TREE-LIKE GRAPHINGS OF COUNTABLE BOREL EQUIVALENCE RELATIONS

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Abstract. We present a streamlined exposition of a construction by R. Chen, A. Poulin, R. Tao, and A. Tserunyan, which proves the treeability of equivalence relations generated by any locally-finite Borel graph such that each component is a quasi-tree. More generally, we show that if each component of a locally-finite Borel graph admits a *finitely-separating Borel family of cuts*, then we may 'canonically' replace each component of the graph by a tree of special ultrafilter-like objects on cuts called *orientations*; moreover, if the cuts are *dense towards ends*, then the union of these trees is a Borel treeing.

The purpose of this note is to provide a streamlined proof of the main result in [CPTT23] in order to better understand the general formalism developed therein. We attempt to make this note relatively self-contained, but nevertheless, we urge the reader to refer to the original paper for more detailed background/discussions and some generalizations of the results we have selected to include here.

Treeings of equivalence relations. A *countable Borel equivalence relation (CBER)* on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$ with each class countable. We are interested in special types of *graphings* of a CBER $E \subseteq X^2$, i.e. a Borel graph $G \subseteq X^2$ whose connectedness relation is precisely E. For instance, a graphing of E such that each component is a tree is called a *treeing* of E, and the CBERs that admit treeings are said to be *treeable*. The main results of [CPTT23] provide new sufficient criteria for treeability of certain classes of CBERs, and in particular, they prove the following

Theorem A (Section 4, [CPTT23, Theorem 1.1])**.** *If a CBER E admits a locally-finite graphing such that each component is a quasi-tree,*[1] *then E is treeable.*

Roughly speaking, the existence of a quasi-isometry $G|C \to T_C$ to a simplicial tree T_C for each component $C \subseteq X$ induces a collection $\mathcal{H}(C) \subseteq 2^C$ of 'cuts' (subsets $H \subseteq C$ with finite boundary such that both H and $C \setminus H$ are connected), which are 'tree-like' in the sense that

(i) $\mathcal{H}(C)$ is *finitely-separating*: each pair $x, y \in C$ is separated by finitely-many $H \in \mathcal{H}(C)$, and
(ii) $\mathcal{H}(C)$ is *dense towards ends*: $\mathcal{H}(C)$ contains a neighborhood basis for each end in $G|C$.

Condition (i) allows for an abstract construction of a tree $\mathcal{U}^\circ(\mathcal{H}(C))$ whose vertices are special 'ultrafilters'[2] on $\mathcal{H}(C)$, as outlined in the following diagram: starting from a finitely-separating family of cuts, one constructs a 'dual median graph' $\mathcal{M}(\mathcal{H}(C))$ with said ultrafilters; this median graph has finite 'hyperplanes', which allows one to apply a Borel cycle-cutting algorithm and obtain a 'canonical' spanning tree thereof.

$$\begin{tikzcd}
\text{Borel quasi-treeing} \arrow[d] & \text{Dense family of cuts} \arrow[rr, "\text{Lem. 4.1}", "\text{Section 4 (Theorem 4.2)}"'] & & \text{Borel treeing} & \text{Thm. A} \\
\text{Quasi-tree} \arrow[ur, "\text{Sec. 1.2+2}"] \arrow[r, "\text{Sec. 1.1+2}"] & \text{Finitely-sep. family of cuts} \arrow[r, "\text{Sec. 3.1}"] \arrow[urr, bend left, "\text{Lem. 4.1}"] & \text{Median graph w/ finite hyperplanes} \arrow[r, "\text{Sec. 3.2}"] & \text{`Canonical' spanning tree} \arrow[u] & \text{Thm. B}
\end{tikzcd}$$

Thus we have the following theorem, which can be viewed as a component-wise version of Theorem A.

Theorem B (Propositions 3.3, 3.5, 3.7)**.** *For any finitely-separating family of cuts $\mathcal{H}$ on a connected locally-finite graph, its dual median graph $\mathcal{M}(\mathcal{H})$ has finite hyperplanes, and fixing a proper colouring of the intersection graph of those hyperplanes yields a canonical spanning tree $\mathcal{U}^\circ(\mathcal{H})$ of $\mathcal{M}(\mathcal{H})$.*

Date: December 14, 2024.

2020 *Mathematics Subject Classification.* 03E15, 20F65, 20E08, 37A20.

Key words and phrases. treeable, countable Borel equivalence relation, quasi-tree, ends, cuts, median graph, pocset.

This work was partially supported by McGill University's SURA (Science Undergraduate Research Award) grant.

[1] Recall that metric spaces X and Y are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error, and X is a *quasi-tree* if it is quasi-isometric to a simplicial tree; see [Gro93] and [DK18].

[2] As in [CPTT23], we call them *orientations* instead, to avoid confusion with the more standard notion; see Definition 3.1.

In the context of Theorem A, the additional condition (ii) then shows that $\mathcal{M}(\mathcal{H}(C))$ is locally-finite for each component $C \subseteq G$ of a locally-finite graphing G of E, which ensures that $\mathcal{U}^\circ(\mathcal{H}) \coloneqq \bigsqcup_C \mathcal{U}^\circ(\mathcal{H}(C))$ is a standard Borel space. This, in turn, proves Theorem A; see Section 4 for details.

Acknowledgements. I would like to thank Professor Anush Tserunyan for supervising me for this project, for her consistent support, patience, and feedback, and for guiding me through this fulfilling research experience. I also thank the original authors of [CPTT23] for their incredible work and detailed writing.

1. Preliminaries on Pocsets, Ends of Graphs, and Median Graphs

Notation. A *graph* on a set X is a symmetric irreflexive binary relation $G \subseteq X^2$. For $A \subseteq X$, we say that A is *connected* if the induced subgraph $G[A]$ is. We always equip connected graphs with their *path metric* d, and let $\mathrm{Ball}_r(x)$ be the closed ball of radius r around x; more generally, we let $\mathrm{Ball}_r(A) \coloneqq \bigcup_{x \in A} \mathrm{Ball}_r(x)$.

For a subset $A \subseteq X$, we let $\partial_{\mathsf{iv}} A \coloneqq A \cap \mathrm{Ball}_1(\neg A)$ be its *inner vertex boundary*, $\partial_{\mathsf{ov}} A \coloneqq \partial_{\mathsf{iv}}(\neg A)$ be its *outer vertex boundary*, and let $\partial_{\mathsf{ie}} A \coloneqq G \cap (\partial_{\mathsf{ov}} A \times \partial_{\mathsf{iv}} A)$ and $\partial_{\mathsf{oe}} A \coloneqq \partial_{\mathsf{ie}}(\neg A)$ respectively be its *inward* and *outward edge boundaries*. Let $\partial_{\mathsf{v}} A \coloneqq \partial_{\mathsf{iv}} A \sqcup \partial_{\mathsf{ov}} A$ be the *(total) vertex boundary* of A.

Finally, for $x, y \in X$, the *interval* $[x, y]$ between x, y is the union of all geodesics between x, y, consisting of exactly those $z \in X$ with $d(x, z) + d(z, y) = d(x, y)$. We say that $A \subseteq X$ is *convex* if $[x, y] \subseteq A$ for all $x, y \in A$. For vertices $x, y, z \in X$, we write x—y—z for $y \in [x, z]$. For all $w, x, y, z \in X$, observe that

$$(w\text{—}x\text{—}y \text{ and } w\text{—}y\text{—}z) \quad \Leftrightarrow \quad (w\text{—}x\text{—}z \text{ and } x\text{—}y\text{—}z),$$

and both sides occur iff there is a geodesic from w to x to y to z, which we write as w—x—y—z.

1.1. Profinite pocsets of cuts.

In the context of Theorem A, the construction starts by identifying a profinite pocset $\mathcal{H}$ of 'cuts' in each component of the graphing, which we first study abstractly. The finitely-separating subpocsets of 2^X are well-known in metric geometry as *wallspaces*; see, e.g., [Nic04] and [CN05].

Definition 1.1. A *pocset* $(\mathcal{H}, \leq, \neg, 0)$ is a poset $(\mathcal{H}, \leq)$ equipped with an order-reversing involution $\neg : \mathcal{H} \to \mathcal{H}$ and a least element $0 \neq \neg 0$ such that 0 is the only lower-bound of $H, \neg H$ for every $H \in \mathcal{H}$. We call the elements in $\mathcal{H}$ *half-spaces*.

