TREE-LIKE GRAPHINGS OF COUNTABLE BOREL EQUIVALENCE RELATIONS

ZHAOSHEN ZHAI

ABSTRACT. We present a streamlined exposition of a construction presented recently by R. Chen, A. Poulin, R. Tao, and A. Tserunyan, where it is proven that every locally-finite Borel graph with each component a quasi-tree induces a canonical treeable equivalence relation. Write some more details...

0. INTRODUCTION

The purpose of this note is to provide a streamlined proof of a particular case of a construction presented in [CPTT23], in order to better understand the general formalism developed therein. We attempt to make this note self-contained, but nevertheless urge the reader to refer to the original paper for more detailed discussions and some generalizations of the results we have selected to include here.

0.1. Treeings of equivalence relations.
A *countable Borel equivalence relation (CBER)* on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$ with each class countable. We are interested in special types of *graphings* on a CBER $E \subseteq X^2$, i.e. a Borel graph $G \subseteq X^2$ whose connectedness relation is precisely E. For instance, a graphing of E such that each component is a tree is called a *treeing* of E, and the CBERs that admit treeings are said to be *treeable*. The main results of [CPTT23] provide new sufficient criteria for treeability of certain classes of CBERs, and in particular, they prove the following

Theorem A ([CPTT23, Theorem 1.1])**.** *If a CBER admits a locally-finite graphing whose components are quasi-trees, then it is treeable.*

Recall that metric spaces X and Y are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error, and X is a *quasi-tree* if it is quasi-isometric to a simplicial tree; see [Gro93] and [DK18].

0.2. Outline of the proof.
Roughly speaking, the existence of a quasi-isometry $G|C \to T_C$ to a simplicial tree T_C for each component $C \subseteq X$ induces a collection $\mathcal{H}(C)$ of 'cuts' (subsets $H \subseteq C$ with finite boundary such that both H and $C \setminus H$ are connected), which are 'tree-like' in the sense that

1. $\mathcal{H}(C)$ is *finitely-separating*: each pair $x, y \in C$ is separated by finitely-many $H \in \mathcal{H}(C)$, and
2. $\mathcal{H}(C)$ is *dense towards ends*: each end in $G|C$ has a neighborhood basis in $\mathcal{H}(C)$.

By Condition (1), these cuts have the structure of a profinite pocset with non-trivial points isolated, which in turn provide exactly the data to construct a 'median graph' whose vertices are 'ultrafilters'[1] thereof. Condition (2) then ensures that this graph has finite 'hyperplanes', which allows us to apply a Borel 'cycle-cutting' algorithm and obtain a canonical spanning tree. Each step above can be done in a uniform way to each component $C \subseteq G$, giving us the desired treeing of the CBER.

$$\text{Quasi-tree} \xrightarrow{1} \begin{array}{c} \text{Dense family} \\ \text{of cuts} \end{array} \xrightarrow{2.1+2.2} \begin{array}{c} \text{Pocset w/} \\ \text{non-triv. pts.} \\ \text{isolated} \end{array} \xrightarrow{3.1+3.2} \begin{array}{c} \text{Median graph} \\ \text{w/ finite} \\ \text{hyperplanes} \end{array} \xrightarrow{4} \begin{array}{c} \text{Canonical} \\ \text{spanning tree} \end{array}$$

Write some more stuff to tie things together...

Remark. We follow [CPTT23, Convention 2.7], where for a family $\mathcal{H} \subseteq 2^X$ of subsets of a fixed set X, we write $\mathcal{H}^* := \mathcal{H} \setminus \{\varnothing, X\}$ for the *non-trivial* elements of $\mathcal{H}$.

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[1]As in [CPTT23], we call them *orientations* instead, to avoid confusion with the more standard notion; see Definition 3.1.

1. Graphs with Dense Families of Cuts

1.1. **Ends of graphs.** Let (X, G) be a connected locally-finite graph, which, in the context of Theorem A, will stand for a single component of the locally-finite graphing of a CBER.

Definition 1.1. For a subset $A \subseteq X$, we let $\partial_{\mathsf{iv}} A \coloneqq A \cap \mathrm{Ball}_1(\neg A)$ be its *inner vertex boundary*, $\partial_{\mathsf{ov}} A \coloneqq \partial_{\mathsf{iv}}(\neg A)$ be its *outer vertex boundary*, and let $\partial_{\mathsf{ie}} A \coloneqq G \cap (\partial_{\mathsf{ov}} A \times \partial_{\mathsf{iv}} A)$ and $\partial_{\mathsf{oe}} A \coloneqq \partial_{\mathsf{ie}}(\neg A)$ respectively be its *inner* and *outer edge boundaries*. Let $\partial_{\mathsf{v}} A \coloneqq \partial_{\mathsf{iv}} A \cup \partial_{\mathsf{ov}} A$ be the *(total) vertex boundary* of A.

Let $\mathcal{H}_{\partial < \infty}(X) \subseteq 2^X$ be the Boolean algebra of all $A \subseteq X$ with finite vertex boundary, which we refer to as *cuts* in X. Later, we will restrict to the connected and co-connected cuts $\mathcal{H}_{\partial < \infty}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$.

Definition 1.2. The *end compactification* of (X, G) is the Stone space $\widehat{X}$ of $\mathcal{H}_{\partial < \infty}(X)$, whose non-principal ultrafilters are the *ends* of (X, G).

We identify $X \hookrightarrow \widehat{X}$ via principal ultrafilter map $x \mapsto p_x$, so $\widehat{X} \setminus X$ is the set of ends of G. By definition, $\widehat{X}$ admits a basis of clopen sets of the form $\widehat{A} \coloneqq \{p \in \widehat{X} : A \in p\}$ for each $A \in \mathcal{H}_{\partial < \infty}(X)$.

Definition 1.3. A family $\mathcal{H} \subseteq \mathcal{H}_{\partial < \infty}(X)$ of cuts is *dense towards ends* of (X, G) if $\mathcal{H}$ contains a neighborhood basis for every end in $\widehat{X} \setminus X$.

In other words, $\mathcal{H}$ is dense towards ends if for every $p \in \widehat{X} \setminus X$ and every (clopen) neighborhood $\widehat{A} \ni p$, where $A \in \mathcal{H}_{\partial < \infty}(X)$, there is some $H \in \mathcal{H}$ with $p \in \widehat{H} \subseteq \widehat{A}$; it is useful to note that $\widehat{H} \subseteq \widehat{A}$ iff $H \subseteq A$.

