TREE OF ORIENTATIONS ON A NESTED COLLECTION OF SETS

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Let $\mathcal{C} \subseteq 2^X$ be a collection of non-empty subsets of a set X. With the definitions in Section 1, we prove the following

Theorem A (Propositions 2.5, 2.6)**.** *If $\mathcal{C}$ is nested, then the graph $\mathcal{T}_{\mathcal{C}}$, whose:*
- *Vertices are finitely-based orientations on $\mathcal{C}$; and whose*
- *Neighbors of $\mathcal{U} \in V(\mathcal{T}_{\mathcal{C}})$ are $\mathcal{U} \triangle \{A, A^c\}$ for every minimal $A \in \mathcal{U}$ with $A^c \in \mathcal{C}$;*

is acyclic. Furthermore, if $\mathcal{C}$ is closed under complements, then $\mathcal{T}_{\mathcal{C}}$ is a tree.

In particular, this applies to when (X, G) is a graph and $\mathcal{C}$ is a nested collection of cuts on X. No further assumptions (like local-finiteness) is needed.

1. Preliminaries

Definition 1.1. A collection $\mathcal{C}$ is said to be *nested* if every $C_1, C_2 \in \mathcal{C}$ has an empty corner, i.e., $C_1^i \cap C_2^j = \varnothing$ for some $i, j \in \{1, -1\}$, where $C^i := C$ if $i = 1$ and $C^i := C^c$ if $i = -1$.

1.1. **Orientations.** Since we do not assume that $\mathcal{C}$ is closed under complements, we slightly modify the definition of orientations, as follows.

Definition 1.2. An *orientation* on $\mathcal{C}$ is a subset $\mathcal{U} \subseteq \mathcal{C}$ such that
1. *(Upward-closure).* If $A \in \mathcal{U}$ and $B \in \mathcal{C}$ contains A, then $B \in \mathcal{U}$.
2. *(Ultra).* If $A, A^c \in \mathcal{C}$, then either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$, but not both.

Remark 1.3. This coincides with the standard definition when $\mathcal{C}$ is a subpocset of 2^X.

Lemma 1.4. *If $\mathcal{U} \subseteq \mathcal{C}$ is an orientation, then for any $\subseteq$-minimal $A \in \mathcal{U}$ with $A^c \in \mathcal{C}$, so is $\mathcal{U} \triangle \{A, A^c\}$.*

Proof. That $\mathcal{V} := \mathcal{U} \triangle \{A, A^c\} = \mathcal{U} \cup \{A^c\} \setminus \{A\}$ is upward-closed follows from $\subseteq$-minimality of A. Now, if $B, B^c \in \mathcal{C}$ and $B^c \notin \mathcal{V}$, we need to show that $B \in \mathcal{V}$.

To this end, note that $B^c \notin \mathcal{V}$ implies $A \neq B$ and either $B = A^c$ or $B^c \notin \mathcal{U}$. The former case follows since $A^c \in \mathcal{V}$, and for the latter, we have $B \in \mathcal{U} \setminus \{A\}$ since $\mathcal{U}$ is ultra. ■

Remark 1.5. In the above notations, clearly $\mathcal{U} \neq \mathcal{U} \triangle \{A, A^c\}$. Furthermore, for any other such orientation $\mathcal{U}'$ and $A' \in \mathcal{U}'$, that $\mathcal{U} = \mathcal{U}'$ and $\mathcal{U} \triangle \{A, A^c\} = \mathcal{U}' \triangle \{A', A'^c\}$ together imply $A = A'$.

1.2. **Finitely-based orientations.**

Definition 1.6. A *base* for an orientation $\mathcal{U} \subseteq \mathcal{C}$ is a $\subseteq$-minimal subset $\mathcal{B} \subseteq \mathcal{U}$ such that $\mathcal{U} = \uparrow\mathcal{B}$, where

$$\uparrow\mathcal{B} := \bigcup_{B \in \mathcal{B}} \uparrow B := \bigcup_{B \in \mathcal{B}} \{A \in \mathcal{C} : A \supseteq B\}.$$

Remark 1.7. If $\mathcal{C}$ is nested, then every $\subseteq$-minimal $B \in \mathcal{C}$ induces an orientation $\uparrow B := \{A \in \mathcal{C} : A \supseteq B\}$, called a *principal* orientation. Indeed, $\uparrow B$ is clearly upward-closed, and if $A, A^c \in \mathcal{C}$, then, by $\subseteq$-minimality of B and nestedness of $\mathcal{C}$, either $A \supseteq B$ or $A^c \supseteq B$ (but clearly not both).

This construction generalizes to any collection $\mathcal{B} \subseteq \mathcal{C}$ with each $B \in \mathcal{B}$ being $\subseteq$-minimal in $\mathcal{C}$, so that $\uparrow\mathcal{B}$ is an orientation on $\mathcal{C}$.

Definition 1.8. An orientation $\mathcal{U} \subseteq \mathcal{C}$ is said to be *finitely-based* if it admits a finite base.

Remark 1.9. If $\mathcal{U} = \uparrow\{B_1, \dots, B_n\}$ is finitely-based, then for any $\subseteq$-minimal $A \in \mathcal{U}$ with $A^c \in \mathcal{C}$, so is the orientation $\mathcal{V} := \mathcal{U} \triangle \{A, A^c\}$. Indeed, we have $A = B_i$ for some $1 \leq i \leq n$, and $\mathcal{V} = \uparrow(\{A^c\} \cup \{B_j\}_{j \neq i})$.