A *profinite pocset* is a pocset $\mathcal{H}$ equipped with a compact topology making $\neg$ continuous and is *totally order-disconnected*, in the sense that if $H \not\leq K$, then there is a clopen upward-closed $U \subseteq \mathcal{H}$ with $H \in U \not\ni K$.

We are primarily interested in subpocsets of $(2^X, \subseteq, \neg, \varnothing)$ for a fixed set X, where $\neg A \coloneqq A^c$ for $A \subseteq X$, which is profinite if equipped with the product topology of the discrete space 2.

Remark. We follow [CPTT23, Convention 2.7], where for a family $\mathcal{H} \subseteq 2^X$ of subsets of a fixed set X, we write $\mathcal{H}^* \coloneqq \mathcal{H} \setminus \{\varnothing, X\}$ for the *non-trivial* elements of $\mathcal{H}$.

The following proposition gives a sufficient criterion for subpocsets of 2^X to be profinite. We also show in this case that every non-trivial element $H \in \mathcal{H}^*$ is isolated, which will be important in Section 3.1.

Lemma 1.2. *If $\mathcal{H} \subseteq 2^X$ is a finitely-separating pocset, then $\mathcal{H}$ is closed and non-trivial elements are isolated.*

Proof. It suffices to show that the limit points of $\mathcal{H}$ are trivial, so let $A \in 2^X \setminus \{\varnothing, X\}$. Fix $x \in A \not\ni y$. Since $\mathcal{H}$ is finitely-separating, there are finitely-many $H \in \mathcal{H}$ with $x \in H \not\ni y$, and for each such $H \in \mathcal{H} \setminus \{A\}$, there is either some $x_H \in A \setminus H$ or $y_H \in H \setminus A$. The family of all subsets $B \subseteq X$ containing x and each x_H, but not y or any y_H, is then a clopen neighborhood of A disjoint from $\mathcal{H} \setminus \{A\}$, as desired. ∎

Our main method of identifying the finitely-separating pocsets in graphs is the following

Lemma 1.3. *Let $\mathcal{H} \subseteq 2^X$ be a pocset in a connected graph (X, G). If each $x \in X$ is on the vertex boundary of finitely-many half-spaces in $\mathcal{H}$, then $\mathcal{H}$ is finitely-separating. The converse holds too if (X, G) is locally-finite.*

Proof. Any $H \in \mathcal{H}$ separating $x, y \in X$ separates some edge on any fixed path between x and y, and there are only finitely-many such H for each edge. If (X, G) is locally-finite, then each $x \in X$ is separated from each of its finitely-many neighbors by finitely-many $H \in \mathcal{H}$. ∎

In the case that $\mathcal{H}$ is a pocset consisting of connected co-connected half-spaces with finite vertex boundary, finite-separation also controls the degree of 'non-nestedness' of $\mathcal{H}$.

Definition 1.4. For a connected locally-finite graph (X, G), we let $\mathcal{H}_{\mathrm{conn}}(X)$ and $\mathcal{H}_{\partial<\infty}(X)$ respectively denote the subpocset of connected co-connected half-spaces in 2^X and the half-spaces in 2^X with finite-vertex boundary. A *cut* in (X, G) is a half-space $H \in \mathcal{H}_{\partial<\infty}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$.

Definition 1.5. Let $\mathcal{H} \subseteq 2^X$ be a pocset. Two half-spaces $H, K \in \mathcal{H}$ are *nested* if $\neg^i H \cap \neg^j K = \varnothing$ for some $i, j \in \{0, 1\}$, where $\neg^0 H \coloneqq H$ and $\neg^1 H \coloneqq \neg H$. We say that $\mathcal{H}$ is *nested* if every pair $H, K \in \mathcal{H}$ is nested.

Lemma 1.6. *For a pocset $\mathcal{H}$ of finitely-separating cuts, each $H \in \mathcal{H}$ is non-nested with finitely-many others.*

Proof. Fix $H \in \mathcal{H}$ and let $K \in \mathcal{H}$ be non-nested with H. By connectedness, the non-empty sets $H \cap K$ and $\neg H \cap K$ are joined by a path in K, so $\partial_{\mathsf{v}} H \cap K \neq \varnothing$; similarly, $\partial_{\mathsf{v}} H \cap \neg K \neq \varnothing$. For each $x \in \partial_{\mathsf{v}} H \cap K$ and $y \in \partial_{\mathsf{v}} H \cap \neg K$, any fixed path p_{xy} between them contains some $z \in \partial_{\mathsf{v}} K \cap p_{xy}$; thus, any $K \in \mathcal{H}$ non-nested with H contains some $z \in \partial_{\mathsf{v}} K \cap p_{xy}$.

Then, since there are finitely-many such $x, y \in \partial_{\mathsf{v}} H$, for each of which there are finitely-many $z \in p_{xy}$, for each of which there are finitely-many $K \in \mathcal{H}$ with $z \in \partial_{\mathsf{v}} K$ (by Lemma 1.3, since $\mathcal{H}$ is finitely-separating), there can only be finitely-many $K \in \mathcal{H}$ non-nested with H. ■

1.2. **Ends of graphs.** Let (X, G) be a connected locally-finite graph, and consider the Boolean algebra of finite vertex boundary half-spaces $\mathcal{H}_{\partial<\infty}(X) \subseteq 2^X$.

Definition 1.7. The *end compactification* of (X, G) is the Stone space $\widehat{X}$ of $\mathcal{H}_{\partial<\infty}(X)$, whose non-principal ultrafilters are the *ends* of (X, G). We let $\varepsilon(X)$ denote the set of ends of (X, G).

Identifying $X \hookrightarrow \widehat{X}$ via principal ultrafilter map $x \mapsto p_x$, we have $\varepsilon(X) = \widehat{X} \setminus X$. By definition, $\widehat{X}$ admits a basis of clopen sets containing $\widehat{A} \coloneqq \{p \in \widehat{X} : A \in p\}$ for each $A \in \mathcal{H}_{\partial<\infty}(X)$; abusing notation, we write $p \in A$, and say A *contains* p, when $p \in \widehat{A}$. Since $A \subseteq B$ iff $\widehat{A} \subseteq \widehat{B}$, we also write $p \in A \subseteq B$ for $p \in \widehat{A} \subseteq \widehat{B}$.

Lemma 1.8. *A finite-boundary subset $A \in \mathcal{H}_{\partial<\infty}(X)$ is infinite iff it contains an end in (X, G).*

Proof. The converse direction follows since ends are non-principal. If A is infinite, then by local-finiteness of (X, G), Kőnig's Lemma furnishes some infinite ray $(x_n) \subseteq A$. Then, A is contained in the filter

$$p \coloneqq \{H \in \mathcal{H}_{\partial<\infty}(X) : \forall^\infty n (x_n \in H)\},$$

which is ultra since $H \in p$ are of *finite*-boundary, and is non-principal since it contains cofinite sets. ■

Definition 1.9. A pocset $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty}(X)$ is *dense towards ends* of (X, G) if $\mathcal{H}$ contains a neighborhood basis for every end in $\varepsilon(X)$.

In other words, $\mathcal{H}$ is dense towards ends if for every $p \in \varepsilon(X)$ and every (clopen) neighborhood $A \ni p$, where $A \in \mathcal{H}_{\partial<\infty}(X)$, there is some $H \in \mathcal{H}$ with $p \in H \subseteq A$.

We will show in Section 2 that certain half-spaces $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty}(X)$ induced by a locally-finite quasi-tree (X, G) is dense towards ends. It will also be important that these half-spaces be cuts, in that witnesses to density can also be found in $\mathcal{H} \cap \mathcal{H}_{\mathrm{conn}}(X)$. The following lemma takes care of this.

