Tree-like graphings of countable Borel equivalence relations

An exposition to

Tree-like graphings, wallings, and median graphings of equivalence relations

by Ruiyuan Chen, Antoine Poulin, Ran Tao, and Anush Tserunyan

Zhaoshen Zhai

October 1, 2024

Countable Borel equivalence relations

Definition

A *countable Borel equivalence relation (CBER)* on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$ such that each E-class is countable.

Countable Borel equivalence relations

Definition

A *countable Borel equivalence relation (CBER)* on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$ such that each E-class is countable.

Example

Any Borel action $\Gamma \curvearrowright X$ of a countable (discrete) group on a standard Borel space induces its *orbit equivalence relation* E_Γ^X, which is a CBER.

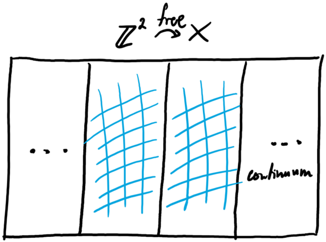
Countable Borel equivalence relations

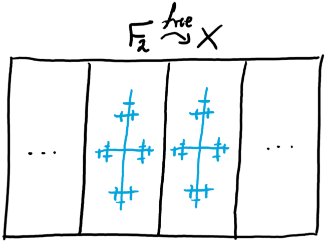
Definition

A *countable Borel equivalence relation (CBER)* on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$ such that each E-class is countable.

Example

Any Borel action $\Gamma \curvearrowright X$ of a countable (discrete) group on a standard Borel space induces its *orbit equivalence relation* E_Γ^X, which is a CBER.





Smooth and hyperfinite CBERs

Example (Smooth)

- Identity relation $=$ on a standard Borel space, say $\mathbb{R}$ or $2^{\mathbb{N}}$.

Smooth and hyperfinite CBERs

Example (Smooth)

- Identity relation $=$ on a standard Borel space, say $\mathbb{R}$ or $2^{\mathbb{N}}$.
- $\mathbb{Z}$-coset equivalence on $\mathbb{R}$: $x E_{\mathbb{Z}}^{\mathbb{R}} y$ iff $x - y \in \mathbb{Z}$.

Smooth and hyperfinite CBERs

Example (Smooth)

- Identity relation $=$ on a standard Borel space, say $\mathbb{R}$ or $2^{\mathbb{N}}$.
- $\mathbb{Z}$-coset equivalence on $\mathbb{R}$: $xE_{\mathbb{Z}}^{\mathbb{R}}y$ iff $x - y \in \mathbb{Z}$.

Example (Hyperfinite)

E_0 on $2^{\mathbb{N}}$, where xE_0y iff $\exists n \in \mathbb{N}, \forall m \geq n : x_m = y_m$.

Smooth and hyperfinite CBERs

Example (Smooth)

- Identity relation $=$ on a standard Borel space, say $\mathbb{R}$ or $2^{\mathbb{N}}$.
- $\mathbb{Z}$-coset equivalence on $\mathbb{R}$: $x E_{\mathbb{Z}}^{\mathbb{R}} y$ iff $x - y \in \mathbb{Z}$.

Example (Hyperfinite)

E_0 on $2^{\mathbb{N}}$, where $x E_0 y$ iff $\exists n \in \mathbb{N}, \forall m \geq n : x_m = y_m$.

This CBER is *hyperfinite*: $E_0 = \bigcup_n F_n$ for an increasing sequence $F_0 \subseteq F_1 \cdots$ of finite Borel equivalence relations:

Smooth and hyperfinite CBERs

Example (Smooth)

- Identity relation $=$ on a standard Borel space, say $\mathbb{R}$ or $2^{\mathbb{N}}$.
- $\mathbb{Z}$-coset equivalence on $\mathbb{R}$: $xE_{\mathbb{Z}}^{\mathbb{R}}y$ iff $x-y\in\mathbb{Z}$.

Example (Hyperfinite)

E_0 on $2^{\mathbb{N}}$, where xE_0y iff $\exists n\in\mathbb{N},\forall m\geq n: x_m=y_m$.

