# TREE-LIKE GRAPHINGS OF COUNTABLE BOREL EQUIVALENCE RELATIONS

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Abstract. We present a streamlined exposition of a construction by R. Chen, A. Poulin, R. Tao, and A. Tserunyan, which proves the treeability of equivalence relations generated by any locally-finite Borel graph such that each component is a quasi-tree. More generally, we show that if each component of a locally-finite Borel graph admits a *finitely-separating* Borel family of *cuts*, then we may 'canonically' construct a forest of special ultrafilters; moreover, if the cuts are *dense towards ends*, then this forest is a Borel treeing.

The purpose of this note is to provide a streamlined proof of a particular case of a construction presented in [CPTT23], in order to better understand the general formalism developed therein. We attempt to make this note self-contained, but nevertheless urge the reader to refer to the original paper for more detailed discussions and some generalizations of the results we have selected to include here.

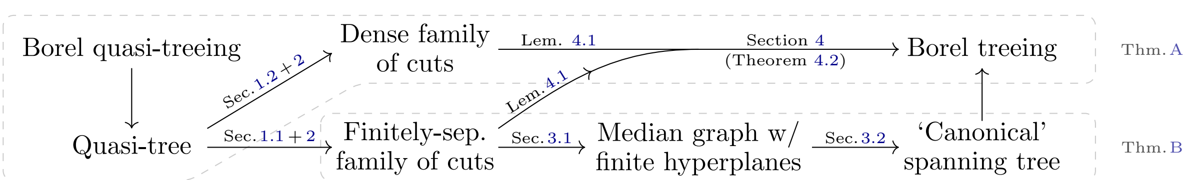
**Treeings of equivalence relations.** A *countable Borel equivalence relation (CBER)* on a standard Borel space $X$ is a Borel equivalence relation $E \subseteq X^2$ with each class countable. We are interested in special types of *graphings* of a CBER $E \subseteq X^2$, i.e. a Borel graph $G \subseteq X^2$ whose connectedness relation is precisely $E$. For instance, a graphing of $E$ such that each component is a tree is called a *treeing* of $E$, and the CBERs that admit treeings are said to be *treeable*. The main results of [CPTT23] provide new sufficient criteria for treeability of certain classes of CBERs, and in particular, they prove the following

**Theorem A** (Section 4, [CPTT23, Theorem 1.1])**.** *If a CBER $E$ admits a locally-finite graphing such that each component is a quasi-tree,*[1] *then $E$ is treeable.*

Roughly speaking, the existence of a quasi-isometry $G|C \to T_C$ to a simplicial tree $T_C$ for each component $C \subseteq X$ induces a collection $\mathcal{H}(C) \subseteq 2^C$ of 'cuts' (subsets $H \subseteq C$ with finite boundary such that both $H$ and $C \setminus H$ are connected), which are 'tree-like' in the sense that

1. $\mathcal{H}(C)$ is *finitely-separating*: each pair $x, y \in C$ is separated by finitely-many $H \in \mathcal{H}(C)$, and
2. $\mathcal{H}(C)$ is *dense towards ends*: each end in $G|C$ has a neighborhood basis in $\mathcal{H}(C)$.

Condition (1) allows for an abstract construction of a tree whose vertices are special 'ultrafilters'[2] on $\mathcal{H}(C)$, as outlined in the following diagram: starting from a finitely-separating family of cuts, one constructs a 'dual median graph' $\mathcal{M}(\mathcal{H}(C))$ with said ultrafilters; this median graph has finite 'hyperplanes', which allows one to apply a Borel cycle-cutting algorithm and obtain a 'canonical' spanning tree thereof.




Thus we have the following theorem, which can be viewed as a component-wise version of Theorem A.

**Theorem B** (Propositions 3.3, 3.5, 3.7)**.** *For any finitely-separating family of cuts $\mathcal{H}$ on a connected locally-finite graph, its dual median graph $\mathcal{M}(\mathcal{H})$ has finite hyperplanes, and fixing a Borel colouring on the intersection graph of those hyperplanes yields a canonical spanning tree of $\mathcal{M}(\mathcal{H})$.*

The additional condition (2) then shows that $\mathcal{M}(\mathcal{H})$ is locally-finite (Lemma 4.1), which ensures that such a spanning tree can be constructed uniformly for each component $C \subseteq G$; see Section 4 for details.

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[1] Recall that metric spaces $X$ and $Y$ are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error, and $X$ is a *quasi-tree* if it is quasi-isometric to a simplicial tree; see [Gro93] and [DK18].

[2] As in [CPTT23], we call them *orientations* instead, to avoid confusion with the more standard notion; see Definition 3.1.

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## 1. Preliminaries on Pocsets, Ends of Graphs, and Median Graphs

**Notation.** A *graph* on a set $X$ is a symmetric irreflexive binary relation $G \subseteq X^2$. For $A \subseteq X$, we say that $A$ is *connected* if the induced subgraph $G[A]$ is. We always equip connected graphs with their *path metric* $d$, and let $\mathrm{Ball}_r(x)$ be the closed ball of radius $r$ around $x$; more generally, we let $\mathrm{Ball}_r(A) \coloneqq \bigcup_{x\in A} \mathrm{Ball}_r(x)$.

For a subset $A \subseteq X$, we let $\partial_{\mathsf{iv}}A \coloneqq A \cap \mathrm{Ball}_1(\neg A)$ be its *inner vertex boundary*, $\partial_{\mathsf{ov}}A \coloneqq \partial_{\mathsf{iv}}(\neg A)$ be its *outer vertex boundary*, and let $\partial_{\mathsf{ie}}A \coloneqq G \cap (\partial_{\mathsf{ov}}A \times \partial_{\mathsf{iv}}A)$ and $\partial_{\mathsf{oe}}A \coloneqq \partial_{\mathsf{ie}}(\neg A)$ respectively be its *inner* and *outer edge boundaries*. Let $\partial_{\mathsf{v}}A \coloneqq \partial_{\mathsf{iv}}A \sqcup \partial_{\mathsf{ov}}A$ be the *(total) vertex boundary* of $A$.

Finally, for $x, y \in X$, the *interval* $[x, y]$ between $x, y$ is the union of all geodesics between $x, y$, consisting of exactly those $z \in X$ with $d(x, z) + d(z, y) = d(x, y)$. We say that $A \subseteq X$ is *convex* if $[x, y] \subseteq A$ for all $x, y \in A$. For vertices $x, y, z \in X$, we write $x$—$y$—$z$ for $y \in [x, z]$. By the triangle inequality, we have for all $w, x, y, z \in X$ that

$$(w\text{—}x\text{—}y \text{ and } w\text{—}y\text{—}z) \quad \Leftrightarrow \quad (w\text{—}x\text{—}z \text{ and } x\text{—}y\text{—}z),$$

and both sides occur iff there is a geodesic from $w$ to $x$ to $y$ to $z$, which we write as $w$—$x$—$y$—$z$.

### 1.1. Profinite pocsets of cuts.

The construction starts by identifying a profinite pocset $\mathcal{H}$ of 'cuts' in each component of the graphing, which we first study abstractly. The finitely-separating subpocsets of $2^X$ are well-known in metric geometry as *wallspaces*; see, e.g., [Nic04] and [CN05].

**Definition 1.1.** A *pocset* $(\mathcal{H}, \leq, \neg, 0)$ is a poset $(\mathcal{H}, \leq)$ equipped with an order-reversing involution $\neg : \mathcal{H} \to \mathcal{H}$ and a least element $0 \neq \neg 0$ such that $0$ is the only lower-bound of $H, \neg H$ for every $H \in \mathcal{H}$. We call the elements in $\mathcal{H}$ *half-spaces*, and the edge boundaries $\partial_{\mathsf{ie}}H$ for $H \in \mathcal{H}$ *hyperplanes*.

A *profinite pocset* is a pocset $\mathcal{H}$ equipped with a compact topology making $\neg$ continuous and is *totally order-disconnected*, in the sense that if $H \not\leq K$, then there is a clopen upward-closed $U \subseteq \mathcal{H}$ with $H \in U \not\ni K$.

We are primarily interested in subpocsets of $(2^X, \subseteq, \neg, \varnothing)$ for a fixed set $X$, which is profinite if equipped with the product topology of the discrete space 2. Indeed, $2^X$ admits a base of *cylinder sets* − which are finite intersections of sets of the form $\pi_x^{-1}(i)$ where $x \in X$, $i \in \{0, 1\}$, and $\pi_x : 2^X \to 2$ is the projection − making $\neg$ continuous since cylinders are clopen. Finally, for $H \not\leq K$, let $U$ be the upward-closure of a clopen neighborhood $U_0 \ni H$ separating it from $K$, which is clopen since $\neg U_0$ is a finite union of cylinders.

**Remark.** We follow [CPTT23, Convention 2.7], where for a family $\mathcal{H} \subseteq 2^X$ of subsets of a fixed set $X$, we write $\mathcal{H}^* \coloneqq \mathcal{H} \setminus \{\varnothing, X\}$ for the *non-trivial* elements of $\mathcal{H}$.