Lemma 1.10. *If a subpocset $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty}$ is dense towards ends, then there is a subpocset $\mathcal{H}' \subseteq \mathcal{H}_{\partial<\infty} \cap \mathcal{H}_{\mathrm{conn}}$, which is also dense towards ends, such that every $H' \in \mathcal{H}'$ has $\partial_{\mathsf{ie}} H' \subseteq \partial_{\mathsf{ie}} H$ for some $H \in \mathcal{H}$.*

Proof. A first attempt is to let $\mathcal{H}'$ be the connected components H'_0 of elements $H \in \mathcal{H}$, but this fails since $\neg H'_0$ is not necessarily connected. Instead, we further take a component $\neg H'$ of $\neg H'_0$, which is co-connected since H' contains H'_0 and the other components of $\neg H'_0$, each of which is connected to H'_0 via $\partial_{\mathsf{ie}} H'_0$. Formally,

$$\mathcal{H}' \coloneqq \{H' \subseteq X : H \in \mathcal{H} \text{ and } H'_0 \in H/G \text{ and } \neg H' \in \neg H'_0/G\},$$

where H/G denotes the G-components of H. Clearly $\partial_{\mathsf{ie}} H' \subseteq \partial_{\mathsf{ie}} H'_0 \subseteq \partial_{\mathsf{ie}} H$, and since $H' \in \mathcal{H}_{\mathrm{conn}}(X)$, it remains to show that $\mathcal{H}'$ is dense towards ends.

Fix an end $p \in \varepsilon(X)$ and a neighborhood $p \in A \in \mathcal{H}_{\partial<\infty}(X)$. Let $B \supseteq \partial_{\mathsf{v}} A$ be finite connected, which can be obtained by adjoining paths between its components. Then $\neg B \in p$ since p is non-principal, so there is $H \in \mathcal{H}$ with $p \in H \subseteq \neg B$. Since $H \in \mathcal{H}_{\partial<\infty}(X)$, it has finitely-many connected components, so exactly one of them belongs to p, say $p \in H'_0 \subseteq H$. Note that $B \subseteq \neg H \subseteq \neg H'_0$, so since B is connected, there is a unique component $\neg H' \subseteq \neg H'_0$ containing B.

Observe that $H' \in \mathcal{H}'$ and $p \in H'$. Lastly, since H' is connected and is disjoint from $\partial_{\mathsf{v}} A \subseteq B$, and since $H' \subseteq \neg A$ would imply $\neg H' \in p$, this forces $H' \subseteq A$, and hence $p \in H' \subseteq A$ as desired. ■

1.3. **Median graphs and projections.** Starting from a profinite pocset $\mathcal{H}$ with every non-trivial element isolated, we construct in Section 3.1 its dual median graph $\mathcal{M}(\mathcal{H})$.

We devote this section and the next to study some basic properties of median graphs and their projections, which will be used in Section 3.2 to construct a spanning tree certain median graphs. For more comprehensive references of median graphs, and their general theory, see [Rol98] and [Bow22].

Definition 1.11. A *median graph* is a connected graph (X, G) such that for any $x, y, z \in X$, the intersection

$$[x, y] \cap [y, z] \cap [x, z]$$

is a singleton, whose element $\langle x, y, z\rangle$ is called the *median* of x, y, z. Thus we have a ternary *median* operation $\langle\cdot,\cdot,\cdot\rangle : X^3 \to X$, and a *median homomorphism* $f : (X, G) \to (Y, H)$ is a map preserving said operation.

Lemma 1.12. *For any $\varnothing \neq A \subseteq X$ and $x \in X$, there is a unique point in* $\operatorname{cvx}(A)$ *between x and every point in A, called the* projection *of x towards A, denoted* $\operatorname{proj}_A(x)$.

Moreover, we have $\bigcap_{a\in A}[x, a] = [x, \operatorname{proj}_A(x)]$, and for any y in this set, we have $\operatorname{proj}_A(y) = \operatorname{proj}_A(x)$.

Proof. To show existence, pick any $a_0 \in A$. Given $a_n \in \operatorname{cvx}(A)$, if there exists $a \in A$ with $a_n \notin [x, a]$, set $a_{n+1} \coloneqq \langle x, a, a_n\rangle \in \operatorname{cvx}(A)$. Then a_0—a_1—$\cdots$—a_n—x for all n, so this sequence terminates in at most $d(a_0, x)$ steps at a point in $\operatorname{cvx}(A)$ between x and every point in A. For uniqueness, if there exist two such points $a, b \in \operatorname{cvx}(A)$, then x—a—b and x—b—a, forcing $a = b$.

Finally, if x—y—$\operatorname{proj}_A(x)$ and $a \in A$, then x—$\operatorname{proj}_A(x)$—a and hence x—y—a. Conversely, let x—y—a for all $a \in A$. Since $[y, a] \subseteq [x, a]$ for all a, we see that

$$\operatorname{proj}_A(y) \in \operatorname{cvx}(A) \cap \bigcap_{a\in A}[y, a] \subseteq \operatorname{cvx}(A) \cap \bigcap_{a\in A}[x, a]$$

and hence $\operatorname{proj}_A(y) = \operatorname{proj}_A(x)$ by uniqueness. But since y—$\operatorname{proj}_A(y)$—a, we have x—y—$\operatorname{proj}_A(y)$, and hence x—y—$\operatorname{proj}_A(x)$ as desired. ■

Remark 1.13. It follows from the proof above that for any median homomorphism $f : (X, G) \to (Y, H)$, we have $f(\operatorname{proj}_A(x)) = \operatorname{proj}_{f(A)}(f(x))$ for any $\varnothing \neq A \subseteq X$ and $x \in X$. Indeed, we have

$$\operatorname{proj}_A(x) = \langle x, a_m, \ldots, \langle x, a_2, \langle x, a_1, a_0\rangle\rangle \ldots\rangle$$

for some $m \leq d(a_0, x)$ and $a_0, \ldots, a_m \in A$, and this is preserved by f.

For $A \coloneqq \{a, b\}$, we have $\operatorname{proj}_A(x) = \langle a, b, x\rangle$, and hence $\operatorname{cvx}(A) = \operatorname{proj}_A(X) = \langle a, b, X\rangle = [a, b]$.

Lemma 1.14. *For each $x, y \in X$,* $\operatorname{cone}_x(y)$ *is convex, and if xGy, then* $\operatorname{cone}_x(y) \sqcup \operatorname{cone}_y(x) = X$.

Proof. Fix $a, b \in \operatorname{cone}_x(y)$ and a—c—b. It suffices to show that x—y—$\langle a, c, x\rangle$, for then x—y—c since we have x—$\langle a, c, x\rangle$—c. Indeed, it follows from the following observations.

- x—y—$\langle a, b, x\rangle$, since $\langle a, b, x\rangle = \operatorname{proj}_{\{a,b\}}(x)$ and so $[x, \langle a, b, x\rangle] = [x, a] \cap [x, b] \ni y$ by Lemma 1.12.
- x—$\langle a, b, x\rangle$—$\langle a, c, x\rangle$, which follows from $\langle a, b, x\rangle$—$\langle a, c, x\rangle$—a, since x—$\langle a, b, x\rangle$—a by definition. Indeed, we have $\langle a, c, x\rangle$ is in both $[a, x]$ and $[a, c] \subseteq [a, b]$, and since $\operatorname{proj}_{\{b,x\}}(a) = \langle a, b, x\rangle$, we have again by Lemma 1.12 that $[\langle a, b, x\rangle, a] = [a, x] \cap [a, b] \ni \langle a, c, x\rangle$.

Finally, take $z \in X$ and consider $w \coloneqq \langle x, y, z\rangle \subseteq [x, y]$. Either $w = x$ or $w = y$ (but not both), giving us the desired partition. ■

Remark 1.15. In particular, this shows that if xGy, then $\operatorname{cone}_x(y) \in \mathcal{H}^*_{\operatorname{cvx}}(X)$. The convexity of cones also shows, in the situation of Lemma 1.12, that $\operatorname{proj}_A = \operatorname{proj}_{\operatorname{cvx}(A)}$, i.e., $\operatorname{proj}_A(x)$ is also between x and every point in $\operatorname{cvx}(A)$: indeed, note that $\operatorname{cone}_x(\operatorname{proj}_A(x))$ is convex and contains A, so it contains $\operatorname{cvx}(A)$ too.

Lemma 1.16. $\operatorname{proj}_A : X \twoheadrightarrow \operatorname{cvx}(A)$ *is a median homomorphism with* $\operatorname{proj}_A \circ \operatorname{cvx} = \operatorname{cvx} \circ \operatorname{proj}_A$.

Proof. The second claim follows from the first since, by Remark 1.13, we have

$$f(\operatorname{cvx}(B)) = f(\operatorname{proj}_B(X)) = \operatorname{proj}_{f(B)}(f(X)) = \operatorname{cvx}(f(B))$$

for all median homomorphisms $f : X \twoheadrightarrow Y$ and $B \subseteq X$, so it in particular applies to $f \coloneqq \operatorname{proj}_A$.