This CBER is *hyperfinite*: $E_0=\bigcup_n F_n$ for an increasing sequence $F_0\subseteq F_1\cdots$ of finite Borel equivalence relations:

$$xF_ny \quad \leftrightarrow \quad \forall m\geq n: x_m=y_m.$$

Smooth and hyperfinite CBERs

Example (Smooth)

- Identity relation $=$ on a standard Borel space, say $\mathbb{R}$ or $2^{\mathbb{N}}$.
- $\mathbb{Z}$-coset equivalence on $\mathbb{R}$: $x E_{\mathbb{Z}}^{\mathbb{R}} y$ iff $x - y \in \mathbb{Z}$.

Example (Hyperfinite)

E_0 on $2^{\mathbb{N}}$, where $x E_0 y$ iff $\exists n \in \mathbb{N}, \forall m \geq n : x_m = y_m$.

This CBER is *hyperfinite*: $E_0 = \bigcup_n F_n$ for an increasing sequence $F_0 \subseteq F_1 \cdots$ of finite Borel equivalence relations:

$$x F_n y \quad \leftrightarrow \quad \forall m \geq n : x_m = y_m.$$

Theorem (Slaman-Steel, Weiss)

Let E be a CBER on a standard Borel space X. TFAE:

1. *E is hyperfinite.* *$E = \bigcup_n F_n$ where $F_0 \subseteq F_1 \subseteq \cdots$ are FBERs.*
2. *E is induced by a Borel $\mathbb{Z}$-action.* *$E = E_{\mathbb{Z}}^{X}$ for some $\mathbb{Z} \curvearrowright X$.*

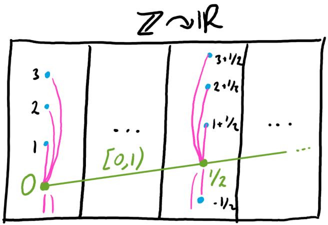
Smooth and Hyperfinite CBERs

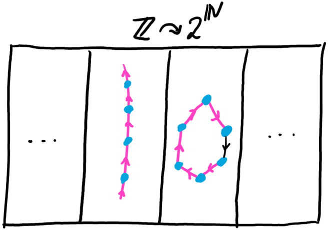
Example (Smooth)

- Identity relation $=$ on a standard Borel space, say $\mathbb{R}$ or $2^{\mathbb{N}}$.
- $\mathbb{Z}$-coset equivalence on $\mathbb{R}$: $x E_{\mathbb{Z}}^{\mathbb{R}} y$ iff $x - y \in \mathbb{Z}$.

Example (Hyperfinite)

E_0 on $2^{\mathbb{N}}$, where $x E_0 y$ iff $\exists n \in \mathbb{N}, \forall m \geq n : x_m = y_m$.





Graphing of a CBER

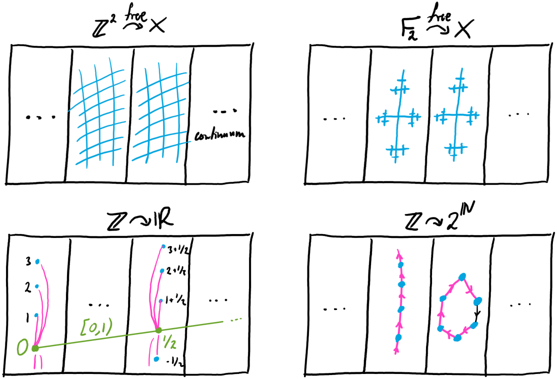
Definition

A *graphing* of a CBER E on X is a Borel graph $G \subseteq X^2$ whose connected relation is E, i.e., $xEy \leftrightarrow xG \cdots Gy$ for all $x, y \in X$.

