With k input positions:

Firing rate at x_i: $\sum_{l=1}^{k} f(\theta_l - x_i)$

$$\int \cdots \int \prod_{l=1}^{k} ds_l \sum_{\mathbf{c}_1 \in \mathbb{N}^n} \sum_{\mathbf{c}_2 \in \mathbb{N}^n} \prod_{i,j=1}^{n} \frac{(d_1 \sum_{l=1}^{k} f_1(s_l - \frac{i}{n}))^{c_{1,i}}}{c_{1,i}!} \frac{(d_2 \sum_{l=1}^{k} f_2(s_l - \frac{j}{n}))^{c_{2,i}}}{c_{2,i}!} e^{-d_1 \sum_i \sum_l f_1(s_l - \frac{i}{n}) - d_2 \sum_j \sum_l f_2(s_l - \frac{j}{n})}$$
$$\ln\left(\frac{\prod_{i,j=1}^{n} (d_1 \sum_{l=1}^{k} f_1(s_l - \frac{i}{n}))^{c_{1,i}} (d_2 \sum_{l=1}^{k} f_2(s_l - \frac{j}{n}))^{c_{2,i}}}{\int \cdots \int \prod_{l=1}^{k} ds_l \prod_{i,j=1}^{n} (d_1 \sum_{l=1}^{k} f_1(s_l - \frac{i}{n}))^{c_{1,i}} (d_2 \sum_{l=1}^{k} f_2(s_l - \frac{j}{n}))^{c_{2,i}}}\right)$$

Poissonian Limit:
Normalize $\tilde{f}_1 = \frac{f_1}{\bar{f}_1}$, $\tilde{f}_2 = \frac{f_2}{\bar{f}_2}$,

$$\begin{aligned}
&\sum_{n_1,n_2} \frac{1}{n_1! n_2!} \int \cdots \int \prod_{l=1}^{k} ds_l \prod_{i=1}^{n_1} dx_i \prod_{j=1}^{n_2} dy_j \Bigg\{ \prod_{i,j} d_1 \left(\sum_l f_1(s_l - x_i) \right) d_2 \left(\sum_l f_2(s_l - y_j) \right) e^{-k\left(d_1 \bar{f}_1 + d_2 \bar{f}_2\right)} \\
&\ln\left(\frac{\prod_{i,j} (d_1 \left(\sum_l f_1(s_l - x_i)\right) d_2 \left(\sum_l f_2(s_l - y_j)\right))}{\int \cdots \int \prod_{l=1}^{k} ds'_l \prod_{i,j} (d_1 \left(\sum_l f_1(s'_l - x_i)\right) d_2 \left(\sum_l f_2(s'_l - y_j)\right))}\right)\Bigg\} \\
= \; & e^{-k\left(d_1 \bar{f}_1 + d_2 \bar{f}_2\right)} \sum_{n_1,n_2} \frac{\left(d_1 \bar{f}_1\right)^{n_1} \left(d_2 \bar{f}_2\right)^{n_2}}{n_1! n_2!} \int \cdots \int \prod_{l=1}^{k} ds_l \prod_{i=1}^{n_1} dx_i \prod_{j=1}^{n_2} dy_j \Bigg\{ \prod_{i,j} \left(\sum_l \tilde{f}_1(s_l - x_i) \right) \left(\sum_l \tilde{f}_2(s_l - y_j) \right) \\
&\ln\left(\frac{\prod_{i,j} \left(\sum_l \tilde{f}_1(s_l - x_i)\right) \left(\sum_l \tilde{f}_2(s_l - y_j)\right)}{\int \cdots \int \prod_{l=1}^{k} ds'_l \prod_{i,j} \left(\sum_l \tilde{f}_1(s'_l - x_i)\right) \left(\sum_l \tilde{f}_2(s'_l - y_j)\right)}\right)\Bigg\} \\
= \; & e^{-k\left(d_1 \bar{f}_1 + d_2 \bar{f}_2\right)} \sum_{n_1,n_2} \frac{\left(d_1 \bar{f}_1\right)^{n_1} \left(d_2 \bar{f}_2\right)^{n_2}}{n_1! n_2!} \int \cdots \int \prod_{l=1}^{k} ds_l \prod_{i=1}^{n_1} dx_i \left(\sum_l \tilde{f}_1(s_l - x_i) \right) (k^{n_2}) \ln\left(\prod_{i=1}^{n_1} \left(\sum_l \tilde{f}_1(s_l - x_i) \right) \right) \\
&+ e^{-k\left(d_1 \bar{f}_1 + d_2 \bar{f}_2\right)} \sum_{n_1,n_2} \frac{\left(d_1 \bar{f}_1\right)^{n_1} \left(d_2 \bar{f}_2\right)^{n_2}}{n_1! n_2!} \int \cdots \int \prod_{l=1}^{k} ds_l \prod_{j=1}^{n_2} dy_j \, (k^{n_1}) \left(\sum_l \tilde{f}_2(s_l - y_j) \right) \ln\left(\prod_{j=1}^{n_2} \left(\sum_l \tilde{f}_2(s_l - y_j) \right) \right) \\
&- e^{-k\left(d_1 \bar{f}_1 + d_2 \bar{f}_2\right)} \sum_{n_1,n_2} \frac{\left(d_1 \bar{f}_1\right)^{n_1} \left(d_2 \bar{f}_2\right)^{n_2}}{n_1! n_2!} \int \cdots \int \prod_{i=1}^{n_1} dx_i \prod_{j=1}^{n_2} dy_j A(x_1, ..., x_{n_1}, y_1, ..., y_{n_2}) \ln\left(A(x_1, ..., x_{n_1}, y_1, ..., y_{n_2})\right) \\
= \; & e^{-k d_1 \bar{f}_1} \sum_{n_1} \frac{\left(d_1 \bar{f}_1\right)^{n_1}}{n_1!} \int \cdots \int \prod_{l=1}^{k} ds_l \prod_{i=1}^{n_1} dx_i \left(\sum_l \tilde{f}_1(s_l - x_i) \right) \ln\left(\prod_{i=1}^{n_1} \left(\sum_l \tilde{f}_1(s_l - x_i) \right) \right) \\
&+ e^{-k d_2 \bar{f}_2} \sum_{n_2} \frac{\left(d_2 \bar{f}_2\right)^{n_2}}{n_2!} \int \cdots \int \prod_{l=1}^{k} ds_l \prod_{j=1}^{n_2} dy_j \left(\sum_l \tilde{f}_2(s_l - y_j) \right) \ln\left(\prod_{j=1}^{n_2} \left(\sum_l \tilde{f}_2(s_l - y_j) \right) \right) \\
&- e^{-k\left(d_1 \bar{f}_1 + d_2 \bar{f}_2\right)} \sum_{n_1,n_2} \frac{\left(d_1 \bar{f}_1\right)^{n_1} \left(d_2 \bar{f}_2\right)^{n_2}}{n_1! n_2!} \int \cdots \int \prod_{i=1}^{n_1} dx_i \prod_{j=1}^{n_2} dy_j A(x_1, ..., x_{n_1}, y_1, ..., y_{n_2}) \ln\left(A(x_1, ..., x_{n_1}, y_1, ..., y_{n_2})\right)
\end{aligned}$$

