## Category Theory - Exercises 1 (Functors, Naturality, etc.)

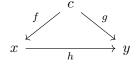
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## **Abstract**

Most of these exercises are taken directly from Emily Riehl, "Category Theory in Context". I invite you to choose a problem, work out a solution, and post it on #gromov or Coauthor for others to benefit from. Coloured exercises are important and more worth doing.

- 1. Write out the definition of a category and all the axioms that it must satisfy.
- 2. Write out the definition of small and locally small. Give an example of a small category and a locally small (but not small) category.
- **3.** Write out the definition of a functor  $F: C \to D$  between categories C and D.
- **4.** Write out the definition of a natural transformation  $\alpha: F \Rightarrow G$  between two functors  $F, G: C \to D$ . Write out the definition of a natural isomorphism.
- **5.** Try to think of an example of a mathematical isomorphism that is not natural. Use Google or StackExchange if necessary.
- 6. Write out the definition of an epimorphism and a monomorphism.
- 7. Show for any category C and any object  $c \in C$ , there exists a category c/C whose objects are morphisms  $f:c \to x$  with domain c, where a morphism from  $f:c \to x$  to  $g:c \to y$  is a map  $h:x \to y$  such that



commutes, i.e. g=hf. (Hint: show that this satisfies the axioms of a category.)

- 8. What are the monomorphisms in the category of fields?
- 9. We say that a morphism  $s: x \to y$  is a **split monomorphism** if there exists some  $r: y \to x$  such that  $rs = 1_x$ .
  - (a) Prove that a morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism.
  - (b) Argue by duality that a split monomorphism that is an epimorphism is also an isomorphism.
- 10. What is a functor between groups G and H, regarded as one-object categories BG and BH?
- 11. Any partially ordered set/poset  $(P, \leq)$  can be made into a category like so. For elements  $x, y \in P$ , draw an arrow  $x \to y$  if  $x \leq y$ . Notice that if  $x \to y \to z$ , (i.e,  $x \leq y, y \leq z$ ) then we have an arrow  $x \to z$  since  $x \leq z$ . This guarantees compositionality and we thus get a category.
  - (a) What is a functor between two posets  $(X, \leq)$  and  $(Y, \subseteq)$ ?
- 12. Functors preserve isomorphisms: for  $F: C \to D$  a functor, if  $f: x \to y$  is an isomorphism in C, then  $Ff: Fx \to Fy$  is an isomorphism in D.
  - Find an example to demonstrate that functors need not **reflect** isomorphisms: that is, find a functor  $F: C \to D$  and a morphism f in C so that Ff is an isomorphism in D but f is not an isomorphism in C.
- 13. Consider the category **Group**, whose objects are groups and morphisms are group homomorphisms. (Note this is not the same thing as the one-object category group BG for a fixed group G.) Show that the construction of the set of conjugacy classes of elements of a group is functorial, defining a functor Conj : Group  $\rightarrow$  Set.
  - Conclude that any pair of groups whose sets of conjugacy classes of elements have differing cardinalities cannot be isomorphic.
- 14. Fix a group G and consider the one-object category BG. As seen in the notes, a group action of G on some set  $X \in \text{Set}$  consists of a functor  $F : BG \to \text{Set}$  such that  $F(\cdot) = X$ , where we use  $\cdot$  to represent the object of BG.

Show that natural transformations  $\alpha: F \Rightarrow H$ , where  $F, H: BG \to \operatorname{Set}$  are group actions on some sets X and Y, correspond to G-equivariant maps: i.e. maps  $f: X \to Y$  such that

$$f(g \cdot x) = g \cdot f(x)$$

15. (!!!) The Yoneda lemma says that for any functor  $F: C \to \mathbf{Set}$ , and any object  $c \in C$ , there is a bijection

$$\operatorname{Hom}_{[C,Set]}(C(c,-),F)\cong Fc$$

in Set that associates a natural transformation  $\alpha:C(c,-)\Rightarrow F$  to the element  $\alpha_c(1_c)\in Fc$ .

Prove that this is a bijection by proving this function is injective and surjective. (**Very important, try not to skip.**)

- 16. Write out what it means for two categories C and D to be equivalent.
- 17. Show that equivalence of categories defines an equivalence relation.
- 18. Write out what it means for a functor  $F: C \to D$  to be:
  - (a) full
  - (b) faithful
  - (c) essentially surjective on objects
- 19. Write out the definition of initial and terminal object.