TREE-LIKE GRAPHINGS

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Abstract. We present a streamlined exposition of a construction presented in [CPTT23], where it is proven that every locally-finite Borel graph with each component a quasi-tree induces a canonical treeable equivalence relation. Write some more details...

1. Introduction

The purpose of this note is to provide a streamlined proof of a particular case of a construction presented in [CPTT23], in order to better understand the general formalism developed therein. We attempt to make this note self-contained, but nevertheless urge the reader to refer to the original paper for more detailed discussion and some generalizations of the results we have selected to include here.

1.1. **Treeings of equivalence relations.** A *countable Borel equivalence relation (CBER)* on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$ with each class countable. We are interested in special types of *graphings* on a CBER $E \subseteq X^2$, i.e. a Borel graph $G \subseteq X^2$ whose connectedness relation is precisely E. For instance, a graphing of E such that each component is a tree is called a *treeing* of E, and the CBERs that admit treeings are said to be *treeable*. The main results of [CPTT23] provide new sufficient criteria for treeability of certain classes of CBERs, and in particular, they prove the following

Theorem A ([CPTT23, Theorem 1.1]). *If a CBER admits a locally-finite graphing whose components are quasi-trees, then it is treeable.*

Recall that metric spaces X and Y are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error, and X is a *quasi-tree* if it is quasi-isometric to a simplicial tree; see [Gro93] and [DK18].

1.2. **Outline of the proof.** Roughly speaking, the existence of a quasi-isometry $G|C \to T_C$ to a simplicial tree T_C for each component $C \subseteq X$ induces a collection $\mathcal{H}(C)$ of 'cuts' (subsets $H \subseteq C$ with finite boundary such that both H and $C \setminus H$ are connected), which are 'tree-like' in the sense that

1. $\mathcal{H}(C)$ is *finitely-separating*: each pair $x, y \in C$ is separated by finitely-many $H \in \mathcal{H}(C)$, and
2. $\mathcal{H}(C)$ is *dense towards ends*: each end in $G|C$ has a neighborhood basis in $\mathcal{H}(C)$.

By Condition (1), these cuts have the structure of a profinite pocset with non-trivial points isolated, which in turn provide exactly the data to construct a 'median graph' whose vertices are 'ultrafilters'[1] thereof. Condition (2) then ensures that this graph has finite 'hyperplanes', which allows us to apply a Borel 'cycle-cutting' algorithm and obtain a canonical spanning tree thereof. Each step above can be done in a uniform way to each component $C \subseteq G$, giving us the desired treeing of the CBER.

Quasi-tree	$\xrightarrow{2}$	Dense family of cuts	$\xrightarrow{3.1}$	Pocset w/ non-triv. pts. isolated	$\xrightarrow{3.2+3.3}$	Median graph w/ finite hyperplanes	$\xrightarrow{4}$	Canonical spanning tree

Write some more stuff to tie things together...

Remark. We follow [CPTT23, Convention 2.7], where for a family $\mathcal{H} \subseteq 2^X$ of subsets of a fixed set X, we write $\mathcal{H}^* := \mathcal{H} \setminus \{\varnothing, X\}$ for the *non-trivial* elements of $\mathcal{H}$.

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[1]As in [CPTT23], we call them *orientations* instead, to avoid confusion with the more standard notion; see Definition 3.5.

2. Graphs with Dense Families of Cuts

2.1. **Ends of graphs.** Let (X, G) be a connected locally-finite graph, which, in the context of Theorem A, will stand for a single component of the locally-finite graphing of a CBER.

Definition 2.1. For a subset $A \subseteq X$, we let $\partial_{\mathsf{iv}} A \coloneqq A \cap \mathrm{Ball}_1(\neg A)$ be its *inner vertex boundary*, $\partial_{\mathsf{ov}} A \coloneqq \partial_{\mathsf{iv}}(\neg A)$ be its *outer vertex boundary*, and let $\partial_{\mathsf{ie}} A \coloneqq G \cap (\partial_{\mathsf{ov}} A \times \partial_{\mathsf{iv}} A)$ and $\partial_{\mathsf{oe}} A \coloneqq \partial_{\mathsf{ie}}(\neg A)$ respectively be its *inner* and *outer edge boundaries*. Let $\partial_{\mathsf{v}} A \coloneqq \partial_{\mathsf{iv}} A \cup \partial_{\mathsf{ov}} A$ be the *(total) vertex boundary* of A.

