Tree-like graphings of countable Borel equivalence relations

An exposition to

Tree-like graphings, wallings, and median graphings of equivalence relations

by Ruiyuan Chen, Antoine Poulin, Ran Tao, and Anush Tserunyan

Zhaoshen Zhai

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Countable Borel equivalence relations

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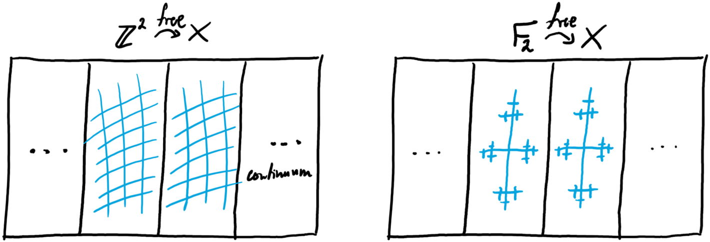

Smooth and hyperfinite CBERs

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This CBER is *hyperfinite*: $E_0 = \bigcup_n F_n$ for an increasing sequence $F_0 \subseteq F_1 \cdots$ of finite Borel equivalence relations:

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Theorem (Slaman-Steel, Weiss)

Let E be a CBER on a standard Borel space X. TFAE:

1. *E is hyperfinite.* *$E = \bigcup_n F_n$ where $F_0 \subseteq F_1 \subseteq \cdots$ are FBERs.*
2. *E is induced by a Borel $\mathbb{Z}$-action.* *$E = E_{\mathbb{Z}}^X$ for some $\mathbb{Z} \curvearrowright X$.*

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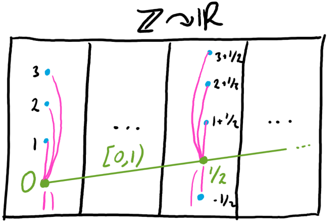

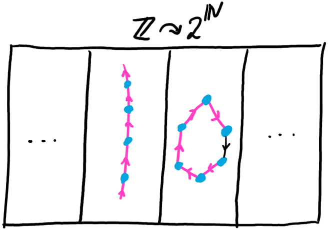

Graphing of a CBER

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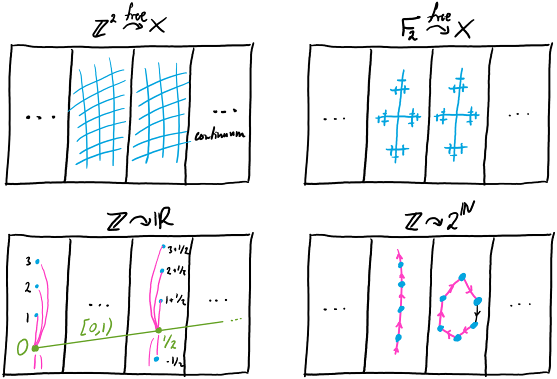
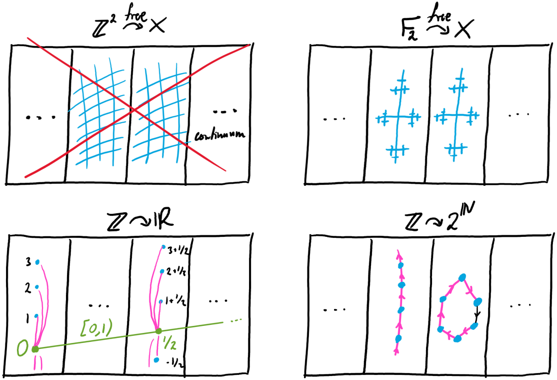


Treeings and treeability

Definition

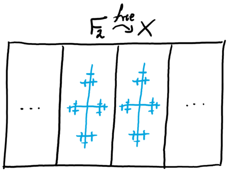
A *treeing* of a CBER E is an acyclic graphing, and a CBER E is said to be *treeable* if it admits a treeing.



Treeable CBERs

Example (Free Actions)

Free actions of a free group $F_r \curvearrowright X$.



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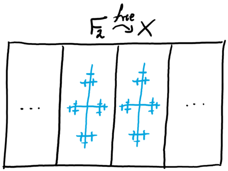

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Free actions of virtually-free groups are treeable.

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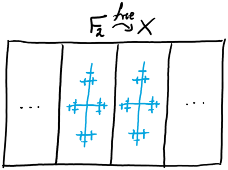

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Every finitely-generated group whose Cayley graph is a quasi-tree is virtually-free, and hence treeable.

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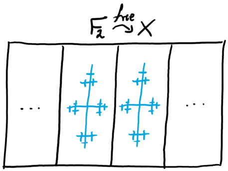

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Theorem (GdlH90)

Every finitely-generated group whose Cayley graph is a quasi-tree is virtually-free, and hence treeable.

Question (Robin Tucker-Drob; 2015)

Is the class of treeable CBERs robust under quasi-isometries?