To this end, let x—y—$z \in X$ and set $w \coloneqq \langle \operatorname{proj}_A(x), \operatorname{proj}_A(y), \operatorname{proj}_A(z)\rangle \in \operatorname{cvx}(A)$. It suffices to show that y—w—a for all $a \in A$, for then $w = \operatorname{proj}_A(y)$ and hence $\operatorname{proj}_A(x)$—$\operatorname{proj}_A(y)$—$\operatorname{proj}_A(z)$. But we have y—$\operatorname{proj}_A(y)$—a already, so it further suffices to show that y—w—$\operatorname{proj}_A(y)$. For this, we note that

$$x\text{—}\operatorname{proj}_A(x)\text{—}\operatorname{proj}_A(y) \quad \text{and} \quad \operatorname{proj}_A(x)\text{—}w\text{—}\operatorname{proj}_A(y),$$

so x—w—$\mathrm{proj}_A(y)$, and similarly z—w—$\mathrm{proj}_A(y)$. Thus, it follows that

$$\begin{aligned} w \in [\mathrm{proj}_A(y), x] \cap [\mathrm{proj}_A(y), z] &= [\mathrm{proj}_A(y), \mathrm{proj}_{\{x,z\}}(\mathrm{proj}_A(y))] && \text{Lemma 1.12} \\ &= [\mathrm{proj}_A(y), \mathrm{proj}_{[x,z]}(\mathrm{proj}_A(y))] && \text{Remark 1.15} \\ &\subseteq [\mathrm{proj}_A(y), y], && \text{Lemma 1.12} \end{aligned}$$

where the second equality follows from $\mathrm{cvx}(\{x, z\}) = [x, z]$, and hence $\mathrm{proj}_{\{x,z\}} = \mathrm{proj}_{[x,z]}$. ■

1.4. **Convex half-spaces of median graphs.** We now use projections to explore the geometry of convex half-spaces in a median graph (X, G). For the axiomatics of convex structures, see [vdV93].

For a convex co-convex half-space $H \in \mathcal{H}^*_{\mathsf{cvx}}(X)$, we call the inward edge boundary $\partial_{\mathsf{ie}} H$ a *hyperplane.*

Proposition 1.17. *Each edge $(x, y) \in G$ is on a unique hyperplane, namely the inward boundary of* $\mathrm{cone}_x(y)$*, and conversely, each half-space $H \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ is* $\mathrm{cone}_x(y)$ *for every $(x, y) \in \partial_{\mathsf{ie}} H$.*

Thus, hyperplanes are equivalence classes of edges. Furthermore, this equivalence relation is generated by parallel sides of squares (i.e., 4-cycles).

Proof. We have $\mathrm{cone}_x(y) \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ by Lemma 1.14, and clearly $(x, y) \in \partial_{\mathsf{ie}} \mathrm{cone}_x(y)$. Conversely, take $H \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ and any $(x, y) \in \partial_{\mathsf{ie}} H$. Then $H = \mathrm{cone}_x(y)$, for if $z \in H \cap \neg\, \mathrm{cone}_x(y)$, then $z \in \mathrm{cone}_y(x)$, and hence $x \in [y, z] \subseteq H$ by convexity of H, a contradiction; if $z \in \mathrm{cone}_x(y) \cap \neg H$, then $[x, z] \subseteq \neg H$ by convexity of $\neg H$, and hence $y \notin H$, a contradiction.

Finally, parallel edges of a strip of squares generate the same hyperplane since, for a given square, each vertex is between its neighbors and hence any hyperplane containing an edge contains its opposite edge. On the other hand, let $(a, b), (c, d) \in \partial_{\mathsf{ie}} H$ for some $H \in \mathcal{H}^*_{\mathsf{cvx}}(X)$. Note that $\partial_{\mathsf{ov}} H = \mathrm{proj}_{\neg H}(H)$ is convex since H is, and since $\mathrm{proj}_{\neg H}$ preserves convexity by Lemma 1.16, any geodesic between $a, c \in \partial_{\mathsf{ov}} H$ lies in $\partial_{\mathsf{ov}} H$. Matching this geodesic via $\partial_{\mathsf{ie}} H : \partial_{\mathsf{ov}} H \to \partial_{\mathsf{iv}} H$ gives us a geodesic between b, d in $\partial_{\mathsf{iv}} H$, which together with the matching forms the desired strip of squares. ■

Corollary 1.18. *Two half-spaces $H, K \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ are non-nested iff there is an embedding $\{0, 1\}^2 \hookrightarrow X$ of the Hamming cube into the four corners $\neg^i H \cap \neg^j K$.*

In particular, if $H, K \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ are non-nested, then $\partial_{\mathsf{v}} H \cap \partial_{\mathsf{v}} K \neq \varnothing$.

Proof. Let H, K be non-nested and take $x_1 \in H \cap K$ and $x_2 \in H \cap \neg K$. Since H is connected, any geodesic between x_1, x_2 crosses an edge $(x'_1, x'_2) \in \partial_{\mathsf{oe}} K$ in H. Similarly, there is an edge $(y'_1, y'_2) \in \partial_{\mathsf{oe}} K$ in $\neg H$, so we may slide both edges along $\partial_{\mathsf{oe}} K$ to obtain the desired square (see Proposition 1.17).

Conversely, the half-spaces cutting the square are clearly non-nested. ■

Lemma 1.19 (Helly)**.** *Any finite intersection of pairwise-intersecting non-empty convex sets is non-empty.*

Proof. For pairwise-intersecting convex sets H_1, H_2, H_3, pick any $x \in H_1 \cap H_2$, $y \in H_1 \cap H_3$ and $z \in H_2 \cap H_3$; their median $\langle x, y, z\rangle$ then lies in $H_1 \cap H_2 \cap H_3$.

Suppose that it holds for some $n \geq 3$ and let $H_1, \dots, H_{n+1} \subseteq X$ pairwise-intersect. Then $\{H_i \cap H_{n+1}\}_{i \leq n}$ is a family of n pairwise-intersecting convex sets, so $\bigcap_{i \leq n+1} H_i = \bigcap_{i \leq n} (H_i \cap H_{n+1})$ is non-empty. ■

Lastly, we have some useful finiteness conditions on convex half-spaces; the former implies that $\mathcal{H}_{\mathsf{cvx}}(X)$ is finitely-separating, and the latter allows us to replace finite sets with their convex hulls.

Lemma 1.20. *Any two disjoint convex sets $\varnothing \neq A, B \subseteq X$ can be separated by a half-space $A \subseteq H \subseteq \neg B$, and furthermore we have $d(A, B) = |\{H \in \mathcal{H}_{\mathsf{cvx}}(X) : A \subseteq H \subseteq \neg B\}|$.*

Proof. Pick a geodesic $A \ni x_0 G x_1 G \cdots G x_n \in B$, where $n \coloneqq d(A, B)$. Then $H \coloneqq \mathrm{cone}_{x_1}(x_0)$, which is a half-space by Lemma 1.14, separates A, B since $x_0 = \mathrm{proj}_A(x_n)$, and thus we have $A \subseteq \mathrm{cone}_{x_n}(x_0) \subseteq \mathrm{cone}_{x_1}(x_0)$ and $B \subseteq \mathrm{cone}_{x_0}(x_n) \subseteq \mathrm{cone}_{x_0}(x_1)$.

Moreover, each such half-space $A \subseteq H \subseteq \neg B$ satisfies $x_i \in H \not\ni x_{i+1}$ for a unique $i < n$, and conversely each pair (x_i, x_{i+1}) has a unique half-space separating them, so we have the desired bijection. ■

Lemma 1.21. *Every interval $[x, y]$ is finite. More generally, if $A \subseteq X$ is finite, then so is* $\mathrm{cvx}(A)$.

Proof. The singletons $\{x\}$ and $\{y\}$ are convex, so there are finitely-many half-spaces $H \subseteq [x, y]$. But each $z \in [x, y]$ is determined uniquely by those half-spaces containing it, so $[x, y]$ is finite.