Graphing of a CBER

Definition

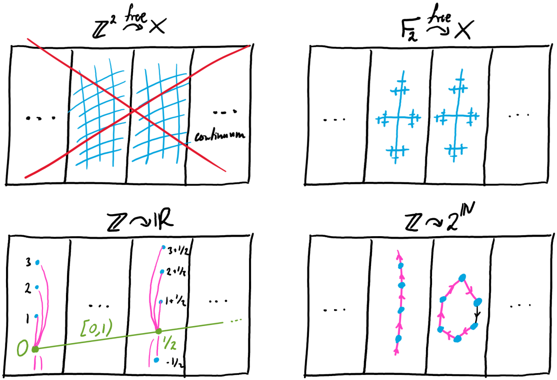
A *graphing* of a CBER E on X is a Borel graph $G \subseteq X^2$ whose connected relation is E, i.e., $xEy \leftrightarrow xG \cdots Gy$ for all $x, y \in X$.



Treeings and treeability

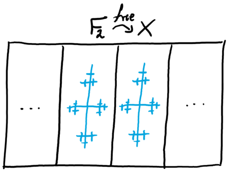
Definition

A *treeing* of a CBER E is an acyclic graphing, and a CBER E is said to be *treeable* if it admits a treeing.

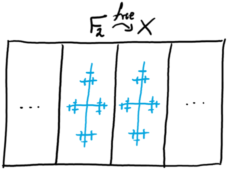


Example (Free Actions)

Free actions of a free group $F_r \curvearrowright X$.



Treeable CBERs



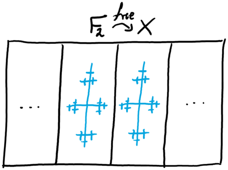
Example (Free Actions)

Free actions of a free group $F_r \curvearrowright X$.

Theorem (JKL02)

Free actions of virtually-free groups are treeable.

Treeable CBERs



Example (Free Actions)

Free actions of a free group $F_r \curvearrowright X$.

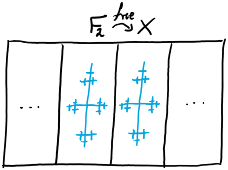
Theorem (JKL02)

Free actions of virtually-free groups are treeable.

Theorem (GdlH90)

Every finitely-generated group whose Cayley graph is a quasi-tree is virtually-free, and hence treeable.

Treeable CBERs



Example (Free Actions)

Free actions of a free group $F_r \curvearrowright X$.

Theorem (JKL02)

Free actions of virtually-free groups are treeable.

Theorem (GdlH90)

Every finitely-generated group whose Cayley graph is a quasi-tree is virtually-free, and hence treeable.

Question (Robin Tucker-Drob; 2015)

Is the class of treeable CBERs robust under quasi-isometries?

Main result

Theorem (Chen, Poulin, Tao, Tserunyan; 2023+)

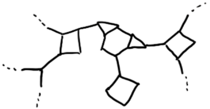
If a CBER E admits a locally-finite graphing such that each component is a quasi-tree, then E is treeable.

Main result

Theorem (Chen, Poulin, Tao, Tserunyan; 2023+)

If a CBER E admits a locally-finite graphing such that each component is a quasi-tree, then E is treeable.

Two metric spaces X, Y are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error; X is a *quasi-tree* if it is quasi-isometric to a tree.



