1D FDFD

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1 Forward on Sign Conventions

The engineering convention = negative sign convention. In this case, the plane wave is:

$$e^{i\omega t \pm kz} \tag{1}$$

where the negative sign is a forward wave. Maxwell's equations in this sign convention are:

$$\nabla \times \mathbf{E} = -i\omega \mathbf{B} \tag{2}$$

$$\nabla \times \mathbf{H} = i\omega \epsilon_0 \epsilon \mathbf{E} \tag{3}$$

The physics convention is the positive sign convention.In this case, the plane wave is:

$$e^{-i\omega t \pm kz} \tag{4}$$

where the POSITIVE sign is a forward wave. Likewise, the Maxwell's equations are:

$$\nabla \times \mathbf{E} = i\omega \mathbf{B} \tag{5}$$

$$\nabla \times \mathbf{H} = -i\omega \epsilon_0 \epsilon \mathbf{E} \tag{6}$$

2 Motivations

This is to supplement our explorations using plane wave expansion method and analytic continuation.

Specifically, this question is trying to determine the eigenvalue k given a frequency ω (rather the converse problem).

3 Maxwell's Equations in 1D TM polarization

Starting with the usual Maxwell's equations

$$\nabla \times \mathbf{H} = i\omega \epsilon_0 \epsilon_r \mathbf{E} \tag{7}$$

$$\nabla \times \mathbf{E} = -i\omega \mu_0 \mathbf{H} \tag{8}$$

In 1D, we will say $\partial_x = 0$ and $\partial_y = 0$. The curl operator becomes:

$$[-\partial_z H_y, \partial_z H_x, 0] = i\omega \epsilon_0 \epsilon_r [E_x, E_y, E_z] \tag{9}$$

where ϵ_r is a tensor and say it is diagonal:

$$\epsilon_r = \begin{bmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} \tag{10}$$

Inverting the tensor ϵ_r, we get:

$$\left[- \frac{1}{\epsilon_{xx}} \partial_z H_y, \frac{1}{\epsilon_{yy}} \partial_z H_x, 0 \right] = i\omega\epsilon_0 [E_x, E_y, E_z] \tag{11}$$

For the curl-E equation:

$$[-\partial_z E_y, \partial_z E_x, 0] = -i\omega\mu_0 [H_x, H_y, H_z] \tag{12}$$

Even in 1D, we have four nonzero components, E_x, E_y, H_x, H_y. Suppose we substitute our H_y and H_x.

$$\left[- \frac{1}{\epsilon_{xx}} \partial_z \partial_z E_x, -\frac{1}{\epsilon_{yy}} \partial_z \partial_z E_y, 0 \right] = \omega^2 \epsilon_0 \mu_0 [E_x, E_y, E_z] \tag{13}$$

We could also solve into H_x and H_y

$$\left[\left(- \partial_z \frac{1}{\epsilon_{xx}} \partial_z \right) H_x, \left(- \partial_z \frac{1}{\epsilon_{yy}} \partial_z \right) H_y, 0 \right] = -\omega^2 \epsilon_0 \mu_0 [H_x, H_y, H_z] \tag{14}$$

Clearly, solving the E-components would be MUCH easier. Let's suppose:

$$E_x(z) = E_{0x}(z) e^{ikz} \tag{15}$$

$$-\frac{1}{\epsilon_{xx}} \partial_z^2 \big(E_{0x}(z) e^{ikz} \big) = \frac{\omega^2}{c^2} E_{0x}(z) e^{ikz} \tag{16}$$

Let's evaluate $\partial_z^2 \big(E_{0x}(z) e^{ikz} \big)$ The chain rule gives:

$$\partial_z \big((\partial_z E_{0x}(z)) e^{ikz} + ikE_{0x}(z) e^{ikz} \big) \tag{17}$$

$$\big(\partial_z^2 + 2ik\partial_z - k^2 \big) E_{0x}(z) e^{ikz} \tag{18}$$

Putting this into the PDE gives:

$$-\frac{1}{\epsilon_{xx}} \big(\partial_z^2 + 2ik\partial_z - k^2 \big) E_{0x}(z) e^{ikz} = \frac{\omega^2}{c^2} E_{0x} e^{ikz} \tag{19}$$