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2. The graph $\mathcal{T}_{\mathcal{C}}$

Fix a nested collection of non-empty subsets of a set X. Using Lemma 1.4 and Remarks 1.5 and 1.9, we construct a graph $\mathcal{T}_{\mathcal{C}}$ whose:

- *Vertices* of $\mathcal{T}_{\mathcal{C}}$ are finitely-based orientations on $\mathcal{C}$.
- *Neighbors* of $\mathcal{U} \in V(\mathcal{T}_{\mathcal{C}})$ are $\mathcal{U} \triangle \{A, A^c\}$ for every minimal $A \in \mathcal{U}$ with $A^c \in \mathcal{C}$.

The goal of this section is to establish Theorem A, stating that $\mathcal{T}_{\mathcal{C}}$ is acyclic (Proposition 2.5), and furthermore, $\mathcal{T}_{\mathcal{C}}$ is a tree when $\mathcal{C}$ is closed under complements (Proposition 2.6).

Definition 2.1. Fix $\mathcal{U}_0 \in V(\mathcal{T}_{\mathcal{C}})$ and $n \in \mathbb{N}$. A sequence $(A_i)_{i<n} \subseteq \mathcal{C}$ is said to *induce a path from* $\mathcal{U}_0$ if $(\mathcal{U}_i)_{i<n}$, defined by $\mathcal{U}_i \coloneqq \mathcal{U}_{i-1} \triangle \left\{A_{i-1}, A_{i-1}^c\right\}$ for every $1 \leq i < n$, is a path in $\mathcal{T}_{\mathcal{C}}$ with each $A_i \in \mathcal{U}_i$.

Remark 2.2. Any path in $\mathcal{T}_{\mathcal{C}}$ is induced by its sequence of flipped basis elements.

Lemma 2.3. *Let $n \geq 3$. A path in $\mathcal{T}_{\mathcal{C}}$ from $\mathcal{U}_0$ induced by $(A_i)_{i<n}$ has no backtracking iff $A_i \neq A_{i-1}^c$ for every $1 \leq i < n$.*

Proof. Take $2 \leq i \leq n$. It suffices to show that $\mathcal{U}_{i-2} = \mathcal{U}_i$ iff $A_{i-1} = A_{i-2}^c$.

($\Rightarrow$). We have by definition that $\mathcal{U}_i = \mathcal{U}_{i-2} \cup \left\{A_{i-1}^c, A_{i-2}^c\right\} \setminus \{A_{i-1}, A_{i-2}\}$, so since $A_{i-2} \in \mathcal{U}_{i-2} = \mathcal{U}_i$, we have $A_{i-2} = A_{i-1}^c$ as desired.

($\Leftarrow$). Again by definition, by noting that the basis-flipping cancels out. ■

Lemma 2.4. *If $(A_i)_{i<n}$ induces a path in $\mathcal{T}_{\mathcal{C}}$ with no backtracking, then $(A_i)_{i<n}$ is strictly increasing.*

Proof. By Lemma 2.3, we have $A_i \neq A_{i-1}^c$ for every $1 \leq i < n$. Thus, since $A_i \in \mathcal{U}_i = \mathcal{U}_{i-1} \cup \left\{A_{i-1}^c\right\} \setminus \{A_{i-1}\}$, we see that $A_i \in \mathcal{U}_{i-1}$. Clearly $A_i \neq A_{i-1}$. It suffices to remove the three cases when $A_i \subseteq A_{i-1}$, $A_{i-1} \subseteq A_i^c$, and $A_i^c \subseteq A_{i-1}$, since then nestedness of $\mathcal{C}$ gives us $A_{i-1} \subsetneq A_i$, as desired.

- If $A_i \subseteq A_{i-1}$, then $A_{i-1} \in \mathcal{U}_i$, contradicting the definition of $\mathcal{U}_i$.
- If $A_{i-1} \subseteq A_i^c$, then $A_i^c \in \mathcal{U}_{i-1}$ by upward-closure of $\mathcal{U}_{i-1}$, a contradiction.
- If $A_i^c \subseteq A_{i-1}$, then $A_{i-1} \in \mathcal{U}_{i+1}$ by upward-closure of $\mathcal{U}_{i+1} \ni A_i^c$. But since $A_{i-1} \neq A_i^c$, we have by definition of $\mathcal{U}_{i+1}$ that $A_{i-1} \in \mathcal{U}_i$, a contradiction. ■

Proposition 2.5. *$\mathcal{T}_{\mathcal{C}}$ is acyclic.*

Proof. Let $(\mathcal{U}_i)_{i<n}$ be a cycle in $\mathcal{T}_{\mathcal{C}}$ induced by $(A_i)_{i<n}$. Since cycles are non-backtracking, we have $A_0 \subsetneq A_0$ by Lemma 2.4, a contradiction. ■

Proposition 2.6. *If $\mathcal{C}$ is chain-vanishing, then $\mathcal{T}_{\mathcal{C}}$ is connected (and hence a tree).*

Proof. ■

Proposition 2.7. *If $\mathcal{C}$ is closed under complements, then $\mathcal{T}_{\mathcal{C}}$ is connected (and hence a tree).*

Proof. If $\mathcal{U}, \mathcal{U}' \in V(\mathcal{T}_{\mathcal{C}})$ are two finitely-based orientations on $\mathcal{C}$, then swapping their basis elements one by one as in Remark 1.9 gives us a path between $\mathcal{U}$ and $\mathcal{U}'$; the closure of $\mathcal{C}$ is needed to ensure that those basis elements induce a path between the vertices, in that their complement lies in $\mathcal{C}$. ■