The goal of this section is to show that certain cuts $\mathcal{H} \subseteq \mathcal{H}_{\partial < \infty}$ induced in a (locally-finite) quasi-tree is dense towards ends. It will also be important that these cuts be connected, in that witnesses to density can also be found in $\mathcal{H} \cap \mathcal{H}_{\mathrm{conn}}$, consisting of those cuts that are connected and co-connected.

Lemma 1.4. *If a subpocset $\mathcal{H} \subseteq \mathcal{H}_{\partial < \infty}$ is dense towards ends, then there is a subpocset $\mathcal{H}' \subseteq \mathcal{H}_{\partial < \infty} \cap \mathcal{H}_{\mathrm{conn}}$, which is also dense towards ends, such that every $H' \in \mathcal{H}'$ has $\partial_{\mathsf{ie}} H' \subseteq \partial_{\mathsf{ie}} H$ for some $H \in \mathcal{H}$.*

Proof. A first attempt is to let $\mathcal{H}'$ be the connected components H_0' of elements in $\mathcal{H}$, but this fails since $\neg H_0'$ is not necessarily connected. Instead, we further take a component of $\neg H_0'$, whose complement clearly co-connected, and is connected since it consists of H_0' and the other components of $\neg H_0'$, each of which is connected to H_0' via $\partial_{\mathsf{ie}} H_0'$. Formally, we let

$$\mathcal{H}' \coloneqq \{H' \subseteq X : H \in \mathcal{H} \text{ and } H_0' \in H/G \text{ and } \neg H' \in \neg H_0'/G\},$$

where H/G denotes the G-components of H. Clearly $\partial_{\mathsf{ie}} H' \subseteq \partial_{\mathsf{ie}} H_0' \subseteq \partial_{\mathsf{ie}} H$, so it remains to show that $\mathcal{H}'$ is dense towards ends.

Fix an end $U \in \widehat{X} \setminus X$ with $U \in \widehat{A}$ for some $A \in \mathcal{H}_{\partial < \infty}(X)$. Let $B \supseteq \partial_{\mathsf{v}} A$ be finite connected, which can be obtained by adjoining paths between its components. Then $\neg B \in U$ since U is non-principal, so there is $H \in \mathcal{H}$ with $U \in \widehat{H} \subseteq \neg \widehat{B}$. We now use the above recipe to find the desired $H' \in \mathcal{H}'$:

(1) Since $H \in \mathcal{H}_{\partial < \infty}(X)$, it has finitely-many connected components, so exactly one of them belongs to U, say $U \in \widehat{H_0'} \subseteq \widehat{H}$. Note that $B \subseteq \neg H \subseteq \neg H_0'$.
(2) Since B is connected, there is a unique component $\neg H' \subseteq \neg H_0'$ containing B.

Observe that $H' \in \mathcal{H}'$ and $U \in \widehat{H'}$. Lastly, since H' is connected and is disjoint from $\partial_{\mathsf{v}} A \subseteq B$, and since $H' \subseteq \neg A$ would imply $\neg H' \in U$, this forces $H' \subseteq A$, and hence $U \in \widehat{H'} \subseteq \widehat{A}$ as desired. ∎

1.2. **Dense cuts induced by quasi-trees.** If X is a quasi-tree − and thus does not have arbitrarily long cycles − we expect that there is some finite bound $R < \infty$ such that the ends in $\widehat{X} \setminus X$ are 'limits' of cuts $\mathcal{H}_{\mathrm{diam}(\partial) \leq R}(X) \subseteq \mathcal{H}_{\partial < \infty}(X)$ with boundary diameter bounded by R. We show that this is indeed the case, in the sense that $\mathcal{H}_{\mathrm{diam}(\partial) \leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ is dense towards ends of (X, G).

Lemma 1.5. *Let $f : (X, G) \to (Y, T)$ be a coarse-equivalence between connected graphs. For a fixed $H \in \mathcal{H}_{\partial < \infty}(Y)$, $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$ is uniformly bounded in terms of $\mathrm{diam}(\partial_{\mathsf{v}} H)$.*

Proof. Since f is bornologous, let $S < \infty$ be such that xGx' implies $d(f(x), f(x')) \leq S$, so that for any $(x, x') \in \partial_{\mathsf{ie}} f^{-1}(H)$, there is a path of length $\leq S$ between $f(x) \notin H$ and $f(x') \in H$. Thus both $d(f(x), \partial_{\mathsf{v}} H)$ and $d(f(x'), \partial_{\mathsf{v}} H)$ are bounded by S, so $f(\partial_{\mathsf{v}} f^{-1}(H)) \subseteq \mathrm{Ball}_S(\partial_{\mathsf{v}} H)$ and hence

$$\mathrm{diam}(f(\partial_{\mathsf{v}} f^{-1}(H))) \leq \mathrm{diam}(\partial_{\mathsf{v}} H) + 2S.$$

That f is a coarse-*equivalence* gives us a uniform bound of $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$ in terms of $\mathrm{diam}(\partial_{\mathsf{v}} H)$. ∎

In particular, if $\mathrm{diam}(\partial_{\mathsf{v}} H)$ is itself also uniformly bounded, then so is $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$.

Proposition 1.6. *The class of connected locally-finite graphs in which $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}$ is dense towards ends for some $R < \infty$ is invariant under coarse equivalence.*

Proof. Let (X, G), (Y, T) be connected locally-finite graphs, $f : X \to Y$ be a coarse equivalence with quasi-inverse $g : Y \to X$, and suppose $\mathcal{H}_{\mathrm{diam}(\partial)\leq S}(Y)$ is dense towards ends for some $S < \infty$. By Lemma 1.5, pick some $R < \infty$ so that for any $H \in \mathcal{H}_{\mathrm{diam}(\partial)\leq S}(Y)$, we have $f^{-1}(H) \in \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$.

Fix an end $U \in \widehat{X} \setminus X$ with $U \in \widehat{A}$ for some $A \in \mathcal{H}_{\partial<\infty}(X)$. We need to find some $B \in \mathcal{H}_{\partial<\infty}(Y)$ such that $\widehat{f}(U) \in \widehat{B}$ and $f^{-1}(B) \subseteq A$, for then $\widehat{f}(U) \in \widehat{H}$ for some $B \supseteq H \in \mathcal{H}_{\mathrm{diam}(\partial)\leq S}(Y)$, and hence we have

$$U \in \widehat{f^{-1}(H)} \subseteq \widehat{f^{-1}(B)} \subseteq \widehat{A}$$

with $f^{-1}(H) \in \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$. For convenience, let $D < \infty$ be the uniform distance $d(1_X, g \circ f)$.