The following proposition gives a sufficient criteria for subpocsets of $2^X$ to be profinite. We also show in this case that every non-trivial element $H \in \mathcal{H}^*$ is isolated, which will be important in Section 3.1.

**Lemma 1.2.** *If $\mathcal{H} \subseteq 2^X$ is a finitely-separating pocset, then $\mathcal{H}$ is closed and non-trivial elements are isolated.*

*Proof.* It suffices to show that the limit points of $\mathcal{H}$ are trivial, so let $A \in 2^X \setminus \{\varnothing, X\}$. Fix $x \in A \not\ni y$. Since $\mathcal{H}$ is finitely-separating, there are finitely-many $H \in \mathcal{H}$ with $x \in H \not\ni y$, and for each such $H \in \mathcal{H} \setminus \{A\}$, we have either some $x_H \in A \setminus H$ or $y_H \in H \setminus A$. Let $U \subseteq 2^X$ be the family of all subsets $B \subseteq X$ containing $x$ and each $x_H$, but not $y$ or any $y_H$.

This is the desired neighborhood isolating $A \in U$. Indeed, it is (cl)open since it is the *finite* intersection of cylinders prescribed by the $x_H$'s and $y_H$'s, and it is disjoint from $\mathcal{H} \setminus \{A\}$ by construction. ■

Our main method of identifying the finitely-separating pocsets is as follows.

**Lemma 1.3.** *Let $\mathcal{H} \subseteq 2^X$ be a pocset in a connected graph $(X, G)$. If each $x \in X$ is on the vertex boundary of finitely-many half-spaces, then $\mathcal{H}$ is finitely-separating. The converse holds too if $(X, G)$ is locally-finite.*

*Proof.* Any $H \in \mathcal{H}$ separating $x, y \in X$ separates some edge on any fixed path between $x$ and $y$, and there are only finitely-many such $H$ for each edge. If $(X, G)$ is locally-finite, then each $x \in X$ is separated from each of its finitely-many neighbors by finitely-many $H \in \mathcal{H}$. ■

In the case that $\mathcal{H}$ is a pocset consisting of connected co-connected half-spaces with finite vertex boundary, finite-separation also controls the degree of 'non-nestedness' of $\mathcal{H}$.

**Definition 1.4.** A *cut* in a connected locally-finite graph $(X, G)$ is a half-space $H \in \mathcal{H}_{\partial<\infty}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$.

**Definition 1.5.** Let $\mathcal{H} \subseteq 2^X$ be a pocset. Two half-spaces $H, K \in \mathcal{H}$ are *nested* if $\neg^i H \cap \neg^j K = \varnothing$ for some $i, j \in \{0, 1\}$, where $\neg^0 H := H$ and $\neg^1 H := \neg H$. We say that $\mathcal{H}$ is *nested* if every pair $H, K \in \mathcal{H}$ is nested.

**Lemma 1.6.** *In a pocset $\mathcal{H}$ of finitely-separating cuts, each $H \in \mathcal{H}$ is non-nested with finitely-many others.*

*Proof.* Fix $H \in \mathcal{H}$ and let $K \in \mathcal{H}$ be non-nested with $H$. By connectedness, the non-empty sets $H \cap K$ and $\neg H \cap K$ are joined by a path in $K$, so $\partial_{\mathsf{v}} H \cap K \neq \varnothing$; similarly, $\partial_{\mathsf{v}} H \cap \neg K \neq \varnothing$. For each $x \in \partial_{\mathsf{v}} H \cap K$ and $y \in \partial_{\mathsf{v}} H \cap \neg K$, any fixed path $p_{xy}$ between them contains some $z \in \partial_{\mathsf{v}} K \cap p_{xy}$; thus, any $K \in \mathcal{H}$ non-nested with $H$ contains some $z \in \partial_{\mathsf{v}} K \cap p_{xy}$.

Then, since there are finitely-many such $x, y \in \partial_{\mathsf{v}} H$, for each of which there are finitely-many $z \in p_{xy}$, for each of which there are finitely-many $K \in \mathcal{H}$ with $z \in \partial_{\mathsf{v}} K$ (by Lemma 1.3, since $\mathcal{H}$ is finitely-separating), there can only be finitely-many $K \in \mathcal{H}$ non-nested with $H$. ■

1.2. **Ends of graphs.** Let $(X, G)$ be a connected locally-finite graph, and consider the Boolean algebra of finite vertex boundary half-spaces $\mathcal{H}_{\partial<\infty}(X) \subseteq 2^X$.

**Definition 1.7.** The *end compactification* of $(X, G)$ is the Stone space $\widehat{X}$ of $\mathcal{H}_{\partial<\infty}(X)$, whose non-principal ultrafilters are the *ends* of $(X, G)$.

We identify $X \hookrightarrow \widehat{X}$ via principal ultrafilter map $x \mapsto p_x$, so $\varepsilon(X) := \widehat{X} \setminus X$ is the set of ends of $G$. By definition, $\widehat{X}$ admits a basis of clopen sets of the form $\widehat{A} := \{p \in \widehat{X} : A \in p\}$ for each $A \in \mathcal{H}_{\partial<\infty}(X)$.

**Lemma 1.8.** *A finite-boundary subset $A \in \mathcal{H}_{\partial<\infty}(X)$ is infinite iff it contains an end in $(X, G)$.*

*Proof.* The converse direction follows since ends are non-principal. If $A$ is infinite, then by local-finiteness of $(X, G)$, Kőnig's Lemma furnishes some infinite ray $(x_n) \subseteq A$. Then, $A$ is contained in the filter

$$p := \{H \in \mathcal{H}_{\partial<\infty}(X) : \forall^\infty n (x_n \in H)\},$$

which is ultra since $H \in p$ are of *finite*-boundary, and is non-principal since it contains cofinite sets. ■

**Definition 1.9.** A pocset $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty}(X)$ is *dense towards ends* of $(X, G)$ if $\mathcal{H}$ contains a neighborhood basis for every end in $\varepsilon(X)$.

In other words, $\mathcal{H}$ is dense towards ends if for every $p \in \varepsilon(X)$ and every (clopen) neighborhood $\widehat{A} \ni p$, where $A \in \mathcal{H}_{\partial<\infty}(X)$, there is some $H \in \mathcal{H}$ with $p \in \widehat{H} \subseteq \widehat{A}$; it is useful to note that $\widehat{H} \subseteq \widehat{A}$ iff $H \subseteq A$, so we will abuse notation and write $p \in H \subseteq A$ for the above condition.

We show in Section 2 that certain half-spaces $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty}$ induced by a locally-finite quasi-tree is dense towards ends. It will also be important that these half-spaces be cuts, in that witnesses to density can also be found in $\mathcal{H} \cap \mathcal{H}_{\mathrm{conn}}$. The following lemma takes care of this.

**Lemma 1.10.** *If a subpocset $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty}$ is dense towards ends, then there is a subpocset $\mathcal{H}' \subseteq \mathcal{H}_{\partial<\infty} \cap \mathcal{H}_{\mathrm{conn}}$, which is also dense towards ends, such that every $H' \in \mathcal{H}'$ has $\partial_{\mathsf{ie}} H' \subseteq \partial_{\mathsf{ie}} H$ for some $H \in \mathcal{H}$.*

*Proof.* A first attempt is to let $\mathcal{H}'$ be the connected components $H'_0$ of elements in $\mathcal{H}$, but this fails since $\neg H'_0$ is not necessarily connected. Instead, we further take a component of $\neg H'_0$, whose complement clearly co-connected, and is connected since it consists of $H'_0$ and the other components of $\neg H'_0$, each of which is connected to $H'_0$ via $\partial_{\mathsf{ie}} H'_0$. Formally, we let

$$\mathcal{H}' := \{H' \subseteq X : H \in \mathcal{H} \text{ and } H'_0 \in H/G \text{ and } \neg H' \in \neg H'_0/G\},$$

where $H/G$ denotes the $G$-components of $H$. Clearly $\partial_{\mathsf{ie}} H' \subseteq \partial_{\mathsf{ie}} H'_0 \subseteq \partial_{\mathsf{ie}} H$, and since $H' \in \mathcal{H}_{\mathrm{conn}}(X)$, it remains to show that $\mathcal{H}'$ is dense towards ends.

Fix an end $p \in \varepsilon(X)$ and a neighborhood $p \in A \in \mathcal{H}_{\partial<\infty}(X)$. Let $B \supseteq \partial_{\mathsf{v}} A$ be finite connected, which can be obtained by adjoining paths between its components. Then $\neg B \in p$ since $p$ is non-principal, so there is $H \in \mathcal{H}$ with $p \in H \subseteq \neg B$. We now use the above recipe to find the desired $H' \in \mathcal{H}'$:

(1) Since $H \in \mathcal{H}_{\partial<\infty}(X)$, it has finitely-many connected components, so exactly one of them belongs to $p$, say $p \in H'_0 \subseteq H$. Note that $B \subseteq \neg H \subseteq \neg H'_0$.
(2) Since $B$ is connected, there is a unique component $\neg H' \subseteq \neg H'_0$ containing $B$.