Main result

Theorem (Chen, Poulin, Tao, Tserunyan; 2023+)

If a CBER E admits a locally-finite graphing such that each component is a quasi-tree, then E is treeable.

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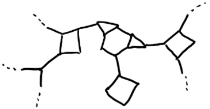
Two metric spaces X, Y are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error; X is a *quasi-tree* if it is quasi-isometric to a tree.



Game plan

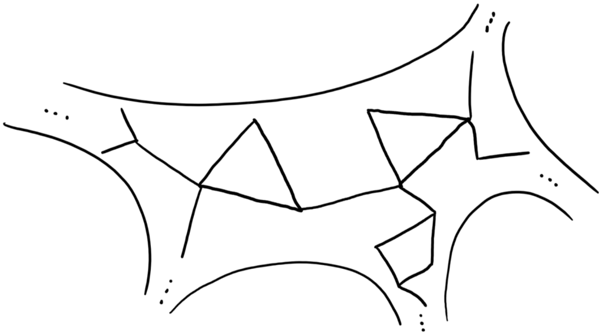
Quasi-treeing

Treeing

Game plan

Quasi-treeing → Quasi-tree

Treeing



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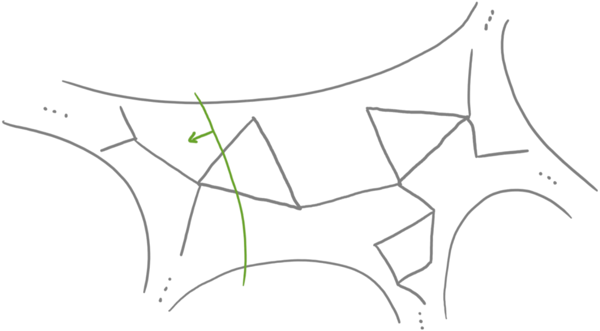
$$\begin{array}{c}\text{Quasi-treeing}\\ \downarrow \\ \text{Quasi-tree} \longrightarrow \text{Finitely-sep. family of cuts}\end{array}$$



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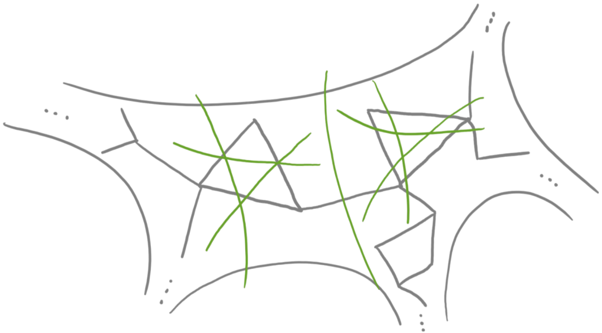
Quasi-treeing ↓ Quasi-tree → Finitely-sep. family of cuts

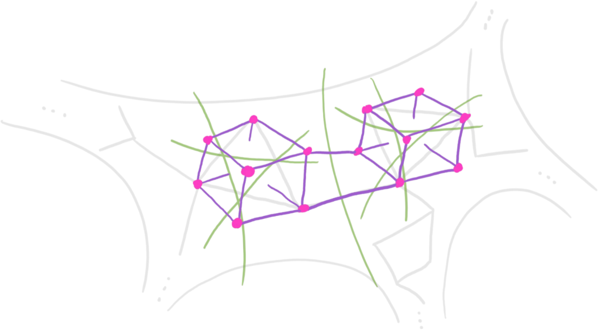
Treeing



Game plan

Quasi-treeing $\longrightarrow$ Treeing

Quasi-treeing
$\downarrow$
Quasi-tree $\longrightarrow$ Finitely-sep. family of cuts $\longrightarrow$ Median graph w/ finite hyperplanes



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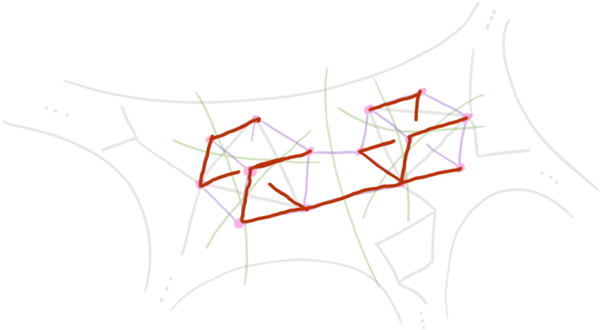

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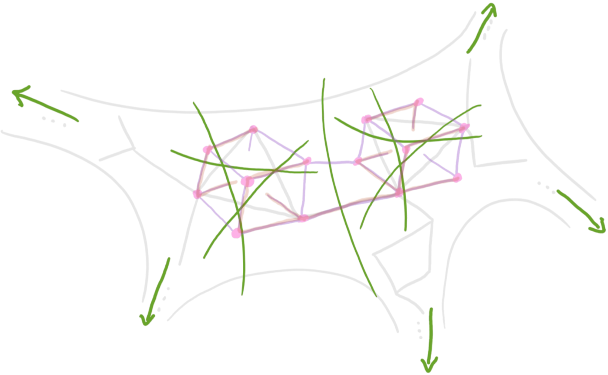

