QUESTIONS IN LEMMA 2.61

ZHAOSHEN ZHAI

1. Results from the Paper

I'll first reference some previous results, and then reproduce Lemma 2.61 and its proof here.

Corollary 2.34. *Every interval* $[x, y]$ *in a median graph is finite. More generally, a convex hull* $\operatorname{cvx}(\{x_0, \ldots, x_n\})$ *of finitely many points is finite.*

Lemma 2.58. *For a median graph* (X, G) *with finite hyperplanes,* $\mathcal{H}_{\partial<\infty}(X) \subseteq 2^X$ *is the Boolean subalgebra generated by* $\mathcal{H}_{cvx}(X)$.

Lemma 2.61. *For a locally-finite median graph* (X, G) *with finite hyperplanes, each end has a neighborhood basis of half-spaces.*

Proof. Let $U \in \widehat{X} \setminus X$ be an end, $\widehat{A} \ni U$ be a clopen neighborhood where $A \in \mathcal{H}_{\partial<\infty}(X)$. By Corollary 2.34, $\operatorname{cvx}(\partial_{\mathsf{ov}} A) \subseteq X$ is finite. By Lemma 2.58 (applied to it and each point on its outer boundary), it is a finite intersection of half-spaces. One such half-space H must not contain U, since $U \notin \operatorname{cvx}(\partial_{\mathsf{ov}} A)$[1]. Then $U \in \neg\widehat{H}$, whence $A \cap \neg H \neq \varnothing$, whence $\neg H \subseteq A$ since $\partial_{\mathsf{ov}} A \cap \neg H = \varnothing$. ■

2. Discussion and Questions

I'll give some more details in the proof. Let $U \in \widehat{A}$ be as above.

Proof. ... $Y := \operatorname{cvx}(\partial_{\mathsf{ov}} A)$ is finite, so there are $H_0, \ldots, H_n \in \mathcal{H}_{\mathrm{cvx}}(X)$ such that $Y = \bigcap_{i<n} H_i$. Then there is some $i < n$ such that $H := H_i \notin U$, since otherwise we have $Y \in U$, a contradiction since Y is finite and U is a *non-principal* ultrafilter. Thus $\neg H \in U$, so $U \in \neg\widehat{H}$. Since $A \in U$, we have $A \cap \neg H \neq \varnothing$. We claim that $\neg H \subseteq A$, so that $U \in \neg\widehat{H} \subseteq \widehat{A}$ as desired.

Indeed, fix some $x_0 \in A \cap \neg H$. If there is some $x \in \neg A \cap \neg H$, then the interval $[x_0, x]$ contains some $y \in \partial_{\mathsf{ov}} A$. Since $\neg H$ is convex, we see that $y \in [x_0, x] \subseteq \neg H$, so that $y \in \partial_{\mathsf{ov}} A \cap \neg H$. This is absurd, since $\partial_{\mathsf{ov}} A \subseteq Y \subseteq H$. ■

Question 2.1. Lemma 2.58 only guarantees that every $A \in \mathcal{H}_{\partial<\infty}(X)$ is a finite *boolean combination* of half-spaces in X, not finite *intersections* thereof.

Assuming that $Y = \bigcap_{i<n} H_i$, I think the rest of the proof applies also for non-locally-finite graphs, so somehow this should be justified by local-finiteness of G.

Question 2.2. Why not set $Y := \partial_{\mathsf{ov}} A$ directly? That is, write $\partial_{\mathsf{ov}} A$ as a finite intersection of half-spaces? The rest of the proof should work with this modification ($\partial_{\mathsf{ov}} A$ is itself finite, so $\partial_{\mathsf{ov}} A \notin U$). I don't see how convexity of Y plays a role here.

Question 2.3. In the original proof: "By Lemma 2.58 (applied to *it* and each point on its outer boundary), it is a finite intersection of half-spaces". Is 'it' here referring to $\operatorname{cvx}(\partial_{\mathsf{ov}} A)$? It would make more sense if the lemma is applied to A and the *finitely-many* points in $\partial_{\mathsf{ov}} A$, and maybe this is where local-finiteness of G is used? This also motivated me to consider Question 2.2, by writing $\partial_{\mathsf{ov}} A$ as a finite intersection of half-spaces, instead of $\operatorname{cvx}(\partial_{\mathsf{ov}} A)$.

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[1] I think this should be $\operatorname{cvx}(\partial_{\mathsf{ov}} A) \notin U$.