

Tree-like graphings of countable Borel equivalence relations

An exposition to

Tree-like graphings, wallings, and median graphings of equivalence relations

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Countable Borel equivalence relations

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Any Borel action $\Gamma \curvearrowright X$ of a countable (discrete) group on a standard Borel space induces its *orbit equivalence relation* E_Γ^X , which is a CBER.

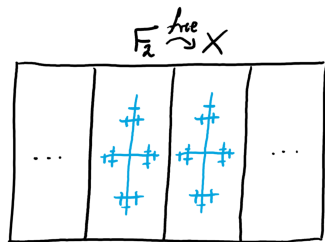
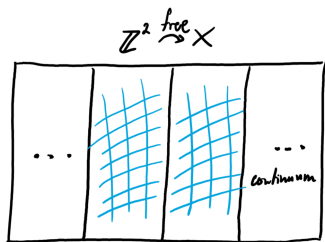
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Smooth and hyperfinite CBERs

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Theorem (Slaman-Steel, Weiss)

Let E be a CBER on a standard Borel space X . TFAE:

1. E is hyperfinite. $E = \bigcup_n F_n$ where $F_0 \subseteq F_1 \subseteq \dots$ are FBERs.
2. E is induced by a Borel \mathbb{Z} -action. $E = E_{\mathbb{Z}}^X$ for some $\mathbb{Z} \curvearrowright X$.

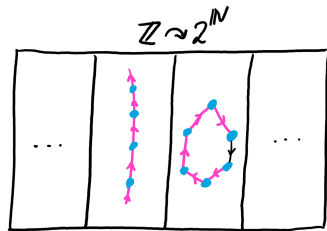
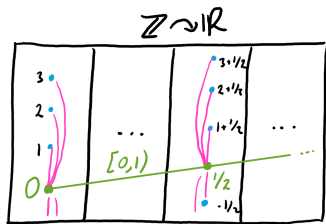
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Graphing of a CBER

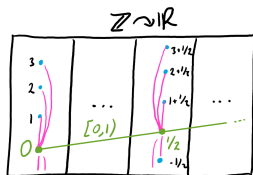
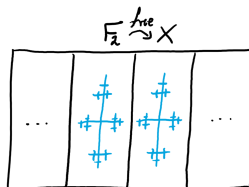
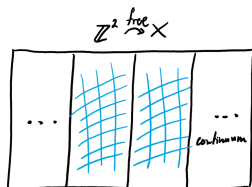
Definition

A *graphing* of a CBER E on X is a Borel graph $G \subseteq X^2$ whose connectedness relation is E ($xEy \leftrightarrow xG \cdots Gy$ for all $x, y \in X$).

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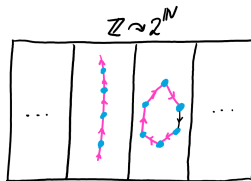
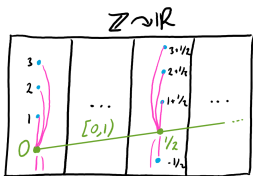
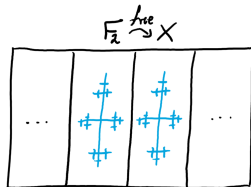
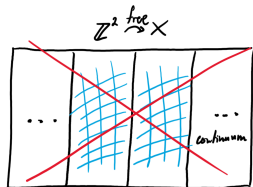
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Treeing of a CBER

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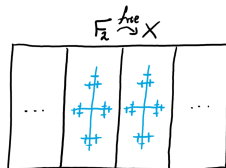
A *treeing* of a CBER E is an acyclic graphing, and a CBER E is said to be *treeable* if it admits a treeing.



Treeable CBERs

Example

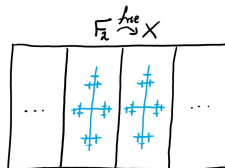
Free actions of a free group $F_r \curvearrowright X$.



Treeable CBERs

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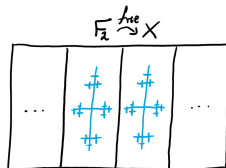
Theorem (JKL02)

Free actions of virtually-free groups are treeable.

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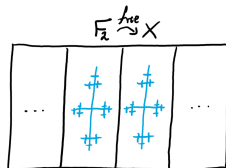
Theorem (GdlH90)

Every finitely-generated group whose Cayley graph is a quasi-tree is virtually-free, and hence treeable.

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Every finitely-generated group whose Cayley graph is a quasi-tree is virtually-free, and hence treeable.

Question (Robin Tucker-Drob; 2015)

Is the class of treeable CBERs robust under quasi-isometries?

Main result

Theorem (Chen, Poulin, Tao, Tserunyan; 2023+)

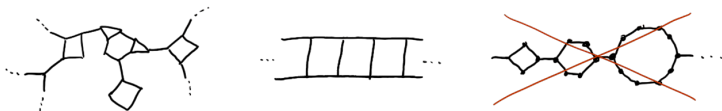
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Main result

Theorem (Chen, Poulin, Tao, Tserunyan; 2023+)

If a CBER E admits a locally-finite graphing such that each component is a quasi-tree, then E is treeable.

