TREE OF ORIENTATIONS ON A NESTED COLLECTION OF SETS

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Let $H \subseteq 2^X$ be a sub-pocset for some fixed set X (so that, in particular, H is closed under complements). With the definitions in Section 1, we prove the following

Theorem A (Propositions 2.5, 2.6)**.** *If H is nested, then the graph $\mathcal{T}_H$, whose:*

- *Vertices are finitely-based orientations on H;*
- *Edges are pairs $\{p, q\}$ such that $q = p \triangle \{h, \neg h\}$ for some minimal $h \in H$;*

is acyclic. Furthermore, if H is finitely-separating (or more generally, chain-vanishing), then $\mathcal{T}_H$ is a tree.

In particular, this applies to when (X, G) is a graph and H is a nested collection of cuts on X. No further assumptions on X (like local-finiteness) is needed.

1. Preliminaries

Let $H \subseteq 2^X$ be a sub-pocset for some fixed set X, whose elements $h \in H$ are called *half-spaces*.

Definition 1.1. Two elements $h, k \in H$ are *nested* if $h^i \cap k^j = \varnothing$ for some $i, j \in \{1, -1\}$, where $h^i \coloneqq h$ for $i = 1$ and $h^i \coloneqq h^c$ otherwise. We say that H is *nested* if every pair $h, k \in H$ are nested.

1.1. Orientations.

We give the standard definition of orientations on H and characterize them as 'consistent assignments of half-spaces to hyperplanes'.

Definition 1.2. An *orientation* on H is a subset $U \subseteq H$ such that

1. *(Upward-closure).* If $h \in U$ and $k \in H$ contains h, then $k \in U$.
2. *(Ultra).* For each $h \in H$, exactly one of h, h^c is contained in U.

Consider the equivalence relation $\sim$ on H generated by $h \sim h^c$ for all $h \in H$, whose classes are called *hyperplanes* $\delta h \coloneqq \{h, h^c\}$ where $\delta : H \to H/{\sim}$ is the projection. We show that an orientation on H is just a choice $\varphi : H/{\sim} \to H$ of a half-space for each hyperplane, that is consistent in the sense below.

Proposition 1.3. *An orientation $p \subseteq H$ is exactly the data of a function $\varphi : H/{\sim} \to H$ such that $\varphi(\delta h) \in \delta h$ and $\varphi(\delta h) \not\subseteq \varphi(\delta k)^c$ for every $h, k \in H$.*

Proof. Given an orientation $p \subseteq H$, let $\varphi_p(\delta h) \coloneqq h^i \in U$ for the unique $i \in \{1, -1\}$. That $\varphi_p(\delta h) \in \delta h$ is clear, and if $\varphi_p(\delta h) \subseteq \varphi_p(\delta k)^c$, then U contains both $\varphi_p(\delta k)$ and $\varphi_p(\delta k)^c$ by upward-closure, a contradiction.

Conversely, given such a function $\varphi : H/{\sim} \to H$, let $p_\varphi \coloneqq \operatorname{im} \varphi \subseteq H$. This is ultra since if $h \in H$ and $h^c \notin p_\varphi$, then $\varphi(\delta h) \in \delta h = \{h, h^c\}$ implies $h \in p_\varphi$. Furthermore, if $p_\varphi \ni h \subseteq k$, then $k^c \in p_\varphi$ implies $\varphi(\delta h) = h \subseteq k = \varphi(\delta k)^c$, a contradiction, so $k \in p_\varphi$ by the above.

