THE GEORGE WASHINGTON UNIVERSITY
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K-coloring Problem NP-complete

CSCI 6212 Design & Analysis of Algorithm

Team 8

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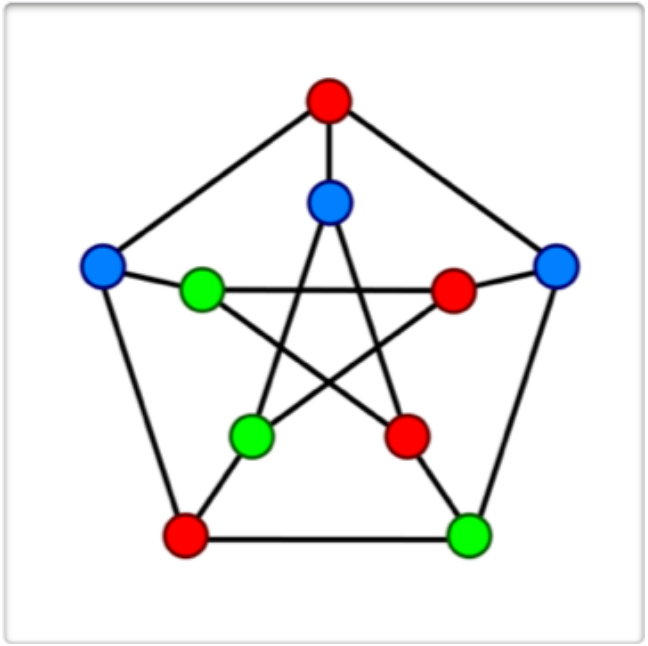
K-coloring Problem

Given a graph G(V, E), the k-coloring problem asks whether G can be properly colored using at most k colors. We say that a coloring is proper if adjacent vertices receives different colors:$\forall(u, v) \in E : c(u) \neq c(v)$.

K-coloring Problem is Important

- Applications
  - Sudoku
  - Map coloring
  - GSM network



NP-completeness Proof Outline

3-coloring problem is NP-complete

- 3-coloring problem is in NP
- Reduction from 3-SAT to 3-COLORING

K-coloring problem is np-complete

- K-coloring problem is in NP
- Reduction from 3-COLOURING to K-COLORING

Reduce 3-SAT to 3-Coloring

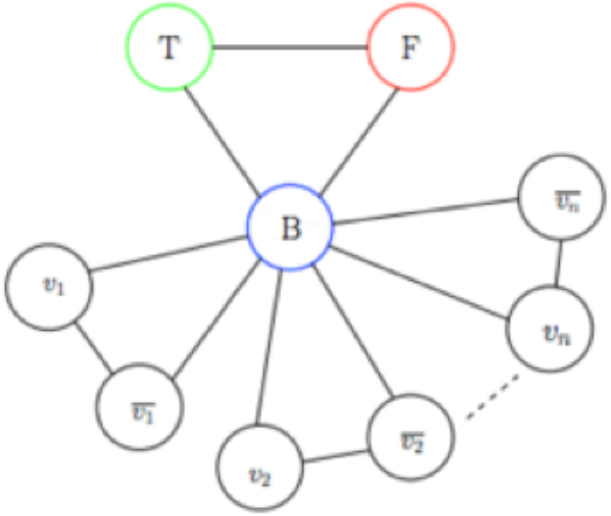
Let φ be a 3-SAT instance and C1, C2, ..., Cm be the clauses of φ defined over the variables{x1, x2, ..., xn}.

The graph G (V,E) that we will construct needs to capture two constrains that based on the 3-SAT:

1. Assign the truth or false assignment for {x1, x2, ..., xn } via the colors of G.
2. Capture the satisfiability of every clause of φ.