This is again, a quadratic eigenvalue problem (even the 1D problem is quadratic, wow). The quadratic eigenvalue problem is given as (moving everything to the RHS):

$$M\lambda^2 + \lambda C + K = 0 \tag{20}$$

$$M = -\frac{1}{\epsilon_{xx}} \tag{21}$$

$$C = \frac{1}{\epsilon_{xx}} 2i\partial_z \tag{22}$$

$$K = \frac{1}{\epsilon_{xx}} \partial_z^2 + \frac{\omega^2}{c^2} \tag{23}$$

4 Comparison with PWEM

Let's verify that this works. First, we show the FDFD spectrum

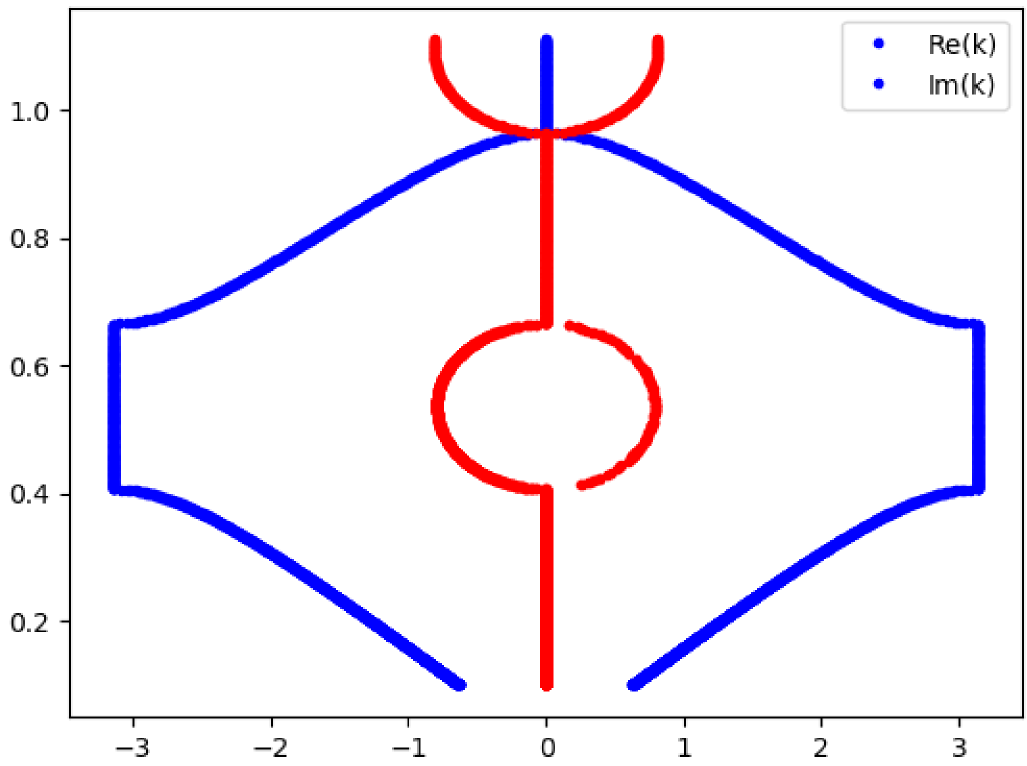

Figure 1: FDFD spectrum of a 1D Bragg mirror

5 Derivation with a Bloch Boundary Condition

Starting with the usual Maxwell's equations AGAIN

$$\nabla \times \mathbf{H} = i\omega\epsilon_0\epsilon_r\mathbf{E} \tag{24}$$

$$\nabla \times \mathbf{E} = -i\omega\mu_0\mathbf{H} \tag{25}$$

In 1D, we NOW say THAT $\partial_x = 0$ BUT $\partial_y = iK_y$. The curl operator becomes:

$$[-\partial_z H_y, \partial_z H_x, 0] = i\omega\epsilon_0\epsilon_r[E_x, E_y, E_z] \tag{26}$$

