TREE-LIKE GRAPHINGS

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Abstract. We present a streamlined exposition of a construction presented in [CPTT23], where it is proven that every locally-finite Borel graph with each component a quasi-tree induces a canonical treeable equivalence relation. Write some more details...

1. Introduction

The purpose of this note is to provide a streamlined proof of a particular case of a construction presented in [CPTT23], in order to better understand the general formalism developed therein. We attempt to make this note self-contained, but nevertheless urge the reader to refer to the original paper for more detailed discussion and generalizations of the results we have selected to include here.

1.1. Treeings of equivalence relations.

A *countable Borel equivalence relation (CBER)* on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$ with each class countable. We are interested in special types of *graphings* on a CBER $E \subseteq X^2$, i.e. a Borel graph $G \subseteq X^2$ whose connectedness relation is precisely E. For instance, a graphing of E such that each component is a tree is called a *treeing* of E, and the CBERs that admit a treeing are said to be *treeable*. The main results of [CPTT23] is to provide new sufficient criteria for treeability of CBERs, and in particular, they prove the following

Theorem A ([CPTT23, Theorem 1.1])**.** *If a CBER admits a locally-finite graphing whose components are quasi-trees, then it is treeable.*

Recall that metric spaces X and Y are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error, and X is a *quasi-tree* if it is quasi-isometric to a simplicial tree; see [Gro93] and [DK18].

1.2. Outline of the proof.

Roughly speaking, the existence of a quasi-isometry $G|C \to T_C$ to a simplicial tree T_C for each component $C \subseteq X$ induces a collection $\mathcal{H}(C)$ of 'cuts' (subsets $H \subseteq C$ with finite boundary such that both H and $C \setminus H$ are connected), which are 'tree-like' in the sense that

1. $\mathcal{H}(C)$ is *finitely-separating*: each pair $x, y \in C$ are separated by finitely-many $H \in \mathcal{H}(C)$, and
2. $\mathcal{H}(C)$ is *dense towards ends*: each end in $G|C$ has a neighborhood basis in $\mathcal{H}(C)$.

These cuts have the structure of a pocset with non-trivial points isolated, which in turn provide exactly the data to construct a 'median graph' whose vertices are 'ultrafilters' thereof. Condition (2) then ensures that this graph has finite 'hyperplanes', which allows us to apply a Borel 'cycle-cutting' algorithm and obtain a canonical spanning tree thereof. Each step above can be done in a uniform way to each component $C \subseteq G$, giving us the desired treeing of the CBER.

Quasi-tree $\xrightarrow{2.1}$ Dense family of cuts $\xrightarrow{2.2}$ Pocset w/ non-triv. pts. isolated $\xrightarrow{2.3}$ Median graph w/ finite hyperplanes $\xrightarrow{2.4}$ Canonical spanning tree

Remark. We follow [CPTT23, Convention 2.7], where for a family $\mathcal{H} \subseteq 2^X$ of subsets of a fixed set X, we write $\mathcal{H}^* := \mathcal{H} \setminus \{\varnothing, X\}$ for the *non-trivial* elements of $\mathcal{H}$.

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2. Detailed Constructions

2.1. Graphs with dense families of cuts.

Let (X, G) be a locally-finite graph.

2.1.1. *End compactification of graphs.*

2.2. Pocsets of cuts.

The family $\mathcal{H}_{\operatorname{diam}\partial\leq R}(X)$ of cuts have the structure of a 'profinite pocset', which we first study abstractly. We then deduce some properties of the pocset induced by a *dense* family of cuts.

2.2.1. *Abstract profinite pocsets.*

Definition 2.1. A *pocset* $(P, \leq, \neg, 0)$ is a poset $(P, \leq)$ equipped with an order-reversing involution $\neg : P \to P$ and a least element $0 \neq \neg 0$ such that 0 is the only lower-bound of $p, \neg p$ for every $p \in P$.

A *profinite pocset* is a pocset P equipped with a compact topology making $\neg$ continuous and is *totally order-disconnected*, in the sense that if $p \not\leq q$, then there is a clopen upward-closed $U \subseteq P$ with $p \in U \not\ni q$.

Remark 2.2. Such a topology is automatically Hausdorff and zero-dimensional.

