

# TREE OF ORIENTATIONS ON A NESTED COLLECTION OF SETS

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Let  $H \subseteq 2^X$  be a sub-pocset for some fixed set  $X$ . With the definitions in Section 1, we prove the following

**Theorem A** (Corollary 2.5). *If  $H$  is nested, then the graph  $\mathcal{T}_H$ , whose:*

- *Vertices are orientations  $p \subseteq H$ ;*
- *Edges are pairs  $\{p, q\}$  such that  $q = p \triangle \{h, h^c\}$  for some  $\subseteq$ -minimal  $h \in H$ ;*

*is acyclic.*

In particular, this applies to when  $(X, G)$  is a graph and  $H$  is a nested collection of cuts on  $X$ . No further assumptions on  $X$  (like local-finiteness) is needed.

**Theorem B** (Proposition 2.8). *If  $H$  is nested and interval-finite, then the induced subgraph of  $\mathcal{T}_H$  consisting only the based orientations is a tree.*

This applies to when  $(X, H)$  is a walling; hence, a nested walling induces a tree of based orientations.

## 1. PRELIMINARIES

Let  $H \subseteq 2^X$  be a sub-pocset for some fixed set  $X$ , whose elements  $h \in H$  we call *half-spaces*.

**Definition 1.1.** Two elements  $h, k \in H$  are *nested* if  $h^i \cap k^j = \emptyset$  for some  $i, j \in \{1, -1\}$ , where  $h^i := h$  for  $i = 1$  and  $h^i := h^c$  otherwise. We say that  $H$  is *nested* if every pair  $h, k \in H$  are nested.

**1.1. Orientations.** We give the standard definition of orientations on  $H$  and characterize them as ‘consistent assignments of half-spaces to hyperplanes’.

**Definition 1.2.** An *orientation* on  $H$  is a subset  $U \subseteq H$  such that

1. (*Upward-closure*). If  $h \in U$  and  $k \in H$  contains  $h$ , then  $k \in U$ .
2. (*Ultra*). For each  $h \in H$ , exactly one of  $h, h^c$  is contained in  $U$ .

Consider the equivalence relation  $\sim$  on  $H$  generated by  $h \sim h^c$  for all  $h \in H$ , whose classes are called *hyperplanes*  $\partial h := \{h, h^c\}$  where  $\partial : H \rightarrow H/\sim$  is the projection. We show that an orientation on  $H$  is just a section  $\varphi : H/\sim \rightarrow H$  of  $\partial$ , which gives a consistent choice of half-space for each hyperplane.

**Proposition 1.3.** *An orientation  $p \subseteq H$  is exactly the data of a section  $\varphi$  of  $\partial$  such that  $\varphi(\partial h) \not\subseteq \varphi(\partial k)^c$  for every  $h, k \in H$ .*

*Proof.* Given an orientation  $p \subseteq H$ , let  $\varphi_p(\partial h) := h^i \in U$  for the unique  $i \in \{1, -1\}$ . That  $\varphi_p(\partial h) \in \partial h$  is clear, and if  $\varphi_p(\partial h) \subseteq \varphi_p(\partial k)^c$ , then  $U$  contains both  $\varphi_p(\partial k)$  and  $\varphi_p(\partial k)^c$  by upward-closure, a contradiction.

Conversely, given such a section  $\varphi : H/\sim \rightarrow H$ , let  $p_\varphi := \text{im } \varphi \subseteq H$ . This is ultra since if  $h \in H$  and  $h^c \notin p_\varphi$ , then  $\varphi(\partial h) \in \partial h = \{h, h^c\}$  implies  $h \in p_\varphi$ . Furthermore, if  $p_\varphi \ni h \subseteq k$ , then  $k^c \in p_\varphi$  implies  $\varphi(\partial h) = h \subseteq k = \varphi(\partial k)^c$ , a contradiction, so  $k \in p_\varphi$  by the above.

Finally, given an orientation  $p \subseteq H$ , we have  $h \in p$  iff  $\varphi_p(\partial h) = h$ , which occurs iff  $h \in \text{im } \varphi_p = p_{\varphi_p}$ . Thus  $p_{\varphi_p} = p$ . Conversely, given  $\varphi : H/\sim \rightarrow H$ , and  $h \in H$ , we have  $\varphi_{p_\varphi}(\partial h) = h^i$  iff  $h^i \in p_\varphi = \text{im } \varphi$ , which occurs iff  $\varphi(\partial h) = h^i$ . Thus  $\varphi_{p_\varphi} = \varphi$  too, as desired. ■

**Definition 1.4.** A *base* for an orientation  $p \subseteq H$  is a  $\subseteq$ -minimal subset  $p_0 \subseteq p$  such that  $p = \uparrow p_0$ , where

$$\uparrow p_0 := \bigcup_{h \in p_0} \uparrow h := \bigcup_{h \in p_0} \{k \in H : k \supseteq h\}.$$

We say that  $p$  is *based* if it admits a basis.

