Problem Set 1 | Question 9 (Github)

Zhaoshen Zhai | October 18, 2023

Exercise 9. We say that a morphism $s : x \to y$ is a **split monomorphism** if there exists some $r : y \to x$ such that $rs = 1_x$.

(a) Prove that a morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism.

(b) Argue by duality that a split monomorphism that is an epimorphism is also an isomorphism.

Solution. A *split epimorphism* is defined dually as a morphism $r : y \to x$ such that $rs = 1_x$ for some $s : x \to y$. Assuming that our category C is locally-small, we prove (a).

Let $f : x \to y$ be a monomorphism, which is equivalent to requiring that for every $w \in C$, the post-composition $f_* : C(w, x) \to C(w, y)$ is an injection. We claim that f is a split epimorphism iff for every $w \in C$, (the same) f_* is a surjection. Thus f_* is a bijection (of sets), so f is an isomorphism as desired.

- Indeed, suppose that f is a split epimorphism and fix $w \in C$. Let $s : y \to x$ be such that $fs = 1_y$. For any $g : w \to y$, let $h \coloneqq sg : w \to x$ and observe that $fh = f(sg) = (fs)g = 1_y g = g$. Conversely, let $w \coloneqq y$ and consider $1_y : y \to y$. That f_* is a surjection furnishes some $s : y \to x$ such that $fs = 1_y$, as desired.

$$\begin{tikzcd}
 & w \arrow[d, dashed, "\exists h"'] \arrow[dr, "g"] & \\
y \arrow[r, "s"] \arrow[rr, bend right, "1_y"'] & x \arrow[r, "f"] & y
\end{tikzcd}$$

This proves (a). For (b), we apply (a) to the opposite category C^{op}. More precisely, let $f : x \to y$ be an epimorphism that is also a split monomorphism. Then $f^{\mathrm{op}} : y \to x$ is a monomorphism that is also a split epimorphism, so f^{op} is an isomorphism in C^{op} by (a). This occurs iff $f : x \to y$ is an isomorphism in C, as desired. ■