\delta_{s_j}(u)}{f(u)}-1\right)\right](v) &= \frac{\delta\left(P[f](s)\right)}{\delta f}(v)\left(\frac{\sum_{j=1}^{n}\delta_{s_j}(u)}{f(u)}-1\right)+P[f](s)\frac{\delta}{\delta f}\left(\frac{\sum_{j=1}^{n}\delta_{s_j}(u)}{f(u)}-1\right)(v) \\
&= P[f](s)\left(\frac{\sum_{j=1}^{n}\delta_{s_j}(u)}{f(u)}-1\right)\left(\frac{\sum_{l=1}^{n}\delta_{s_l}(v)}{f(v)}-1\right) \\
&\quad +P[f](s)\left(-\delta_u(v)\frac{\sum_{j=1}^{n}\delta_{s_j}(u)}{f(u)^2}\right) \\
&= P[f](s)\left[\left(\frac{\sum_{j=1}^{n}\delta_{s_j}(u)}{f(u)}-1\right)\left(\frac{\sum_{l=1}^{n}\delta_{s_l}(v)}{f(v)}-1\right)-\delta_u(v)\frac{\sum_{j=1}^{n}\delta_{s_j}(u)}{f(u)^2}\right] \qquad (16)
\end{aligned}
$$

Compute saparately,

$$
\begin{aligned}
& \sum_n \frac{1}{n!}\int\cdots\int\prod ds_i P[f](s)\frac{\sum_{j=1}^{n}\delta_{s_j}(u)}{f(u)}\ln\left(W[f](s)\right) \\
=\ & \sum_n \frac{1}{n!}\sum_{j=1}^{n}\int\cdots\int\prod_{i\neq j} ds_i e^{-\int_0^1 f(s)ds}\ln\left(\int_0^1\prod_{i\neq j}\frac{f(s_i-\theta)}{f(s_i)}\frac{f(u-\theta)}{f(u)}d\theta\right) \\
=\ & \sum_n \frac{1}{(n-1)!}\int\cdots\int\prod_{i=1}^{n-1} ds_i e^{-\int_0^1 f(s)ds}\ln\left(\int_0^1\prod_{i=1}^{n-1}\frac{f(s_i-\theta)}{f(s_i)}\frac{f(u-\theta)}{f(u)}d\theta\right) \\
=\ & E_N\left[\ln\left(\int_0^1\prod_{i=1}^{n}\frac{f(s_i-\theta)}{f(s_i)}\frac{f(u-\theta)}{f(u)}d\theta\right)\right] \\
=\ & E_N\left[\ln\left(W[f](s_1,...,s_n,u)\right)\right] \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
& \sum_n \frac{1}{n!}\int\cdots\int\prod ds_i P[f](s)\frac{\sum_{j=1}^{n}\delta_{s_j}(u)}{f(u)}\frac{\sum_{l=1}^{n}\delta_{s_l}(v)}{f(v)}\ln\left(W[f](s)\right) \\
=\ & \sum_n \frac{1}{n!}\sum_{j=1}^{n}\int\cdots\int\prod_{i\neq j} ds_i f(s_i)e^{-\int_0^1 f(s)ds}\frac{\sum_{l\neq j}\delta_{s_l}(v)+\delta_{s_j}(v)}{f(v)}\ln\left(\int_0^1\prod_{i\neq j}\frac{f(s_i-\theta)}{f(s_i)}\frac{f(u-\theta)}{f(u)}d\theta\right) \\
=\ & \sum_n \frac{1}{(n-1)!}\int\cdots\int\prod_{i=1}^{n-1} ds_i f(s_i)e^{-\int_0^1 f(s)ds}\frac{\sum_{l=1}^{n-1}\delta_{s_l}(v)}{f(v)}\ln\left(\int_0^1\prod_{i=1}^{n-1}\frac{f(s_i-\theta)}{f(s_i)}\frac{f(u-\theta)}{f(u)}d\theta\right) \\
& +\frac{\delta_u(v)}{f(v)}\sum_n \frac{1}{n!}\sum_{j=1}^{n}\int\cdots\int\prod_{i\neq j} ds_i f(s_i)e^{-\int_0^1 f(s)ds}\ln\left(\int_0^1\prod_{i\neq j}\frac{f(s_i-\theta)}{f(s_i)}\frac{f(u-\theta)}{f(u)}d\theta\right) \\
=\ & E_N\left[\ln\left(\int_0^1\prod_{i=1}^{n}\frac{f(s_i-\theta)}{f(s_i)}\frac{f(u-\theta)}{f(u)}\frac{f(v-\theta)}{f(v)}d\theta\right)\right]+\frac{\delta_u(v)}{f(v)}E_N\left[\ln\left(\int_0^1\prod_{i=1}^{n}\frac{f(s_i-\theta)}{f(s_i)}\frac{f(u-\theta)}{f(u)}d\theta\right)\right] \\
=\ & E_N\left[\ln\left(W[f](s_1,...,s_n,u,v)\right)\right]+\frac{\delta_u(v)}{f(v)}E_N\left[\ln\left(W[f](s_1,...,s_n,u)\right)\right] \qquad (18)
\end{aligned}
$$