Observe that $H' \in \mathcal{H}'$ and $p \in H'$. Lastly, since $H'$ is connected and is disjoint from $\partial_{\mathsf{v}} A \subseteq B$, and since $H' \subseteq \neg A$ would imply $\neg H' \in p$, this forces $H' \subseteq A$, and hence $p \in H' \subseteq A$ as desired. ■

1.3. **Median graphs and projections.** Starting from a profinite pocset $\mathcal{H}$ with every non-trivial element isolated, we construct in Section 3.1 its dual median graph $\mathcal{M}(\mathcal{H})$.

We devote this section and the next to study some basic properties of median graphs and their projections, which will be used in Section 3.2 to construct a spanning tree certain median graphs. For more comprehensive references of median graphs, and their general theory, see [Rol98] and [Bow22].

**Definition 1.11.** A *median graph* is a connected graph $(X, G)$ such that for any $x, y, z \in X$, the intersection

$$[x, y] \cap [y, z] \cap [x, z]$$

is a singleton, whose element $\langle x, y, z\rangle$ is called the *median* of $x, y, z$. Thus we have a ternary median operation $\langle\cdot,\cdot,\cdot\rangle : X^3 \to X$, and a *median homomorphism* $f : (X, G) \to (Y, H)$ is a map preserving said operation.

**Lemma 1.12.** *For any $\varnothing \neq A \subseteq X$ and $x \in X$, there is a unique point in* $\mathrm{cvx}(A)$ *between $x$ and every point in $A$, called the* projection *of $x$ towards $A$, denoted* $\mathrm{proj}_A(x)$*.*

*Moreover, we have $\bigcap_{a\in A}[x, a] = [x, \mathrm{proj}_A(x)]$, and for any $y$ in this set, we have* $\mathrm{proj}_A(y) = \mathrm{proj}_A(x)$*.*

*Proof.* To show existence, pick any $a_0 \in A$. Given $a_n \in \mathrm{cvx}(A)$, if there exists $a \in A$ with $a_n \notin [x, a]$, set $a_{n+1} \coloneqq \langle x, a, a_n\rangle \in \mathrm{cvx}(A)$. Then $a_0$—$a_1$—$\cdots$—$a_n$—$x$ for all $n$, so this sequence terminates in at most $d(a_0, x)$ steps at a point in $\mathrm{cvx}(A)$ between $x$ and every point in $A$. For uniqueness, if there exist two such points $a, b \in \mathrm{cvx}(A)$, then $x$—$a$—$b$ and $x$—$b$—$a$, forcing $a = b$.

Finally, if $x$—$y$—$\mathrm{proj}_A(x)$ and $a \in A$, then $x$—$\mathrm{proj}_A(x)$—$a$ and hence $x$—$y$—$a$. Conversely, let $x$—$y$—$a$ for all $a \in A$. Since $[y, a] \subseteq [x, a]$ for all $a$, we see that

$$\mathrm{proj}_A(y) \in \mathrm{cvx}(A) \cap \bigcap_{a\in A}[y, a] \subseteq \mathrm{cvx}(A) \cap \bigcap_{a\in A}[x, a]$$

and hence $\mathrm{proj}_A(y) = \mathrm{proj}_A(x)$ by uniqueness. But since $y$—$\mathrm{proj}_A(y)$—$a$, we have $x$—$y$—$\mathrm{proj}_A(y)$, and hence $x$—$y$—$\mathrm{proj}_A(x)$ as desired. ∎

**Remark 1.13.** It follows from the proof above that for any median homomorphism $f : (X, G) \to (Y, H)$, we have $f(\mathrm{proj}_A(x)) = \mathrm{proj}_{f(A)}(f(x))$ for any $\varnothing \neq A \subseteq X$ and $x \in X$. Indeed, we have

$$\mathrm{proj}_A(x) = \langle x, a_m, \ldots, \langle x, a_2, \langle x, a_1, a_0\rangle\rangle \ldots\rangle$$

for some $m \leq d(a_0, x)$ and $a_0, \ldots, a_m \in A$, and this is preserved by $f$.

For $A \coloneqq \{a, b\}$, we have $\mathrm{proj}_A(x) = \langle a, b, x\rangle$, and hence $\mathrm{cvx}(A) = \mathrm{proj}_A(X) = \langle a, b, X\rangle = [a, b]$.

**Lemma 1.14.** *For each $x, y \in X$,* $\mathrm{cone}_x(y)$ *is convex, and if $xGy$, then* $\mathrm{cone}_x(y) \sqcup \mathrm{cone}_y(x) = X$*.*

*Proof.* Fix $a, b \in \mathrm{cone}_x(y)$ and $a$—$c$—$b$. It suffices to show that $x$—$y$—$\langle a, c, x\rangle$, for then $x$—$y$—$c$ since we have $x$—$\langle a, c, x\rangle$—$c$. Indeed, it follows from the following observations.

- $x$—$y$—$\langle a, b, x\rangle$, since $\langle a, b, x\rangle = \mathrm{proj}_{\{a,b\}}(x)$ and so $[x, \langle a, b, x\rangle] = [x, a] \cap [x, b] \ni y$ by Lemma 1.12.
- $x$—$\langle a, b, x\rangle$—$\langle a, c, x\rangle$, which follows from $\langle a, b, x\rangle$—$\langle a, c, x\rangle$—$a$, since $x$—$\langle a, b, x\rangle$—$a$ by definition. Indeed, we have $\langle a, c, x\rangle$ is in both $[a, x]$ and $[a, c] \subseteq [a, b]$, and since $\mathrm{proj}_{\{b,x\}}(a) = \langle a, b, x\rangle$, we have again by Lemma 1.12 that $[\langle a, b, x\rangle, a] = [a, x] \cap [a, b] \ni \langle a, c, x\rangle$.

Finally, take $z \in X$ and consider $w \coloneqq \langle x, y, z\rangle \subseteq [x, y]$. Either $w = x$ or $w = y$ (but not both), giving us the desired partition. ∎

**Remark 1.15.** In particular, this shows that if $xGy$, then $\mathrm{cone}_x(y) \in \mathcal{H}^*_{\mathrm{cvx}}(X)$. The convexity of cones also shows, in the situation of Lemma 1.12, that $\mathrm{proj}_A = \mathrm{proj}_{\mathrm{cvx}(A)}$, i.e., $\mathrm{proj}_A(x)$ is also between $x$ and every point in $\mathrm{cvx}(A)$: indeed, note that $\mathrm{cone}_x(\mathrm{proj}_A(x))$ is convex and contains $A$, so it contains $\mathrm{cvx}(A)$ too.

**Lemma 1.16.** $\mathrm{proj}_A : X \twoheadrightarrow \mathrm{cvx}(A)$ *is a median homomorphism with* $\mathrm{proj}_A \circ \mathrm{cvx} = \mathrm{cvx} \circ \mathrm{proj}_A$*.*

*Proof.* The second claim follows from the first since, by Remark 1.13, we have

$$f(\mathrm{cvx}(B)) = f(\mathrm{proj}_B(X)) = \mathrm{proj}_{f(B)}(f(X)) = \mathrm{cvx}(f(B))$$

for all median homomorphisms $f : X \twoheadrightarrow Y$ and $B \subseteq X$, so it in particular applies to $f \coloneqq \mathrm{proj}_A$.

To this end, let $x$—$y$—$z \in X$ and set $w \coloneqq \langle \mathrm{proj}_A(x), \mathrm{proj}_A(y), \mathrm{proj}_A(z)\rangle \in \mathrm{cvx}(A)$. It suffices to show that $y$—$w$—$a$ for all $a \in A$, for then $w = \mathrm{proj}_A(y)$ and hence $\mathrm{proj}_A(x)$—$\mathrm{proj}_A(y)$—$\mathrm{proj}_A(z)$. But we have $y$—$\mathrm{proj}_A(y)$—$a$ already, so it further suffices to show that $y$—$w$—$\mathrm{proj}_A(y)$. For this, we note that

$$x\text{—}\mathrm{proj}_A(x)\text{—}\mathrm{proj}_A(y) \quad \text{and} \quad \mathrm{proj}_A(x)\text{—}w\text{—}\mathrm{proj}_A(y),$$

so $x$—$w$—$\mathrm{proj}_A(y)$, and similarly $z$—$w$—$\mathrm{proj}_A(y)$. Thus, it follows that

$$\begin{aligned} w \in [\mathrm{proj}_A(y), x] \cap [\mathrm{proj}_A(y), z] &= [\mathrm{proj}_A(y), \mathrm{proj}_{\{x,z\}}(\mathrm{proj}_A(y))] && \text{Lemma 1.12} \\ &= [\mathrm{proj}_A(y), \mathrm{proj}_{[x,z]}(\mathrm{proj}_A(y))] && \text{Remark 1.15} \\ &\subseteq [\mathrm{proj}_A(y), y], && \text{Lemma 1.12} \end{aligned}$$

where the second equality follows from $\mathrm{cvx}(\{x, z\}) = [x, z]$, and hence $\mathrm{proj}_{\{x,z\}} = \mathrm{proj}_{[x,z]}$. ■

1.4. **Convex half-spaces of median graphs.** We now use projections to explore the geometry of convex half-spaces in median graphs. For the axiomatics of convex structures, see [vdV93].