To this end, note that $\widehat{f}(U) \in \widehat{B}$ iff $U \in \widehat{f^{-1}(B)}$. Since $U \in \widehat{A}$, the latter can occur if $|A \triangle f^{-1}(B)| < \infty$, and so we need to find such a $B \in \mathcal{H}_{\partial<\infty}(Y)$ with the additional property that $f^{-1}(B) \subseteq A$.

> *Attempt 1.* Set $B := g^{-1}(A) \in \mathcal{H}_{\partial<\infty}(Y)$. Then $f^{-1}(B) \subseteq \mathrm{Ball}_D(A)$ since if $(g \circ f)(x) \in A$, then
> $$d(x, A) \leq d(x, (g \circ f)(x)) \leq d(1_X, g \circ f) = D.$$
> By local-finiteness of G, we see that $A \triangle f^{-1}(B) = A \setminus f^{-1}(B)$ is finite, as desired.

However, it is *not* the case that $f^{-1}(B) \subseteq A$. To remedy this, we 'shrink' A by D to A' so that $\mathrm{Ball}_D(A') \subseteq A$, and take $B := g^{-1}(A')$ instead. Indeed, $A' := \neg \mathrm{Ball}_D(\neg A) \subseteq A$ works, since $f^{-1}(B) \subseteq \mathrm{Ball}_D(A')$ as before, so $A' \triangle f^{-1}(B) = A' \setminus f^{-1}(B)$ is finite. Also, $A \triangle A'$ is finite since $x \in A \triangle A'$ iff $x \in A$ and $d(x, \neg A) \leq D$, so $A \triangle f^{-1}(B)$ is finite too. It remains to show that $\mathrm{Ball}_D(A') \subseteq A$, for then $f^{-1}(B) \subseteq A$ as desired.

Indeed, if $y \in \mathrm{Ball}_D(A')$, then by the (reverse) triangle-inequality we have $d(y, \neg A) \geq d(x, \neg A) - d(x, y)$ for all $x \in A'$. But $d(x, \neg A) > D$, strictly, so $d(y, \neg A) > D - D = 0$, and hence $y \in A$. ∎

Corollary 1.7. *If (X, G) is a locally-finite quasi-tree, then the subpocset $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ is dense towards ends for some $R < \infty$.*

Proof. Observe that $\mathcal{H}_{\mathrm{diam}(\partial)\leq 2}(T)$ is dense towards ends for any tree T, so Proposition 1.6 proves the density of $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$ for some $R < \infty$. By Lemma 1.4, there is a subpocset $\mathcal{H}' \subseteq \mathcal{H}_{\partial<\infty}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ dense towards ends such that for every $H' \in \mathcal{H}'$, we have $\partial_{\mathsf{ie}} H' \subseteq \partial_{\mathsf{ie}} H$ for some $H \in \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$. Hence we have $H' \in \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$, so the result follows. ∎

2. Pocsets of Dense Families of Cuts

2.1. Pocsets of cuts. The family $\mathcal{H}_{\mathrm{diam}\,\partial\leq R}(X)$ of cuts have the structure of a 'profinite pocset', which we first study abstractly. We then deduce some properties of the pocset induced by a *dense* family of cuts.

Definition 2.1. A *pocset* $(\mathcal{H}, \leq, \neg, 0)$ is a poset $(\mathcal{H}, \leq)$ equipped with an order-reversing involution $\neg : \mathcal{H} \to \mathcal{H}$ and a least element $0 \neq \neg 0$ such that 0 is the only lower-bound of $H, \neg H$ for every $H \in \mathcal{H}$. We call the elements in $\mathcal{H}$ *half-spaces*.

A *profinite pocset* is a pocset $\mathcal{H}$ equipped with a compact topology making $\neg$ continuous and is *totally order-disconnected*, in the sense that if $H \not\leq K$, then there is a clopen upward-closed $U \subseteq \mathcal{H}$ with $H \in U \not\ni K$.

Remark 2.2. Such a topology is automatically Hausdorff and zero-dimensional.

We are primarily interested in subpocsets of $(2^X, \subseteq, \neg, \varnothing)$, which is profinite if equipped with the product topology of the discrete space 2. Indeed, 2^X admits a base of *cylinder sets* — which are finite intersections of sets of the form $\pi_x^{-1}(i)$ where $x \in X$, $i \in \{0, 1\}$, and $\pi_x : 2^X \to 2$ is the projection — making $\neg$ continuous since cylinders are clopen. Finally, for $H \not\leq K$, let U be the upward-closure of a clopen neighborhood $U_0 \ni H$ separating it from K, which is clopen since $\neg U_0$ is a finite union of cylinders.

The following proposition gives a sufficient criteria for subpocsets of 2^X to be profinite. We also show in this case that every non-trivial element $H \in \mathcal{H}^*$ is isolated, which will important in Section 3.1.

Lemma 2.3. *Let X be a set and $\mathcal{H} \subseteq 2^X$ be a subpocset. If $\mathcal{H}$ is finitely-separating, then $\mathcal{H} \subseteq 2^X$ is closed and every non-trivial element is isolated.*

Proof. It suffices to show that the limit points of $\mathcal{H}$ are trivial, so let $A \in 2^X \setminus \{\varnothing, X\}$. Fix $x \in A \not\ni y$. Since $\mathcal{H}$ is finitely-separating, there are finitely-many $H \in \mathcal{H}$ with $x \in H \not\ni y$, and for each such $H \in \mathcal{H} \setminus \{A\}$, we have either some $x_H \in A \setminus H$ or $y_H \in H \setminus A$. Let $U \subseteq 2^X$ be the family of all subsets $B \subseteq X$ containing x and each x_H but not y or any y_H.

This is the desired neighborhood isolating $A \in U$. Indeed, it is (cl)open since it is the *finite* intersection of cylinders prescribed by the x_H's and y_H's, and it is disjoint from $\mathcal{H} \setminus \{A\}$ by construction. ∎

In the case when $\mathcal{H} \subseteq \mathcal{H}_{\partial < \infty}$ is a subpocset of cuts in a graph (X, G), we can deduce that $\mathcal{H} \cap \mathcal{H}_{\mathrm{conn}} \subseteq 2^X$ is closed. Hence, the above lemma refines to the following

Lemma 2.4. *Let $\mathcal{H} \subseteq \mathcal{H}_{\partial < \infty}$ be a subpocset of cuts. If $\mathcal{H}$ is finitely-separating, then $\mathcal{H} \cap \mathcal{H}_{\mathrm{conn}}$ is closed and every non-trivial element is isolated.*

Proof. It suffices to show that no $H \in \mathcal{H}^*$ is a limit point of $\mathcal{H}_{\mathrm{conn}}(X)$, for then $\mathcal{H} \cap \mathcal{H}_{\mathrm{conn}}(X)$ is closed, so fix $H \in \mathcal{H}^*$. Let $U \subseteq 2^X$ be the family of all subsets $B \subseteq X$ containing $\partial_{\mathrm{iv}} H$ and disjoint from $\partial_{\mathrm{ov}} H$, which clearly contains H, and is clopen since $\partial_{\mathrm{v}} H$ is finite.