Let $\mathcal{H}_{\partial<\infty}(X) \subseteq 2^X$ be the Boolean algebra of all $A \subseteq X$ with finite vertex boundary, called *cuts* in X.

Definition 2.2. The *end compactification* of (X, G) is the Stone space $\widehat{X}$ of $\mathcal{H}_{\partial<\infty}(X)$, whose non-principal ultrafilters are the *ends* of (X, G).

We identify $X \hookrightarrow \widehat{X}$ via principal ultrafilter map $x \mapsto p_x$, so $\widehat{X} \setminus X$ is the set of ends of G. By definition, $\widehat{X}$ admits a basis of clopen sets of the form $\widehat{A} \coloneqq \{p \in \widehat{X} : A \in p\}$ for each $A \in \mathcal{H}_{\partial<\infty}(X)$.

Definition 2.3. A family $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty}(X)$ of cuts is *dense towards ends* of (X, G) if $\mathcal{H}$ contains a neighborhood basis for every end in $\widehat{X} \setminus X$.

In other words, $\mathcal{H}$ is dense towards ends if for every $p \in \widehat{X} \setminus X$ and every (clopen) neighborhood $\widehat{A} \ni p$, where $A \in \mathcal{H}_{\partial<\infty}(X)$, there is some $H \in \mathcal{H}$ with $p \in \widehat{H} \subseteq \widehat{A}$; it is useful to note that $\widehat{H} \subseteq \widehat{A}$ iff $H \subseteq A$.

2.2. **Dense cuts induced by quasi-trees.** If X is a quasi-tree – and thus does not have arbitrary long cycles – we expect that there is some finite bound $R < \infty$ such that the ends in $\widehat{X} \setminus X$ are 'limits' of cuts $\mathcal{H}_{\mathrm{diam}(\partial)\le R}(X) \subseteq \mathcal{H}_{\partial<\infty}(X)$ with boundary diameter bounded by R. We show that this is indeed the case, in the sense that $\mathcal{H}_{\mathrm{diam}(\partial)\le R}(X)$ is dense towards ends of (X, G).

Lemma 2.4. *Let $f : (X, G) \to (Y, T)$ be a coarse-equivalence between connected graphs. For a fixed $H \in \mathcal{H}_{\partial<\infty}(Y)$, $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$ is uniformly bounded in terms of $\mathrm{diam}(\partial_{\mathsf{v}} H)$.*

Proof. Since f is bornologous, let $S < \infty$ be such that xGx' implies $d(f(x), f(x')) \le S$, so that for any $(x, x') \in \partial_{\mathsf{ie}} f^{-1}(H)$, there is a path of length $\le S$ between $f(x) \notin H$ and $f(x') \in H$. Thus both $d(f(x), \partial_{\mathsf{v}} H)$ and $d(f(x'), \partial_{\mathsf{v}} H)$ are bounded by S, so $f(\partial_{\mathsf{v}} f^{-1}(H)) \subseteq \mathrm{Ball}_S(\partial_{\mathsf{v}} H)$ and hence

$$\mathrm{diam}(f(\partial_{\mathsf{v}} f^{-1}(H))) \le \mathrm{diam}(\partial_{\mathsf{v}} H) + 2S.$$

That f is a coarse-*equivalence* gives us a uniform bound of $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$ in terms of $\mathrm{diam}(\partial_{\mathsf{v}} H)$. ∎

In particular, if $\mathrm{diam}(\partial_{\mathsf{v}} H)$ is itself also uniformly bounded, then so is $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$.

