Tree-like graphings of countable Borel equivalence relations

An exposition to

Tree-like graphings, wallings, and median graphings of equivalence relations

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Countable Borel Equivalence Relations

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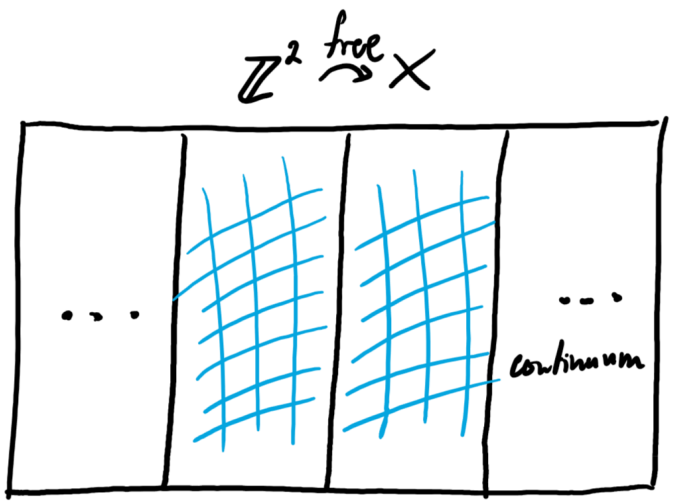

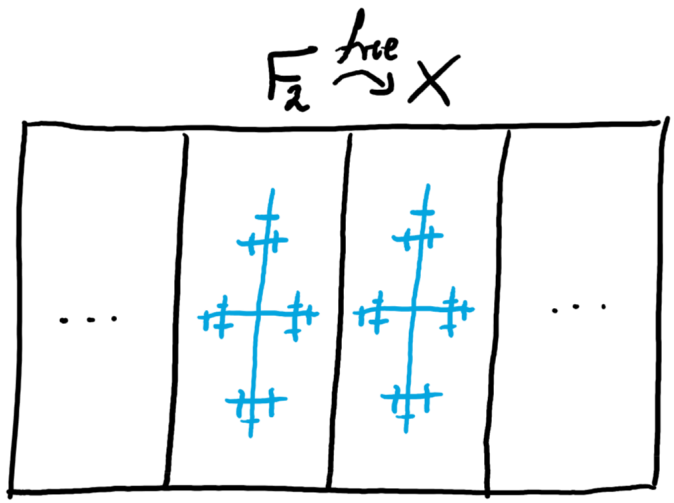

Smooth and Hyperfinite CBERs

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Theorem (Slaman-Steel, Weiss)

Let E be a CBER on a standard Borel space X. TFAE:

1. *E is hyperfinite.* $E = \bigcup_n F_n$ *where* $F_0 \subseteq F_1 \subseteq \cdots$ *are FBERs.*
2. *E is induced by a Borel $\mathbb{Z}$-action.* $E = E_{\mathbb{Z}}^X$ *for some* $\mathbb{Z} \curvearrowright X$.

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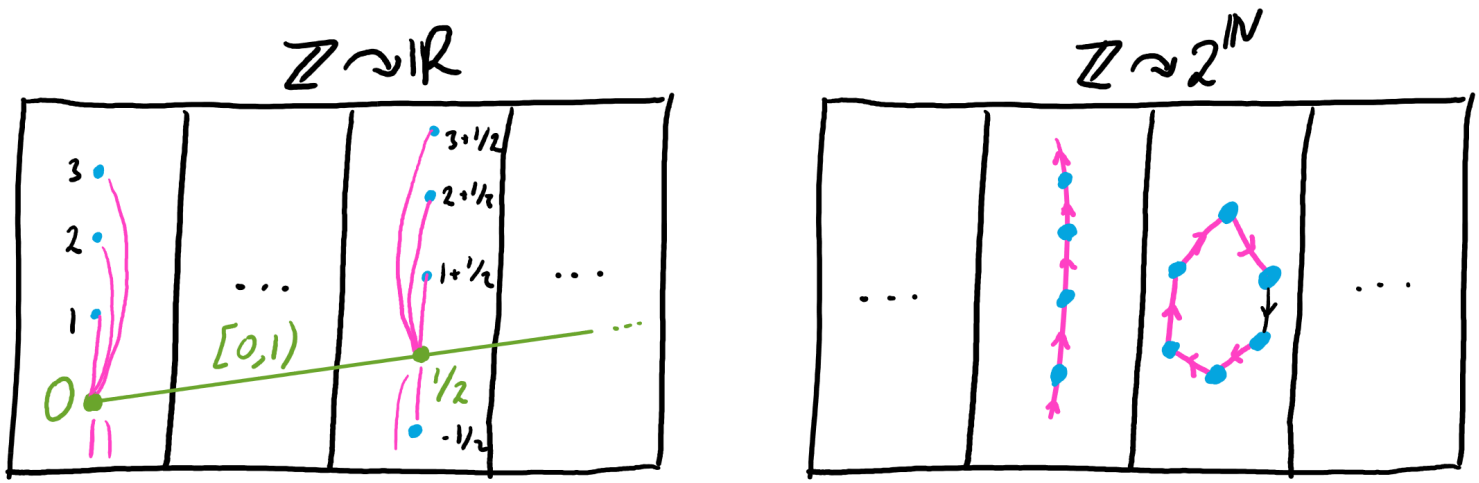