**Proposition 1.17.** *Each edge $(x, y) \in G$ is on a unique hyperplane, namely the inward boundary of $\mathrm{cone}_x(y)$, and conversely, each half-space $H \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ is $\mathrm{cone}_x(y)$ for every $(x, y) \in \partial_{\mathsf{ie}} H$.*

*Thus, hyperplanes are equivalence classes of edges. Furthermore, this equivalence relation is generated by parallel sides of squares (i.e., 4-cycles).*

*Proof.* We have $\mathrm{cone}_x(y) \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ by Lemma 1.14, and clearly $(x, y) \in \partial_{\mathsf{ie}} \mathrm{cone}_x(y)$. Conversely, take $H \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ and any $(x, y) \in \partial_{\mathsf{ie}} H$. Then $H = \mathrm{cone}_x(y)$, for if $z \in H \cap \neg\, \mathrm{cone}_x(y)$, then $z \in \mathrm{cone}_y(x)$, and hence $x \in [y, z] \subseteq H$ by convexity of $H$, a contradiction; if $z \in \mathrm{cone}_x(y) \cap \neg H$, then $[x, z] \subseteq \neg H$ by convexity of $\neg H$, and hence $y \notin H$, a contradiction.

Finally, parallel edges of a strip of squares generate the same hyperplane since, for a given square, each vertex is between its neighbors and hence any hyperplane containing an edge contains its opposite edge. On the other hand, let $(a, b), (c, d) \in \partial_{\mathsf{ie}} H$ for some $H \in \mathcal{H}^*_{\mathsf{cvx}}(X)$. Note that $\partial_{\mathsf{ov}} H = \mathrm{proj}_{\neg H}(H)$ is convex since $H$ is, and since $\mathrm{proj}_{\neg H}$ preserves convexity by Lemma 1.16, any geodesic between $a, c \in \partial_{\mathsf{ov}} H$ lies in $\partial_{\mathsf{ov}} H$. Matching this geodesic via $\partial_{\mathsf{ie}} H : \partial_{\mathsf{ov}} H \to \partial_{\mathsf{iv}} H$ gives us a geodesic between $b, d$ in $\partial_{\mathsf{iv}} H$, which together with the matching forms the desired strip of squares. ■

**Corollary 1.18.** *Two half-spaces $H, K \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ are non-nested iff there is an embedding $\{0,1\}^2 \hookrightarrow X$ of the Hamming cube into the four corners $\neg^i H \cap \neg^j K$.*

*In particular, if $H, K \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ are non-nested, then $\partial_{\mathsf{v}} H \cap \partial_{\mathsf{v}} K \neq \varnothing$.*

*Proof.* Let $H, K$ be non-nested and take $x_1 \in H \cap K$ and $x_2 \in H \cap \neg K$. Since $H$ is connected, any geodesic between $x_1, x_2$ crosses an edge $(x'_1, x'_2) \in \partial_{\mathsf{oe}} K$ in $H$. Similarly, there is an edge $(y'_1, y'_2) \in \partial_{\mathsf{oe}} K$ in $\neg H$, so we may slide both edges along $\partial_{\mathsf{oe}} K$ to obtain the desired square (see Proposition 1.17).

Conversely, the half-spaces cutting the square are clearly non-nested. ■

**Lemma 1.19** (Helly)**.** *Any finite intersection of pairwise-intersecting non-empty convex sets is non-empty.*

*Proof.* For pairwise-intersecting convex sets $H_1, H_2, H_3$, pick any $x \in H_1 \cap H_2$, $y \in H_1 \cap H_3$ and $z \in H_2 \cap H_3$; their median $\langle x, y, z\rangle$ then lies in $H_1 \cap H_2 \cap H_3$.

Suppose that it holds for some $n \geq 3$ and let $H_1, \dots, H_{n+1} \subseteq X$ pairwise-intersect. Then $\{H_i \cap H_{n+1}\}_{i \leq n}$ is a family of $n$ pairwise-intersecting convex sets, so $\bigcap_{i \leq n+1} H_i = \bigcap_{i \leq n} (H_i \cap H_{n+1})$ is non-empty. ■

Lastly, we have some useful finiteness conditions on convex half-spaces; the former implies that $\mathcal{H}_{\mathsf{cvx}}(X)$ is finitely-separating, and the latter allows us to replace finite sets with their convex hulls.

**Lemma 1.20.** *Any two disjoint convex sets $\varnothing \neq A, B \subseteq X$ can be separated by a half-space $A \subseteq H \subseteq \neg B$, and furthermore we have $d(A, B) = |\{H \in \mathcal{H}_{\mathsf{cvx}}(X) : A \subseteq H \subseteq \neg B\}|$.*

*Proof.* Pick a geodesic $A \ni x_0 G x_1 G \cdots G x_n \in B$, where $n := d(A, B)$. Then $H := \mathrm{cone}_{x_1}(x_0)$, which is a half-space by Lemma 1.14, separates $A, B$ since $x_0 = \mathrm{proj}_A(x_n)$, and thus we have $A \subseteq \mathrm{cone}_{x_n}(x_0) \subseteq \mathrm{cone}_{x_1}(x_0)$ and $B \subseteq \mathrm{cone}_{x_0}(x_n) \subseteq \mathrm{cone}_{x_0}(x_1)$.

Moreover, each such half-space $A \subseteq H \subseteq \neg B$ satisfies $x_i \in H \not\ni x_{i+1}$ for a unique $i < n$, and conversely each pair $(x_i, x_{i+1})$ has a unique half-space separating them, so we have the desired bijection. ■

**Lemma 1.21.** *Every interval $[x, y]$ is finite. More generally, if $A \subseteq X$ is finite, then so is $\mathrm{cvx}(A)$.*

*Proof.* The singletons $\{x\}$ and $\{y\}$ are convex, so there are finitely-many half-spaces $H \subseteq [x, y]$. But each $z \in [x, y]$ is determined uniquely by those half-spaces containing it, so $[x, y]$ is finite.

Let $A := \{x_0, \dots, x_n\}$. Since $\mathrm{cvx}(A) = \mathrm{proj}_A(X)$, we have by Remark 1.13 that points in $\mathrm{cvx}(A)$ are of the form $\langle x, x_n, \dots, \langle x, x_2, \langle x, x_1, x_0\rangle\rangle \dots\rangle$, which is finite by induction using that intervals are finite. ■

## 2. Graphs with Dense Families of Cuts

Let $(X, G)$ be a connected locally-finite quasi-tree, which, in the context of Theorem A, stands for a single component of the locally-finite graphing of the CBER. For Theorem B to apply, we need to first identify a family of finitely-separating cuts therein, and we do so in such a way that the cuts are dense towards ends.

Since $X$ is a quasi-tree, and thus does not have arbitrarily long cycles, we expect that there is some finite bound $R < \infty$ such that the ends in $\varepsilon(X)$ are 'limits' of cuts $\mathcal{H} := \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ with boundary diameter bounded by $R$. We show that this is indeed the case, in the sense that $\mathcal{H}$ is dense towards ends.

**Lemma 2.1.** *Let $f : (X, G) \to (Y, T)$ be a coarse-equivalence between connected graphs. For a fixed $H \in \mathcal{H}_{\partial<\infty}(Y)$, $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$ is uniformly bounded in terms of $\mathrm{diam}(\partial_{\mathsf{v}} H)$.*

*Proof.* Since $f$ is bornologous, let $S < \infty$ be such that $xGx'$ implies $d(f(x), f(x')) \leq S$, so that for any $(x, x') \in \partial_{\mathsf{ie}} f^{-1}(H)$, there is a path of length $\leq S$ between $f(x) \notin H$ and $f(x') \in H$. Thus both $d(f(x), \partial_{\mathsf{v}} H)$ and $d(f(x'), \partial_{\mathsf{v}} H)$ are bounded by $S$, so $f(\partial_{\mathsf{v}} f^{-1}(H)) \subseteq \mathrm{Ball}_S(\partial_{\mathsf{v}} H)$ and hence

$$\mathrm{diam}(f(\partial_{\mathsf{v}} f^{-1}(H))) \leq \mathrm{diam}(\partial_{\mathsf{v}} H) + 2S.$$

That $f$ is a coarse-*equivalence* gives us a uniform bound of $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$ in terms of $\mathrm{diam}(\partial_{\mathsf{v}} H)$. ∎

In particular, if $\mathrm{diam}(\partial_{\mathsf{v}} H)$ is itself also uniformly bounded, then so is $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$.