Let $A \coloneqq \{x_0, \dots, x_n\}$. Since $\mathrm{cvx}(A) = \mathrm{proj}_A(X)$, we have by Remark 1.13 that points in $\mathrm{cvx}(A)$ are of the form $\langle x, x_n, \dots, \langle x, x_2, \langle x, x_1, x_0\rangle\rangle \dots\rangle$, which is finite by induction using that intervals are finite. ■

2. Graphs with Dense Families of Cuts

Let (X, G) be a connected locally-finite quasi-tree, which, in the context of Theorem A, stands for a single component of the locally-finite graphing of the CBER. For Theorem B to apply, we need to first identify a family of finitely-separating cuts therein, and we do so in such a way that the cuts are dense towards ends.

Since X is a quasi-tree, and thus does not have arbitrarily long cycles, we expect that there is some finite bound $R < \infty$ such that the ends in $\varepsilon(X)$ are 'limits' of cuts $\mathcal{H} \coloneqq \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ with boundary diameter bounded by R. We show that this is indeed the case, in the sense that $\mathcal{H}$ is dense towards ends.

Lemma 2.1. *If $f : (X, G) \to (Y, T)$ is a coarse-equivalence between connected graphs, then $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$ is uniformly bounded in terms of $\mathrm{diam}(\partial_{\mathsf{v}} H)$ for any $H \in \mathcal{H}_{\partial<\infty}(Y)$.*

Proof. Since f is coarse, there exists $S < \infty$ be such that xGx' implies $d(f(x), f(x')) \leq S$, so that for any $(x, x') \in \partial_{\mathsf{ie}} f^{-1}(H)$, there is a path of length $\leq S$ between $f(x) \notin H$ and $f(x') \in H$. Thus both $d(f(x), \partial_{\mathsf{v}} H)$ and $d(f(x'), \partial_{\mathsf{v}} H)$ are bounded by S, so $f(\partial_{\mathsf{v}} f^{-1}(H)) \subseteq \mathrm{Ball}_S(\partial_{\mathsf{v}} H)$ and hence

$$\mathrm{diam}(f(\partial_{\mathsf{v}} f^{-1}(H))) \leq \mathrm{diam}(\partial_{\mathsf{v}} H) + 2S.$$

That f is a coarse-*equivalence* gives us a uniform bound of $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$ in terms of $\mathrm{diam}(\partial_{\mathsf{v}} H)$. ■

In particular, if $\mathrm{diam}(\partial_{\mathsf{v}} H)$ is itself also uniformly bounded, then so is $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$.

Proposition 2.2. *The class of connected locally-finite graphs in which $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}$ is dense towards ends for some $R < \infty$ is invariant under coarse-equivalence.*

Proof. Let (X, G), (Y, T) be connected locally-finite graphs, $f : X \to Y$ be a coarse-equivalence with quasi-inverse $g : Y \to X$, and suppose $\mathcal{H}_{\mathrm{diam}(\partial)\leq S}(Y)$ is dense towards ends for some $S < \infty$. By Lemma 2.1, pick some $R < \infty$ so that for any $H \in \mathcal{H}_{\mathrm{diam}(\partial)\leq S}(Y)$, we have $f^{-1}(H) \in \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$.

Fix an end $p \in \varepsilon(X)$ and a neighborhood $p \in A \in \mathcal{H}_{\partial<\infty}(X)$. We need to find some $B \in \mathcal{H}_{\partial<\infty}(Y)$ such that $f(p) \in B$ and $f^{-1}(B) \subseteq A$, for then $f(p) \in H$ for some $B \supseteq H \in \mathcal{H}_{\mathrm{diam}(\partial)\leq S}(Y)$, and hence we have

$$p \in f^{-1}(H) \subseteq f^{-1}(B) \subseteq A$$

with $f^{-1}(H) \in \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$. For convenience, let $D < \infty$ be the uniform distance $d(1_X, g \circ f)$.

To this end, note that $f(p) \in B$ iff $p \in f^{-1}(B)$. Since $p \in A$, the latter can occur if $|A \triangle f^{-1}(B)| < \infty$, and so we need to find such a $B \in \mathcal{H}_{\partial<\infty}(Y)$ with the additional property that $f^{-1}(B) \subseteq A$.

> *Attempt 1.* Set $B \coloneqq g^{-1}(A) \in \mathcal{H}_{\partial<\infty}(Y)$. Then $f^{-1}(B) \subseteq \mathrm{Ball}_D(A)$ since if $(g \circ f)(x) \in A$, then
>
> $$d(x, A) \leq d(x, (g \circ f)(x)) \leq d(1_X, g \circ f) = D.$$
>
> By local-finiteness of G, we see that $A \triangle f^{-1}(B) = A \setminus f^{-1}(B)$ is finite, as desired.

However, it is *not* the case that $f^{-1}(B) \subseteq A$. To remedy this, we 'shrink' A by D to A' so that $\mathrm{Ball}_D(A') \subseteq A$, and take $B \coloneqq g^{-1}(A')$ instead. Indeed, $A' \coloneqq \neg\,\mathrm{Ball}_D(\neg A) \subseteq A$ works, since $f^{-1}(B) \subseteq \mathrm{Ball}_D(A')$ as before, so $A' \triangle f^{-1}(B) = A' \setminus f^{-1}(B)$ is finite. Also, $A \triangle A'$ is finite since $x \in A \triangle A'$ iff $x \in A$ and $d(x, \neg A) \leq D$, so $A \triangle f^{-1}(B)$ is finite too. It remains to show that $\mathrm{Ball}_D(A') \subseteq A$, for then $f^{-1}(B) \subseteq A$ as desired.

Indeed, if $y \in \mathrm{Ball}_D(A')$, then by the (reverse) triangle-inequality we have $d(y, \neg A) \geq d(x, \neg A) - d(x, y)$ for all $x \in A'$. But $d(x, \neg A) > D$, strictly, so $d(y, \neg A) > D - D = 0$, and hence $y \in A$. ■

Corollary 2.3. *If (X, G) is a locally-finite quasi-tree, then the subpocset $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ is dense towards ends for some $R < \infty$.*

Proof. Observe that $\mathcal{H}_{\mathrm{diam}(\partial)\leq 2}(T)$ is dense towards ends for any tree T, so Proposition 2.2 proves the density of $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$ for some $R < \infty$. By Lemma 1.10, there is a subpocset $\mathcal{H}' \subseteq \mathcal{H}_{\partial<\infty}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ dense towards ends such that for every $H' \in \mathcal{H}'$, we have $\partial_{\mathsf{ie}} H' \subseteq \partial_{\mathsf{ie}} H$ for some $H \in \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$. Hence we have $H' \in \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$, so the result follows. ■

The cuts $\mathcal{H} \coloneqq \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ obtained here will be our starting point for Theorem B, so we need to show that it is finitely-separating. Indeed, by local-finiteness of (X, G), the R-ball around any fixed $x \in X$ is finite, so since any cut $H \in \mathcal{H}$ with $x \in \partial_{\mathsf{v}} H$ is contained in said R-ball, there are finitely-many such cuts. Thus, by Lemma 1.3, $\mathcal{H}$ is finitely-separating.

Furthermore, we have by Lemma 1.2 that $\mathcal{H} \subseteq 2^X$ is closed and every non-trivial element is isolated, and those conditions allow for the construction in Theorem B to continue in Section 3.1.

Finally, we will need to represent 'density towards ends' in a Borel manner, which ultimately is to ensure that the bounds $R < \infty$ can be obtained uniformly across all components; see Section 4 for details. For this, we will need the following

Definition 2.4. Let $\mathcal{H} \subseteq 2^X$ be a pocset. For $x, y \in X$, write $x \sim_{\mathcal{H}} y$ if x and y are contained in exactly the same half-spaces in $\mathcal{H}$, which induces an equivalence relation on X whose classes $[x]_{\mathcal{H}}$ are called *$\mathcal{H}$-blocks*.

Proposition 2.5. *If $\mathcal{H}$ is a finitely-separating pocset of cuts on a connected locally-finite graph, then $\mathcal{H}$ is dense towards ends iff (i) each $\mathcal{H}$-block is finite, and (ii) each $H \in \mathcal{H}^*$ has only finitely-many successors.*

Proof. If $\mathcal{H}$ is dense towards ends, we will cover (certain closed sets in) $\varepsilon(X)$ by cuts in $\mathcal{H}$, which contains a finite subcover. We will show that a certain Boolean combination of this subcover, which has finite-boundary, is finite using Lemma 1.8, giving us the desired finiteness claims (i) and (ii). Below are the details.