Reduce 3-SAT to 3-Coloring

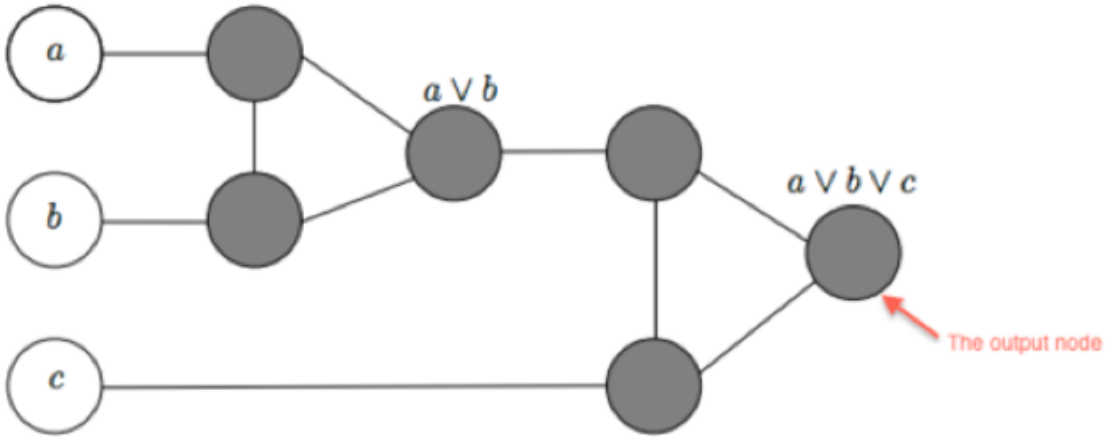
- We first create a triangle in G with three vertices {T,F,B}.
- Then we add two vertices vi, $\overline{\text{vi}}$ for every xi and create a triangle B



Reduce 3-SAT to 3-Coloring

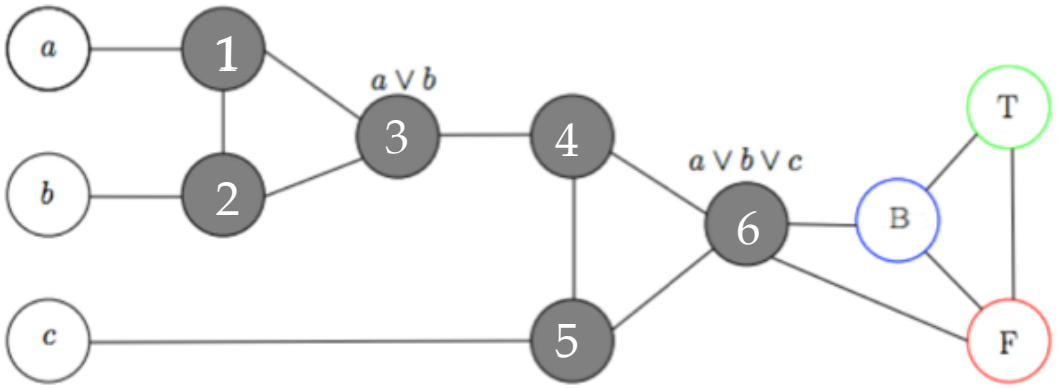
- If G is 3-colourable,then either vi or $\overline{\text{vi}}$ gets the color T, and we just assign the truth to vi. Now we need to add constraints of the clause.

We introduce the OR-gadget:



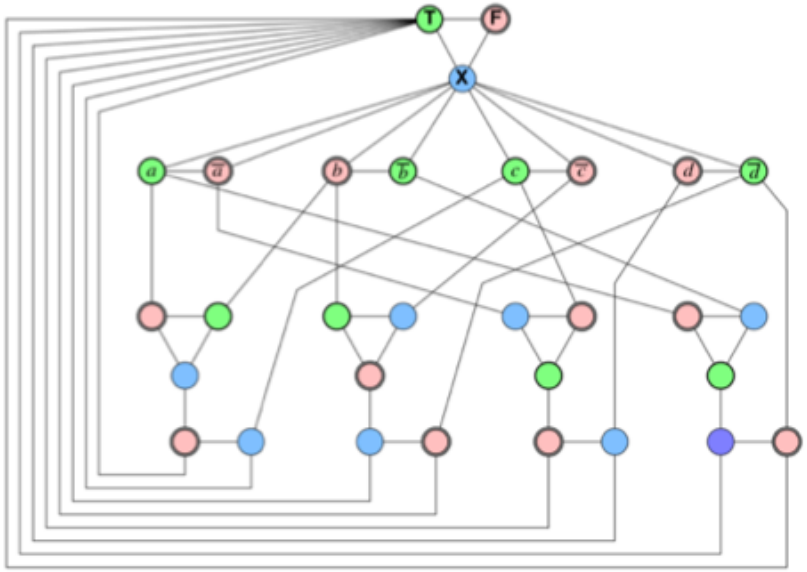
Reduce 3-SAT to 3-Coloring

- If one of the a,b,c is colored T, then there exists a valid 3-coloring if the OR-gadget where the out put node is colored T, thus satisfy the requirement of the clause.