Two metric spaces X, Y are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error; X is a *quasi-tree* if it is quasi-isometric to a tree.



Game plan

Quasi-treeing

Treeing

Game plan

Quasi-treeing



Quasi-tree

Treeing



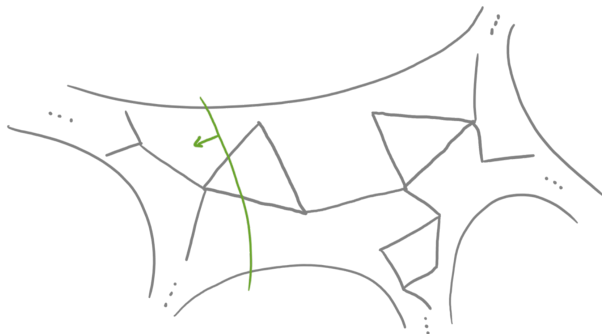
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Quasi-treeing

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family of cuts



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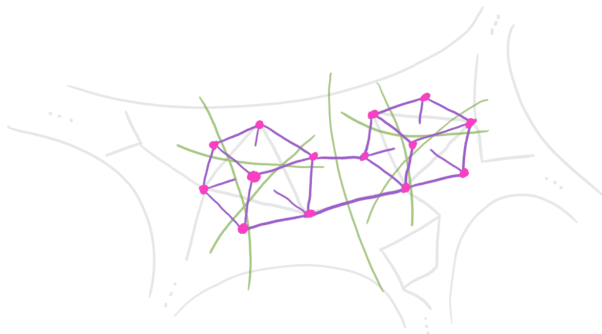
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Median graph w/
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Game plan

Quasi-treeing

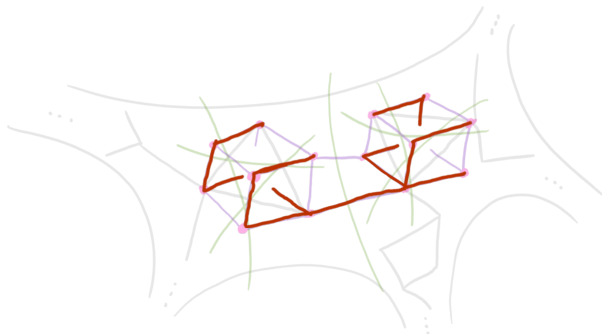
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Quasi-tree

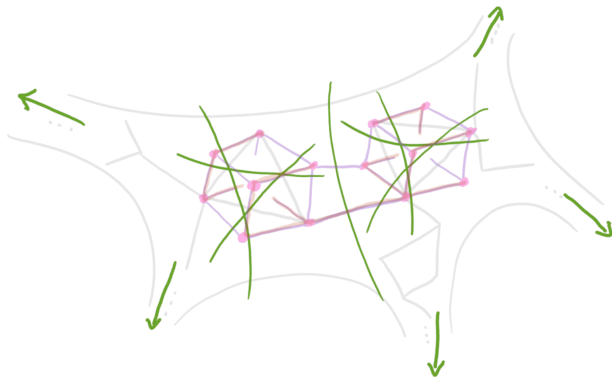
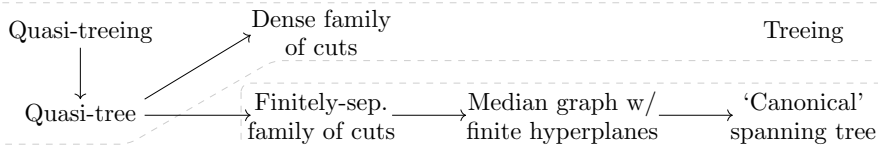
Finitely-sep.
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Median graph w/
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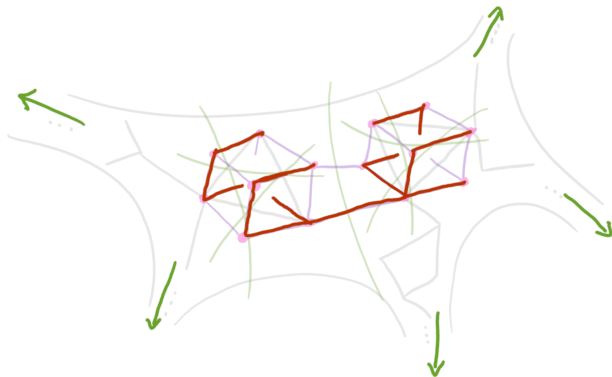
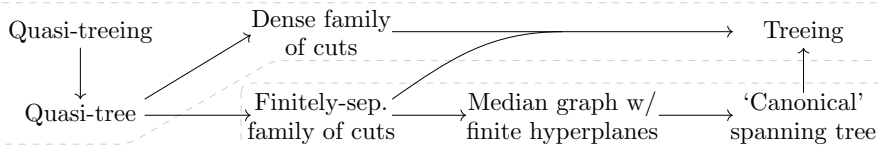
'Canonical'
spanning tree



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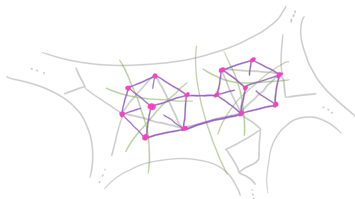
Definition

Such a family \mathcal{H} is *finitely-separating* if for each $x, y \in X$, there are finitely-many $H \in \mathcal{H}$ with $x \in H \not\ni y$.