Game plan

Quasi-treeing

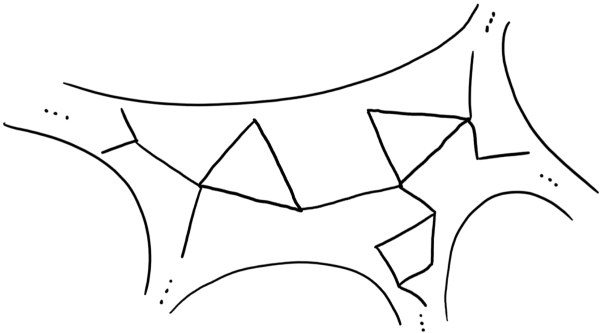
Treeing

Game plan

Quasi-treeing $\longrightarrow$ Treeing

$\downarrow$

Quasi-tree

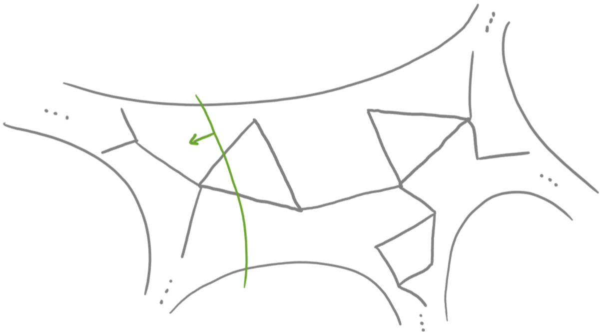


Game plan

Quasi-treeing

Treeing

Quasi-tree $\longrightarrow$ Finitely-sep. family of cuts



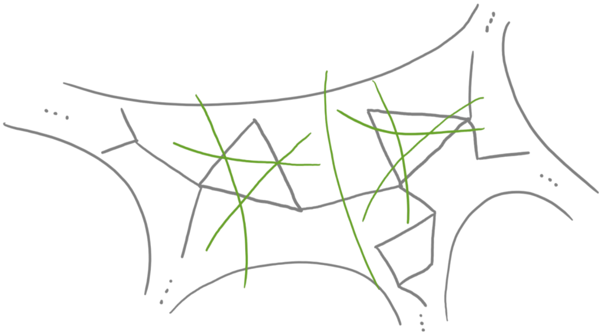
Game plan

Quasi-treeing

Treeing

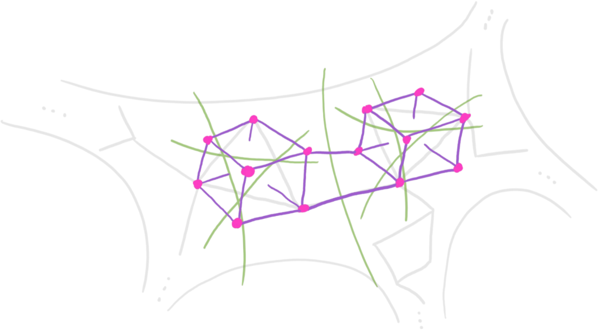
Quasi-tree

Finitely-sep. family of cuts



Game plan

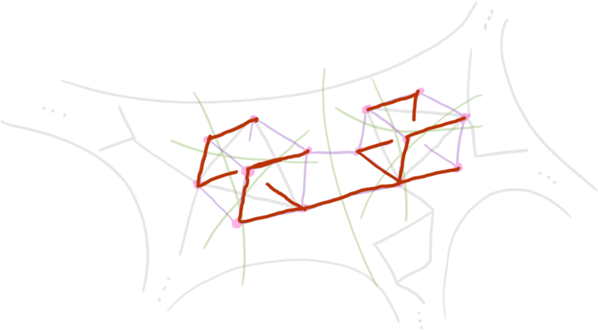
$$\begin{tikzcd}
\text{Quasi-treeing} \arrow[d] & & & \text{Treeing} \\
\text{Quasi-tree} \arrow[r] & \text{Finitely-sep. family of cuts} \arrow[r] & \text{Median graph w/ finite hyperplanes} &
\end{tikzcd}$$



Game plan

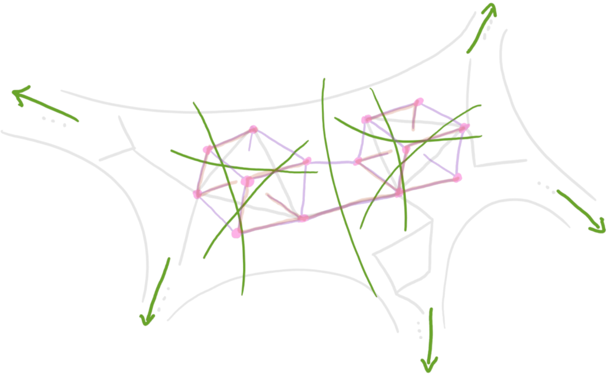
Quasi-treeing → Quasi-tree → Finitely-sep. family of cuts → Median graph w/ finite hyperplanes → 'Canonical' spanning tree