where

$$A(x_1, ..., x_{n_1}, y_1, ..., y_{n_2}) \coloneqq \int \cdots \int \prod_{l=1}^{k} ds'_l \prod_{i,j} \left(\sum_l \tilde{f}_1(s'_l - x_i) \right) \left(\sum_l \tilde{f}_2(s'_l - y_j) \right)$$

and

$$\int \cdots \int \prod_{i=1}^{n_1} dx_i \prod_{j=1}^{n_2} dy_j A(x_1, ..., x_{n_1}, y_1, ..., y_{n_2}) = k^{n_1+n_2}.$$

Normalize A such that :

$$
\begin{aligned}
& e^{-k\left(d_1\bar{f}_1+d_2\bar{f}_2\right)}\sum_{n_1,n_2}\frac{\left(d_1\bar{f}_1\right)^{n_1}\left(d_2\bar{f}_2\right)^{n_2}}{n_1!n_2!}\int\cdots\int\prod_{i=1}^{n_1}dx_i\prod_{j=1}^{n_2}dy_j k^{n_1+n_2}\tilde{A}(x_1,...,x_{n_1},y_1,...,y_{n_2})\ln\left(k^{n_1+n_2}\tilde{A}(x_1,...,x_{n_1},y_1,...,y_{n_2}\right. \\
= & e^{-k\left(d_1\bar{f}_1+d_2\bar{f}_2\right)}\sum_{n_1,n_2}\frac{\left(kd_1\bar{f}_1\right)^{n_1}\left(kd_2\bar{f}_2\right)^{n_2}}{n_1!n_2!}\int\cdots\int\prod_{i=1}^{n_1}dx_i\prod_{j=1}^{n_2}dy_j\tilde{A}(x_1,...,x_{n_1},y_1,...,y_{n_2})\ln\left(\tilde{A}(x_1,...,x_{n_1},y_1,...,y_{n_2})\right) \\
& +e^{-k\left(d_1\bar{f}_1+d_2\bar{f}_2\right)}\sum_{n_1,n_2}\frac{\left(kd_1\bar{f}_1\right)^{n_1}\left(kd_2\bar{f}_2\right)^{n_2}}{n_1!n_2!}\ln\left(k^{n_1+n_2}\right) \\
\leq & 0+k\ln k\left(d_1\bar{f}_1+d_2\bar{f}_2\right)
\end{aligned}
$$