Suppose that there is some $B \in U \cap \mathcal{H}_{\mathrm{conn}}(X)$. Then $H \subseteq B$, since if $x \in H \cap \neg B$, then there is a path in $\neg B$ to some $x' \in \partial_{\mathrm{ov}} H \subseteq \neg B$, which passes through $\partial_{\mathrm{iv}} H \subseteq B$, a contradiction. The converse is similar. ∎

Example 2.5. For a connected graph (X, G), a subpocset $\mathcal{H} \subseteq 2^X$ is finitely-separating if each $x \in X$ is on the boundary of finitely-many half-spaces. Indeed, any $H \in \mathcal{H}$ separating $x, y \in X$ separates some edge on any fixed path between x and y, and there are only finitely-many such H for each edge.

In particular, if (X, G) is locally-finite[2], then $\mathcal{H}_{\mathrm{diam}(\partial) \leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ for any fixed $R < \infty$ (see Section 1.2) is finitely-separating. Indeed, since the R-ball around any fixed $x \in X$ is finite, any $H \in \mathcal{H}_{\mathrm{diam}(\partial) \leq R}(X)$ with $x \in \partial_{\mathrm{v}} H$ is contained in said R-ball, so there are finitely-many such half-spaces.

2.2. **Finiteness conditions on $\mathcal{H}$ induced by dense cuts.** Let (X, G) be a connected locally-finite graph and consider a family $\mathcal{H} \subseteq \mathcal{H}_{\partial < \infty}$ of cuts that are dense towards ends of G. TODO

3. The Dual Median Graph with Finite Hyperplanes

3.1. **The dual median graph of a pocset.** Let $\mathcal{H}$ be a profinite subpocset with every non-trivial element isolated; for instance, if $\mathcal{H}$ is finitely-separating, and in particular the cuts $\mathcal{H}_{\mathrm{diam}(\partial) \leq R}(X)$ for some locally-finite connected graph (X, G). We present a classical construction in geometric group theory (see [Dun79], [Rol98], [Sag95], and [NR03] for applications) of a 'tree-like' graph associated to such a pocset.

Definition 3.1. An *orientation* on $\mathcal{H}$ is an upward-closed subset $U \subseteq \mathcal{H}$ containing exactly one of $H, \neg H$ for each $H \in \mathcal{H}$. We let $\mathcal{U}(\mathcal{H})$ denote the set of all orientations on $\mathcal{H}$ and let $\mathcal{U}^\circ(\mathcal{H})$ denote the clopen ones.

Intuitively, an orientation is a 'maximally consistent' choice of half-spaces[3].

Example 3.2. Each $x \in X$ induces its *principal orientation* $\widehat{x} := \{H \in \mathcal{H} : x \in H\} = \mathcal{H} \cap \pi_x^{-1}(1)$ — which is clearly clopen in $\mathcal{H}$ — and gives us a canonical map $X \to \mathcal{U}^\circ(\mathcal{H})$. However, this map is *not necessarily* injective, and we call a fiber $[x]_{\mathcal{H}} := \{y \in X : \widehat{x} = \widehat{y}\}$ thereof an *$\mathcal{H}$-block*.

The goal of this section is to canonically construct a graph whose vertices are clopen orientations on $\mathcal{H}$.

Definition 3.3. A *median graph* is a connected graph (X, G) such that for any $x, y, z \in X$, the intersection

$$[x, y] \cap [y, z] \cap [x, z]$$

is a singleton, whose element $\langle x, y, z \rangle$ is called the *median* of x, y, z. Thus we have a ternary median operation $\langle \cdot, \cdot, \cdot \rangle : X^3 \to X$, and a *median homomorphism* $f : (X, G) \to (Y, H)$ is a map preserving said operation.

Some basic properties of median graphs are given in Section 4.2. For more comprehensive references and their general theory, see [Rol98] and [Bow22].

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[2]In this case, $\mathcal{H}$ being finitely-separating implies that each $x \in X$ is separated from each of its finitely-many neighbors by finitely-many $H \in \mathcal{H}$. Thus, the previous condition is necessary.

[3]This can be formalized by letting $\sim$ be the equivalence relation on $\mathcal{H}$ given by $H \sim \neg H$. Letting $\partial : \mathcal{H} \to \mathcal{H}/\sim$ denote the quotient map, orientations $U \subseteq \mathcal{H}$ then correspond precisely to sections $\varphi : \mathcal{H}/\sim \to \mathcal{H}$ of ∂ such that $\varphi(\partial H) \not\subseteq \neg\varphi(\partial K)$ for every $H, K \in \mathcal{H}$; the latter condition rules out 'orientations' of the form $\leftarrow\!\mid \ \mid\!\mapsto$.

Proposition 3.4. *Let $\mathcal{H}$ be a profinite pocset with every non-trivial element isolated. Then the graph $\mathcal{M}(\mathcal{H})$, whose vertices are clopen orientations $\mathcal{U}^\circ(\mathcal{H})$ and whose edges are pairs $\{U, V\}$ with $V = U \bigtriangleup \{H, \neg H\}$ for some $\subseteq$-minimal $H \in U \setminus \{\varnothing, X\}$, is a median graph with path metric $d(U, V) = |U \bigtriangleup V|/2$ and medians*

$$\langle U, V, W \rangle \coloneqq \{H \in \mathcal{H} : H \text{ belongs to at least two of } U, V, W\}$$
$$= (U \cap V) \cup (V \cap W) \cup (U \cap W).$$

Proof. First, $V \coloneqq U \bigtriangleup \{H, \neg H\}$ as above is clopen since $H, \neg H \in \mathcal{H}^*$ are isolated (whence $\{H\}, \{\neg H\}$ are clopen), and it is an orientation by $\subseteq$-minimality of H. That $\mathcal{M}(\mathcal{H})$ is connected follows from the following claim by noting that $U \bigtriangleup V \not\ni \varnothing, X$ is clopen, so it is a compact set of isolated points, whence finite.

