Tree-like graphings of countable Borel equivalence relations

An exposition to

Tree-like graphings, wallings, and median graphings of equivalence relations

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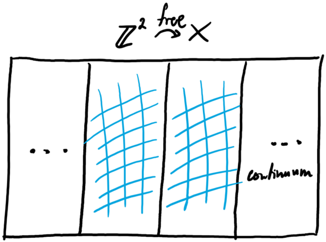

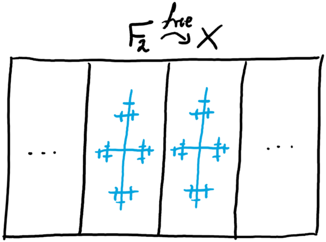

Smooth and Hyperfinite CBERs

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Theorem (Slaman-Steel, Weiss)

Let E be a CBER on a standard Borel space X. TFAE:

1. *E is hyperfinite.* *$E = \bigcup_n F_n$ where $F_0 \subseteq F_1 \subseteq \cdots$ are FBERs.*
2. *E is induced by a Borel $\mathbb{Z}$-action.* *$E = E_{\mathbb{Z}}^{X}$ for some $\mathbb{Z} \curvearrowright X$.*

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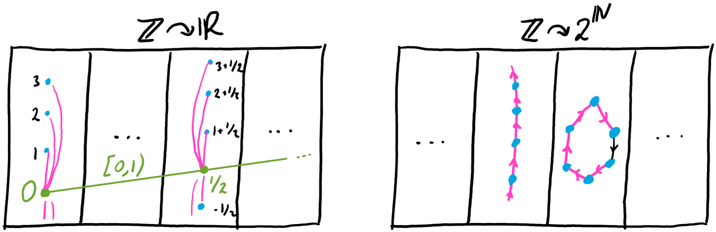

Graphing of a CBER

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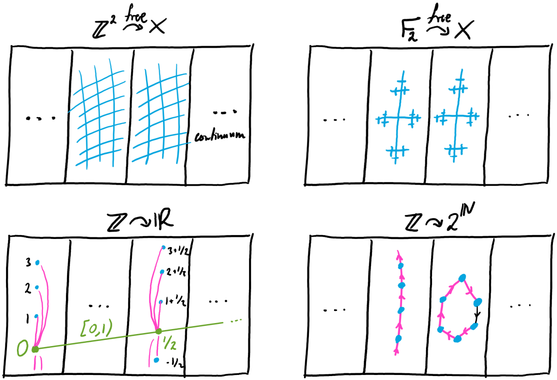
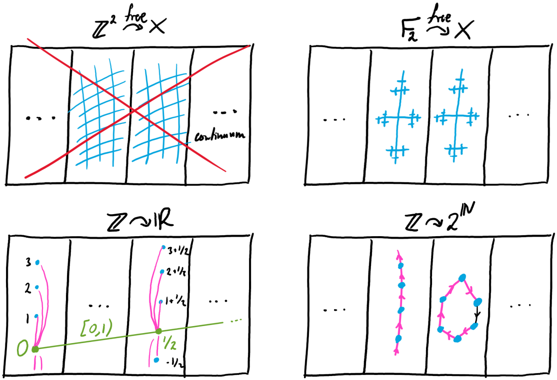


Treeings and Treeability

Definition

A *treeing* of a CBER E is an acyclic graphing, and a CBER E is said to be *treeable* if it admits a treeing.



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Theorem (GdlH90)

Every finitely-generated group whose Cayley graph is a quasi-tree is virtually-free, and hence treeable.

The End

Thank you!