Category Theory - Exercises 1 (Functors, Naturality, etc.)

Tasmin Chu

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Abstract

Most of these exercises are taken directly from Emily Riehl, "Category Theory in Context". I invite you to choose a problem, work out a solution, and post it on #gromov or Coauthor for others to benefit from. Coloured exercises are important and more worth doing.

1. Write out the definition of a category and all the axioms that it must satisfy.
2. Write out the definition of small and locally small. Give an example of a small category and a locally small (but not small) category.
3. Write out the definition of a functor $F : C \to D$ between categories C and D.
4. Write out the definition of a natural transformation $\alpha : F \Rightarrow G$ between two functors $F, G : C \to D$. Write out the definition of a natural isomorphism.
5. Try to think of an example of a mathematical isomorphism that is not natural. Use Google or StackExchange if necessary.
6. Write out the definition of an epimorphism and a monomorphism.
7. Show for any category C and any object $c \in C$, there exists a category c/C whose objects are morphisms $f : c \to x$ with domain c, where a morphism from $f : c \to x$ to $g : c \to y$ is a map $h : x \to y$ such that

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

commutes, i.e. $g = hf$. (Hint: show that this satisfies the axioms of a category.)

8. What are the monomorphisms in the category of fields?

9. We say that a morphism $s : x \to y$ is a **split monomorphism** if there exists some $r : y \to x$ such that $rs = 1_x$.

   (a) Prove that a morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism.

   (b) Argue by duality that a split monomorphism that is an epimorphism is also an isomorphism.

10. What is a functor between groups G and H, regarded as one-object categories BG and BH?

11. Any partially ordered set/poset $(P, \leq)$ can be made into a category like so. For elements $x, y \in P$, draw an arrow $x \to y$ if $x \leq y$. Notice that if $x \to y \to z$, (i.e, $x \leq y, y \leq z$) then we have an arrow $x \to z$ since $x \leq z$. This guarantees compositionality and we thus get a category.

   (a) What is a functor between two posets $(X, \leq)$ and $(Y, \subseteq)$?

12. Functors preserve isomorphisms: for $F : C \to D$ a functor, if $f : x \to y$ is an isomorphism in C, then $Ff : Fx \to Fy$ is an isomorphism in D.

    Find an example to demonstrate that functors need not **reflect** isomorphisms: that is, find a functor $F : C \to D$ and a morphism f in C so that Ff is an isomorphism in D but f is not an isomorphism in C.

13. Consider the category **Group**, whose objects are groups and morphisms are group homomorphisms. (Note this is not the same thing as the one-object category group BG for a fixed group G.) Show that the construction of the set of conjugacy classes of elements of a group is functorial, defining a functor $\mathrm{Conj} : \mathrm{Group} \to \mathrm{Set}$.

    Conclude that any pair of groups whose sets of conjugacy classes of elements have differing cardinalities cannot be isomorphic.

14. Fix a group G and consider the one-object category BG. As seen in the notes, a group action of G on some set $X \in \mathrm{Set}$ consists of a functor $F : BG \to \mathrm{Set}$ such that $F(\cdot) = X$, where we use $\cdot$ to represent the object of BG.

    Show that natural transformations $\alpha : F \Rightarrow H$, where $F, H : BG \to \mathrm{Set}$ are group actions on some sets X and Y, correspond to G-equivariant maps: i.e. maps $f : X \to Y$ such that

$$f(g \cdot x) = g \cdot f(x)$$

15. (!!!) The Yoneda lemma says that for any functor $F : C \to \text{Set}$, and any object $c \in C$, there is a bijection

$$\text{Hom}_{[C,Set]}(C(c,-),F) \cong Fc$$

in Set that associates a natural transformation $\alpha : C(c,-) \Rightarrow F$ to the element $\alpha_c(1_c) \in Fc$.

Prove that this is a bijection by proving this function is injective and surjective. (**Very important, try not to skip.**)

16. Write out what it means for two categories C and D to be equivalent.

17. Show that equivalence of categories defines an equivalence relation.

18. Write out what it means for a functor $F : C \to D$ to be:

    (a) full

    (b) faithful

    (c) essentially surjective on objects

19. Write out the definition of initial and terminal object.