Problem Set 1 | Question 9 (GitHub)

Zhaoshen Zhai | October 18, 2023

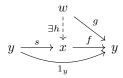
Exercise 9. We say that a morphism $s: x \to y$ is a **split monomorphism** if there exists some $r: y \to x$ such that $rs = 1_x$.

- (a) Prove that a morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism.
- (b) Argue by duality that a split monomorphism that is an epimorphism is also an isomorphism.

Solution. A split epimorphism is defined dually as a morphism $r: y \to x$ such that $rs = 1_x$ for some $s: x \to y$. Assuming that our category C is locally-small, we prove (a).

Let $f: x \to y$ be a monomorphism, which is equivalent to requiring that for every $w \in C$, the post-composition $f_*: C(w,x) \to C(w,y)$ is an injection. We claim that f is a split epimorphism iff for every $w \in C$, (the same) f_* is a surjection. Thus f_* is a bijection (of sets), so f is an isomorphism as desired.

• Indeed, suppose that f is a split epimorphism and fix $w \in C$. Let $s: y \to x$ be such that $fs = 1_y$. For any $g: w \to y$, let $h \coloneqq sg: w \to x$ and observe that $fh = f(sg) = (fs)g = 1_yg = g$. Conversely, let $w \coloneqq y$ and consider $1_y: y \to y$. That f_* is a surjection furnishes some $s: y \to x$ such that $fs = 1_y$, as desired.



This proves (a). For (b), we apply (a) to the opposite category C^{op} . More precisely, let $f: x \to y$ be an epimorphism that is also a split monomorphism. Then $f^{\text{op}}: y \to x$ is a monomorphism that is also a split epimorphism, so f^{op} is an isomorphism in C^{op} by (a). This occurs iff $f: x \to y$ is an isomorphism in C, as desired.