Treeing



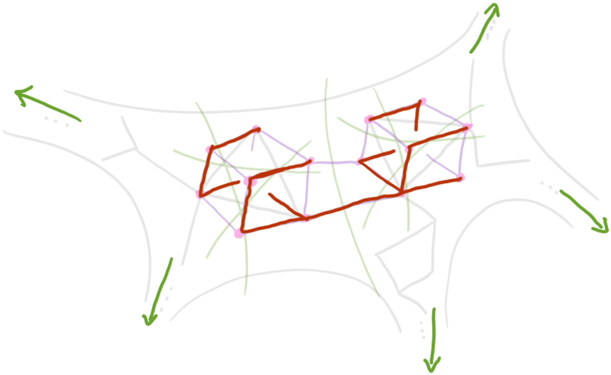
Game plan

$$
\begin{tikzcd}
\text{Quasi-treeing} \arrow[d] & \text{Dense family of cuts} & & \text{Treeing} \\
\text{Quasi-tree} \arrow[ur] \arrow[r] & \text{Finitely-sep. family of cuts} \arrow[r] & \text{Median graph w/ finite hyperplanes} \arrow[r] & \text{`Canonical' spanning tree}
\end{tikzcd}
$$



Game plan

$$\begin{tikzcd}
\text{Quasi-treeing} \arrow[d] & \text{Dense family of cuts} \arrow[rr] & & \text{Treeing} \\
\text{Quasi-tree} \arrow[ur] \arrow[r] & \text{Finitely-sep. family of cuts} \arrow[urr, bend left] \arrow[r] & \text{Median graph w/ finite hyperplanes} \arrow[r] & \text{`Canonical' spanning tree} \arrow[u]
\end{tikzcd}$$



Definition

A *cut* in a connected locally-finite graph (X, G) is a connected co-connected subset $H \subseteq X$ with finite boundary.

Finitely-separating cuts

Definition

A *cut* in a connected locally-finite graph (X, G) is a connected co-connected subset $H \subseteq X$ with finite boundary.

Let $\mathcal{H}$ be a family of cuts such that if $H \in \mathcal{H}$, then $\neg H \in \mathcal{H}$.

Finitely-separating cuts

Definition

A *cut* in a connected locally-finite graph (X, G) is a connected co-connected subset $H \subseteq X$ with finite boundary.

Let $\mathcal{H}$ be a family of cuts such that if $H \in \mathcal{H}$, then $\neg H \in \mathcal{H}$.

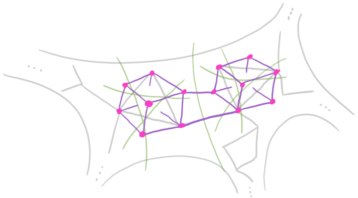
Definition

Such a family $\mathcal{H}$ is *finitely-separating* if for each $x, y \in X$, there are finitely-many $H \in \mathcal{H}$ with $x \in H \not\ni y$.

Orientations

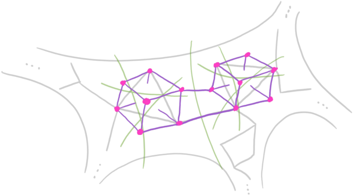
Definition

An *orientation* on $\mathcal{H}$ is an upward-closed subset $U \subseteq \mathcal{H}$ containing exactly one of $H, \neg H$ for every $H \in \mathcal{H}$.



Definition

An *orientation* on $\mathcal{H}$ is an upward-closed subset $U \subseteq \mathcal{H}$ containing exactly one of $H, \neg H$ for every $H \in \mathcal{H}$.

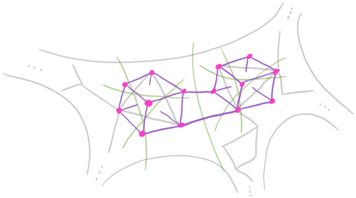


We'll only consider the orientations that are *based*, in the sense that each $H \in U$ contains a minimal $H_0 \in U$.