where maximum is achieved when $\tilde{A}\equiv 1$, i.e. f_1 and f_2 randomly takes f_+ and f_-. Therefore

$$
\begin{aligned}
& I[f_1,f_2] \\
\leq & e^{-kd_1\bar{f}_1}\sum_{n_1}\frac{\left(d_1\bar{f}_1\right)^{n_1}}{n_1!}\int\cdots\int\prod_{l=1}^{k}ds_l\prod_{i=1}^{n_1}dx_i\left(\sum_l\tilde{f}_1(s_l-x_i)\right)\ln\left(\prod_{i=1}^{n_1}\left(\sum_l\tilde{f}_1(s_l-x_i)\right)\right) \\
& +e^{-kd_2\bar{f}_2}\sum_{n_2}\frac{\left(d_2\bar{f}_2\right)^{n_2}}{n_2!}\int\cdots\int\prod_{l=1}^{k}ds_l\prod_{j=1}^{n_2}dy_j\left(\sum_l\tilde{f}_2(s_l-y_j)\right)\ln\left(\prod_{j=1}^{n_2}\left(\sum_l\tilde{f}_2(s_l-y_j)\right)\right) \\
& -k\ln k\left(d_1\bar{f}_1+d_2\bar{f}_2\right) \\
= & e^{-kd_1\bar{f}_1}\sum_{n_1}\frac{\left(d_1\bar{f}_1\right)^{n_1}}{n_1!}n_1k^{n_1-1}\int\cdots\int\prod_{l=1}^{k}ds_l\left(\sum_{l=1}^{k}\tilde{f}_1(s_l)\right)\ln\left(\sum_{l=1}^{k}\tilde{f}_1(s_l)\right) \\
& +e^{-kd_2\bar{f}_2}\sum_{n_2}\frac{\left(d_2\bar{f}_2\right)^{n_2}}{n_2!}n_2k^{n_2-1}\int\cdots\int\prod_{l=1}^{k}ds_l\left(\sum_{l=1}^{k}\tilde{f}_2(s_l)\right)\ln\left(\sum_{l=1}^{k}\tilde{f}_2(s_l)\right) \\
& -k\ln k\left(d_1\bar{f}_1+d_2\bar{f}_2\right) \\
= & e^{-kd_1\bar{f}_1}\sum_{n_1}\frac{\left(d_1\bar{f}_1\right)^{n_1}}{n_1!}n_1k^{n_1-1}\left[k\int\cdots\int\prod_{l=1}^{k}ds_l\left(\frac{\sum_{l=1}^{k}\tilde{f}_1(s_l)}{k}\right)\ln\left(\frac{\sum_{l=1}^{k}\tilde{f}_1(s_l)}{k}\right)+k\ln k\right] \\
& +e^{-kd_2\bar{f}_2}\sum_{n_2}\frac{\left(d_2\bar{f}_2\right)^{n_2}}{n_2!}n_2k^{n_2-1}\left[k\int\cdots\int\prod_{l=1}^{k}ds_l\left(\frac{\sum_{l=1}^{k}\tilde{f}_2(s_l)}{k}\right)\ln\left(\frac{\sum_{l=1}^{k}\tilde{f}_2(s_l)}{k}\right)+k\ln k\right] \\
& -k\ln k\left(d_1\bar{f}_1+d_2\bar{f}_2\right) \\
= & -d_1\bar{f}_1H\left[F_1\right]-d_2\bar{f}_2H\left[F_2\right]
\end{aligned}
$$

where $F_1\left(s_1,...,s_k\right):=\frac{\sum_{l=1}^{k}\tilde{f}_1(s_l)}{k}=\frac{\sum_{l=1}^{k}f_1(s_l)}{k\bar{f}_1}$, $F_2\left(s_1,...,s_k\right):=\frac{\sum_{l=1}^{k}\tilde{f}_2(s_l)}{k}=\frac{\sum_{l=1}^{k}f_2(s_l)}{k\bar{f}_2}$,
both maximized when F_1, F_2 reaches the upper and lower bounds, i.e. $f_i(s_l)=f_+,\forall s_l$, or $f_i(s_l)=f_-,\forall s_l$, thus

$$
I\leq -d_1\bar{f}_1H\left[\frac{f_+}{\bar{f}_1},\frac{f_-}{\bar{f}_1}\right]-d_2\bar{f}_2H\left[\frac{f_+}{\bar{f}_2},\frac{f_-}{\bar{f}_2}\right]
$$