In the above correspondence, a base for  $\varphi : H/\sim \rightarrow H$  is a function  $\varphi_0 \subseteq \varphi$  where  $\text{dom } \varphi_0$  is a minimal subset of hyperplanes for which  $\varphi_0$  extends uniquely to  $\varphi$ .

**1.2. Flipping basis elements.** We now investigate the behaviour of an orientation when its choice on a single half-space is modified. Although the proofs work without the characterization in Proposition 1.3, it makes orientations more intuitive to me, and so will be phrased this way.

**Lemma 1.5.** *Let  $p \subseteq H$  be an orientation. Then  $q := p \triangle \{h, h^c\}$  is an orientation iff  $h \in p$  is  $\subseteq$ -minimal. Furthermore, if  $p$  is based, then so is  $q$ .*

*Proof.* First, note that  $\varphi_q(\partial h) = \varphi_p(\partial h)^c = h^c$  and  $\varphi_q = \varphi_p$  away from  $\partial h$ .

( $\Rightarrow$ ). If  $q$  is an orientation and  $p \ni k \subsetneq h$ , then we have a contradiction since  $\partial k \neq \partial h$  and hence

$$\varphi_q(\partial k) = \varphi_p(\partial k) = k \subseteq h = \varphi_q(\partial h)^c.$$

( $\Leftarrow$ ). Conversely, let  $h \in p$  be  $\subseteq$ -minimal and suppose that  $q$  is not an orientation. Since only  $\varphi_q(\partial h) = h^c$  differs from  $\varphi_p$ , this can only occur if  $\varphi_q(\partial h) \subseteq \varphi_q(\partial k)^c$  for some  $k \in H$ . But then

$$h^c = \varphi_q(\partial h) \subseteq \varphi_q(\partial k)^c = k^i \notin p$$

for some unique  $i \in \{1, -1\}$ , so that  $p \ni k^{-i} \subseteq h$  and contradicts that  $h \in p$  is  $\subseteq$ -minimal.

Finally, suppose that  $p$  is based. ???.

**Remark 1.6.** If  $h \in p$  is  $\subseteq$ -minimal and  $p_0 \subseteq p$  is any basis, then  $h \in p_0$ . Indeed, if  $h \notin p_0$ , then there is some  $h_0 \in p_0 \subseteq p$  with  $h \supset h_0$ , contradicting the minimality of  $h$ .

Conversely, if  $p_0 \subseteq p$  is a basis, then every  $h \in p_0$  is  $\subseteq$ -minimal in  $p$ . Indeed, if  $h \supset k$  for some  $k \in p$ , then there is some  $k_0 \in p_0$  with  $k \supseteq k_0$ , so  $h \supset k_0$ . Thus  $p = \uparrow(p_0 \setminus \{h\})$ , contradicting the minimality of  $p_0$ .

## 2. THE GRAPHS $\mathcal{T}_H$ AND $\mathcal{T}_H^\circ$

Let  $H \subseteq 2^X$  be a sub-pocset as before. Using Lemma 1.5, we construct a graph  $\mathcal{T}_H$  whose:

- *Vertices* are orientations  $p \subseteq H$ ;
- *Edges* are pairs  $\{p, q\}$  such that  $q = p \triangle \{h, h^c\}$  for some  $\subseteq$ -minimal  $h \in H$ .

The goal of this section is to establish Theorem A, stating that if  $H$  is nested, then  $\mathcal{T}_H$  is acyclic (Corollary 2.5), and Theorem B, that the induced subgraph  $\mathcal{T}_H^\circ \subseteq \mathcal{T}_H$  consisting of based orientations is a tree when  $H$  is also interval-finite (Proposition 2.8).

For the rest of this section, we let  $H \subseteq 2^X$  be a *nested* sub-pocset.

**2.1. Acyclicity of  $\mathcal{T}_H$ .** Essentially,  $\mathcal{T}_H$  is acyclic because any path in  $\mathcal{T}_H$  (without backtracking) is induced by a sequence of the flipped half-spaces, and those form a *strictly*-increasing chain.

**Example 2.1.** This fails if  $H$  is non-nested. Indeed, let  $X := 2 \times 2 := \{(x, y) : x, y \in 2\}$  and consider the set  $H$  of all fibers  $h_y := \{(x, y) : x \in 2\}$  for each  $y \in 2$  and  $h_x := \{(x, y) : y \in 2\}$  for each  $x \in 2$ . All orientations on  $H$  are principal – of the form  $\{h^x, h_y\}$  for  $x, y \in 2$  – and the resulting graph  $\mathcal{T}_H$  is a square.