Combined together,

$$
\sum_n \frac{1}{n!}\int\cdots\int\prod ds_i \frac{\delta}{\delta f}\left[P[f](s)\left(\frac{\sum_{j=1}^{n}\delta_{s_j}(u)}{f(u)}-1\right)\right](v)\ln\left(W[f](s)\right)
$$

$$
\begin{aligned}
&= \sum_n \frac{1}{n!}\int\cdots\int\prod ds_i P[f](s)\ln\left(W[f](s)\right)\left[\frac{\sum_{j=1}^n \delta_{s_j}(u)}{f(u)}\frac{\sum_{l=1}^n \delta_{s_l}(v)}{f(v)} - \frac{\sum_{l=1}^n \delta_{s_l}(u)}{f(u)} - \frac{\sum_{l=1}^n \delta_{s_l}(v)}{f(v)} + 1 - \delta_u(v)\frac{\sum_{j=1}^n \delta_{s_j}(u)}{f(u)^2}\right] \\
&= E_N\left[\ln\left(W[f](s_1,...,s_n,u,v)\right)\right] + \frac{\delta_u(v)}{f(v)}E_N\left[\ln\left(W[f](s_1,...,s_n,u)\right)\right] \\
&\quad -E_N\left[\ln\left(W[f](s_1,...,s_n,u)\right)\right] - E_N\left[\ln\left(W[f](s_1,...,s_n,v)\right)\right] + E_N\left[\ln\left(W[f](s_1,...,s_n)\right)\right] \\
&\quad -\frac{\delta_u(v)}{f(u)}E_N\left[\ln\left(W[f](s_1,...,s_n,u)\right)\right] \\
&= E_N\left[\ln\left(W[f](s_1,...,s_n,u,v)\right) - \ln\left(W[f](s_1,...,s_n,u)\right) - \ln\left(W[f](s_1,...,s_n,v)\right) + \ln\left(W[f](s_1,...,s_n)\right)\right] \qquad (19) \\
&= E_N\left[\ln\left(\int_0^1 \prod_{i=1}^n \frac{f(s_i-\theta)}{f(s_i)}\frac{f(u-\theta)}{f(u)}\frac{f(v-\theta)}{f(v)}d\theta\right) + \ln\left(\int_0^1 \prod_{i=1}^n \frac{f(s_i-\theta)}{f(s_i)}d\theta\right)\right] \\
&\quad -E_N\left[\ln\left(\int_0^1 \prod_{i=1}^n \frac{f(s_i-\theta)}{f(s_i)}\frac{f(u-\theta)}{f(u)}d\theta\right) + \ln\left(\int_0^1 \prod_{i=1}^n \frac{f(s_i-\theta)}{f(s_i)}\frac{f(v-\theta)}{f(v)}d\theta\right)\right] \\
&= E_N\left[\ln\left(\int_0^1 \frac{f(u-\theta)}{f(u)}\frac{f(v-\theta)}{f(v)}X_{\theta,N}d\theta\right) + \ln\left(\int_0^1 X_{\theta,N}d\theta\right) - \ln\left(\int_0^1 \frac{f(u-\theta)}{f(u)}X_{\theta,N}d\theta\right) - \ln\left(\int_0^1 \frac{f(v-\theta)}{f(v)}X_{\theta,N}d\theta\right)\right] \qquad (20) \\
&= E_N\left[\ln\left(\frac{\int_0^1 \frac{f(u-\theta)}{f(u)}\frac{f(v-\theta)}{f(v)}X_{\theta,N}d\theta\cdot\int_0^1 X_{\theta,N}d\theta}{\int_0^1 \frac{f(u-\theta)}{f(u)}X_{\theta,N}d\theta\cdot\int_0^1 \frac{f(v-\theta)}{f(v)}X_{\theta,N}d\theta}\right)\right] \qquad (21)
\end{aligned}
$$

where $X_{\theta,N} = \prod_{i=1}^n \frac{f(s_i-\theta)}{f(s_i)} = e^{\int_0^1 \frac{f(s-\theta)}{f(s)}dN(s)}$.