Proposition 2.5. *The class of connected locally-finite graphs in which $\mathcal{H}_{\mathrm{diam}(\partial)\le R}$ is dense towards ends for some $R < \infty$ is invariant under coarse equivalence.*

Proof. Let (X, G), (Y, T) be connected locally-finite graphs, $f : X \to Y$ be a coarse equivalence with quasi-inverse $g : Y \to X$, and suppose $\mathcal{H}_{\mathrm{diam}(\partial)\le S}(Y)$ is dense towards ends for some $S < \infty$. By Lemma 2.4, pick some $R < \infty$ so that for any $H \in \mathcal{H}_{\mathrm{diam}(\partial)\le S}(Y)$, we have $f^{-1}(H) \in \mathcal{H}_{\mathrm{diam}(\partial)\le R}(X)$.

Fix an end $U \in \widehat{X} \setminus X$ with $U \in \widehat{A}$ for some $A \in \mathcal{H}_{\partial<\infty}(X)$. We need to find some $B \in \mathcal{H}_{\partial<\infty}(Y)$ such that $\widehat{f}(U) \in \widehat{B}$ and $f^{-1}(B) \subseteq A$, for then $\widehat{f}(U) \in \widehat{H}$ for some $B \supseteq H \in \mathcal{H}_{\mathrm{diam}(\partial)\le S}(Y)$, and hence we have

$$U \in \widehat{f^{-1}(H)} \subseteq \widehat{f^{-1}(B)} \subseteq \widehat{A}$$

with $f^{-1}(H) \in \mathcal{H}_{\mathrm{diam}(\partial)\le R}(X)$. For convenience, let $D < \infty$ be the uniform distance $d(1_X, g \circ f)$.

To this end, note that $\widehat{f}(U) \in \widehat{B}$ iff $U \in \widehat{f^{-1}(B)}$. Since $U \in \widehat{A}$, the latter can occur if $|A \triangle f^{-1}(B)| < \infty$, and so we need to find such a $B \in \mathcal{H}_{\partial<\infty}(Y)$ with the additional property that $f^{-1}(B) \subseteq A$.

Attempt 1. Set $B \coloneqq g^{-1}(A) \in \mathcal{H}_{\partial<\infty}(Y)$. Then $f^{-1}(B) \subseteq \mathrm{Ball}_D(A)$ since if $(g \circ f)(x) \in A$, then

$$d(x, A) \le d(x, (g \circ f)(x)) \le d(1_X, g \circ f) = D.$$

By local-finiteness of G, we see that $A \triangle f^{-1}(B) = A \setminus f^{-1}(B)$ is finite, as desired.

However, it is *not* the case that $f^{-1}(B) \subseteq A$. To remedy this, we 'shrink' A by D to A' so that $\mathrm{Ball}_D(A') \subseteq A$, and take $B := g^{-1}(A')$ instead. Indeed, $A' := \neg \mathrm{Ball}_D(\neg A) \subseteq A$ works, since $f^{-1}(B) \subseteq \mathrm{Ball}_D(A')$ as before, so $A' \triangle f^{-1}(B) = A' \setminus f^{-1}(B)$ is finite. Also, $A \triangle A'$ is finite since $x \in A \triangle A'$ iff $x \in A$ and $d(x, \neg A) \leq D$, so $A \triangle f^{-1}(B)$ is finite too. It remains to show that $\mathrm{Ball}_D(A') \subseteq A$, for then $f^{-1}(B) \subseteq A$ as desired.

Indeed, if $y \in \mathrm{Ball}_D(A')$, then by the (reverse) triangle-inequality we have $d(y, \neg A) \geq d(x, \neg A) - d(x, y)$ for all $x \in A'$. But $d(x, \neg A) > D$, strictly, so $d(y, \neg A) > D - D = 0$, and hence $y \in A$. ■

Corollary 2.6. *If (X, G) is a locally-finite quasi-tree, then $\mathcal{H}_{\mathrm{diam}(\partial) \leq R}(X)$ is dense towards ends for some $R < \infty$.*

Proof. Observe that $\mathcal{H}_{\mathrm{diam}(\partial) \leq 2}(T)$ is dense towards ends for any tree T, and invoke Proposition 2.5. ■

3. Pocsets and Orientations

3.1. **Pocsets of cuts.** The family $\mathcal{H}_{\mathrm{diam}\,\partial \leq R}(X)$ of cuts have the structure of a 'profinite pocset', which we first study abstractly. We then deduce some properties of the pocset induced by a *dense* family of cuts.