We are primarily interested in subpocsets of $(2^X, \subseteq, \neg, \varnothing)$, which is profinite if equipped with the product topology of the discrete space 2. Indeed, 2^X admits a base of *cylinder sets* – which are finite intersections of sets of the form $\pi_x^{-1}(i)$ where $x \in X$, $i \in \{0, 1\}$, and $\pi_x : 2^X \to 2$ is the projection – making $\neg$ continuous since cylinders are clopen. Show that it is totally order-disconnected.

The following proposition gives a sufficient criteria for subpocsets of 2^X to be profinite. We also show in this case that every non-trivial element $H \in \mathcal{H}^*$ is isolated, which will important in Section 2.3.

Proposition 2.3. *Let X be a set and $\mathcal{H} \subseteq 2^X$ be a subpocset. If $\mathcal{H}$ is finitely-separating, then $\mathcal{H} \subseteq 2^X$ is closed and every non-trivial element is isolated.*

Proof. It suffices to show that the limit points of $\mathcal{H}$ are trivial, so let $A \in 2^X \setminus \{\varnothing, X\}$. Fix $x \in A \not\ni y$. Since $\mathcal{H}$ is finitely-separating, there are finitely-many $H \in \mathcal{H}$ with $x \in H \not\ni y$, and for each such $H \in \mathcal{H} \setminus \{A\}$, we have either some $x_H \in A \setminus H$ or $y_H \in H \setminus A$. Let $U \subseteq 2^X$ be the family of all subsets $B \subseteq X$ containing x and each x_H but not y or any y_H.

This is the desired neighborhood isolating $A \in U$. Indeed, it is (cl)open since it is the *finite* intersection of cylinders prescribed by the x_H's and y_H's, and it is disjoint from $\mathcal{H} \setminus \{A\}$ by construction. ■

2.2.2. *The pocset induced by a dense family cuts.*

2.3. The dual median graph of a pocset.

Let $\mathcal{H} \subseteq 2^X$ be a finitely-separating pocset, whose elements we call *half-spaces*.

Definition 2.4. An *orientation* on $\mathcal{H}$ is an upward-closed subset $U \subseteq \mathcal{H}$ containing exactly one of $H, \neg H$ for each $H \in \mathcal{H}$. We let $\mathcal{U}(\mathcal{H})$ denote the set of all orientations on $\mathcal{H}$, and let $\mathcal{U}^\circ(\mathcal{H}) \subseteq \mathcal{U}(\mathcal{H})$ denote the clopen orientations on $\mathcal{H}$.

Remark 2.5. Intuitively, an orientation is a 'maximally consistent' choice of a half-space from each hyperplane. To formalize this, let $\sim$ be the equivalence relation on $\mathcal{H}$ given by $H \sim \neg H$, and let $\partial : \mathcal{H} \to \mathcal{H}/{\sim}$ be the quotient map. Orientations $U \subseteq \mathcal{H}$ then correspond precisely to sections $\varphi : \mathcal{H}/{\sim} \to \mathcal{H}$ of ∂ such that $\varphi(\partial H) \not\subseteq \neg\varphi(\partial K)$ for every $H, K \in \mathcal{H}$; the latter condition rules out 'orientations' of the form $\leftarrow\!\!\mid\,\mid\!\!\rightarrow$.

Theorem 2.6.

2.4. Spanning trees of median graphs with finite hyperplanes.

A. Tree of orientations on a nested pocset of sets

Let $\mathcal{H} \subseteq 2^X$ be a finitely-separating subpocset (see Definition 2.2.1), so every non-trivial element is isolated by Proposition 2.3. As shown in Theorem 2.6, the clopen orientations $\mathcal{U}^\circ(\mathcal{H})$ form the vertices of a median graph, whose edges are given by 'minimal base-flipping'.

We devote this appendix to show that if $\mathcal{H}$ is *nested*, then $\mathcal{U}^\circ(\mathcal{H})$ is in fact a tree, and so we may bypass the Borel cycle-cutting algorithm in Section 2.4 and obtain a treeing directly.

References

[CPTT23] Ruiyuan Chen, Antoine Poulin, Ran Tao, and Anush Tserunyan, *Tree-like graphings, wallings, and median graphings of equivalence relations* (2023), available at https://arxiv.org/abs/2308.13010.

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