For the second term, use previous conclusion (11) and apply the same trick,

$$
\frac{\delta(\ln W[f](s))}{\delta f}(v) = \frac{1}{f(v)}\left[\frac{\sum_{j=1}^n \prod_i f(s_i - s_j + v)}{\int_0^1 \prod_i f(s_i-\theta)d\theta} - \sum_{j=1}^n \delta_{s_j}(v)\right]
$$

$$
\begin{aligned}
& f(u)f(v)\sum_n \frac{1}{n!}\int\cdots\int\prod ds_i P[f](s)\left(\frac{\sum_{j=1}^n \delta_{s_j}(u)}{f(u)} - 1\right)\frac{\delta}{\delta f}\left(\ln\left(W[f](s)\right)\right)(v) \\
&= \sum_n \frac{1}{n!}\int\cdots\int\prod ds_i P[f](s)\left(\sum_{j=1}^n \delta_{s_j}(u) - f(u)\right)\left(\frac{\sum_{l=1}^n \prod_i f(s_i - s_l + v)}{\int_0^1 \prod_i f(s_i-\theta)d\theta} - \sum_{l=1}^n \delta_{s_l}(v)\right) \\
&= \sum_n \frac{1}{n!}\int\cdots\int\prod ds_i P[f](s)\left(\sum_{j=1}^n \delta_{s_j}(u) - f(u)\right)\frac{\sum_{l=1}^n \int_0^1 d\theta\left(\delta_{s_l - v}(\theta) - \delta_{s_l}(v)\right)\prod_i f(s_i-\theta)}{\int_0^1 \prod_i f(s_i-\theta)d\theta} \\
&= \sum_n \frac{1}{n!}\int\cdots\int\prod ds_i e^{-\int_0^1 f(s)ds}\left(\sum_{j=1}^n \delta_{s_j}(u) - f(u)\right)\frac{\sum_{l=1}^n \int_0^1 d\theta\left(\delta_{s_l - v}(\theta) - \delta_{s_l}(v)\right)\prod_i f(s_i-\theta)\prod_i f(s_i)}{\int_0^1 \prod_i f(s_i-\theta)d\theta} \\
&= \sum_n \frac{1}{n!}\int\cdots\int\prod ds_i e^{-\int_0^1 f(s)ds}\frac{\sum_{j=1}^n\sum_{l=1}^n \int_0^1 d\theta\delta(u-s_j)\left(\delta(v-s_l+\theta) - \delta(v-s_l)\right)\prod_i f(s_i-\theta)\prod_i f(s_i)}{\int_0^1 \prod_i f(s_i-\theta)d\theta} \qquad (22)
\end{aligned}
$$

where the $f(u)$ term is zero from the proof of Lemma1. Aiming to simplify equation (22), we show that the integral can be averaged by

$$
\begin{aligned}
& \int\cdots\int\prod ds_i e^{-\int_0^1 f(s)ds}\frac{\int_0^1 d\theta\delta(u-s_j)\left(\delta(v-s_l+\theta) - \delta(v-s_l)\right)\prod_i f(s_i-\theta)\prod_i f(s_i)}{\int_0^1 \prod_i f(s_i-\theta)d\theta} \\
&= \int\cdots\int\prod ds_i' e^{-\int f(s)ds}\frac{\int_0^1 d\theta\delta(u-s_j'-\theta)\left(\delta(v-s_l') - \delta(v-s_l'-\theta)\right)\prod_i f(s_i')\prod_i f(s_i'+\theta)}{\int_0^1 \prod_i f(s_i'+\theta-\gamma)d\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&= \int\cdots\int\prod ds_i' e^{-\int f(s)ds}\frac{\int_0^1 d\theta'\delta(u-s_j'+\theta')\left(\delta(v-s_l')-\delta(v-s_l'+\theta')\right)\prod_i f(s_i')\prod_i f(s_i'-\theta')}{\int_0^1\prod_i f(s_i'-\gamma)d\gamma} \\
&= \int\cdots\int\prod ds_i P[f](s)\frac{\int_0^1 d\theta\delta(u-s_j+\theta)\left(\delta(v-s_l)-\delta(v-s_l+\theta)\right)\prod_i f(s_i-\theta)}{\int_0^1\prod_i f(s_i-\gamma)d\gamma} \qquad (23)
\end{aligned}
$$