Inverting the tensor ϵ_r, we get:

$$\left[-\frac{1}{\epsilon_{xx}}\partial_z H_y, \frac{1}{\epsilon_{yy}}\partial_z H_x, 0\right] = i\omega\epsilon_0[E_x, E_y, E_z] \tag{27}$$

For the curl-E equation:

$$[-\partial_z E_y, \partial_z E_x, 0] = -i\omega\mu_0[H_x, H_y, H_z] \tag{28}$$

Even in 1D, we have four nonzero components, E_x, E_y, H_x, H_y. Suppose we substitute our H_y and H_x.

$$\left[-\frac{1}{\epsilon_{xx}}\partial_z\partial_z E_x, -\frac{1}{\epsilon_{yy}}\partial_z\partial_z E_y, 0\right] = \omega^2\epsilon_0\mu_0[E_x, E_y, E_z] \tag{29}$$

6 Maxwell's Equations in 1D TM polarization

Let's focus on this polarization

$$\left[\left(-\partial_z\frac{1}{\epsilon_{xx}(z)}\partial_z\right)H_x, \left(-\partial_z\frac{1}{\epsilon_{yy}(z)}\partial_z\right)H_y, 0\right] = -\omega^2\epsilon_0\mu_0[H_x, H_y, H_z] \tag{30}$$

Suppose $H_x(z) = H_{0x}(z)e^{ikz}$. This is harder because we have to apply the chain rule twice.

$$\partial_z(H_{0x}(z)e^{ikz}) = \big(\partial_z(H_{0x}(z)) + ikH_{0x}(z)\big)e^{ikz} \tag{31}$$

Let's call this expression $(f(z))$, then we're responsible for:

$$\partial_z\left(\frac{1}{\epsilon_{xx}(z)}f(z)\right) = \left(\frac{1}{\epsilon_{xx}(z)}\partial_z f(z) - \frac{1}{\epsilon_{xx}(z)^2}(\partial_z\epsilon_{xx}(z))f(z)\right) \tag{32}$$

$$\partial_z(f(z)) = \partial_z\left(\big(\partial_z(H_{0x}(z)) + ikH_{0x}(z)\big)e^{ikz}\right) \tag{33}$$

$$\partial_z(f(z)) = \big(\partial_z^2(H_{0x}(z)) - k^2H_{0x}(z) + 2ikH_{0x}(z)\big)e^{ikz} \tag{34}$$

Alright, let's simplify everything in the end. First, we can factor out $1/\epsilon_{xx}$ factor

$$\frac{1}{\epsilon_{xx}}\left(A(z)k^2 + B(z)k + C(z)\right)H_{0x}(z) \tag{35}$$

$$A(z) = -1 - \frac{1}{\epsilon_{xx}}(\partial_z\epsilon_{xx}(z))\bigg(1\bigg) \tag{36}$$

$$B(z) = 2i - \frac{1}{\epsilon_{xx}}(\partial_z\epsilon_{xx}(z))\bigg(i\bigg) \tag{37}$$

$$C(z) = \partial_z^2 - \frac{1}{\epsilon_{xx}}(\partial_z\epsilon_{xx}(z))\bigg(\partial_z\bigg) \tag{38}$$

Much more complicated...

6.1 Numerical Verification

7 FDFD Eigensolver for the IMI-MIM System

This situation is different because we want to solve for modes propagating in z but there is variation in a transverse direction!

8 Conclusions

One of the most interesting conclusions we can draw from the 1D eigen study is that even in 1D, solving the frequency to k eigenproblem is a quadratic eigenvalue problem.