Claim ([Sag95, Theorem 3.3])**.** There is a path between U, V iff $U \bigtriangleup V$ is finite, in which case
$$d(U, V) = |U \bigtriangleup V|/2 = |U \setminus V| = |V \setminus U|.$$

Proof. If $(U_i)_{i<n}$ is a path from $U =: U_0$ to $V =: U_{n-1}$, then, letting $\{H_i, \neg H_i\} \coloneqq U_i \bigtriangleup U_{i-1}$ for all $1 \leq i < n$ gives us a sequence $(H_i)_{i<n}$ inducing[a] this path, whence $U \bigtriangleup V$ consists of $\{H_i\}_{i<n}$ and their complements. Thus $U \bigtriangleup V = 2n = 2d(U, V)$, as desired.

Conversely, if $U \bigtriangleup V = \{H_1, \ldots, H_n\} \sqcup \{K_1, \ldots, K_m\}$ with $U \setminus V = \{H_i\}$ and $V \setminus U = \{K_j\}$, then $\neg H_i \in V \setminus U$ and $\neg K_j \in U \setminus V$ for all $i < n$ and $j < m$, so $n = m$ and $V = U \cup \{\neg H_i\} \setminus \{H_i\}$. We claim that there is a permutation $\sigma \in S_n$ such that $(H_{\sigma(i)})$ induces a path from $U =: U_0$, which is the desired path from U to V. Choose a minimal $H \in \{H_i\}$, which is also minimal in U: if $K \subseteq H$ for some $K \in U$, then $\neg H \subseteq \neg K$, and hence $\neg K \in V$, so $K = H_i \subseteq H$ for some i, forcing $K = H$. Set $U_1 \coloneqq U \bigtriangleup \{H, \neg H\}$, which is a clopen orientation. Continuing in this manner by choosing a minimal element in $\{H_i\} \setminus \{H\}$ — and so on — gives us the desired path with $d(U, V) = n$. □

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[a]In the sense that $U_i = U_{i-1} \bigtriangleup \{H_i, \neg H_i\}$ and $H_i \in U_i$ for each $1 \leq i < n$; see [Tse20, Definition 2.20].

Finally, we show that $\mathcal{M}(\mathcal{H})$ is a median graph. Fix $U, V, W \in \mathcal{U}^\circ(\mathcal{H})$, and note that for any $M \in \mathcal{U}^\circ(\mathcal{H})$, we have by the triangle inequality that $M \in [U, V]$ iff $(U \setminus M) \cup (M \setminus V) \subseteq U \setminus V$, which clearly occurs iff $U \cap V \subseteq M \subseteq U \cup V$. Thus, a vertex M lies in the triple intersection $[U, V] \cap [V, W] \cap [U, W]$ iff

$$(U \cap V) \cup (V \cap W) \cup (U \cap W) \subseteq M \subseteq (U \cup V) \cap (V \cup W) \cap (U \cup W).$$

Note that the two sides coincide, so $M = \langle U, V, W \rangle$ — which is clopen if U, V, W are — is as claimed. ∎

Given such a pocset $\mathcal{H}$, the graph $\mathcal{M}(\mathcal{H})$ constructed above is called the *dual*[4] median graph of $\mathcal{H}$. An important special case of this construction is when $\mathcal{H}$ is *nested*, in which case $\mathcal{M}(\mathcal{H})$ is a tree:

Definition 3.5. Let $\mathcal{H} \subseteq 2^X$ be a pocset. Two half-spaces $H, K \in \mathcal{H}$ are *nested* if $\neg^i H \cap \neg^j K = \varnothing$ for some $i, j \in \{0, 1\}$, where $\neg^0 H \coloneqq H$ and $\neg^1 H \coloneqq \neg H$. We say that $\mathcal{H}$ is *nested* if every pair $H, K \in \mathcal{H}$ is nested.

Lemma 3.6. *Let $\mathcal{H}$ be a profinite pocset with non-trivial points isolated. If $\mathcal{H}$ is nested and $(H_i)_{i<n} \subseteq \mathcal{H}$ induces a path in $\mathcal{M}(\mathcal{H})$ with no backtracking, then (H_i) is strictly increasing.*

Proof. Let $(U_i)_{i<n}$ be the induced path. By definition, we have $U_{i+1} = U_{i-1} \cup \{\neg H_i, \neg H_{i-1}\} \setminus \{H_i, H_{i-1}\}$, so $H_i \neq \neg H_{i-1}$, lest $U_{i+1} = U_{i-1}$. Thus, since $H_i \in U_i = U_{i-1} \cup \{\neg H_{i-1}\} \setminus \{H_{i-1}\}$, we see that $H_i \in U_{i-1}$. Clearly $H_i \neq H_{i-1}$. It suffices to remove the three cases when $\neg H_i \subseteq H_{i-1}$, $H_{i-1} \subseteq \neg H_i$, and $H_i \subseteq H_{i-1}$, since then nestedness of $\mathcal{H}$ gives us $H_{i-1} \subsetneq H_i$, as desired.

Indeed, if $\neg H_i \subseteq H_{i-1}$, then $H_{i-1} \in U_{i+1}$ by upward-closure of $U_{i+1} \ni \neg H_i$. But since $H_{i-1} \neq \neg H_i$, we have by definition of U_{i+1} that $H_{i-1} \in U_i$, a contradiction. The other cases are similar. ∎

Corollary 3.7. *If $\mathcal{H}$ is nested, then $\mathcal{M}(\mathcal{H})$ is a tree.*

Proof. A cycle in $\mathcal{M}(\mathcal{H})$ is induced by a *strictly* increasing chain $H_0 \subset \cdots \subset H_n \subset H_0$, which is absurd. ∎

Nonetheless, in the general non-nested case, there is a sufficient criteria for the existence of a *canonical* spanning tree of $\mathcal{M}(\mathcal{H})$, which we now explore.

3.2. Finiteness conditions on $\mathcal{M}(\mathcal{H})$ induced by ???

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[4]The name is justified by a Stone-type duality between median graphs with median homomorphisms and profinite pocsets whose non-trivial points are isolated with continuous maps, where from a median graph one can construct a canonical pocset of 'convex' half-spaces (see [CPTT23, Section 2.D] for details).

4. Median Graphs, Convex Half-spaces, and Canonical Spanning Trees

Throughout this section, let (X, G) be a median graph, which, in the context of Theorem A, is the dual median graph of the pocset $\mathcal{H}_{\mathrm{diam}(\partial) \leq R}$ given by the quasi-isometry.

The presence of the median operation gives rise to a canonical finitely-separating pocset of *convex* (and co-convex) half-spaces $\mathcal{H}_{\mathrm{cvx}}(X) \subseteq 2^X$ with a very rigid structure. To study them, we need to first understand the geometry of *projections* therein.