(22)+(23):

$$
\begin{aligned}
& 2f(u)f(v)\sum_n\frac{1}{n!}\int\cdots\int\prod ds_i P[f](s)\left(\frac{\sum_{j=1}^n\delta_{s_j}(u)}{f(u)}-1\right)\frac{\delta}{\delta f}\left(\ln\left(W[f](s)\right)\right)(v) \\
= \;& 2f(u)f(v)\sum_n\frac{1}{n!}\int\cdots\int\prod ds_i P[f](s)\frac{-\sum_j\sum_l\int d\theta\left(\delta(u-s_j+\theta)-\delta(u-s_j)\right)\left(\delta(v-s_l+\theta)-\delta(v-s_l)\right)\prod_i f(s_i-\theta)}{\int_0^1\prod_i f(s_i-\theta)d\theta} \\
= \;& -2f(u)f(v)E_N\left[\frac{\sum_{j=1}^n\sum_{l=1}^n\int_0^1 d\theta\left(\delta(u-s_j+\theta)-\delta(u-s_j)\right)\left(\delta(v-s_l+\theta)-\delta(v-s_l)\right)\prod_i f(s_i-\theta)}{\int_0^1\prod_i f(s_i-\theta)d\theta}\right] \qquad (24)
\end{aligned}
$$

Combining the two terms:

$$
\begin{aligned}
\frac{\delta^2 I[f]}{\delta f^2}(u,v) &= -\sum_n\frac{1}{n!}\int\cdots\int\prod ds_i\frac{\delta}{\delta f}\left[P[f](s)\left(\frac{\sum_{j=1}^n\delta_{s_j}(u)}{f(u)}-1\right)\right](v)\ln\left(W[f](s)\right) \\
&\quad -\sum_n\frac{1}{n!}\int\cdots\int\prod ds_i P[f](s)\left(\frac{\sum_{j=1}^n\delta_{s_j}(u)}{f(u)}-1\right)\frac{\delta}{\delta f}\left(\ln\left(W[f](s)\right)\right)(v) \\
&= -E_N\left[\ln\left(\frac{\int_0^1\frac{f(u-\theta)}{f(u)}\frac{f(v-\theta)}{f(v)}X_{\theta,N}d\theta\cdot\int_0^1 X_{\theta,N}d\theta}{\int_0^1\frac{f(u-\theta)}{f(u)}X_{\theta,N}d\theta\cdot\int_0^1\frac{f(v-\theta)}{f(v)}X_{\theta,N}d\theta}\right)\right] \\
&\quad +\frac{1}{f(u)f(v)}E_N\left[\frac{\sum_{j=1}^n\sum_{l=1}^n\int_0^1 d\theta\delta(u-s_j)\left(\delta(v-s_l+\theta)-\delta(v-s_l)\right)\prod_i f(s_i-\theta)}{\int_0^1\prod_i f(s_i-\theta)d\theta}\right] \qquad (25) \\
&= -E_N\left[\ln\left(\frac{\int_0^1\frac{f(u-\theta)}{f(u)}\frac{f(v-\theta)}{f(v)}X_{\theta,N}d\theta\cdot\int_0^1 X_{\theta,N}d\theta}{\int_0^1\frac{f(u-\theta)}{f(u)}X_{\theta,N}d\theta\cdot\int_0^1\frac{f(v-\theta)}{f(v)}X_{\theta,N}d\theta}\right)\right] \\
&\quad +\frac{1}{2f(u)f(v)}E_N\left[\frac{\sum_j^n\sum_l^n\int d\theta\left(\delta(u-s_j+\theta)-\delta(u-s_j)\right)\left(\delta(v-s_l+\theta)-\delta(v-s_l)\right)\prod_i f(s_i-\theta)}{\int_0^1\prod_i f(s_i-\theta)d\theta}\right] \qquad (26)
\end{aligned}
$$

where $X_{\theta,N}=\prod_{i=1}^n\frac{f(s_i-\theta)}{f(s_i)}=e^{\int_0^1\frac{f(s-\theta)}{f(s)}dN(s)}$.