For (i), fix an $\mathcal{H}$-block $[x]_{\mathcal{H}}$ and let $A := \neg\{x\}$. By density, we can cover each end $p \in \varepsilon(X)$ with some $H_p \subseteq A$, which gives us a finite subcover $\{H_i\}_{i<n}$ of $\varepsilon(X)$. Note that $\bigcap_{i<n} \neg H_i \in \mathcal{H}_{\partial<\infty}(X)$ contains $[x]_{\mathcal{H}}$, and is finite since it contains no ends in $\varepsilon(X)$.

For (ii), fix $H \in \mathcal{H}^*$ and let $\mathcal{K} \subseteq \mathcal{H}^*$ be the collection of all successors of H. By density, we can cover each end $p \in \varepsilon(\neg H)$ by some $H_p \subseteq \neg H$ in $\mathcal{H}$, which in turn is contained in $\neg K_\alpha$ for some $K_\alpha \in \mathcal{K}$; this gives us a finite subcover $\{\neg K_i\}_{i<n}$ of $\varepsilon(\neg H)$. Every successor K of H not in $\{K_i\}_{i<n}$ is non-nested with at least one K_i, and by Lemma 1.6, each K_i is non-nested with finitely-many other half-spaces. Thus $\mathcal{K}$ is finite.

Conversely, fix an end $p \in \varepsilon(X)$ and a neighborhood $p \in A \in \mathcal{H}_{\partial<\infty}(X)$. Since $\mathcal{H}$ consists of connected sets, it suffices to find some $H \in \mathcal{H}$ containing $\partial_{\mathsf{v}} A$ but not p, for then $p \in \neg H \subseteq A$ as desired.

> **Observation.** Finitely-many $H, \neg H \in \mathcal{H}$ may be removed from $\mathcal{H}$ and it will still satisfy the conditions (i) and (ii) above. Indeed, that (ii) still holds is obvious. For (i), we may remove a single pair, since with $\mathcal{H}' := \mathcal{H} \setminus \{H, \neg H\}$, the map $X/\mathcal{H} \to X/\mathcal{H}'$ sending $[x]_{\mathcal{H}} \to [x]_{\mathcal{H}'}$ is surjective and at-most 2-to-1.

Thus, for each of the finitely-many $x, y \in \partial_{\mathsf{v}} A$, we may remove the finitely-many half-spaces separating them, so we may assume that $\mathcal{H} = \mathcal{H}_A \sqcup \neg\mathcal{H}_A$ is partitioned into the half-spaces in $\mathcal{H}_A$ containing $\partial_{\mathsf{v}} A$, and its complements which are disjoint from $\partial_{\mathsf{v}} A$. Towards a contradiction, assume that each $H \in \mathcal{H}_A$ contains p.

Since $\mathcal{H}$-blocks are finite, there exists some $H_0 \in \mathcal{H}_A^*$, which we may take to be minimal by Lemma 1.2. There are finitely-many half-spaces non-nested with H_0 by Lemma 1.6, so by the above observation, we may assume without loss of generality that there are none. By minimality of H_0, all successors $K \supset \neg H_0$ lie in $\mathcal{H}_A^*$, so the finite intersection $B := H_0 \cap \bigcap_{K \supset \neg H_0} K$ contains p and is hence infinite. Since $\mathcal{H}$-blocks are finite and B is a union of $\mathcal{H}$-blocks, it contains infinitely-many $\mathcal{H}$-blocks; any two such $\mathcal{H}$-blocks are separated by some $H \in \mathcal{H}_A^*$ nested with H_0, which forces $H \subset H_0$, and contradicts the minimality of H_0. ■

3. The Dual Median Graph of a Profinite Pocset and its Spanning Trees

3.1. **Construction of the dual median graph.** We present a classical construction in geometric group theory of a median graph $\mathcal{M}(\mathcal{H})$ associated to a profinite pocset $\mathcal{H}$ with every non-trivial element isolated; see [Dun79], [Rol98], [Sag95], and [NR03] for various other applications of this construction.

In the context of Theorem A, this will be applied to the pocset $\mathcal{H} := \mathcal{H}_{\mathrm{diam}(\partial) \leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ of cuts in a locally-finite graph (X, G), and will also be the first step in the construction in Theorem B.

Definition 3.1. An *orientation* on $\mathcal{H}$ is an upward-closed subset $U \subseteq \mathcal{H}$ containing exactly one of $H, \neg H$ for each $H \in \mathcal{H}$. We let $\mathcal{U}(\mathcal{H})$ denote the set of all orientations on $\mathcal{H}$ and let $\mathcal{U}^\circ(\mathcal{H})$ denote the clopen ones.

Intuitively, an orientation is a 'maximally consistent' choice of half-spaces[3] in $\mathcal{H}$.

Example 3.2. Each $x \in X$ induces its *principal orientation* $\widehat{x} := \{H \in \mathcal{H} : x \in H\}$, which is clopen in $\mathcal{H}$, and gives us a *principal orientations* map $X \to \mathcal{U}^\circ(\mathcal{H})$. However, this map is *not necessarily* injective, and its fibers $[x]_{\mathcal{H}} := \{y \in X : \widehat{x} = \widehat{y}\}$ are precisely the $\mathcal{H}$-blocks as in Definition 2.4.

[3]This can be formalized by letting $\sim$ be the equivalence relation on $\mathcal{H}$ given by $H \sim \neg H$. Letting $\partial : \mathcal{H} \to \mathcal{H}/{\sim}$ denote the quotient map, orientations $U \subseteq \mathcal{H}$ then correspond precisely to sections $\varphi : \mathcal{H}/{\sim} \to \mathcal{H}$ of ∂ such that $\varphi(\partial H) \not\subseteq \neg\varphi(\partial K)$ for every $H, K \in \mathcal{H}$; the latter condition rules out 'orientations' of the form $\leftarrow\!\mid\;\mid\!\rightarrow$.

Proposition 3.3. *Let $\mathcal{H}$ be a profinite pocset with every non-trivial element isolated. Then the graph $\mathcal{M}(\mathcal{H})$, whose vertices are clopen orientations $\mathcal{U}^\circ(\mathcal{H})$ and whose edges are pairs $\{U, V\}$ with $V = U \triangle \{H, \neg H\}$ for some minimal $H \in U \setminus \{\neg 0\}$, is a median graph with path metric $d(U, V) = |U \triangle V|/2$ and medians*

$$\begin{aligned}\langle U, V, W\rangle &:= \{H \in \mathcal{H} : H \textit{ belongs to at least two of } U, V, W\} \\ &= (U \cap V) \cup (V \cap W) \cup (U \cap W).\end{aligned}$$

Proof. First, $V := U \triangle \{H, \neg H\}$ as above is clopen since $H, \neg H \in \mathcal{H}^*$ are isolated (whence $\{H\}, \{\neg H\}$ are clopen), and it is an orientation by minimality of H. That $\mathcal{M}(\mathcal{H})$ is connected follows from the following claim by noting that $U \triangle V \not\ni 0, \neg 0$ is clopen, so it is a compact set of isolated points, whence finite.

Claim ([Sag95, Theorem 3.3]). *There is a path between U, V iff $U \triangle V$ is finite, in which case*

$$d(U, V) = |U \triangle V|/2 = |U \setminus V| = |V \setminus U|.$$

Proof. If $(U_i)_{i<n}$ is a path from $U =: U_0$ to $V =: U_{n-1}$, then, letting $\{H_i, \neg H_i\} := U_i \triangle U_{i-1}$ for all $1 \leq i < n$ gives us a sequence $(H_i)_{i<n}$ inducing[a] this path, whence $U \triangle V$ consists of $\{H_i\}_{i<n}$ and their complements. Thus $U \triangle V = 2n = 2d(U, V)$, as desired.

Conversely, if $U \triangle V = \{H_1, \ldots, H_n\} \sqcup \{K_1, \ldots, K_m\}$ with $U \setminus V = \{H_i\}$ and $V \setminus U = \{K_j\}$, then $\neg H_i \in V \setminus U$ and $\neg K_j \in U \setminus V$ for all $i < n$ and $j < m$, so $n = m$ and $V = U \cup \{\neg H_i\} \setminus \{H_i\}$. We claim that there is a permutation $\sigma \in S_n$ such that $(H_{\sigma(i)})$ induces a path from $U =: U_0$ to $V =: U_{n-1}$. Choose a minimal $H \in \{H_i\}_{i \leq n}$, which is also minimal in U: if $K \subseteq H$ for some $K \in U$, then $\neg H \subseteq \neg K$, and hence $\neg K \in V$, so $K = H_i \subseteq H$ for some i, forcing $K = H$. Set $U_1 := U \triangle \{H, \neg H\}$, which is a clopen orientation, and continuing inductively gives us the desired path with $d(U, V) = n$. □

[a] In the sense that $U_i = U_{i-1} \triangle \{H_i, \neg H_i\}$ and $H_i \in U_i$ for each $1 \leq i < n$; see [Tse20, Definition 2.20].