Orientations

Definition

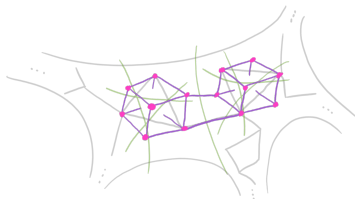
An *orientation* on \mathcal{H} is an upward-closed subset $U \subseteq \mathcal{H}$ containing exactly one of $H, \neg H$ for every $H \in \mathcal{H}$.



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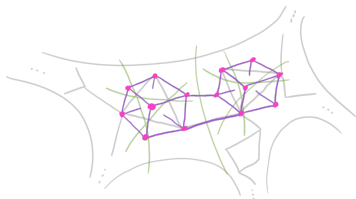


We'll only consider the orientations that are *based*, in the sense that each $H \in U$ contains a minimal $H_0 \in U$.

The dual median graph

Definition

A *median graph* is a connected graph (X, G) such that for each $x, y, z \in X$, the intersection $[x, y] \cap [x, z] \cap [y, z]$ is a singleton, called the *median* of x, y, z , and is denoted by $\langle x, y, z \rangle$.



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Theorem (Sageev 95)

If \mathcal{H} is finitely-separating, then the graph $\mathcal{M}(\mathcal{H})$:

- Vertices are based orientations on \mathcal{H} ;
- Neighbors of U are $U \triangle \{H, \neg H\}$ for each minimal $H \in U \setminus \{-0\}$;

is a median graph.

Ends of graphs

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The *end compactification* of a connected locally-finite (X, G) is the Stone space \hat{X} of the Boolean algebra $\mathcal{H}_{\partial < \infty}(X)$, whose non-principal ultrafilters are the *ends* of (X, G) .



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Definition

A family \mathcal{H} of cuts is *dense towards ends* of X if \mathcal{H} contains a neighborhood basis for every end in \hat{X} .

Density towards ends for quasi-trees

Lemma

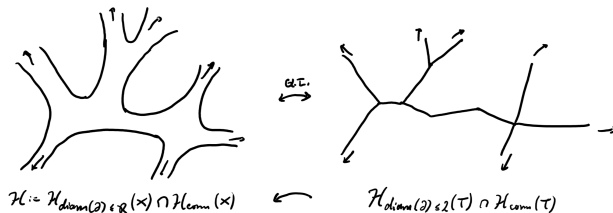
The connected locally-finite graphs in which $\mathcal{H}_{\text{diam}(\partial) \leq R}$ is dense towards ends for some $R < \infty$ is invariant under quasi-isometry.

Corollary

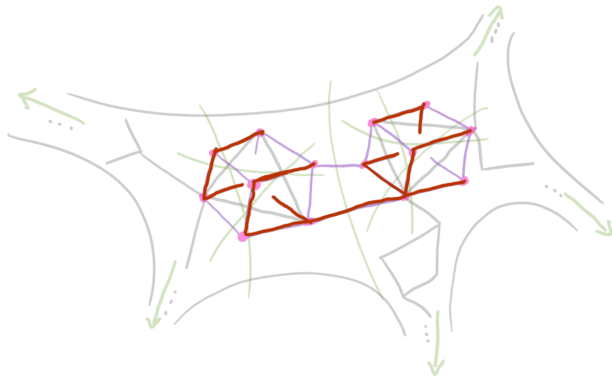
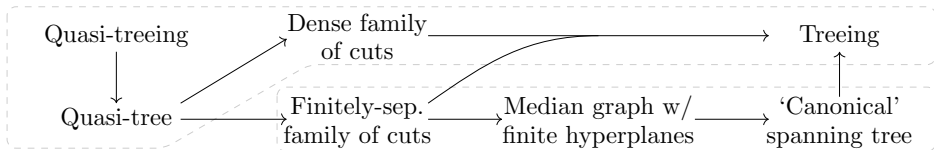
If (X, G) is a locally-finite quasi-tree, then the family

$$\mathcal{H} := \mathcal{H}_{\text{diam}(\partial) \leq R}(X) \cap \mathcal{H}_{\text{conn}}(X)$$

of cuts is dense towards ends for some $R < \infty$.



Wrapping things up...



The End

Thank you!