Definition 3.1. A *pocset* $(\mathcal{H}, \leq, \neg, 0)$ is a poset $(\mathcal{H}, \leq)$ equipped with an order-reversing involution $\neg : \mathcal{H} \to \mathcal{H}$ and a least element $0 \neq \neg 0$ such that 0 is the only lower-bound of $H, \neg H$ for every $H \in \mathcal{H}$.

A *profinite pocset* is a pocset $\mathcal{H}$ equipped with a compact topology making $\neg$ continuous and is *totally order-disconnected*, in the sense that if $H \not\leq K$, then there is a clopen upward-closed $U \subseteq \mathcal{H}$ with $H \in U \not\ni K$.

Remark 3.2. Such a topology is automatically Hausdorff and zero-dimensional.

We are primarily interested in subpocsets of $(2^X, \subseteq, \neg, \varnothing)$, which is profinite if equipped with the product topology of the discrete space 2. Indeed, 2^X admits a base of *cylinder sets* – which are finite intersections of sets of the form $\pi_x^{-1}(i)$ where $x \in X$, $i \in \{0, 1\}$, and $\pi_x : 2^X \to 2$ is the projection – making $\neg$ continuous since cylinders are clopen. Show that it is totally order-disconnected.

The following proposition gives a sufficient criteria for subpocsets of 2^X to be profinite. We also show in this case that every non-trivial element $H \in \mathcal{H}^*$ is isolated, which will important in Section 3.2.

Proposition 3.3. *Let X be a set and $\mathcal{H} \subseteq 2^X$ be a subpocset. If $\mathcal{H}$ is finitely-separating, then $\mathcal{H} \subseteq 2^X$ is closed and every non-trivial element is isolated.*

Proof. It suffices to show that the limit points of $\mathcal{H}$ are trivial, so let $A \in 2^X \setminus \{\varnothing, X\}$. Fix $x \in A \not\ni y$. Since $\mathcal{H}$ is finitely-separating, there are finitely-many $H \in \mathcal{H}$ with $x \in H \not\ni y$, and for each such $H \in \mathcal{H} \setminus \{A\}$, we have either some $x_H \in A \setminus H$ or $y_H \in H \setminus A$. Let $U \subseteq 2^X$ be the family of all subsets $B \subseteq X$ containing x and each x_H but not y or any y_H.

This is the desired neighborhood isolating $A \in U$. Indeed, it is (cl)open since it is the *finite* intersection of cylinders prescribed by the x_H's and y_H's, and it is disjoint from $\mathcal{H} \setminus \{A\}$ by construction. ■

Example 3.4. For a locally-finite connected graph (X, G), a subpocset $\mathcal{H} \subseteq \mathcal{H}_{\partial < \infty}(X)$ of cuts therein is finitely-separating iff each $x \in X$ is on the boundary of finitely-many $H \in \mathcal{H}$.

Indeed, if $\mathcal{H}$ is finitely-separating, then each $x \in X$ is separated from each of its finitely-many neighbors by finitely-many $H \in \mathcal{H}$. Conversely, any $H \in \mathcal{H}$ separating $x, y \in X$ separates some edge on any fixed path between x and y, and there are only finitely-many such H for each edge.

In particular, the subpocset $\mathcal{H}_{\mathrm{diam}(\partial) \leq R}(X)$ for any fixed $R < \infty$ (see Section 2.2) is finitely-separating.

3.2. **The dual median graph of a pocset.** Let $\mathcal{H} \subseteq 2^X$ be a finitely-separating subpocset, whose elements we call *half-spaces*. By Proposition 3.3, $\mathcal{H}$ is profinite and every non-trivial element thereof is isolated.

Definition 3.5. An *orientation* on $\mathcal{H}$ is an upward-closed subset $U \subseteq \mathcal{H}$ containing exactly one of $H, \neg H$ for each $H \in \mathcal{H}$. We let $\mathcal{U}(\mathcal{H})$ denote the set of all orientations on $\mathcal{H}$ and let $\mathcal{U}^\circ(\mathcal{H})$ denote the clopen ones.

Intuitively, an orientation is a 'maximally consistent' choice of half-spaces[2].