Finally, we show that $\mathcal{M}(\mathcal{H})$ is a median graph. Fix $U, V, W \in \mathcal{U}^\circ(\mathcal{H})$, and note that for any $M \in \mathcal{U}^\circ(\mathcal{H})$, we have by the triangle inequality that $M \in [U, V]$ iff $(U \setminus M) \cup (M \setminus V) \subseteq U \setminus V$, which clearly occurs iff $U \cap V \subseteq M \subseteq U \cup V$. Thus, a vertex M lies in the triple intersection $[U, V] \cap [V, W] \cap [U, W]$ iff

$$(U \cap V) \cup (V \cap W) \cup (U \cap W) \subseteq M \subseteq (U \cup V) \cap (V \cup W) \cap (U \cup W).$$

Note that the two sides coincide, so $M = \langle U, V, W\rangle$ – which is clopen if U, V, W are – is as claimed. ■

Given such a pocset $\mathcal{H}$, the graph $\mathcal{M}(\mathcal{H})$ constructed above is called the *dual*[4] median graph of $\mathcal{H}$. An important special case of this construction is when $\mathcal{H}$ is *nested*, in which case $\mathcal{M}(\mathcal{H})$ is a tree.

Corollary 3.4. *Let $\mathcal{H}$ be a profinite pocset with non-trivial points isolated. If $\mathcal{H}$ is nested, then the median graph $\mathcal{M}(\mathcal{H})$ is acyclic, and hence $\mathcal{M}(\mathcal{H})$ is a tree.*

Proof. Let $(U_i)_{i<n}$ be a cycle in $\mathcal{M}(\mathcal{H})$, say induced by some sequence $(H_i)_{i<n} \subseteq \mathcal{H}$ of half-spaces. We show that (H_i) is *strictly* increasing, so that $H_0 \subset \cdots \subset H_n \subset H_0$, which is absurd.

We have $U_{i+1} = U_{i-1} \cup \{\neg H_i, \neg H_{i-1}\} \setminus \{H_i, H_{i-1}\}$, so $H_i \neq \neg H_{i-1}$ (for otherwise $U_{i+1} = U_{i-1}$). Since $H_i \in U_i = U_{i-1} \cup \{\neg H_{i-1}\} \setminus \{H_{i-1}\}$, we see that $H_i \in U_{i-1}$, and since $H_i \neq H_{i-1}$, it suffices by nestedness of $\mathcal{H}$ to remove the three cases when $\neg H_i \subseteq H_{i-1}$, $H_{i-1} \subseteq \neg H_i$, and $H_i \subseteq H_{i-1}$.

Indeed, if $\neg H_i \subseteq H_{i-1}$, then $H_{i-1} \in U_{i+1}$ by upward-closure of $U_{i+1} \ni \neg H_i$. But since $H_{i-1} \neq \neg H_i$, we have by definition of U_{i+1} that $H_{i-1} \in U_i$, a contradiction. The other cases are similar. ■

Nonetheless, in the general non-nested case, $\mathcal{M}(\mathcal{H})$ still admits a *canonical* spanning tree if we fix a proper colouring of $\mathcal{H}^*_{\mathrm{cvx}}(\mathcal{M}(\mathcal{H}))$ into its nested sub-pocsets (Proposition 3.7). For this, we will need the following

Proposition 3.5. *The dual median graph $\mathcal{M}(\mathcal{H})$ of a pocset of finitely-separating cuts has finite hyperplanes.*

Proof. Fix $K \in \mathcal{H}^*_{\mathrm{cvx}}(\mathcal{M}(\mathcal{H}))$, which by Proposition 1.17 is of the form $K = \mathrm{cone}_V(U)$ for some (and hence any) $(U, V) \in \partial_{\mathsf{ie}} K$, and we have by Proposition 3.3 that $V = U \triangle \{H, \neg H\}$ for some (non-trivial) minimal $H \in U$. We claim that any other edge $(U', V') \in \partial_{\mathsf{ie}} K$ can be reached from (U, V) by simultaneously flipping only the half-spaces $H'_0, \ldots, H'_n \in \mathcal{H}^*$ non-nested with H, of which there are finitely-many by Lemma 1.6.

[4] The name comes from a Stone-type duality between the categories {median graphs, median homomorphisms} and {profinite pocsets with non-trivial points isolated, continuous maps}, where from a median graph X one can construct a canonical pocset $\mathcal{H}_{\mathrm{cvx}}(X)$ of convex half-spaces (see [CPTT23, Section 2.D] for details).

Since the edges $(U, V), (U', V')$ induce the same hyperplane $\partial_{\mathsf{ie}} K$, it suffices by Proposition 1.17 to prove this for when (U', V') is an edge of a square parallel to (U, V), in which case there is some minimal $H' \in \mathcal{H}^*$ flipping both U to U' and V to V'. Note that H, H' are non-nested by minimality, and $H \in U'$ is still minimal since $U' = U \triangle \{H', \neg H'\}$, and the only way it is not is if $\neg H' \subseteq H$, which contradicts the minimality of H. Thus, H induces the edge from U' to V' with H' non-nested with H, as desired. ■

3.2. **Canonical spanning trees.** We now present the Borel cycle-cutting algorithm that can be preformed on any countable median graph with finite hyperplanes. Applying this algorithm to the dual median graph of a finitely-separating family of cuts, which has finite hyperplanes by Proposition 3.5, proves Theorem B.

Lemma 3.6. *For any subpocset $\mathcal{H} \subseteq \mathcal{H}_{\mathrm{cvx}}(X)$ of convex co-convex half-spaces in a median graph (X, G), the principal orientations map $X \to \mathcal{U}^\circ(\mathcal{H})$ is surjective.*

Proof. Let $U \in \mathcal{U}^\circ(\mathcal{H})$, we need to find some $x \in X$ with $U = \widehat{x}$. Since $U \subseteq \mathcal{H}$ is clopen, there is a finite set $A \subseteq X$ − which we may assume to be convex by Lemma 1.21 − such that for all $H \in \mathcal{H}$, we have $H \in U$ iff there is $H' \in U$ with $H \cap A = H' \cap A$. Note that $H \cap A \neq \varnothing$ for every $H \in U$, since otherwise $\varnothing \in U$. Furthermore, $H \cap H' \neq \varnothing$ for every $H, H' \in U$, since otherwise we have $H \subseteq \neg H'$, and so $\neg H' \in U$.

By Lemma 1.19, the intersection $(H \cap A) \cap (H' \cap A) = H \cap H' \cap A$ is non-empty, and applying it again furnishes some $x \in \bigcap_{H \in U} H \cap A$ in X. Thus $U \subseteq \widehat{x}$, so $U = \widehat{x}$ since both are orientations. ■

This induces a *G-adjacency* graph $X/\mathcal{H} \cong \mathcal{M}(\mathcal{H})$; explicitly, two $\mathcal{H}$-blocks $[x]_{\mathcal{H}}, [y]_{\mathcal{H}}$ are G-adjacent if $(\widehat{x}, \widehat{y}) \in \mathcal{M}(\mathcal{H})$. Note that $\mathcal{M}(\mathcal{H})$ may be constructed as in Proposition 3.3 since $\mathcal{H} \subseteq \mathcal{H}_{\mathrm{cvx}}(X)$ is finitely-separating by Lemma 1.20. In particular, if $\mathcal{H}$ is nested, then $X/\mathcal{H}$ is a tree by Corollary 3.4.