4.1. **Projections in median graphs.** For vertices $x, y, z \in X$, we write x—y—z for $y \in [x, z]$. By the triangle inequality, we have for all $w, x, y, z \in X$ that

$$(w\text{—}x\text{—}y \text{ and } w\text{—}y\text{—}z) \quad \Leftrightarrow \quad (w\text{—}x\text{—}z \text{ and } x\text{—}y\text{—}z),$$

and both sides occur iff there is a geodesic from w to x to y to z, which we write as w—x—y—z.

Lemma 4.1. *For any $\varnothing \neq A \subseteq X$ and $x \in X$, there is a unique point in $\mathrm{cvx}(A)$ between x and every point in A, called the* projection *of x towards A, denoted $\mathrm{proj}_A(x)$.*

Moreover, we have $\bigcap_{a \in A}[x, a] = [x, \mathrm{proj}_A(x)]$, and for any y in this set, we have $\mathrm{proj}_A(y) = \mathrm{proj}_A(x)$.

Proof. To show existence, pick any $a_0 \in A$. Given $a_n \in \mathrm{cvx}(A)$, if there exists $a \in A$ with $a_n \notin [x, a]$, set $a_{n+1} := \langle x, a, a_n \rangle \in \mathrm{cvx}(A)$. Then a_0—a_1—$\cdots$—a_n—x for all n, so this sequence terminates in at most $d(a_0, x)$ steps at a point in $\mathrm{cvx}(A)$ between x and every point in A. For uniqueness, if there exist two such points $a, b \in \mathrm{cvx}(A)$, then x—a—b and x—b—a, forcing $a = b$.

Finally, if x—y—$\mathrm{proj}_A(x)$ and $a \in A$, then x—$\mathrm{proj}_A(x)$—a and hence x—y—a. Conversely, let x—y—a for all $a \in A$. Since $[y, a] \subseteq [x, a]$ for all a, we see that

$$\mathrm{proj}_A(y) \in \mathrm{cvx}(A) \cap \bigcap_{a \in A}[y, a] \subseteq \mathrm{cvx}(A) \cap \bigcap_{a \in A}[x, a]$$

and hence $\mathrm{proj}_A(y) = \mathrm{proj}_A(x)$ by uniqueness. But since y—$\mathrm{proj}_A(y)$—a, we have x—y—$\mathrm{proj}_A(y)$, and hence x—y—$\mathrm{proj}_A(x)$ as desired.   ∎

Remark 4.2. It follows from the proof above that for any median homomorphism $f : (X, G) \to (Y, H)$, we have $f(\mathrm{proj}_A(x)) = \mathrm{proj}_{f(A)}(f(x))$ for any $\varnothing \neq A \subseteq X$ and $x \in X$. Indeed, we have

$$\mathrm{proj}_A(x) = \langle x, a_m, \ldots, \langle x, a_2, \langle x, a_1, a_0 \rangle \rangle \ldots \rangle$$

for some $m \leq d(a_0, x)$ and $a_0, \ldots, a_m \in A$, and this is preserved by f.

For $A := \{a, b\}$, we have $\mathrm{proj}_A(x) = \langle a, b, x \rangle$, and hence $\mathrm{cvx}(A) = \mathrm{proj}_A(X) = \langle a, b, X \rangle = [a, b]$.

Lemma 4.3. *For each $x, y \in X$, $\mathrm{cone}_x(y)$ is convex, and if xGy, then $\mathrm{cone}_x(y) \sqcup \mathrm{cone}_y(x) = X$.*

Proof. Fix $a, b \in \mathrm{cone}_x(y)$ and a—c—b. It suffices to show that x—y—$\langle a, c, x \rangle$, for then x—y—c since we have x—$\langle a, c, x \rangle$—c. Indeed, it follows from the following observations.

- x—y—$\langle a, b, x \rangle$, since $\langle a, b, x \rangle = \mathrm{proj}_{\{a,b\}}(x)$ and so $[x, \langle a, b, x \rangle] = [x, a] \cap [x, b] \ni y$ by Lemma 4.1.
- x—$\langle a, b, x \rangle$—$\langle a, c, x \rangle$, which follows from $\langle a, b, x \rangle$—$\langle a, c, x \rangle$—a, since x—$\langle a, b, x \rangle$—a by definition. Indeed, we have $\langle a, c, x \rangle$ is in both $[a, x]$ and $[a, c] \subseteq [a, b]$, and since $\mathrm{proj}_{\{b,x\}}(a) = \langle a, b, x \rangle$, we have again by Lemma 4.1 that $[\langle a, b, x \rangle, a] = [a, x] \cap [a, b] \ni \langle a, c, x \rangle$.

Finally, take $z \in X$ and consider $w := \langle x, y, z \rangle \subseteq [x, y]$. Either $w = x$ or $w = y$ (but not both), giving us the desired partition.   ∎

In particular, this shows that if xGy, then $\mathrm{cone}_x(y) \in \mathcal{H}_{\mathrm{cvx}}^*(X)$. The convexity of cones also shows, in the situation of Lemma 4.1, that $\mathrm{proj}_A = \mathrm{proj}_{\mathrm{cvx}(A)}$, i.e., $\mathrm{proj}_A(x)$ is also between x and every point in $\mathrm{cvx}(A)$. Indeed, note that $\mathrm{cone}_x(\mathrm{proj}_A(x))$ is convex and contains A, so it contains $\mathrm{cvx}(A)$ too.

Lemma 4.4. $\mathrm{proj}_A : X \twoheadrightarrow \mathrm{cvx}(A)$ *is a median homomorphism with* $\mathrm{proj}_A \circ \mathrm{cvx} = \mathrm{cvx} \circ \mathrm{proj}_A$.

Proof. The second claim follows from the first since, by Remark 4.2, we have

$$f(\mathrm{cvx}(B)) = f(\mathrm{proj}_B(X)) = \mathrm{proj}_{f(B)}(f(X)) = \mathrm{cvx}(f(B))$$

for all median homomorphisms $f : X \twoheadrightarrow Y$ and $B \subseteq X$, so it in particular applies to $f := \mathrm{proj}_A$.