Example 3.6. Each $x \in X$ induces its *principal orientation* $\widehat{x} := \{H \in \mathcal{H} : x \in H\} = \mathcal{H} \cap \pi_x^{-1}(1)$ – which is clearly clopen in $\mathcal{H}$ – and gives us a canonical map $X \to \mathcal{U}^\circ(\mathcal{H})$. However, this map is *not necessarily* injective, and we call a fiber $[x]_{\mathcal{H}} := \{y \in X : \widehat{x} = \widehat{y}\}$ thereof an *$\mathcal{H}$-block*.

[2] This can be formalized by letting $\sim$ be the equivalence relation on $\mathcal{H}$ given by $H \sim \neg H$. Letting $\partial : \mathcal{H} \to \mathcal{H}/{\sim}$ denote the quotient map, orientations $U \subseteq \mathcal{H}$ then correspond precisely to sections $\varphi : \mathcal{H}/{\sim} \to \mathcal{H}$ of ∂ such that $\varphi(\partial H) \not\subseteq \neg\varphi(\partial K)$ for every $H, K \in \mathcal{H}$; the latter condition rules out 'orientations' of the form $\leftarrow\!\mid\ \mid\!\rightarrow$.

The *clopen* orientations are exactly those which are *finitely-based*, in the sense that there is a $\subseteq$-minimal finite subset $U_0 \subseteq U$ such that $U = \uparrow U_0$, where

$$\uparrow U_0 \coloneqq \bigcup_{H \in U_0} \uparrow H \coloneqq \bigcup_{H \in U_0} \{K \in \mathcal{H} : K \supseteq H\}.$$

Indeed, if U is clopen, then there is a finite subset $A \subseteq X$ such that for all $H \in \mathcal{H}$, we have $H \in U$ iff there is some $K \in U$ with $H \cap A = K \cap A$.

The goal of this section is to canonically construct a graph whose vertices are clopen orientations on $\mathcal{H}$.

Definition 3.7. A *median graph* is a connected graph (X, G) such that for any $x, y, z \in X$, the intersection

$$[x, y] \cap [y, z] \cap [x, z]$$

is a singleton. Its unique element $\langle x, y, z\rangle$ is called the *median* of x, y, z.

Theorem 3.8. *Let $\mathcal{H} \subseteq 2^X$ be a finitely-separating subpocset. Then the graph $\mathcal{M}(\mathcal{H})$, whose:*

- *Vertices are clopen orientations $\mathcal{U}^\circ(\mathcal{H})$;*
- *Edges are pairs $\{U, V\}$ with $V = U \triangle \{H, \neg H\}$ for some $\subseteq$-minimal $H \in \mathcal{H}^*$;*

is a median graph with path metric $d(U, V) = (U \triangle V)/2$ and medians

$$\langle U, V, W\rangle \coloneqq \{H \in \mathcal{H} : H \text{ belongs to at least two of } U, V, W\}.$$

Proof. First, $V \coloneqq U \triangle \{H, \neg H\}$ as above is a clopen orientation: it is clopen since H is isolated, and an orientation since H is $\subseteq$-minimal.

Claim. There is a path between U, V iff $U \triangle V$ is finite, in which case $d(U, V) = (U \triangle V)/2$.

Proof. Hi □ *(claim)*

■

3.3. Finiteness conditions on $\mathcal{U}^\circ(\mathcal{H})$ induced by dense cuts.

Hi

4. Median Graphs and their Canonical Spanning Trees

TODO

A. Tree of orientations on a nested pocset of sets

Let $\mathcal{H} \subseteq 2^X$ be a finitely-separating subpocset (see Definition 3.1), so every non-trivial element is isolated by Proposition 3.3. As shown in Theorem 3.8, the clopen orientations $\mathcal{U}^\circ(\mathcal{H})$ form the vertices of a median graph, whose edges are given by 'minimal base-flipping'.

We devote this appendix to show that if $\mathcal{H}$ is *nested*, then $\mathcal{U}^\circ(\mathcal{H})$ is in fact a tree, and so we may bypass the Borel cycle-cutting algorithm in Section 4 and obtain a treeing directly.

References

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