Finally, given an orientation $p \subseteq H$, we have $h \in p$ iff $\varphi_p(\partial h) = h$, which occurs iff $h \in \operatorname{im} \varphi_p = p_{\varphi_p}$. Thus $p_{\varphi_p} = p$. Conversely, given $\varphi : H/{\sim} \to H$, and $h \in H$, we have $\varphi_{p_\varphi}(\partial h) = h^i$ iff $h^i \in p_\varphi = \operatorname{im} \varphi$, which occurs iff $\varphi(\partial h) = h^i$. Thus $\varphi_{p_\varphi} = \varphi$ too, as desired. ∎

Definition 1.4. A *base* for an orientation $p \subseteq H$ is a $\subseteq$-minimal subset $p_0 \subseteq p$ such that $p = {\uparrow} p_0$, where

$$\uparrow p_0 \coloneqq \bigcup_{h \in p_0} \uparrow h \coloneqq \bigcup_{h \in p_0} \{k \in H : k \supseteq h\}.$$

We say that p is *finitely-based* if it admits a finite basis.

In the above correspondence, a base for $\varphi : H/{\sim} \to H$ is a function $\varphi_0 \subseteq \varphi$ where $\operatorname{dom} \varphi_0$ is a $\subseteq$-minimal subset of hyperplanes such that φ_0 extends uniquely to φ. Thus, the finitely-based orientations are the ones determined by a choice of half-spaces from finitely-many hyperplanes.

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1.2. **Flipping basis elements.** We now investigate the behaviour of orientation when its choice on a single half-space is modified. Although the proofs work without the characterization in Proposition 1.3, it makes orientations a lot more intuitive to me, and so will be phrased this way.

Lemma 1.5. *If $p \subseteq H$ is an orientation, then $q \coloneqq p \triangle \{h, h^c\}$ is an orientation iff $h \in p$ is $\subseteq$-minimal. Furthermore, if p is finitely-based, then so is q.*

Proof. First, note that $\varphi_q(\partial h) = \varphi_p(h)^c = h^c$ and $\varphi_q = \varphi_p$ away from ∂h.

($\Rightarrow$). If q is an orientation and $p \ni k \subset h$, then $\varphi_q(\partial k) = \varphi_p(\partial k) = k \subseteq h = \varphi_q(\partial h)^c$, a contradiction.

($\Leftarrow$). Let $h \in p$ be $\subseteq$-minimal and suppose that q is not an orientation. Since only $\varphi_q(\partial h) = h^c$ differs from φ_p, this can only occur if $\varphi_q(\partial h) \subseteq \varphi_q(\partial k)^c$ for some $k \in H$. But then

$$h^c = \varphi_q(\partial h) \subseteq \varphi_q(\partial k)^c = k^i \notin p$$

for some unique $i \in \{1, -1\}$, so that $p \ni k^{1-i} \subseteq h$ and contradicts that $h \in p$ is $\subseteq$-minimal.

Finally, suppose that p is finitely-based, as witnessed by some finite $B \subseteq H/\sim$ and $\varphi_0 : B \to H$. ∎

Remark 1.6. In the above notations, clearly $\mathcal{U} \neq \mathcal{U} \triangle \{A, A^c\}$. Furthermore, for any other such orientation $\mathcal{U}'$ and $A' \in \mathcal{U}'$, that $\mathcal{U} = \mathcal{U}'$ and $\mathcal{U} \triangle \{A, A^c\} = \mathcal{U}' \triangle \{A', A'^c\}$ together imply $A = A'$.

1.3. **Finitely-based orientations.**

Definition 1.7. A *base* for an orientation $\mathcal{U} \subseteq P$ is a $\subseteq$-minimal subset $\mathcal{B} \subseteq \mathcal{U}$ such that $\mathcal{U} = \uparrow\mathcal{B}$, where

$$\uparrow\mathcal{B} \coloneqq \bigcup_{B \in \mathcal{B}} \uparrow B \coloneqq \bigcup_{B \in \mathcal{B}} \{A \in P : A \supseteq B\}.$$

Remark 1.8. If P is nested, then every $\subseteq$-minimal $B \in P$ induces an orientation $\uparrow B \coloneqq \{A \in P : A \supseteq B\}$, called a *principal* orientation. Indeed, $\uparrow B$ is clearly upward-closed, and if $A, A^c \in P$, then, by $\subseteq$-minimality of B and nestedness of P, either $A \supseteq B$ or $A^c \supseteq B$ (but clearly not both).