The dual median graph

Definition

A *median graph* is a connected graph (X, G) such that for each $x, y, z \in X$, the intersection $[x, y] \cap [x, z] \cap [y, z]$ is a singleton, called the *median* of x, y, z, and is denoted by $\langle x, y, z \rangle$.



Definition

A *median graph* is a connected graph (X, G) such that for each $x, y, z \in X$, the intersection $[x, y] \cap [x, z] \cap [y, z]$ is a singleton, called the *median* of x, y, z, and is denoted by $\langle x, y, z \rangle$.

Theorem (Sageev 95)

If $\mathcal{H}$ is finitely-separating, then the graph $\mathcal{M}(\mathcal{H})$:

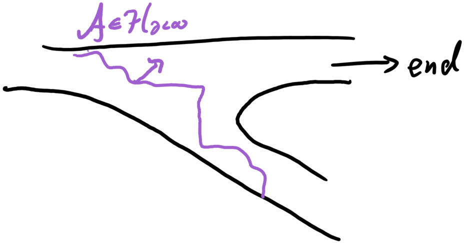
- *Vertices are based orientations on $\mathcal{H}$;*
- *Neighbors of U are $U \triangle \{H, \neg H\}$ for each minimal $H \in U \setminus \{\neg 0\}$;*

is a median graph.

Ends of graphs

Definition

The *end compactification* of a connected locally-finite (X, G) is the Stone space $\widehat{X}$ of the Boolean algebra $\mathcal{H}_{\partial<\infty}(X)$, whose non-principal ultrafilters are the *ends* of (X, G).



Ends of graphs

Definition

The *end compactification* of a connected locally-finite (X, G) is the Stone space $\widehat{X}$ of the Boolean algebra $\mathcal{H}_{\partial<\infty}(X)$, whose non-principal ultrafilters are the *ends* of (X, G).

Definition

A family $\mathcal{H}$ of cuts is *dense towards ends* of X if $\mathcal{H}$ contains a neighborhood basis for every end in $\widehat{X}$.

Density towards ends for quasi-trees

Lemma

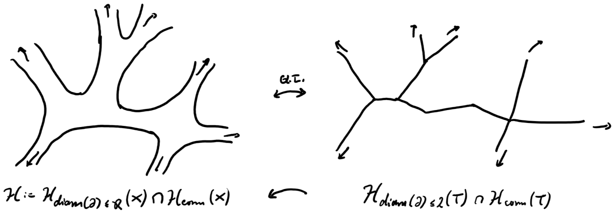
The connected locally-finite graphs in which $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}$ is dense towards ends for some $R < \infty$ is invariant under quasi-isometry.

Corollary

If (X, G) is a locally-finite quasi-tree, then the family

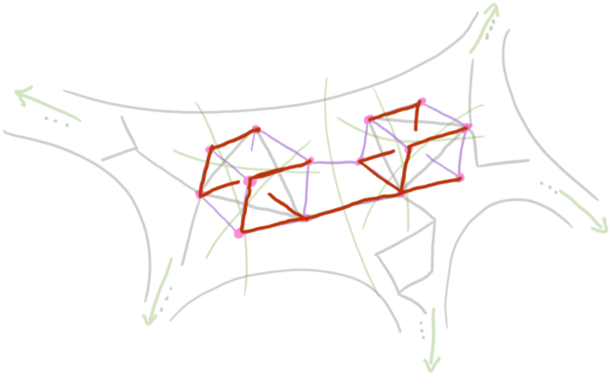
$$\mathcal{H} := \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$$

of cuts is dense towards ends for some $R < \infty$.



Wrapping things up...

$$
\begin{tikzcd}
\text{Quasi-treeing} \arrow[d] & \text{Dense family of cuts} \arrow[rr] & & \text{Treeing} \\
\text{Quasi-tree} \arrow[ur] \arrow[r] & \text{Finitely-sep. family of cuts} \arrow[urr, bend left] \arrow[r] & \text{Median graph w/ finite hyperplanes} \arrow[r] & \text{`Canonical' spanning tree} \arrow[u]
\end{tikzcd}
$$



The End

Thank you!