**Proposition 2.2.** *The class of connected locally-finite graphs in which $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}$ is dense towards ends for some $R < \infty$ is invariant under coarse equivalence.*

*Proof.* Let $(X, G)$, $(Y, T)$ be connected locally-finite graphs, $f : X \to Y$ be a coarse equivalence with quasi-inverse $g : Y \to X$, and suppose $\mathcal{H}_{\mathrm{diam}(\partial)\leq S}(Y)$ is dense towards ends for some $S < \infty$. By Lemma 2.1, pick some $R < \infty$ so that for any $H \in \mathcal{H}_{\mathrm{diam}(\partial)\leq S}(Y)$, we have $f^{-1}(H) \in \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$.

Fix an end $p \in \varepsilon(X)$ and a neighborhood $p \in A \in \mathcal{H}_{\partial<\infty}(X)$. We need to find some $B \in \mathcal{H}_{\partial<\infty}(Y)$ such that $f(p) \in B$ and $f^{-1}(B) \subseteq A$, for then $f(p) \in H$ for some $B \supseteq H \in \mathcal{H}_{\mathrm{diam}(\partial)\leq S}(Y)$, and hence we have

$$p \in f^{-1}(H) \subseteq f^{-1}(B) \subseteq A$$

with $f^{-1}(H) \in \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$. For convenience, let $D < \infty$ be the uniform distance $d(1_X, g \circ f)$.

To this end, note that $f(p) \in B$ iff $p \in f^{-1}(B)$. Since $p \in A$, the latter can occur if $|A \mathbin{\triangle} f^{-1}(B)| < \infty$, and so we need to find such a $B \in \mathcal{H}_{\partial<\infty}(Y)$ with the additional property that $f^{-1}(B) \subseteq A$.

> *Attempt 1.* Set $B := g^{-1}(A) \in \mathcal{H}_{\partial<\infty}(Y)$. Then $f^{-1}(B) \subseteq \mathrm{Ball}_D(A)$ since if $(g \circ f)(x) \in A$, then
>
> $$d(x, A) \leq d(x, (g \circ f)(x)) \leq d(1_X, g \circ f) = D.$$
>
> By local-finiteness of $G$, we see that $A \mathbin{\triangle} f^{-1}(B) = A \setminus f^{-1}(B)$ is finite, as desired.

However, it is *not* the case that $f^{-1}(B) \subseteq A$. To remedy this, we 'shrink' $A$ by $D$ to $A'$ so that $\mathrm{Ball}_D(A') \subseteq A$, and take $B := g^{-1}(A')$ instead. Indeed, $A' := \neg \mathrm{Ball}_D(\neg A) \subseteq A$ works, since $f^{-1}(B) \subseteq \mathrm{Ball}_D(A')$ as before, so $A' \mathbin{\triangle} f^{-1}(B) = A' \setminus f^{-1}(B)$ is finite. Also, $A \mathbin{\triangle} A'$ is finite since $x \in A \mathbin{\triangle} A'$ iff $x \in A$ and $d(x, \neg A) \leq D$, so $A \mathbin{\triangle} f^{-1}(B)$ is finite too. It remains to show that $\mathrm{Ball}_D(A') \subseteq A$, for then $f^{-1}(B) \subseteq A$ as desired.

Indeed, if $y \in \mathrm{Ball}_D(A')$, then by the (reverse) triangle-inequality we have $d(y, \neg A) \geq d(x, \neg A) - d(x, y)$ for all $x \in A'$. But $d(x, \neg A) > D$, strictly, so $d(y, \neg A) > D - D = 0$, and hence $y \in A$. ∎

**Corollary 2.3.** *If $(X, G)$ is a locally-finite quasi-tree, then the subpocset $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ is dense towards ends for some $R < \infty$.*

*Proof.* Observe that $\mathcal{H}_{\mathrm{diam}(\partial)\leq 2}(T)$ is dense towards ends for any tree $T$, so Proposition 2.2 proves the density of $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$ for some $R < \infty$. By Lemma 1.10, there is a subpocset $\mathcal{H}' \subseteq \mathcal{H}_{\partial<\infty}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ dense towards ends such that for every $H' \in \mathcal{H}'$, we have $\partial_{\mathsf{ie}} H' \subseteq \partial_{\mathsf{ie}} H$ for some $H \in \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$. Hence we have $H' \in \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$, so the result follows. ∎

The cuts $\mathcal{H} := \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ obtained here will be our starting point for Theorem B, so we need to show that it is finitely-separating. Indeed, by local-finiteness of $(X, G)$, the $R$-ball around any fixed $x \in X$ is finite, so since any cut $H \in \mathcal{H}$ with $x \in \partial_{\mathsf{v}} H$ is contained in said $R$-ball, there are finitely-many such cuts. Thus, by Lemma 1.3, $\mathcal{H}$ is finitely-separating.

Furthermore, we have by Lemma 1.2 that $\mathcal{H} \subseteq 2^X$ is closed and every non-trivial element is isolated, and those conditions allow for the construction in Theorem B to continue.

Finally, we will need to represent 'density towards ends' in a Borel manner, which ultimately is to ensure that the bounds $R < \infty$ can be obtained uniformly across all components; see Section 4 for details.

**Proposition 2.4.** *If $\mathcal{H}$ is a finitely-separating pocset of cuts on a connected locally-finite graph, then $\mathcal{H}$ is dense towards ends iff (1) each $\mathcal{H}$-block is finite, and (2) each $H \in \mathcal{H}^*$ has finitely-many successors.*

*Proof.* If $\mathcal{H}$ is dense towards ends, we will cover (certain closed sets in) $\varepsilon(X)$ by cuts in $\mathcal{H}$, which cuts down to a finite subcover. We will show that a certain Boolean combination thereof, which has finite-boundary, is finite using Lemma 1.8, giving us the desired finiteness claim. Below are the details.

1. Fix an $\mathcal{H}$-block $[x]_{\mathcal{H}}$ and let $A \coloneqq \neg\{x\}$. By density, we can cover each end $p \in \varepsilon(X)$ with some $H_p \subseteq A$, which gives us a finite subcover $\bigcup_{i<n} H_i$ of $\varepsilon(X)$. Note that $\bigcap_{i<n} \neg H_i \in \mathcal{H}_{\partial<\infty}(X)$ contains $[x]_{\mathcal{H}}$, and is finite since it contains no ends in $\varepsilon(X)$.
2. Fix $H \in \mathcal{H}^*$ and let $K_\alpha \in \mathcal{H}^*$ be the successors of $H$. By density, each end $p \in \varepsilon(\neg H)$ is contained in some $H_p \subseteq \neg H$, which in turn is contained in some $\neg K_\alpha$; this gives us a finite subcover $\bigcup_{i<n} \neg K_i$ of $\varepsilon(\neg H)$. Any other successor $H \subset K \in \mathcal{H}^*$ not in this subcover may be assumed to be nested with each $K_i$, since there are finitely-many non-nested ones by Lemma 1.6. But then $K \not\subseteq K_i \not\subseteq K$ and $K \cap K_i \neq \varnothing$ for all $i < n$, which forces $\neg K \subseteq \neg H \cap \bigcap_{i<n} K_i \in \mathcal{H}_{\partial<\infty}(X)$; the latter contains no ends in $\varepsilon(X)$, so it is finite, and hence there are finitely-many possibilities for such $K$.

Conversely, fix an end $p \in \varepsilon(X)$ and a neighborhood $p \in A \in \mathcal{H}_{\partial<\infty}(X)$. Since $\mathcal{H}$ consists of connected sets, it suffices to find some $H \in \mathcal{H}$ containing $\partial_{\mathsf{v}} A$ but not $p$, for then $p \in \neg H \subseteq A$ as desired.

**Observation.** Finitely-many $H, \neg H \in \mathcal{H}$ may be removed and still satisfy conditions (1) and (2) above. That (2) still holds is obvious; for (1), we may remove a single pair, since with $\mathcal{H}' \coloneqq \mathcal{H} \setminus \{H, \neg H\}$, the map $X/\mathcal{H} \to X/\mathcal{H}'$ sending $[x]_{\mathcal{H}} \to [x]_{\mathcal{H}'}$ is surjective and at-most 2-to-1.

Thus, for each of the finitely-many $x, y \in \partial_{\mathsf{v}} A$, we may remove the finitely-many half-spaces separating them, so we may assume that $\mathcal{H} = \mathcal{H}_A \sqcup \neg\mathcal{H}_A$ is partitioned into the half-spaces in $\mathcal{H}_A$ containing $\partial_{\mathsf{v}} A$, and its complements which are disjoint from $\partial_{\mathsf{v}} A$. By contradiction, if each $H \in \mathcal{H}_A$ contains $p$, we will construct a strictly decreasing chain $(H_n) \subseteq \mathcal{H}_A^*$, which gives an infinite family separating $\partial_{\mathsf{v}} A$ from $\neg H_0$.