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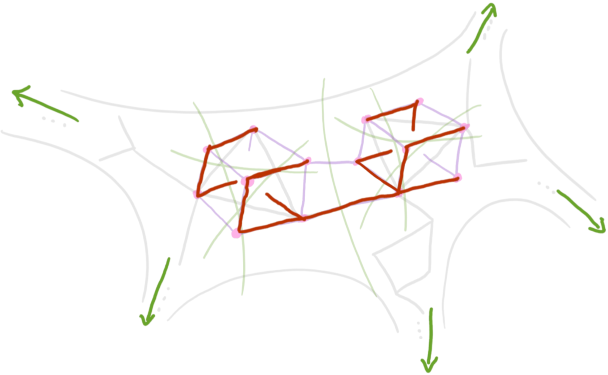

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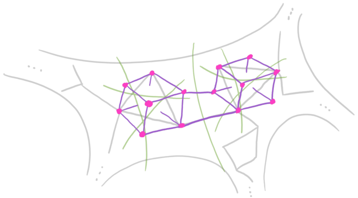
Definition

Such a family $\mathcal{H}$ is *finitely-separating* if for each $x, y \in X$, there are finitely-many $H \in \mathcal{H}$ with $x \in H \not\ni y$.

Orientations

Definition

An *orientation* on $\mathcal{H}$ is an upward-closed subset $U \subseteq \mathcal{H}$ containing exactly one of $H, \neg H$ for every $H \in \mathcal{H}$.



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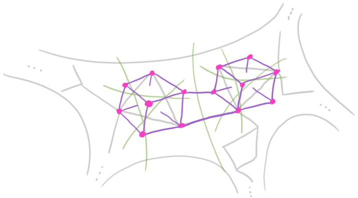
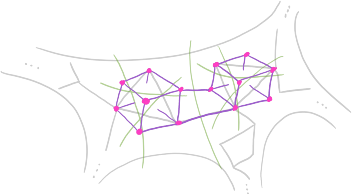


We'll only consider the orientations that are *based*, in the sense that each $H \in U$ contains a minimal $H_0 \in U$.

The dual median graph

Definition

A *median graph* is a connected graph (X, G) such that for each $x, y, z \in X$, the intersection $[x, y] \cap [x, z] \cap [y, z]$ is a singleton, called the *median* of x, y, z, and is denoted by $\langle x, y, z \rangle$.



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Theorem (Sageev 95)

If $\mathcal{H}$ is finitely-separating, then the graph $\mathcal{M}(\mathcal{H})$:

- *Vertices are based orientations on $\mathcal{H}$;*
- *Neighbors of U are $U \triangle \{H, \neg H\}$ for each minimal $H \in U \setminus \{\neg 0\}$;*

is a median graph.

Ends of graphs

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Definition

A family $\mathcal{H}$ of cuts is *dense towards ends* of X if $\mathcal{H}$ contains a neighborhood basis for every end in $\widehat{X}$.

Density towards ends for quasi-trees

Lemma

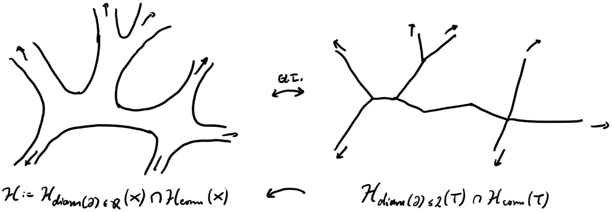
The connected locally-finite graphs in which $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}$ is dense towards ends for some $R < \infty$ is invariant under quasi-isometry.

Corollary

If (X, G) is a locally-finite quasi-tree, then the family

$$\mathcal{H} := \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$$

of cuts is dense towards ends for some $R < \infty$.



Wrapping things up...

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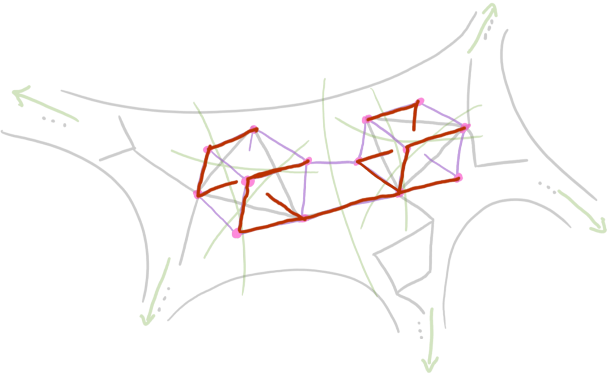

The End

Thank you!