Graphing of a CBER

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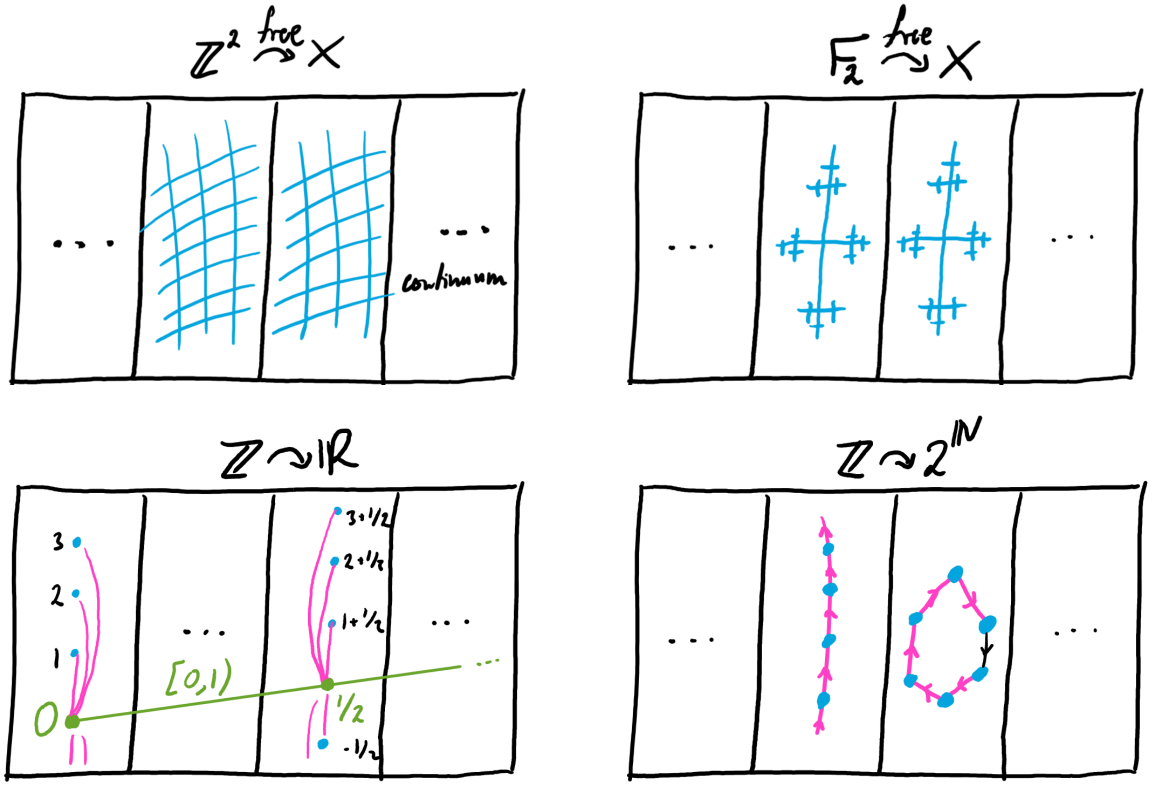
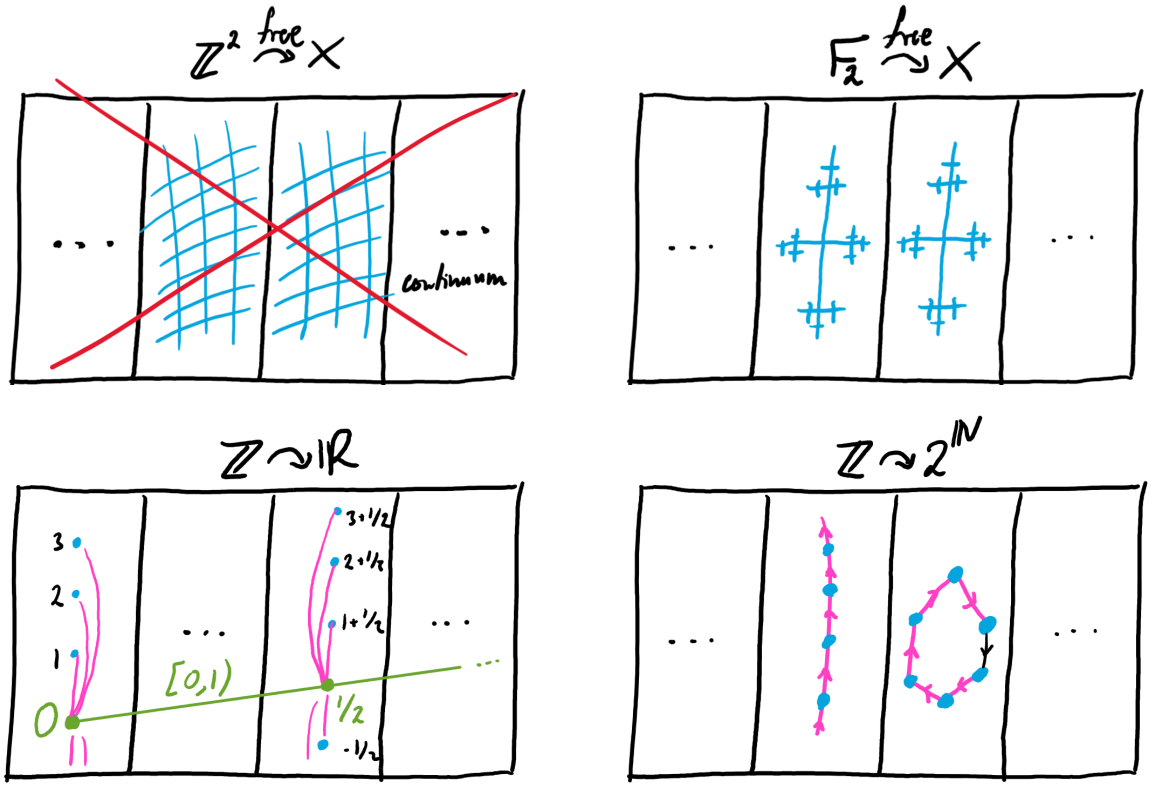


