

Problem Set 1 | Question 9 ([Github](#))

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Exercise 9. We say that a morphism $s : x \rightarrow y$ is a **split monomorphism** if there exists some $r : y \rightarrow x$ such that $rs = 1_x$.

- (a) Prove that a morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism.
- (b) Argue by duality that a split monomorphism that is an epimorphism is also an isomorphism.

Solution. A *split epimorphism* is defined dually as a morphism $r : y \rightarrow x$ such that $rs = 1_x$ for some $s : x \rightarrow y$. Assuming that our category C is locally-small, we prove (a).

Let $f : x \rightarrow y$ be a monomorphism, which is equivalent to requiring that for every $w \in C$, the post-composition $f_* : C(w, x) \rightarrow C(w, y)$ is an injection. We claim that f is a split epimorphism iff for every $w \in C$, (the same) f_* is a surjection. Thus f_* is a bijection (of sets), so f is an isomorphism as desired.

- Indeed, suppose that f is a split epimorphism and fix $w \in C$. Let $s : y \rightarrow x$ be such that $fs = 1_y$. For any $g : w \rightarrow y$, let $h := sg : w \rightarrow x$ and observe that $fh = f(sg) = (fs)g = 1_yg = g$. Conversely, let $w := y$ and consider $1_y : y \rightarrow y$. That f_* is a surjection furnishes some $s : y \rightarrow x$ such that $fs = 1_y$, as desired.

This proves (a). For (b), we apply (a) to the opposite category C^{op} . More precisely, let $f : x \rightarrow y$ be an epimorphism that is also a split monomorphism. Then $f^{\text{op}} : y \rightarrow x$ is a monomorphism that is also a split epimorphism, so f^{op} is an isomorphism in C^{op} by (a). This occurs iff $f : x \rightarrow y$ is an isomorphism in C , as desired. ■