QUESTIONS IN LEMMA 5.19

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1. Results from the Paper

I'll first reference a previous lemma, and then reproduce Lemma 5.19 and its proof here.

Lemma 2.67. *Let (X, G), (Y, H) be connected locally-finite graphs. For a coarse embedding $f : X \to Y$ and $A \in \mathcal{H}_{\partial<\infty}(Y)$, $\operatorname{diam}(\partial_{\mathsf{v}} f^{-1}(A))$ is uniformly bounded in terms of $\operatorname{diam}(\partial_{\mathsf{v}} A)$, and also $f^{-1}(A) \in \mathcal{H}_{\partial<\infty}(X)$.*

Lemma 5.19. *The class of connected locally-finite graphs in which $\mathcal{H}_{\operatorname{diam}(\partial)\leq R}$ is dense towards ends for some $R < \infty$ is invariant under coarse equivalence.*

Proof. Let (X, G), (Y, T) be connected locally-finite graphs, $f : X \to Y$ be a coarse equivalence with quasi-inverse $g : Y \to X$, and suppose $\mathcal{H}_{\operatorname{diam}(\partial)\leq S}(Y)$ is dense towards ends for some $S < \infty$. By Lemma 2.67, pick some $R < \infty$ so that for any $H \in \mathcal{H}_{\operatorname{diam}(\partial)\leq S}(Y)$, we have $f^{-1}(H) \in \mathcal{H}_{\operatorname{diam}(\partial)\leq R}(X)$. Then for any $U \in \widehat{X} \setminus X$ and $A \in \mathcal{H}_{\partial<\infty}(X)$ with $U \in \widehat{A}$, letting $B := \neg \operatorname{Ball}_{d(1_X, g\circ f)}(\neg A)$, we have $f^{-1}(g^{-1}(B)) \subseteq \operatorname{Ball}_{d(1_X, g\circ f)}(B) \subseteq A$, and $A \mathbin{\triangle} B$, $B \mathbin{\triangle} f^{-1}(g^{-1}(B))$ are finite, so $U \in f^{-1}\widehat{(g^{-1}(B))}$, so $\widehat{f}(U) \in \widehat{g^{-1}(B)}$, so there is $g^{-1}(B) \supseteq H \in \mathcal{H}_{\operatorname{diam}(\partial)\leq S}(Y)$ with $\widehat{f}(U) \in H$[1], so $f^{-1}(H) \in \mathcal{H}_{\operatorname{diam}(\partial)\leq R}(X)$ with $U \in \widehat{f^{-1}(H)}$ and $f^{-1}(H) \subseteq f^{-1}(g^{-1}(B)) \subseteq A$. ■

2. Detailed Proof to Check my Understanding

I'll give some details in the proof and rewrite it in a way that I can understand, in order to ask you if my understanding of this proof is correct (as it is a very important step towards a generalization/modification).

Proof. Let (X, G), (Y, T), $f : X \to Y$ and $g : Y \to X$ be as above, $\mathcal{H}_{\operatorname{diam}(\partial)\leq S}(Y)$ be dense towards ends, and $R < \infty$ be so that for any $H \in \mathcal{H}_{\operatorname{diam}(\partial)\leq S}(Y)$, we have $f^{-1}(H) \in \mathcal{H}_{\operatorname{diam}(\partial)\leq R}(X)$.

Fix an end $U \in \widehat{X} \setminus X$ with $U \in \widehat{A}$ for some $A \in \mathcal{H}_{\partial<\infty}(X)$. We need to find some[2] $B \in \mathcal{H}_{\partial<\infty}(Y)$ such that $\widehat{f}(U) \in \widehat{B}$ and $f^{-1}(B) \subseteq A$, for then $\widehat{f}(U) \in \widehat{H}$ for some $B \supseteq H \in \mathcal{H}_{\operatorname{diam}(\partial)\leq S}(Y)$, and hence we have

$$U \in \widehat{f^{-1}(H)} \subseteq \widehat{f^{-1}(B)} \subseteq \widehat{A}$$

with $f^{-1}(H) \in \mathcal{H}_{\operatorname{diam}(\partial)\leq R}(X)$. For convenience, let $D < \infty$ be the uniform distance $d(1_X, g \circ f)$.

To this end, note that $\widehat{f}(U) \in \widehat{B}$ iff $U \in \widehat{f^{-1}(B)}$. Since $U \in \widehat{A}$, the latter can occur if $|A \mathbin{\triangle} f^{-1}(B)| < \infty$, and so we need to find such a $B \in \mathcal{H}_{\partial<\infty}(Y)$ with the additional property that $f^{-1}(B) \subseteq A$.

- *Attempt 1:* Set $B := g^{-1}(A) \in \mathcal{H}_{\partial<\infty}(Y)$. Then $f^{-1}(B) \subseteq \operatorname{Ball}_D(A)$ since if $(g \circ f)(x) \in A$, then
$$d(x, A) \leq d(x, (g \circ f)(x)) \leq d(1_X, g \circ f) = D.$$
By local-finiteness of G, we see that $A \mathbin{\triangle} f^{-1}(B) = A \setminus f^{-1}(B)$ is finite, as desired.

However, it is *not* the case that $f^{-1}(B) \subseteq A$. To remedy this, we 'shrink' A by D to A' so that $\operatorname{Ball}_D(A') \subseteq A$, and take $B := g^{-1}(A')$ instead. Indeed, $A' := \neg \operatorname{Ball}_D(\neg A) \subseteq A$ works, since $f^{-1}(B) \subseteq \operatorname{Ball}_D(A')$ as before, so $A' \mathbin{\triangle} f^{-1}(B) = A' \setminus f^{-1}(B)$ is finite. Also, $A \mathbin{\triangle} A'$ is finite since $x \in A \mathbin{\triangle} A'$ iff $x \in A$ and $d(x, \neg A) \leq D$, so $A \mathbin{\triangle} f^{-1}(B)$ is finite too. It remains to show that $\operatorname{Ball}_D(A') \subseteq A$, for then $f^{-1}(B) \subseteq A$ as desired.

Indeed, if $y \in \operatorname{Ball}_D(A')$, then by the (reverse) triangle-inequality we have $d(y, \neg A) \geq d(x, \neg A) - d(x, y)$ for all $x \in A'$. But $d(x, \neg A) > D$, strictly, so $d(y, \neg A) > D - D = 0$, and hence $y \in A$. ■

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[1] I think this should be $H \in \widehat{f}(U)$, or equivalently $\widehat{f}(U) \in \widehat{H}$.

[2] Warning: My $B \in \mathcal{H}_{\partial<\infty}(X)$ is *not* the same B as in the original proof.