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Note

On the NP-completeness of the k-colorability problem for triangle-free graphs

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Abstract

We show that the question "Is a graph 3-colorable?" remains NP-complete when restricted to the class of triangle-free graphs with maximum degree 4. Likewise the question "Is a triangle-free graph k-colorable?" is shown to be NP-complete for any fixed value of $k \geqslant 4$.

Keywords: Graph theory; Computational complexity

A k-coloring of a graph G with vertex-set V is a mapping $c: V \to \{1, \ldots, k\}$ such that $c(x) \neq c(y)$ holds for every edge xy of G. Karp [7] proved that determining whether a graph admits a k-coloring is NP-complete whenever $k \geqslant 3$. Actually, the question "Is G a 3-colorable graph" remains NP-complete under several restrictions. Garey and Johnson [4] proved:

Theorem 1 (Garey and Johnson [4]). Graph 3-colorability *is* NP-*complete even when restricted to planar graphs with maximum degree four.*

This theorem is not given explicitly but can be obtained by successively using the transformations given in Theorems 4.2 and 4.1 in [4]. Note that the theorem is best of its type in the sense that every connected graph with maximum degree three is 3-colorable, except for the complete graph with four vertices; this is by Brooks' Theorem [1].

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We would like to consider the question of k-coloring triangle-free graphs. This question arose in another context (see [9]). Combining two results from [8] it can be derived that TRIANGLE-FREE GRAPH k-COLORABILITY is NP-complete for each fixed $k \geqslant 3$. Here we prove the following.

Theorem 2. GRAPH 3-COLORABILITY *remains NP-complete when restricted to the class of triangle-free graphs with maximum degree four.*

Likewise,

Theorem 3. *For each $k \geqslant 3$,* GRAPH k-COLORABILITY *remains* NP-*complete among triangle-free graphs where the maximum degree is bounded by a function of k.*

Let us remark that we cannot add the planarity condition in these theorems, because every triangle-free planar graph is 3-colorable [5, 6].

Proof of Theorem 2. The problem is clearly in NP as the general 3-colorability problem is. To prove NP-completeness, by Theorem 1, it suffices to show that every instance of the 3-colorability problem consisting of a graph G with maximum degree four can be polynomially reduced to an instance of the 3-colorability problem consisting of a triangle-free graph G' with maximum degree four in such a way that G is 3-colorable if and only if G' is.

So let G be any graph with maximum degree four. If G contains an induced 4-clique then obviously G is not 3-colorable and we let G' be a fixed 4-chromatic triangle-free graph with maximum degree 4 (for example, the twelve-vertex graph in [2]). Notice that in this case the complexity of the reduction is that of testing whether G contains a 4-clique, which is obviously polynomial in the order of G. From now on we assume that G contains no induced 4-clique.

Consider the following eleven-vertex graph H_0. Start with five vertices $a_1, \ldots, a_5$ with edges a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1. Add five vertices $b_1, \ldots, b_5$ and connect each b_i to the a_j's adjacent to a_i. Add a vertex c and edges cb_2, cb_3, cb_4, cb_5. (Observe that H_0 plus the edge cb_1 is the well-known 4-chromatic Mycielski graph [3, 10]). It is easy to check that H_0 is triangle-free and 3-chromatic. Moreover, in every 3-coloring of H_0 the vertices b_1, b_3 and b_4 have three different colours. Indeed, consider any 3-coloring of H_0. Since the Mycielski graph $H_0 + cb_1$ is not 3-colorable, c and b_1 must share the same color, say color 1. So the two neighbors a_2 and a_5 of b_1 have different colors, say color 2 and 3, respectively, and consequently b_4 and b_3 have colors 3 and 2, respectively.

Now we build a graph H by adding to H_0 a vertex d and edges db_3 and db_4. In H the vertices d and b_1 have degree two and all other vertices have degree three or four. The two vertices d and b_1 will be called the *tips* of H. It results from the preceding discussion that H is a triangle-free 3-chromatic graph and that the two tips have the same color in every 3-coloring of H.

