# TREE-LIKE GRAPHINGS

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ABSTRACT. We present a streamlined exposition of a construction presented in [CPTT23], where it is proven that every locally-finite Borel graph with each component a quasi-tree induces a canonical treeable equivalence relation. Write some more details...

## 1. Introduction

The purpose of this note is to provide a streamlined proof of a particular case of a construction presented in [CPTT23], in order to better understand the general formalism developed therein. We attempt to make this note self-contained, but nevertheless urge the reader to refer to the original paper for more detailed discussion and generalizations of the results we have selected to include here.

1.1. **Treeings of Equivalence Relations.** A *countable Borel equivalence relation (CBER)* on a standard Borel space $X$ is a Borel equivalence relation $E \subseteq X^2$, such that each class is countable. We are interested in special types of *graphings* on a CBER $E \subseteq X^2$, i.e. a Borel graph $G \subseteq X^2$ whose connectedness relation is precisely $E$. For instance, a graphing of $E$ such that each component is a tree is called a *treeing* of $E$, and the CBERs that admit a treeing are said to be *treeable*. The main results of [CPTT23] is to provide new sufficient criteria for treeability of CBERs, and in particular, they prove the following

**Theorem A** ([CPTT23, Theorem 1.1])**.** *If a CBER admits a locally-finite graphing whose components are quasi-trees, then it is treeable.*

Recall that metric spaces $X$ and $Y$ are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error, and $X$ is a *quasi-tree* if it is quasi-isometric to a simplicial tree; see [Gro93] and [DK18].

1.2. **Outline of the Proof.** Roughly speaking, the existence of a quasi-isometry $G|C \to T_C$ to a simplicial tree $T_C$ for each component $C \subseteq G$ induces a collection $\mathcal{H}(C)$ of 'cuts' (subsets $H \subseteq C$ with finite boundary such that both $H$ and $C \setminus H$ are connected), which are 'tree-like' in the sense that

1. $\mathcal{H}(C)$ is a *walling*: each vertex $x \in C$ is on the boundary of finitely-many $H \in \mathcal{H}(C)$, and
2. $\mathcal{H}(C)$ is *dense towards ends*: each end in $G|C$ has a neighborhood basis in $\mathcal{H}(C)$.

These cuts provide exactly the data to construct a 'median graph' whose vertices are 'ultrafilters' thereof, and condition (2) ensures that this graph has finite 'hyperplanes'. This finiteness condition allows us to apply a Borel 'cycle-cutting' algorithm to obtain a subtree thereof. Each step above can be done in a uniform way to each component $C \subseteq G$, giving us the desired treeing of the CBER.

$$\text{Quasi-tree} \xrightarrow{2.1} \text{Dense walling of cuts} \xrightarrow{2.2} \text{Median graph w/ finite hyperplanes} \xrightarrow{2.3} \text{Tree}$$

## 2. Detailed Constructions

2.1. **Dense walling of Cuts.**

2.2. **Median Graph of Orientations.**

2.3. **Canonical Construction of a Sub-treeing.**

## References

[CPTT23] Ruiyuan Chen, Antoine Poulin, Ran Tao, and Anush Tserunyan, *Tree-like graphings, wallings, and median graphings of equivalence relations* (2023), available at `https://arxiv.org/abs/2308.13010`.

[Gro93] Mikhail Gromov, *Asymptotic invariants of infinite groups*, Geometric Group Theory, Vol. 2 (Sussex, 1991), London Mathematical Society Lecture Note Series, vol. 182, Cambridge University Press, 1993.

[DK18] Cornelia Druţu and Michael Kapovich, *Geometric Group Theory*, with appendix by Bogdan Nica, Vol. 63, American Mathematical Society Colloquium Publications, 2018.

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