Let $H_0 \in \mathcal{H}_A^*$ be arbitrary, which exists since $X$ is infinite, so $\mathcal{H}$-blocks being finite forces $\mathcal{H}^* = \mathcal{H}_A^* \sqcup \neg\mathcal{H}_A^*$ to be infinite. There are finitely-many half-spaces non-nested with $H_0$ by Lemma 1.6, so we may assume that there is none. We may further assume that all successors $H$ of $\neg H_0$ lie in $\mathcal{H}_A^*$, for otherwise $\mathcal{H}_A^* \ni \neg H \subset H_0$ and we may take $H_1 \coloneqq \neg H$. Thus, $B \coloneqq H_0 \cap \bigcap_{H \supset \neg H_0} H$ is a *finite* intersection of the minimal ones in $\mathcal{H}_A^*$, and since $p \in H$ for each such $H$ by assumption, $B$ contains $p$ and is infinite. Since $\mathcal{H}$-blocks are finite and $B$ is $\mathcal{H}$-invariant, it contains infinitely-many $\mathcal{H}$-blocks; any two such $\mathcal{H}$-blocks is separated by some $H \in \mathcal{H}_A^*$ nested with $H_0$, and since $H \not\subseteq H_0^c$ and $H$ contains neither $H_0$ nor $\neg H_0$, we can set $H_1 \coloneqq H \subset H_0$. ■

## 3. The Dual Median Graph of a Profinite Pocset and its Spanning Trees

3.1. **Construction of the dual median graph.** We present a classical construction in geometric group theory of a median graph associated to a profinite pocset with every non-trivial element isolated; see [Dun79], [Rol98], [Sag95], and [NR03] for other applications.

In the context of Theorem A, this will be applied to the pocset $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ of cuts in a locally-finite graph $(X, G)$, and is also the first step in the construction in Theorem B.

**Definition 3.1.** An *orientation* on $\mathcal{H}$ is an upward-closed subset $U \subseteq \mathcal{H}$ containing exactly one of $H, \neg H$ for each $H \in \mathcal{H}$. We let $\mathcal{U}(\mathcal{H})$ denote the set of all orientations on $\mathcal{H}$ and let $\mathcal{U}^\circ(\mathcal{H})$ denote the clopen ones.

Intuitively, an orientation is a 'maximally consistent' choice of half-spaces[3].

**Example 3.2.** Each $x \in X$ induces its *principal orientation* $\widehat{x} \coloneqq \{H \in \mathcal{H} : x \in H\} = \mathcal{H} \cap \pi_x^{-1}(1)$ – which is clearly clopen in $\mathcal{H}$ – and gives us a canonical map $X \to \mathcal{U}^\circ(\mathcal{H})$. However, this map is *not necessarily* injective, and we call a fiber $[x]_{\mathcal{H}} \coloneqq \{y \in X : \widehat{x} = \widehat{y}\}$ thereof an *$\mathcal{H}$-block*. This induces an equivalence relation on $X$ by declaring $x \sim_{\mathcal{H}} y$ iff $x$ and $y$ are contained in exactly the same half-spaces in $\mathcal{H}$.

[3]This can be formalized by letting $\sim$ be the equivalence relation on $\mathcal{H}$ given by $H \sim \neg H$. Letting $\partial : \mathcal{H} \to \mathcal{H}/{\sim}$ denote the quotient map, orientations $U \subseteq \mathcal{H}$ then correspond precisely to sections $\varphi : \mathcal{H}/{\sim} \to \mathcal{H}$ of $\partial$ such that $\varphi(\partial H) \not\subseteq \neg\varphi(\partial K)$ for every $H, K \in \mathcal{H}$; the latter condition rules out 'orientations' of the form $\leftarrow\!\mid\ \mid\!\rightarrow$.

**Proposition 3.3.** *Let $\mathcal{H}$ be a profinite pocset with every non-trivial element isolated. Then the graph $\mathcal{M}(\mathcal{H})$, whose vertices are clopen orientations $\mathcal{U}^\circ(\mathcal{H})$ and whose edges are pairs $\{U,V\}$ with $V = U \triangle \{H, \neg H\}$ for some minimal $H \in U \setminus \{\varnothing, X\}$, is a median graph with path metric $d(U,V) = |U \triangle V|/2$ and medians*

$$\begin{aligned}\langle U,V,W\rangle &:= \{H \in \mathcal{H} : H \text{ belongs to at least two of } U,V,W\}\\ &= (U\cap V)\cup(V\cap W)\cup(U\cap W).\end{aligned}$$

*Proof.* First, $V := U \triangle \{H,\neg H\}$ as above is clopen since $H,\neg H \in \mathcal{H}^*$ are isolated (whence $\{H\},\{\neg H\}$ are clopen), and it is an orientation by minimality of $H$. That $\mathcal{M}(\mathcal{H})$ is connected follows from the following claim by noting that $U \triangle V \not\ni \varnothing, X$ is clopen, so it is a compact set of isolated points, whence finite.

**Claim** ([Sag95, Theorem 3.3])**.** *There is a path between $U,V$ iff $U \triangle V$ is finite, in which case*

$$d(U,V) = |U\triangle V|/2 = |U\setminus V| = |V \setminus U|.$$

*Proof.* If $(U_i)_{i<n}$ is a path from $U =: U_0$ to $V =: U_{n-1}$, then, letting $\{H_i,\neg H_i\} := U_i \triangle U_{i-1}$ for all $1 \le i < n$ gives us a sequence $(H_i)_{i<n}$ inducing[a] this path, whence $U \triangle V$ consists of $\{H_i\}_{i<n}$ and their complements. Thus $U \triangle V = 2n = 2d(U,V)$, as desired.

Conversely, if $U \triangle V = \{H_1,\dots,H_n\} \sqcup \{K_1,\dots,K_m\}$ with $U \setminus V = \{H_i\}$ and $V\setminus U = \{K_j\}$, then $\neg H_i \in V \setminus U$ and $\neg K_j \in U \setminus V$ for all $i<n$ and $j<m$, so $n=m$ and $V = U \cup \{\neg H_i\}\setminus\{H_i\}$. We claim that there is a permutation $\sigma \in S_n$ such that $(H_{\sigma(i)})$ induces a path from $U =: U_0$, which is the desired path from $U$ to $V$. Choose a minimal $H \in \{H_i\}$, which is also minimal in $U$: if $K \subseteq H$ for some $K \in U$, then $\neg H \subseteq \neg K$, and hence $\neg K \in V$, so $K = H_i \subseteq H$ for some $i$, forcing $K = H$. Set $U_1 := U \triangle \{H,\neg H\}$, which is a clopen orientation. Continuing in this manner by choosing a minimal element in $\{H_i\}\setminus\{H\}$ – and so on – gives us the desired path with $d(U,V) = n$. □

[a] In the sense that $U_i = U_{i-1} \triangle \{H_i, \neg H_i\}$ and $H_i \in U_i$ for each $1 \le i < n$; see [Tse20, Definition 2.20].

Finally, we show that $\mathcal{M}(\mathcal{H})$ is a median graph. Fix $U,V,W \in \mathcal{U}^\circ(\mathcal{H})$, and note that for any $M \in \mathcal{U}^\circ(\mathcal{H})$, we have by the triangle inequality that $M \in [U,V]$ iff $(U\setminus M)\cup(M\setminus V) \subseteq U \setminus V$, which clearly occurs iff $U \cap V \subseteq M \subseteq U \cup V$. Thus, a vertex $M$ lies in the triple intersection $[U,V]\cap[V,W]\cap[U,W]$ iff

$$(U\cap V)\cup(V\cap W)\cup(U\cap W) \subseteq M \subseteq (U\cup V)\cap(V\cup W)\cap(U\cup W).$$

Note that the two sides coincide, so $M = \langle U,V,W\rangle$ – which is clopen if $U,V,W$ are – is as claimed. ■

Given such a pocset $\mathcal{H}$, the graph $\mathcal{M}(\mathcal{H})$ constructed above is called the *dual*[4] median graph of $\mathcal{H}$. An important special case of this construction is when $\mathcal{H}$ is *nested*, in which case $\mathcal{M}(\mathcal{H})$ is a tree.

**Corollary 3.4.** *Let $\mathcal{H}$ be a profinite pocset with non-trivial points isolated. If $\mathcal{H}$ is nested, then the median graph $\mathcal{M}(\mathcal{H})$ is acyclic, and hence $\mathcal{M}(\mathcal{H})$ is a tree.*

*Proof.* Let $(U_i)_{i<n}$ be a (non-backtracking) cycle in $\mathcal{M}(\mathcal{H})$, say induced by some sequence $(H_i)_{i<n} \subseteq \mathcal{H}$ of half-spaces. We show that $(H_i)$ is *strictly* increasing, so that $H_0 \subset \cdots \subset H_n \subset H_0$, which is absurd.