**Definition 2.2.** Fix  $p_0 \in V(\mathcal{T}_H)$  and  $n \in \mathbb{N}$ . A sequence  $(h_i)_{i < n} \subseteq H$  is said to *induce a path in  $\mathcal{T}_H$  from  $p_0$*  if  $(p_i)_{i < n}$ , defined by  $p_i := p_{i-1} \triangle \{h_{i-1}, h_{i-1}^c\}$  for every  $1 \leq i < n$ , is a path in  $\mathcal{T}_H$  with each  $h_i \in p_i$ .

**Lemma 2.3.** *A path from  $p_0$  induced by  $(h_i)_{i < n}$ ,  $n \geq 3$ , has no backtracking iff  $h_i \neq h_{i-1}^c$  for every  $1 \leq i < n$ .*

*Proof.* Take  $2 \leq i \leq n$ . It suffices to show that  $p_{i-2} = p_i$  iff  $h_{i-1} = h_{i-2}^c$ .

( $\Rightarrow$ ). We have by definition that  $p_i = p_{i-2} \cup \{h_{i-1}^c, h_{i-2}^c\} \setminus \{h_{i-1}, h_{i-2}\}$ , so since  $h_{i-2} \in p_{i-2} = p_i$ , we have  $h_{i-2} = h_{i-1}^c$  as desired.

( $\Leftarrow$ ). Again by definition, by noting that the basis-flipping cancels out.

**Proposition 2.4.** *If  $(h_i)_{i < n}$  induces a path in  $\mathcal{T}_H$  with no backtracking, then  $(h_i)_{i < n}$  is strictly increasing.*

*Proof.* By Lemma 2.3, we have  $h_i \neq h_{i-1}^c$  for every  $1 \leq i < n$ . Thus, since  $h_i \in p_i = p_{i-1} \cup \{h_{i-1}^c\} \setminus \{h_{i-1}\}$ , we see that  $h_i \in p_{i-1}$ . Clearly  $h_i \neq h_{i-1}$ . It suffices to remove the three cases when  $h_i \subseteq h_{i-1}$ ,  $h_{i-1} \subseteq h_i^c$ , and  $h_i^c \subseteq h_{i-1}$ , since then nestedness of  $H$  gives us  $h_{i-1} \subsetneq h_i$ , as desired.

- If  $h_i \subseteq h_{i-1}$ , then  $h_{i-1} \in p_i$ , contradicting the definition of  $p_i$ .
- If  $h_{i-1} \subseteq h_i^c$ , then  $h_i^c \in p_{i-1}$  by upward-closure of  $p_{i-1}$ , a contradiction.
- If  $h_i^c \subseteq h_{i-1}$ , then  $h_{i-1} \in p_{i+1}$  by upward-closure of  $p_{i+1} \ni h_i^c$ . But since  $h_{i-1} \neq h_i^c$ , we have by definition of  $p_{i+1}$  that  $h_{i-1} \in p_i$ , a contradiction.

**Corollary 2.5.**  *$\mathcal{T}_H$  is acyclic.*

**2.2. Connectedness of  $\mathcal{T}_H^\circ$ .** In general,  $\mathcal{T}_H^\circ$  need not be connected, as shown by the following

**Example 2.6** (Tserunyan). Let  $X := \omega_1 := \omega + 1 := \omega \sqcup \{\omega\}$  and let

$$H := \{[0, n] : n < \omega\} \cup \{(n, \omega] : n < \omega\} \cup \{\omega, \{\omega\}\}.$$

We claim that every based orientation  $p \subseteq H$  is principal (i.e.,  $p = \hat{x} := \{h \in H : h \ni x\}$  for some  $x \in X$ ).

*Proof.* For each  $h \in H$ , the hyperplane  $\partial h$  is either  $\{[0, n], (n, \omega]\}$  for some  $n < \omega$ , or  $\{\omega, \{\omega\}\}$ . Consider the set  $N_p := \{n < \omega : [0, n] \in p\}$ , for which we have two cases.