Treeings and Treeability

Definition

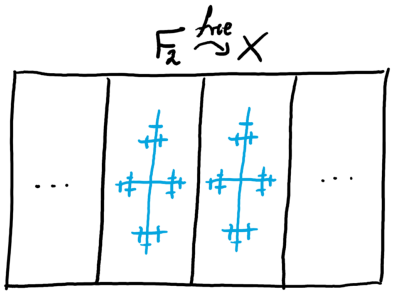
A *treeing* of a CBER E is an acyclic graphing, and a CBER E is said to be *treeable* if it admits a treeing.



Treeable CBERs

Example (Free Actions)

Any free action of a free group $F_r \curvearrowright X$.



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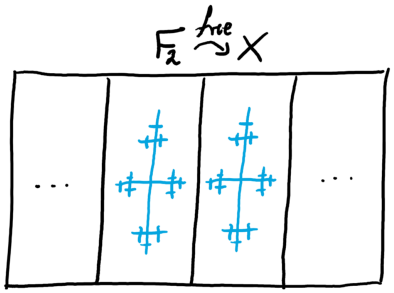

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Free actions of virtually-free groups are treeable.

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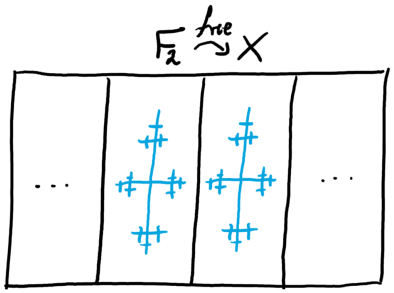

Theorem (JKL02)

Free actions of virtually-free groups are treeable.

Theorem (GdlH90)

Every finitely-generated group whose Cayley graph is a quasi-tree is virtually-free, and hence treeable.

Question (Robin Tucker-Drob; 2015)

Is the class of treeable CBERs robust under quasi-isometries?

Main Result

Theorem (Chen, Poulin, Tao, Tserunyan; 2023+)

If a CBER E admits a locally-finite graphing such that each component is a quasi-tree, then E is treeable.

The End

Thank you!