We have $U_{i+1} = U_{i-1} \cup \{\neg H_i, \neg H_{i-1}\} \setminus \{H_i, H_{i-1}\}$, so $H_i \neq \neg H_{i-1}$ (for otherwise $U_{i+1} = U_{i-1}$). Since $H_i \in U_i = U_{i-1} \cup \{\neg H_{i-1}\}\setminus\{H_{i-1}\}$, we see that $H_i \in U_{i-1}$, and since $H_i \neq H_{i-1}$, it suffices by nestedness of $\mathcal{H}$ to remove the three cases when $\neg H_i \subseteq H_{i-1}$, $H_{i-1} \subseteq \neg H_i$, and $H_i \subseteq H_{i-1}$.

Indeed, if $\neg H_i \subseteq H_{i-1}$, then $H_{i-1} \in U_{i+1}$ by upward-closure of $U_{i+1} \ni \neg H_i$. But since $H_{i-1} \neq \neg H_i$, we have by definition of $U_{i+1}$ that $H_{i-1} \in U_i$, a contradiction. The other cases are similar. ■

Nonetheless, in the general non-nested case, $\mathcal{M}(\mathcal{H})$ still admits a *canonical* spanning tree if we fix a Borel colouring of $\mathcal{H}^*_{\mathrm{cvx}}(\mathcal{M}(\mathcal{H}))$ into its nested sub-pocsets, the existence of which follows from the following

**Proposition 3.5.** *The dual median graph $\mathcal{M}(\mathcal{H})$ of a pocset of finitely-separating cuts has finite hyperplanes.*

*Proof.* Fix $K \in \mathcal{H}^*_{\mathrm{cvx}}(\mathcal{M}(\mathcal{H}))$, which by Proposition 1.17 is of the form $K = \mathrm{cone}_V(U)$ for some (and hence any) $(U,V) \in \partial_{\mathsf{ie}} K$, and we have by Proposition 3.3 that $V = U \triangle \{H,\neg H\}$ for some (non-trivial) minimal $H \in U$. We claim that any other edge $(U',V') \in \partial_{\mathsf{ie}}K$ can be reached from $(U,V)$ by simultaneously flipping only the half-spaces $H_0',\dots,H_n' \in \mathcal{H}^*$ non-nested with $H$, of which there are finitely-many by Lemma 1.6.

[4] The name is justified by a Stone-type duality between {median graphs, median homomorphisms} and {profinite pocsets with non-trivial points isolated, continuous maps}, where from a median graph $X$ one can construct a canonical pocset $\mathcal{H}_{\mathrm{cvx}}(X)$ of convex half-spaces (see [CPTT23, Section 2.D] for details).

Since the edges $(U, V), (U', V')$ induce the same hyperplane $\partial_{\mathsf{ie}} K$, it suffices by Proposition 1.17 to prove this for when $(U', V')$ is an edge of a square parallel to $(U, V)$, in which case there is some minimal $H' \in \mathcal{H}^*$ flipping both $U$ to $U'$ and $V$ to $V'$. Note that $H, H'$ are non-nested since $H' \not\subseteq H$ and $H \not\subseteq H'$ by minimality; if $H \subseteq \neg H'$, then $\neg H' \in U$; and if $\neg H \subseteq H'$, then $H'$ is not minimal in $U$. Moreover, $H \in U'$ is still minimal since $U' = U \triangle \{H', \neg H'\}$, and the only way this can fail is if $\neg H' \subseteq H$, contradicting minimality of $H$. Thus we have $V' = U' \triangle \{H, \neg H\}$, so the induction continues with $(U', V')$ in place of $(U, V)$. ∎

3.2. **Canonical spanning trees.** We now present the Borel cycle-cutting algorithm that can be preformed on any countable median graph with finite hyperplanes. Applying this algorithm to the dual median graph of a finitely-separating family of cuts, which has finite hyperplanes by Proposition 3.5, proves Theorem B.

**Lemma 3.6.** *For any subpocset $\mathcal{H} \subseteq \mathcal{H}_{\mathrm{cvx}}(X)$ on a median graph $(X, G)$, the map $X \to \mathcal{U}^\circ(\mathcal{H})$ is surjective.*

*Proof.* Let $U \in \mathcal{U}^\circ(\mathcal{H})$, we need to find some $x \in X$ with $U = \widehat{x}$. Since $U \subseteq \mathcal{H}$ is clopen, there is a finite set $A \subseteq X$ – which we may assume to be convex by Lemma 1.21 – such that for all $H \in \mathcal{H}$, we have $H \in U$ iff there is $H' \in U$ with $H \cap A = H' \cap A$. Note that $H \cap A \neq \varnothing$ for every $H \in U$, since otherwise $\varnothing \in U$. Furthermore, $H \cap H' \neq \varnothing$ for every $H, H' \in U$, since otherwise we have $H \subseteq \neg H'$, and so $\neg H' \in U$.

By Lemma 1.19, the intersection $(H \cap A) \cap (H' \cap A) = H \cap H' \cap A$ is non-empty, and applying it again furnishes some $x \in \bigcap_{H \in U} H \cap A$ in $X$. Thus $U \subseteq \widehat{x}$, so $U = \widehat{x}$ since both are orientations. ∎

This induces a *$G$-adjacency* graph $X/\mathcal{H} \cong \mathcal{M}(\mathcal{H})$; explicitly, two $\mathcal{H}$-blocks $([x]_{\mathcal{H}}, [y]_{\mathcal{H}})$ are $G$-adjacent if $(\widehat{x}, \widehat{y}) \in \mathcal{M}(\mathcal{H})$. Note that $\mathcal{M}(\mathcal{H})$ may be constructed as in Proposition 3.3 since $\mathcal{H} \subseteq \mathcal{H}_{\mathrm{cvx}}(X)$ is finitely-separating by Lemma 1.20. In particular, if $\mathcal{H}$ is nested, then $X/\mathcal{H}$ is a tree by Corollary 3.4.

**Proposition 3.7.** *If $(X, G)$ is a countable median graph with finite hyperplanes, then fixing any colouring of $\mathcal{H}^*_{\mathrm{cvx}}(X)$ into nested sub-pocsets yields a canonical spanning tree thereof.*

*Proof.* Such a colouring exists, since, by Corollary 1.18, if two half-spaces $H, K \in \mathcal{H}^*_{\mathrm{cvx}}(X)$ are non-nested, then $\partial_{\mathsf{v}} H \cap \partial_{\mathsf{v}} K \neq \varnothing$. Thus, the intersection graph of the boundaries admits a countable colouring, which descends into a colouring $\mathcal{H}^*_{\mathrm{cvx}}(X) = \bigsqcup_{n \in \mathbb{N}} \mathcal{H}^*_n$ such that each $H, \neg H$ receive the same colour and that each $\mathcal{H}_n \coloneqq \mathcal{H}^*_n \cup \{\varnothing, X\}$ is a *nested* subpocset. For each $n \in \mathbb{N}$, let $\mathcal{K}_n \coloneqq \bigcup_{m \geq n} \mathcal{H}_m$.

We shall inductively construct an increasing chain of subforests $T_n \subseteq G$ such that the components of $T_n$ are exactly the $\mathcal{K}_n$-blocks. Then, the increasing union $T \coloneqq \bigcup_n T_n$ is a spanning tree, since each $(x, y) \in G$ lies in a $\mathcal{K}_n$-block for sufficiently large $n$ (namely, the $n$ such that $\mathrm{cone}_x(y) \in \mathcal{H}^*_{n-1}$, since $\mathrm{cone}_x(y)$ and its complement are the only half-spaces separating $x$ and $y$ by Proposition 1.17).

Since each pair of distinct points is separated by a half-space, the $\mathcal{K}_0$-blocks are singletons, so put $T_0 \coloneqq \varnothing$. Suppose that a forest $T_n$ is constructed as required. Note that each $\mathcal{K}_{n+1}$-block $Y \in X/\mathcal{K}_{n+1}$ is not separated by any half-spaces in $\mathcal{H}_m$ for $m > n$, but is separated by $\mathcal{H}_n$ into the $\mathcal{K}_n$-blocks contained in $Y$, which are precisely the $\mathcal{H}_n$-blocks in $Y/\mathcal{H}_n$. Pick an edge from the *finite* hyperplane $\partial_{\mathsf{ie}} H$ for each $H \in \mathcal{H}_n$, which connects a unique pair of $G$-adjacent blocks in $Y/\mathcal{H}_n$. Since each $Y/\mathcal{H}_n$ is a tree by Corollary 3.4, and each pair of $G$-adjacent blocks in $Y/\mathcal{H}_n$ is connected by a single picked edge, the graph $T_{n+1}$ obtained from $T_n$ by adding all such edges is a forest whose components are exactly the $\mathcal{K}_{n+1}$-blocks. ∎

## 4. Borel Treeings of Graphings with Dense Cuts

We finally prove Theorem A, stating that if a CBER $(X, E)$ admits a locally-finite graphing $G$ such that each component is a quasi-tree, then $E$ is treeable. The first step is to identify, for each component $G|C$, a family $\mathcal{H}(C)$ of finitely-separating cuts that is dense towards ends of $G|C$; since each $G|C$ is a quasi-tree, the cuts $\mathcal{H}(C) \coloneqq \mathcal{H}_{\mathrm{diam}(\partial) \leq R_C}(C) \cap \mathcal{H}_{\mathrm{conn}}(C)$ for some $R_C < \infty$ from Section 2 will do. Applying Theorem B then gives us, for each component $G|C$, a median graph $\mathcal{M}(\mathcal{H}(C))$ on $\mathcal{U}^\circ(\mathcal{H}(C))$ with finite hyperplanes.