To this end, let x—y—$z \in X$ and set $w := \langle \mathrm{proj}_A(x), \mathrm{proj}_A(y), \mathrm{proj}_A(z) \rangle \in \mathrm{cvx}(A)$. It suffices to show that y—w—a for all $a \in A$, for then $w = \mathrm{proj}_A(y)$ and hence $\mathrm{proj}_A(x)$—$\mathrm{proj}_A(y)$—$\mathrm{proj}_A(z)$. But we have y—$\mathrm{proj}_A(y)$—a already, so it further suffices to show that y—w—$\mathrm{proj}_A(y)$. For this, we note that

$$x\text{—}\mathrm{proj}_A(x)\text{—}\mathrm{proj}_A(y) \quad \text{and} \quad \mathrm{proj}_A(x)\text{—}w\text{—}\mathrm{proj}_A(y),$$

so x—w—$\mathrm{proj}_A(y)$, and similarly z—w—$\mathrm{proj}_A(y)$. Thus, it follows that

$$
\begin{aligned}
w \in [\mathrm{proj}_A(y), x] \cap [\mathrm{proj}_A(y), z] &= [\mathrm{proj}_A(y), \mathrm{proj}_{\{x,z\}}(\mathrm{proj}_A(y))] &&\text{Lemma 4.1}\\
&= [\mathrm{proj}_A(y), \mathrm{proj}_{[x,z]}(\mathrm{proj}_A(y))] \\
&\subseteq [\mathrm{proj}_A(y), y], &&\text{Lemma 4.1}
\end{aligned}
$$

where the second equality follows from $\mathrm{cvx}(\{x,z\}) = [x,z]$, and hence $\mathrm{proj}_{\{x,z\}} = \mathrm{proj}_{[x,z]}$. ∎

4.2. Convex half-spaces of median graphs.
We now use projections to explore the geometry of convex half-spaces in median graphs. In particular, they are very rigid:

Proposition 4.5. *Each edge $(x, y) \in G$ is on a unique hyperplane, namely the inward boundary of $\mathrm{cone}_x(y)$, and conversely, each half-space $H \in \mathcal{H}^*_{\mathrm{cvx}}(X)$ is $\mathrm{cone}_x(y)$ for every $(x, y) \in \partial_{\mathsf{ie}} H$.*

Thus, hyperplanes are equivalence classes of edges. Furthermore, this equivalence relation is generated by parallel sides of squares (i.e., 4-cycles).

Proof. We have $\mathrm{cone}_x(y) \in \mathcal{H}^*_{\mathrm{cvx}}(X)$ by the above lemma, and clearly $(x, y) \in \partial_{\mathsf{ie}} \mathrm{cone}_x(y)$. Conversely, take $H \in \mathcal{H}^*_{\mathrm{cvx}}(X)$ and any $(x, y) \in \partial_{\mathsf{ie}} H$. Then $H = \mathrm{cone}_x(y)$, for if $z \in H \cap \neg \mathrm{cone}_x(y)$, then $z \in \mathrm{cone}_y(x)$, and hence $x \in [y, z] \subseteq H$ by convexity of H, a contradiction; if $z \in \mathrm{cone}_x(y) \cap \neg H$, then $[x, z] \subseteq \neg H$ by convexity of $\neg H$, and hence $y \notin H$, a contradiction.

Finally, parallel edges of a strip of squares generate the same hyperplane since, for a given square, each vertex is between its neighbors and hence any hyperplane containing an edge contains its opposite edge. On the other hand, let $(a, b), (c, d) \in \partial_{\mathsf{ie}} H$ for some $H \in \mathcal{H}^*_{\mathrm{cvx}}(X)$. Note that $\partial_{\mathsf{ov}} H = \mathrm{proj}_{\neg H}(H)$ is convex since H is, and $\mathrm{proj}_{\neg H}$ preserves convexity by Proposition 4.4, so any geodesic between $a, c \in \partial_{\mathsf{ov}} H$ lies in $\partial_{\mathsf{ov}} H$. Matching this geodesic via $\partial_{\mathsf{ie}} H : \partial_{\mathsf{ov}} H \to \partial_{\mathsf{iv}} H$ gives us a geodesic between b, d in $\partial_{\mathsf{iv}} H$, which together with the matching forms the desired strip of squares. ∎

Corollary 4.6. *Two half-spaces $H, K \in \mathcal{H}^*_{\mathrm{cvx}}(X)$ are non-nested iff there is an embedding $\{0,1\}^2 \hookrightarrow X$ of the Hamming cube into the four corners $\neg^i H \cap \neg^j K$.*

In particular, if $H, K \in \mathcal{H}^*_{\mathrm{cvx}}(X)$ are non-nested, then $\partial_{\mathsf{v}} H \cap \partial_{\mathsf{v}} K \neq \varnothing$.

Proof. Let H, K be non-nested and take $x_1 \in H \cap K$ and $x_2 \in H \cap \neg K$. Since H is connected, any geodesic between x_1, x_2 crosses an edge $(x'_1, x'_2) \in \partial_{\mathsf{oe}} K$ in H. Similarly, there is an edge $(y'_1, y'_2) \in \partial_{\mathsf{oe}} K$ in $\neg H$, so we may slide both edges along $\partial_{\mathsf{oe}} K$ to obtain the desired square (see Proposition 4.5).

Conversely, the half-spaces cutting the square are clearly non-nested. ∎

Lemma 4.7 (Helly). *Any finite intersection of pairwise-intersecting non-empty convex sets is non-empty.*

Proof. For pairwise-intersecting convex sets H_1, H_2, H_3, pick any $x \in H_1 \cap H_2$, $y \in H_1 \cap H_3$ and $z \in H_2 \cap H_3$; their median $\langle x, y, z \rangle$ then lies in $H_1 \cap H_2 \cap H_3$.

Suppose that it holds for some $n \geq 3$ and let $H_1, \dots, H_{n+1} \subseteq X$ pairwise-intersect. Then $\{H_i \cap H_{n+1}\}_{i \leq n}$ is a family of n pairwise-intersecting convex sets, so $\bigcap_{i \leq n+1} H_i = \bigcap_{i \leq n}(H_i \cap H_{n+1})$ is non-empty. ∎

Lastly, we have some useful finiteness conditions on convex half-spaces; the former implies that $\mathcal{H}_{\mathrm{cvx}}(X)$ is finitely-separating, and the latter allows us to replace finite sets with their convex hulls.

Lemma 4.8. *Any two disjoint convex sets $\varnothing \neq A, B \subseteq X$ can be separated by a half-space $A \subseteq H \subseteq \neg B$, and furthermore we have in this case that $d(A, B) = |\{H \in \mathcal{H}_{\mathrm{cvx}}(X) : A \subseteq H \subseteq \neg B\}|$.*

Proof. Pick a geodesic $A \ni x_0 G x_1 G \cdots G x_n \in B$, where $n := d(A, B)$. Then $H := \mathrm{cone}_{x_1}(x_0)$, which is a half-space by Lemma 4.3, separates A, B since $x_0 = \mathrm{proj}_A(x_n)$, and thus we have $A \subseteq \mathrm{cone}_{x_n}(x_0) \subseteq \mathrm{cone}_{x_1}(x_0)$ and $B \subseteq \mathrm{cone}_{x_0}(x_n) \subseteq \mathrm{cone}_{x_0}(x_1)$.

