

Problem Set 1 | Question 9 ([GitHub](#))

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**Exercise 9.** We say that a morphism  $s : x \rightarrow y$  is a **split monomorphism** if there exists some  $r : y \rightarrow x$  such that  $rs = 1_x$ .

- (a) Prove that a morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism.
- (b) Argue by duality that a split monomorphism that is an epimorphism is also an isomorphism.

*Solution.* A *split epimorphism* is defined dually as a morphism  $r : y \rightarrow x$  such that  $rs = 1_x$  for some  $s : x \rightarrow y$ . Assuming that our category  $C$  is locally-small, we prove (a).

Let  $f : x \rightarrow y$  be a monomorphism, which is equivalent to requiring that for every  $w \in C$ , the post-composition  $f_* : C(w, x) \rightarrow C(w, y)$  is an injection. We claim that  $f$  is a split epimorphism iff for every  $w \in C$ , (the same)  $f_*$  is a surjection. Thus  $f_*$  is a bijection (of sets), so  $f$  is an isomorphism as desired.

- Indeed, suppose that  $f$  is a split epimorphism and fix  $w \in C$ . Let  $s : y \rightarrow x$  be such that  $fs = 1_y$ . For any  $g : w \rightarrow y$ , let  $h := sg : w \rightarrow x$  and observe that  $fh = f(sg) = (fs)g = 1_y g = g$ . Conversely, let  $w := y$  and consider  $1_y : y \rightarrow y$ . That  $f_*$  is a surjection furnishes some  $s : y \rightarrow x$  such that  $fs = 1_y$ , as desired.

$$\begin{array}{ccccc}
 & & w & & \\
 & & \downarrow \exists h & \searrow g & \\
 y & \xrightarrow{s} & x & \xrightarrow{f} & y \\
 & \searrow & & \nearrow & \\
 & & 1_y & & 
 \end{array}$$

This proves (a). For (b), we apply (a) to the opposite category  $C^{\text{op}}$ . More precisely, let  $f : x \rightarrow y$  be an epimorphism that is also a split monomorphism. Then  $f^{\text{op}} : y \rightarrow x$  is a monomorphism that is also a split epimorphism, so  $f^{\text{op}}$  is an isomorphism in  $C^{\text{op}}$  by (a). This occurs iff  $f : x \rightarrow y$  is an isomorphism in  $C$ , as desired. ■