- If  $N_p = \emptyset$ , then  $(n, \omega] \in p$  for all  $p$ . We claim that  $p = \hat{\omega}$ . If  $\omega \in p$ , then  $p$  is not based since  $(n, \omega]$ , for  $n$  large enough, does not contain any basis element. Thus  $\{\omega\} \in p$ . Both types of half-spaces contain  $\omega$ , and those are the only half-spaces in  $p$ .
- If  $N_p \neq \emptyset$ , then there is a minimal  $n < \omega$  with  $[0, n] \in p$ ; we claim that  $p = \hat{n}$ . Indeed, we have  $(m, \omega] \in p$  if  $m < n$ ,  $[0, m] \in p$  if  $m \geq n$ , and  $\omega \in p$  by upward-closure. All three types of half-spaces contain  $n$ , and those are the only half-spaces in  $p$ . ■

Now, each orientation  $\hat{n}$  is generated by the basis  $\{[0, n], (n-1, \omega]\}$ , and  $\hat{\omega}$  is generated by  $\{\{\omega\}\}$ . Thus, we see that  $\hat{\alpha}, \hat{\beta}$ , where  $\alpha < \beta \leq \omega$ , is joined by an edge iff  $\beta = \alpha + 1$ , where the  $\subseteq$ -minimal element  $[0, \alpha] \in \hat{\alpha}$  is flipped to  $(\alpha, \omega] = (\beta-1, \omega] \in \hat{\beta}$ . This shows that  $\mathcal{T}_H^\circ$  is disconnected, with  $\omega$  an isolated point.

A sufficient criterion ruling out Example 2.6 is given in Proposition 2.8, which needs the following

**Lemma 2.7.** *There is a path between  $p, q \in V(\mathcal{T}_H)$  iff  $p \triangle q$  is finite, in which case  $d(p, q) = (p \triangle q)/2$ .*

*Proof.* If  $p \triangle q = \{h_1, \dots, h_n\} \sqcup \{k_1, \dots, k_m\}$  with  $p \setminus q = \{h_i\}$  and  $q \setminus p = \{k_j\}$ , then  $h_i^c \in q \setminus p$  and  $k_j^c \in p \setminus q$  for all  $i < n$  and  $j < m$ , so  $n = m$  and  $q = p \cup \{h_i^c\} \setminus \{h_i\}$ . We claim that there is a permutation  $\sigma \in S_n$  such that  $(h_{\sigma(i)})$  induces a path from  $p =: p_0$ , which is the desired path from  $p$  to  $q$ . Choose a minimal  $h \in \{h_i\}$ , which is also minimal in  $p$ : if  $k \subseteq h$  for some  $k \in p$ , then  $h^c \subseteq k^c$ , and hence  $k^c \in q$ , so  $k = h_i \subseteq h$  for some  $i$ , forcing  $k = h$ . Set  $p_1 := p \triangle \{h, h^c\}$ , which is an orientation by Lemma 1.5. Continuing in this manner by choosing a minimal element in  $\{h_i\} \setminus \{h\}$  – and so on – gives us the desired path with  $d(p, q) = n$ .

Conversely, let  $(p_i)_{i < n}$  be a path from  $p = p_0$  to  $q = p_{n-1}$ . Then letting  $h_i := p_i \triangle p_{i-1}$  for all  $1 \leq i < n$  gives us a sequence  $(h_i)_{i < n}$  inducing this path, whence  $p \triangle q$  consists of  $\{h_i\}$  and their complements; hence,  $p \triangle q$  is finite with  $p \triangle q = 2d(p, q)$ . ■

**Proposition 2.8.** *If  $H$  is interval-finite (i.e., for every  $h, h' \in H$ , the interval  $[h, h'] := \{k \in H : h \subseteq k \subseteq h'\}$  is finite), then  $\mathcal{T}_H^\circ$  is connected.*

*Proof.* Fix  $p, q \in V(\mathcal{T}_H^\circ)$ . Since they are based, we can use Remark 1.6 to choose minimal elements  $h_1 \in p \setminus q$  and  $h_2 \in q \setminus p$ , and by assumption there are finitely-many  $k \in H$  with  $h_1 \subseteq k \subseteq h_2^c$ , say  $k_1, \dots, k_n$ . We claim that  $p \triangle q \subseteq \{k_i\} \cup \{k_i^c\}$ , so the result follows from Lemma 2.7.

Indeed, it suffices to show that  $k \in \{k_i\}$  for every  $k \in p \setminus q$ , for then any  $k' \in q \setminus p$  is in  $k' \in \{k_i^c\}$ . Let  $k \in p \setminus q$ , which by minimality of  $h_1$ , implies  $k \not\subseteq h_1$ . Thus, either  $k^c \subseteq h_1$ ,  $k \subseteq h_1^c$ , or  $h_1 \subseteq k$  by nestedness of  $H$ , and the first two cases are impossible since either case forces  $h_1 \in q$ . A similar argument using the observation that  $k^c \in q \setminus p$ , and hence  $k^c \not\subseteq h_2$ , shows that  $k \subseteq h_2^c$ . Thus  $h_1 \subseteq k \subseteq h_2^c$ , as desired. ■