We obtain a triangle-free graph G' with maximum degree four by successively transforming G as follows. Enumerate the vertices of G as $v_1, v_2, \dots, v_n$. Set $G_0 = G$. For each $i = 1, \dots, n$, if the neighborhood $N(v_i)$ of v_i in G_{i-1} has no edge or is isomorphic to $K_{1,3}$ then set $G_i = G_{i-1}$, else let G_i be a graph obtained by:

(i) Partitioning the vertices of $N(v_i)$ into two stable sets $S_1(v_i)$, $S_2(v_i)$ of cardinality at most 2;

(ii) Removing v_i, adding a copy $H(v_i)$ of H, connecting one tip of $H(v_i)$ to the vertices of $S_1(v_i)$ and the other tip of $H(v_i)$ to the vertices of $S_2(v_i)$.

Since any graph with at most four vertices, with no triangle and not isomorphic to a $K_{1,3}$ can be partitioned into two stable sets of size at most two, Step (i) can always be done under the assumptions.

Now let $G' = G_n$. It follows from this construction and the definition of H that G' has maximum degree four. It is clear that the construction of G' is polynomial in the size and order of G. Furthermore, observe that after Step (ii) has been performed, all triangles containing v_i are destroyed and no new triangle is created. So if G' contained a triangle xyz it could only be the case that all three vertices x, y, z are such that their neighborhoods in G are isomorphic to $K_{1,3}$ — but this condition is intrinsically impossible in a graph with maximum degree four. So G' is triangle-free.

There remains to be checked that G is 3-colorable if and only if G' is. First suppose that G' is 3-colored. For each vertex v of G that has been replaced by a copy $H(v)$ of H, it must be that the two tips of $H(v)$ have the same color. Assign this colour to v in G. Now it is clear that we have obtained a proper 3-coloring of the vertices of G. Second, suppose that G is 3-colored. For each vertex v of G that has been replaced in G' by a copy $H(v)$ of H, color the vertices of $H(v)$ with the three colors in such a way that the color assigned to the tips of $H(v)$ is the color of v in G. Again it is clear that we obtain a proper 3-coloring of G'.

So we have obtained a polynomial reduction of GRAPH 3-COLORABILITY WITH MAXIMUM DEGREE FOUR to TRIANGLE-FREE GRAPH 3-COLORABILITY WITH MAXIMUM DEGREE FOUR, and the theorem is proved. □

Proof of Theorem 3. The proof of this theorem works along the same lines as for the preceding theorem. First we will reduce GRAPH k-COLORABILITY to BOUNDED DEGREE GRAPH k-COLORABILITY and then to triangle-free graphs.

For the first step, consider a gadget with two $(k-2)$-cliques A and B and six other vertices x, y, z, t, u, v. Add all edges between each vertex of A and x, y, z, t; add all edges between each vertex of B and y, z, u, v; add edges xy, yz, zu, uv, tu. Observe that in any k-coloring of this gadget the vertices x, z, t, v must have the same color. Now for any integer d build a super-gadget by taking $d-2$ copies of this gadget and identifying the v-vertex of the ith gadget with the x-vertex of the $(i+1)$th gadget. The t-vertices in each gadget, plus the x vertex in the first and the v vertex in the $(d-2)$th, will be called the exits. In every k-coloring of this super-gadget, the d exits must have the same color. Now for any instance G of the GRAPH k-COLORABILITY problem, replace each vertex s of G of degree $d(s)$ by a super-gadget with $d(s)$ exits and connect each of the $d(s)$ neighbors s to one different exit. It is easy to check (as in the proof of Theorem 4.1 in [4]) that G is k-colorable if and only if the new graph is. It is not difficult to see that in the new graph the maximum degree is $2k - 2$. So we have a polynomial reduction from GRAPH k-COLORABILITY to k-COLORABILITY OF GRAPHS WITH MAXIMUM DEGREE $2k - 2$.