Proposition 3.7. *If (X, G) is a countable median graph with finite hyperplanes, then fixing any colouring of $\mathcal{H}^*_{\mathrm{cvx}}(X)$ into nested sub-pocsets yields a canonical spanning tree thereof.*

Proof. Such a colouring exists, since, by Corollary 1.18, if two half-spaces $H, K \in \mathcal{H}^*_{\mathrm{cvx}}(X)$ are non-nested, then $\partial_{\mathsf{v}} H \cap \partial_{\mathsf{v}} K \neq \varnothing$. Thus, the intersection graph of the boundaries admits a countable colouring, which descends into a colouring $\mathcal{H}^*_{\mathrm{cvx}}(X) = \bigsqcup_{n \in \mathbb{N}} \mathcal{H}^*_n$ such that each $H, \neg H$ receive the same colour and that each $\mathcal{H}_n := \mathcal{H}^*_n \cup \{\varnothing, X\}$ is a *nested* subpocset. For each $n \in \mathbb{N}$, let $\mathcal{K}_n := \bigcup_{m \geq n} \mathcal{H}_m$.

We shall inductively construct an increasing chain of subforests $T_n \subseteq G$ such that the components of T_n are exactly the $\mathcal{K}_n$-blocks. Then, the increasing union $T := \bigcup_n T_n$ is a spanning tree, since each $(x, y) \in G$ lies in a $\mathcal{K}_n$-block for sufficiently large n (namely, the n such that $\mathrm{cone}_x(y) \in \mathcal{H}^*_{n-1}$, since $\mathrm{cone}_x(y)$ and its complement are the only half-spaces separating x and y by Proposition 1.17).

Since each pair of distinct points is separated by a half-space, the $\mathcal{K}_0$-blocks are singletons, so put $T_0 := \varnothing$. Suppose that a forest T_n is constructed as required. Note that each $\mathcal{K}_{n+1}$-block $Y \in X/\mathcal{K}_{n+1}$ is not separated by any half-spaces in $\mathcal{H}_m$ for $m > n$, but is separated by $\mathcal{H}_n$ into the $\mathcal{K}_n$-blocks contained in Y, which are precisely the $\mathcal{H}_n$-blocks in $Y/\mathcal{H}_n$. Pick an edge from the *finite* hyperplane $\partial_{\mathsf{ie}} H$ for each $H \in \mathcal{H}_n$, which connects a unique pair of G-adjacent blocks in $Y/\mathcal{H}_n$. Since each $Y/\mathcal{H}_n$ is a tree by Corollary 3.4, and each pair of G-adjacent blocks in $Y/\mathcal{H}_n$ is connected by a single picked edge, the graph T_{n+1} obtained from T_n by adding all such edges is a forest whose components are exactly the $\mathcal{K}_{n+1}$-blocks. ■

4. Borel Treeings of Graphings with Dense Cuts

We finally prove Theorem A, stating that if a CBER (X, E) admits a locally-finite graphing G such that each component is a quasi-tree, then E is treeable. The first step is to identify, for each component $G|C$, a family $\mathcal{H}(C)$ of finitely-separating cuts that is dense towards ends of $G|C$; since each $G|C$ is a quasi-tree, the cuts $\mathcal{H}(C) := \mathcal{H}_{\mathrm{diam}(\partial) \leq R_C}(C) \cap \mathcal{H}_{\mathrm{conn}}(C)$ for some $R_C < \infty$ from Section 2 will do. Applying Theorem B then gives us, for each component $G|C$, a median graph $\mathcal{M}(\mathcal{H}(C))$ on $\mathcal{U}^\circ(\mathcal{H}(C))$ with finite hyperplanes.

The issue lies in making the family $\mathcal{U}^\circ(\mathcal{H}) := \bigsqcup_C \mathcal{U}^\circ(\mathcal{H}(C))$ of *all* clopen orientations on $\mathcal{H} := \bigsqcup_C \mathcal{H}(C)$ into a standard Borel space. *If $\mathcal{U}^\circ(\mathcal{H})$ is standard Borel*, the above partition induces a CBER $\mathcal{E}$ admitting a median graphing $\mathcal{M}(\mathcal{H}) := \bigsqcup_C \mathcal{M}(\mathcal{H}(C))$ with finite hyperplanes, from which one can implement the proof of Proposition 3.7 in a Borel manner (using [KM04, Lemma 7.3] for a countable colouring of the intersection graph of finite hyperplanes therein) to obtain a treeing of $\mathcal{E}$. Finally, E is Borel bireducible with $\mathcal{E}$ via the principal orientations map $X \ni x \mapsto \widehat{x} \in \mathcal{U}^\circ(\mathcal{H})$, so E is also treeable by [JKL02, Proposition 3.3 (ii)].

We will remedy this issue using the fact that the cuts $\mathcal{H}(C)$ are dense towards ends of $G|C$. In particular, we have the following crucial lemma, which, by Proposition 3.3, shows that $\mathcal{M}(\mathcal{H}(C))$ is locally-finite.

Lemma 4.1. *Let $\mathcal{H}$ be a finitely-separating pocset of cuts on a connected locally-finite graph (X, G). If $\mathcal{H}$ is dense towards ends, then each clopen orientation $U \in \mathcal{U}^\circ(\mathcal{H})$ contains finitely-many minimal cuts $H \in \mathcal{H}$.*

Proof. Fix a vertex $U \in \mathcal{U}^\circ(\mathcal{H})$ and let $\mathcal{K} \subseteq U$ be the minimal elements in U. Since $U \subseteq \mathcal{H}$ is clopen, there is a finite set $A \subseteq X$ such that for all $H \in \mathcal{H}$, we have $H \in U$ iff there is $H' \in U$ with $H \cap A = H' \cap A$. Note that $H \cap A \neq \varnothing$ for every $H \in U$, for otherwise $\varnothing \in U$; in particular, we have $\neg H \in U$ for every $H \subseteq \neg A$.

Each end $p \in \varepsilon(X)$ lies in $\neg A$, so density of $\mathcal{H}$ furnishes $H_p \in \mathcal{H}$ with $p \in H_p \subseteq \neg A$, and thus $\neg H_p \in U$. Since U is clopen, we have $K_p \subseteq \neg H_p$ for some $K_p \in \mathcal{K}$. Thus $\{\neg K_p\}$ covers $\varepsilon(X)$, which by compactness contains a finite subcover $\{\neg K_i\}_{i<n}$. We show that there are at-most finitely-many more minimal $K \in U$.

Let $K \in \mathcal{K} \setminus \{K_i\}_{i<n}$ be any other minimal element in U. By Lemma 1.6, each K_i is non-nested with finitely-many other half-spaces, so we may assume without loss of generality that K is nested with every K_i. But $K \not\subseteq K_i \not\subseteq K$ and $K \cap K_i \neq \varnothing$ for all $i < n$, so $\neg K \subseteq \bigcap_{i<n} K_i \in \mathcal{H}_{\partial<\infty}(X)$; the latter contains no ends in $\varepsilon(X)$, so it is finite by Lemma 1.8, and hence $\mathcal{K}$ is finite too. ■

We now describe the encoding of $\mathcal{U}^\circ(\mathcal{H})$ into a standard Borel space. Since cuts have finite edge boundary, we may first represent the space $\mathcal{K}$ of *all* non-trivial cuts of G as a Borel subset of $[G]^{<\infty}$. The subcollection $\mathcal{H} \subseteq \mathcal{K}$ consisting of those cuts with component-wise bounded boundary diameter is also Borel, since each $R_C < \infty$ can be witnessed as the minimal number making $\mathcal{H}_{\mathrm{diam}(\partial)\leq R_C}(C)$ dense towards ends of $G|C$, and the latter is a Borel condition as characterized in Proposition 2.5. Finally, $\mathcal{U}^\circ(\mathcal{H})$ is a Borel subset of $[\mathcal{H}]^{<\infty}$, since we may encode each clopen orientation $U \in \mathcal{U}^\circ(\mathcal{H}(C))$ by its set of minimal elements in $\mathcal{H}(C)$, which is finite by Lemma 4.1. This makes $\mathcal{U}^\circ(\mathcal{H})$ a standard Borel space, and finishes the proof of Theorem A.

The above discussion actually proves the following generalization of Theorem A, which is no longer about quasi-trees; rather, we only require that the (locally-finite) graphing admits a Borel family of 'tree-like' cuts.

Theorem 4.2. *If a CBER (X, E) admits a locally-finite graphing G such that each component $G|C$ admits a family $\mathcal{H}(C)$ of finitely-separating cuts that is dense towards ends of $G|C$, and if $\mathcal{H} \coloneqq \bigsqcup_C \mathcal{H}(C)$ is a Borel subset of the standard Borel space of all cuts of G, then E is treeable.*

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