Moreover, each such half-space $A \subseteq H \subseteq \neg B$ satisfies $x_i \in H \not\ni x_{i+1}$ for a unique $i < n$, and conversely each pair (x_i, x_{i+1}) has a unique half-space separating them, so we have the desired bijection. ∎

Lemma 4.9. *Every interval $[x, y]$ is finite. More generally, if $A \subseteq X$ is finite, then so is $\mathrm{cvx}(A)$.*

Proof. The singletons $\{x\}$ and $\{y\}$ are convex, so there are finitely-many half-spaces $H \subseteq [x, y]$. But each $z \in [x, y]$ is determined uniquely by those half-spaces containing it, so $[x, y]$ is finite.

Let $A := \{x_0, \ldots, x_n\}$. Since $\mathrm{cvx}(A) = \mathrm{proj}_A(X)$, we have by Remark 4.2 that points in $\mathrm{cvx}(A)$ are of the form $\langle x, x_n, \ldots, \langle x, x_2, \langle x, x_1, x_0 \rangle \rangle \rangle$, which is finite by induction using that intervals are finite. ∎

4.3. **Canonical spanning trees.** We now present the Borel cycle-cutting algorithm that can be preformed canonically on any countable median graph with finite hyperplanes. We do so by colouring the half-spaces $\mathcal{H}^*_{\mathrm{cvx}}(X)$ into certain nested half-spaces $\mathcal{H}_n \subseteq \mathcal{H}_{\mathrm{cvx}}(X)$, from which we inductively build a spanning forest by leveraging a tree structure on the $\mathcal{H}_n$-blocks $X/\mathcal{H}_n$. This tree is constructed as follows.

Lemma 4.10. *For any subpocset $\mathcal{H} \subseteq \mathcal{H}_{\mathrm{cvx}}(X)$, the principal orientations map $X \to \mathcal{U}^\circ(\mathcal{H})$ is surjective.*

Proof. Let $U \in \mathcal{U}^\circ(\mathcal{H})$. Since $U \subseteq \mathcal{H}$ is clopen, there is a finite set $A \subseteq X$ − which we may assume to be convex by Corollary 4.9 − such that for all $H \in \mathcal{H}$, we have $H \in U$ iff there is $K \in U$ with $H \cap A = K \cap A$. Note that $K \cap A \neq \varnothing$ for every $K \in U$, since otherwise $\varnothing \in U$. Furthermore, $H \cap K \neq \varnothing$ for every $H, K \in U$, since otherwise we have $H \subseteq \neg K$, and so $\neg K \in U$. By Lemma 4.7, we have $(H \cap A) \cap (K \cap A) = H \cap K \cap A \neq \varnothing$, and applying it again furnishes some $x \in \bigcap_{H \in U} H \cap A$ in X; note that the latter intersection is finite. Thus $U \subseteq \widehat{x}$, so $U = \widehat{x}$ since both are orientations. ∎

This lemma is crucial, in that it induces an isomorphism $X/\mathcal{H} \cong \mathcal{M}(\mathcal{H})$. Here, $\mathcal{H}$ is finitely-separating by Example 2.5 and Proposition 4.5, and so $\mathcal{U}^\circ(\mathcal{H})$ is the vertices of the median graph $\mathcal{M}(\mathcal{H})$ by Lemma 2.3 and Proposition 3.4. The fibers of $X \to \mathcal{U}^\circ(\mathcal{H})$ are the $\mathcal{H}$-blocks, so we have the desired bijection.

Explicitly, two $\mathcal{H}$-blocks $([x]_\mathcal{H}, [y]_\mathcal{H})$ are said to be *G-adjacent* if $(\widehat{x}, \widehat{y}) \in \mathcal{M}(\mathcal{H})$.

Proposition 4.11. *Every countable median graph with finite hyperplanes admits a canonical spanning tree.*

Proof. Let (X, G) be a countable median graph with finite hyperplanes. By Corollary 4.6, if two half-spaces $H, K \in \mathcal{H}^*_{\mathrm{cvx}}(X)$ are non-nested, then $\partial_\vee H \cap \partial_\vee K \neq \varnothing$, so looking at the intersection graph of the boundaries furnishes a countable colouring $\mathcal{H}^*_{\mathrm{cvx}}(X) = \bigsqcup_{n \in \mathbb{N}} \mathcal{H}^*_n$ such that each $H, \neg H$ receive the same color and that each $\mathcal{H}_n := \mathcal{H}^*_n \cup \{\varnothing, X\}$ is a *nested* subpocset. For each $n \in \mathbb{N}$, let $\mathcal{K}_n := \bigcup_{m \geq n} \mathcal{H}_m$.

We shall inductively construct an increasing chain of subforests $T_n \subseteq G$ such that the components of T_n are exactly the $\mathcal{K}_n$-blocks. Then, the increasing union $T := \bigcup_n T_n$ is a spanning tree, since each $(x, y) \in G$ lies in a $\mathcal{K}_n$-block for sufficiently large n (namely, the n such that $\mathrm{cone}_x(y) \in \mathcal{H}^*_{n-1}$, since $\mathrm{cone}_x(y)$ and its complement are the only half-spaces separating x and y by Proposition 4.5).

Since each pair of distinct points is separated by a half-space, the $\mathcal{K}_0 = \mathcal{H}_{\mathrm{cvx}}(X)$-blocks are singletons, so put $T_0 := \varnothing$. Suppose that a forest T_n is constructed as required. Note that each $\mathcal{K}_{n+1}$-block $Y \in X/\mathcal{K}_{n+1}$ is not separated by any half-spaces in $\mathcal{H}_m$ for $m > n$, but is separated (by Proposition 4.5) by $\mathcal{H}_n$ into the $\mathcal{K}_n$-blocks contained in Y, and those correspond precisely to the $\mathcal{H}_n$-blocks in $Y/\mathcal{H}_n$. For each G-adjacent pair $A, B \in Y/\mathcal{H}_n$, which is separated by a unique half-space $H \in \mathcal{H}_n$ by Lemma 4.8, we may pick an edge from the *finite* hyperplane $\partial_{\mathsf{ie}} H$. Since each $Y/\mathcal{H}_n$ is a tree by Corollary 3.7, and a single edge is picked between every pair of G-adjacent blocks therein, the graph T_{n+1} obtained from T_n by adding all such edges is a forest whose components are exactly the $\mathcal{K}_{n+1}$-blocks. ∎

Corollary 4.12. *If a CBER admits a graphing whose components are median graphs, then it has a subtreeing.*

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