The issue lies in making the family $\mathcal{U}^\circ(\mathcal{H}) \coloneqq \bigsqcup_C \mathcal{U}^\circ(\mathcal{H}(C))$ of *all* clopen orientations on $\mathcal{H} \coloneqq \bigsqcup_C \mathcal{H}(C)$ into a standard Borel space. *If $\mathcal{U}^\circ(\mathcal{H})$ is standard Borel*, the above partition induces a CBER $\mathcal{E}$ admitting a median graphing $\mathcal{M}(\mathcal{H}) \coloneqq \bigsqcup_C \mathcal{M}(\mathcal{H}(C))$ with finite hyperplanes, from which one can implement the proof of Proposition 3.7 in a Borel manner (using [KM04, Lemma 7.3] for a countable colouring of the intersection graph of finite hyperplanes therein) to obtain a treeing of $\mathcal{E}$. Finally, $E$ is Borel bireducible with $\mathcal{E}$ via the principal orientations map $X \ni x \mapsto \widehat{x} \in \mathcal{U}^\circ(\mathcal{H})$, so $E$ is also treeable by [JKL02, Proposition 3.3 (ii)].

We will remedy this issue using the fact that the cuts $\mathcal{H}(C)$ are dense towards ends of $G|C$. In particular, we have the following crucial lemma, which, by Proposition 3.3, shows that $\mathcal{M}(\mathcal{H})$ is locally-finite.

**Lemma 4.1.** *Let $\mathcal{H}$ be a finitely-separating pocset of cuts on a connected locally-finite graph $(X, G)$. If $\mathcal{H}$ is dense towards ends, then each clopen orientation $U \in \mathcal{U}^\circ(\mathcal{H})$ contains finitely-many minimal cuts $H \in \mathcal{H}$.*

*Proof.* Fix a vertex $U \in \mathcal{U}^\circ(\mathcal{H})$ and let $K_\alpha \in U$ be its minimal elements. Since $U \subseteq \mathcal{H}$ is clopen, there is a finite set $A \subseteq X$ such that for all $H \in \mathcal{H}$, we have $H \in U$ iff there is $H' \in U$ with $H \cap A = H' \cap A$. Note that $H \cap A \neq \varnothing$ for every $H \in U$, for otherwise $\varnothing \in U$; in particular, we have $\neg H \in U$ for every $H \subseteq \neg A$.

Each end $p \in \varepsilon(X)$ lies in $\neg A$, so density of $\mathcal{H}$ furnishes $H_p \in \mathcal{H}$ with $p \in H_p \subseteq \neg A$, and thus $\neg H_p \in U$. Since $U$ is clopen, we have $K_\alpha \subseteq \neg H_p$ for some $\alpha$. Thus $\varepsilon(X) \subseteq \bigcup_\alpha \neg K_\alpha$, which by compactness cuts down to a finite subcover $\varepsilon(X) \subseteq \bigcup_{i<n} \neg K_i$. We show that there are at-most finitely-many more minimal $K \in U$.

Let $K \neq K_i$ be any other minimal element in $U$. By Lemma 1.6, each $K_i$ is non-nested with finitely-many other half-spaces, so we may assume that $K$ is nested with every $K_i$. But then $K \not\subseteq K_i \not\subseteq K$ and $K \cap K_i \neq \varnothing$ for all $i < n$, which forces $\neg K \subseteq \bigcap_{i<n} K_i \in \mathcal{H}_{\partial<\infty}(X)$; the latter contains no ends in $\varepsilon(X)$, so it is finite by Lemma 1.8, and hence there are at-most finitely-many more minimal $K \in U$. ■

We now describe the encoding of $\mathcal{U}^\circ(\mathcal{H})$ into a standard Borel space. Since cuts have finite edge boundary, we may first represent the space $\mathcal{K}$ of *all* non-trivial cuts of $G$ as a Borel subset of $[G]^{<\infty}$. The subcollection $\mathcal{H} \subseteq \mathcal{K}$ consisting of those cuts with component-wise bounded boundary diameter is also Borel, since each $R_C < \infty$ can be witnessed as the minimal number making $\mathcal{H}_{\mathrm{diam}(\partial) \leq R_C}(C)$ dense towards ends of $G|C$, and the latter is a Borel condition as characterized in Proposition 2.4. Finally, $\mathcal{U}^\circ(\mathcal{H})$ is a Borel subset of $[\mathcal{H}]^{<\infty}$, since we may encode each clopen orientation $U \in \mathcal{U}^\circ(\mathcal{H}(C))$ by its set of minimal elements in $\mathcal{H}(C)$, which is finite by Lemma 4.1. This makes $\mathcal{U}^\circ(\mathcal{H})$ a standard Borel space, and finishes the proof of Theorem A.

The above discussion actually proves the following generalization of Theorem A, which is no longer about quasi-trees; rather, we only require the (locally-finite) graphing to admit a Borel family of 'tree-like' cuts.

**Theorem 4.2.** *If a CBER $(X, E)$ admits a locally-finite graphing $G$ such that each component $G|C$ admits a family $\mathcal{H}(C)$ of finitely-separating cuts that is dense towards ends of $G|C$, and if $\mathcal{H} \coloneqq \bigsqcup_C \mathcal{H}(C)$ is a Borel subset of the standard Borel space of all cuts of $G$, then $E$ is treeable.*

## References


[CPTT23] Ruiyuan Chen, Antoine Poulin, Ran Tao, and Anush Tserunyan, *Tree-like graphings, wallings, and median graphings of equivalence relations* (2023), available at https://arxiv.org/abs/2308.13010.

[Gro93] Mikhail Gromov, *Asymptotic invariants of infinite groups*, Geometric Group Theory, vol. 2 (Sussex, 1991), London Mathematical Society Lecture Note Series, vol. 182, Cambridge University Press, 1993.

[DK18] Cornelia Druţu and Michael Kapovich, *Geometric Group Theory*, with appendix by Bogdan Nica, Vol. 63, American Mathematical Society Colloquium Publications, 2018.

[Nic04] B. Nica, *Cubulating Spaces with Walls*, Algebraic & Geometric Topology **4** (2004), 297-309.

[CN05] I. Chatterji and G.A. Niblo, *From Wall Spaces to CAT(0) Cube Complexes*, Internat. J. Algebra Comput. **15** (2005), no. 5-6, 875–885.

[Rol98] Martin Roller, *Poc sets, median algebras and group actions*, Universität Regensburg (1998), available at https://arxiv.org/abs/1607.07747. Ph.D. thesis.

[Bow22] Brian H. Bowditch, *Median Algebras* (2022), available at https://homepages.warwick.ac.uk/~masgak/papers/median-algebras.pdf.

[vdV93] M. L. J. van de Vel, *Theory of Convex Structures*, North-Holland Mathematical Library, vol. 50, North-Holland Publishing Co., Amsterdam, 1993.

[Dun79] M. J. Dunwoody, *Accessibility and Groups of Cohomological Dimension One*, Proceedings of the London Mathematical Society (3) **38** (1979), no. 2, 193-215.

[Sag95] Michah Sageev, *Ends of Group Pairs and Non-Positively Curved Cube Complexes*, Proceedings of the London Mathematical Society (3) **71** (1995), no. 3, 585-617.

[NR03] G.A. Niblo and L.D. Reeves, *Coxeter Groups act on CAT(0) cube complexes*, Journal of Group Theory **6** (2003), no. 3, 399-413.

[Tse20] Anush Tserunyan, *A descriptive construction of trees and Stallings' theorem*, Trends in Set Theory, Contemporary Mathematics **752** (2020), 191-207.

[KM04] Alexander S. Kechris and Benjamin D. Miller, *Topics in Orbit Equivalence*, Lecture Notes in Mathematics, vol. 1852, Springer-Verlag Berlin, Heidelberg, 2004.

[JKL02] S. Jackson, Alexander S. Kechris, and A. Louveau, *Countable Borel Equivalence Relations*, Journal of Mathematical Logic **2** (2002), no. 1, 1-80.



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