Reduce 3-SAT to 3-Coloring

- Once we add the OR-gadget of every clause, we connect the out-put node of every gadget to the initial triangle.
- So we can approve that 3-SAT instance is satisfiable if and only the graph G constructer above is 3-colorable.



NP-completeness Proof Outline

3-coloring problem is np-complete

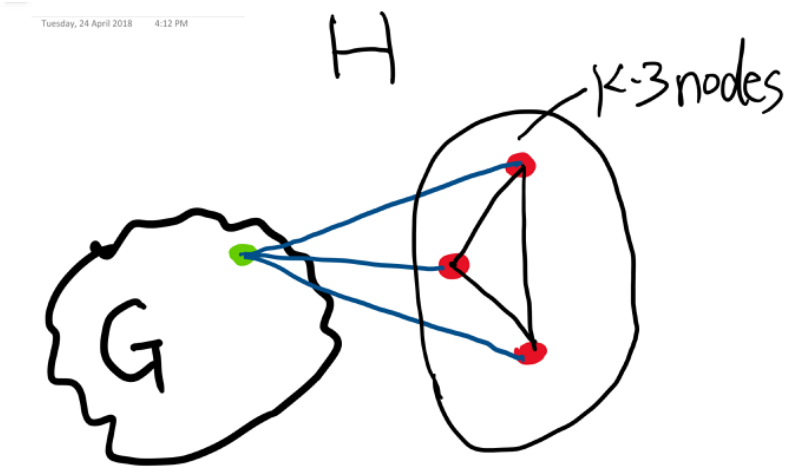
- 3-coloring problem is in NP
- Reduction from 3-SAT to 3-COLORING

K-coloring problem is np-complete

- K-coloring problem is in NP
- Reduction from 3-COLOURING to K-COLORING

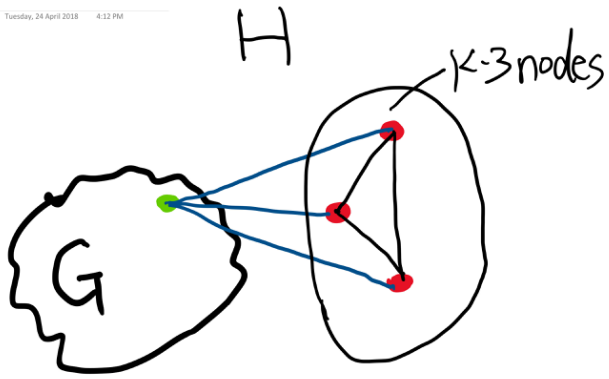
Reduce 3-coloring to K-coloring

- Supposed that we have a graph G, we want to find out whether G satisfy 3-coloring.
- Then create a new graph H by adding k-3 nodes to G. The k-3 nodes are already connected to each other. For each vertex in G, connect it to all the K-3 nodes.
- Now, The problem turns into whether we can use K-color to color the H.



Reduce 3-coloring to K-coloring

- Because the k-3 nodes in graph H are connected to each other, we need to use k-3 colors to color these nodes.
- For each vertex in G, it is connected to all the k-3 nodes, such that these colors are different from the (k-3) colors that we have applied.
- We could just use three colors to color the rest of graph H, which is G.



- So, if G is three colorable, then H is k colorable. And if H is K colorable, G is three colorable.

K-coloring problem is NP-complete

References:

- www.dcs.fmph.uniba.sk/~fduris/VKTI/3color_HamCycle.pdf
- https://math.stackexchange.com/questions/1752064/reducing-graph-3-coloring-to-10-coloring