Recall that the kth Mycielski graph M_k is defined as follows. Let M_2 be the 2-clique. For $k \geqslant 3$, M_k is obtained by taking a copy of M_{k-1}, adding for each vertex v of M_{k-1} a new "shadow" vertex v' and for each edge vu of M_{k-1} a new edge $v'u$, and adding a final vertex w and an edge wv' for each shadow vertex. It is not difficult to check that M_k is a triangle-free k-chromatic graph. (Indeed if there existed a $(k-1)$-coloring of M_k then, in the copy of M_{k-1}, assigning to any vertex v having the same color as w the color of its shadow v' would result in a $(k-2)$-coloring of M_{k-1}.) Moreover, M_k is edge-critical, i.e., removing any edge yields a $k-1$-colorable graph.

Given an integer d, build a graph H_d from M_{k+1} by removing an arbitrary edge xy and duplicating the vertex y $d-2$ times (i.e., create $d-2$ new vertices $y_2, \ldots, y_{d-1}$ and connect each of them to the neighbors of y in $M_{k+1} - xy$). Call $y, y_2, \ldots, y_{d-1}, x$ the tips of H_d. It is clear from the construction that H_d is triangle-free, k-chromatic, and that in every k-coloring of H_d the tips all have the same color (or else a k-coloring of M_{k+1} could be derived). In H_d the maximum degree is at most $3 \cdot 2^{k-1} + d$.

Now consider an instance G of the k-COLORABILITY OF GRAPHS WITH MAXIMUM DEGREE $2k-2$ problem. We want to build a triangle-free graph G', with bounded degree, that is k-colorable if and only if G is k-colorable. Given a vertex v of G, replace v by a copy $H(v)$ of H_d, where d is the degree of v in G, and connect each tip of $H(v)$ to a different neighbor of v in G. Repeat this for every vertex v of G. The resulting graph is called G'. Now it can be argued, like in the proof of the preceding theorem, that G' is k-colorable if and only if G is k-colorable. It is clear that the size and order of G' are polynomial in the size and order of G. Finally, it is not difficult to check that G' is triangle-free and that its maximum degree is $3 \cdot 2^{k-1} + 2k - 2$. So we have a polynomial reduction of GRAPH k-COLORABILITY to TRIANGLE-FREE GRAPH k-COLORING WITH BOUNDED DEGREE, and Theorem 3 follows from Theorem 1. □

References

[1] R.L. Brooks, On coloring the nodes of a network, Proc. Cambridge Phil. Soc. 37 (1941) 194–197.
[2] V. Chvátal, The smallest triangle-free 4-chromatic 4-regular graph, J. Combin. Theory 9 (1970) 93–94.
[3] V. Chvátal, The minimality of the Mycielski graph, in: R.A. Bari and F. Harary, eds., Graphs and Combinatorics, Proc. Capital Conf. on Graph Th. and Combinatorics at G. Washington Univ., June 1973. Lecture Notes in Mathematics, Vol. 406 (Springer, Berlin, 1974) 243–246.
[4] M.R. Garey and D.S. Johnson, Computer and Intractability: A Guide to the Theory of NP-completeness (W.H. Freeman, San Fransisco, 1979).
[5] H. Grötzsch, Ein Dreifarbensatz fur dreikreisfreie Netze auf der Kugel, Wiss. Z. Martin Luther Univ., Halle-Wittenberg, Math. Naturwiss. Reihe, 8 (1958) 109–119.
[6] B. Grünbaum, Grötzsch's theorem on 3-colorings, Michigan Math. J. 10 (1963) 303–310.

[7] R.M. Karp, Reducibility among combinatorial problems, in: R.E. Miller and J.W. Thatcher, eds., Complexity of computer computations (Plenum Press, New York, 1972) 85–104.
[8] L. Lovász, Coverings and colorings of hypergraphs, in: 4th Southeastern Conf. Combinatorics, Graph Theory, and Computing, Utilitas Mathematica (1973) 3–12.
[9] F. Maffray and M. Preissmann, Split-neighborhood graphs, Manuscript, Université Joseph Fourier, Grenoble, France (1993) J. Combin. Theory B, to appear.
[10] J. Mycielski, Sur le coloriage des graphes, Colloq. Math. 3 (1955) 161–162.