This construction generalizes to any collection $\mathcal{B} \subseteq P$ with each $B \in \mathcal{B}$ being $\subseteq$-minimal in P, so that $\uparrow\mathcal{B}$ is an orientation on P.

Definition 1.9. An orientation $\mathcal{U} \subseteq P$ is said to be *finitely-based* if it admits a finite base.

Remark 1.10. If $\mathcal{U} = \uparrow\{B_1, \ldots, B_n\}$ is finitely-based, then for any $\subseteq$-minimal $A \in \mathcal{U}$ with $A^c \in P$, so is the orientation $\mathcal{V} \coloneqq \mathcal{U} \triangle \{A, A^c\}$. Indeed, we have $A = B_i$ for some $1 \leq i \leq n$, and $\mathcal{V} = \uparrow(\{A^c\} \cup \{B_j\}_{j \neq i})$.

2. The graph $\mathcal{T}_P$

Fix a nested collection of non-empty subsets of a set X. Using Lemma 1.5 and Remarks 1.6 and 1.10, we construct a graph $\mathcal{T}_P$ whose:

- *Vertices* of $\mathcal{T}_P$ are finitely-based orientations on P.
- *Neighbors* of $\mathcal{U} \in V(\mathcal{T}_P)$ are $\mathcal{U} \triangle \{A, A^c\}$ for every minimal $A \in \mathcal{U}$ with $A^c \in P$.

The goal of this section is to establish Theorem A, stating that $\mathcal{T}_P$ is acyclic (Proposition 2.5), and furthermore, $\mathcal{T}_P$ is a tree when P is closed under complements (Proposition 2.6).

Definition 2.1. Fix $\mathcal{U}_0 \in V(\mathcal{T}_P)$ and $n \in \mathbb{N}$. A sequence $(A_i)_{i<n} \subseteq P$ is said to *induce a path from* $\mathcal{U}_0$ if $(\mathcal{U}_i)_{i<n}$, defined by $\mathcal{U}_i \coloneqq \mathcal{U}_{i-1} \triangle \left\{A_{i-1}, A_{i-1}^c\right\}$ for every $1 \leq i < n$, is a path in $\mathcal{T}_P$ with each $A_i \in \mathcal{U}_i$.

Remark 2.2. Any path in $\mathcal{T}_P$ is induced by its sequence of flipped basis elements.

Lemma 2.3. *Let $n \geq 3$. A path in $\mathcal{T}_P$ from $\mathcal{U}_0$ induced by $(A_i)_{i<n}$ has no backtracking iff $A_i \neq A_{i-1}^c$ for every $1 \leq i < n$.*

Proof. Take $2 \leq i \leq n$. It suffices to show that $\mathcal{U}_{i-2} = \mathcal{U}_i$ iff $A_{i-1} = A_{i-2}^c$.

($\Rightarrow$). We have by definition that $\mathcal{U}_i = \mathcal{U}_{i-2} \cup \left\{A_{i-1}^c, A_{i-2}^c\right\} \setminus \{A_{i-1}, A_{i-2}\}$, so since $A_{i-2} \in \mathcal{U}_{i-2} = \mathcal{U}_i$, we have $A_{i-2} = A_{i-1}^c$ as desired.

($\Leftarrow$). Again by definition, by noting that the basis-flipping cancels out. ∎

Lemma 2.4. *If $(A_i)_{i<n}$ induces a path in $\mathcal{T}_P$ with no backtracking, then $(A_i)_{i<n}$ is strictly increasing.*

Proof. By Lemma 2.3, we have $A_i \neq A_{i-1}^c$ for every $1 \leq i < n$. Thus, since $A_i \in \mathcal{U}_i = \mathcal{U}_{i-1} \cup \{A_{i-1}^c\} \setminus \{A_{i-1}\}$, we see that $A_i \in \mathcal{U}_{i-1}$. Clearly $A_i \neq A_{i-1}$. It suffices to remove the three cases when $A_i \subseteq A_{i-1}$, $A_{i-1} \subseteq A_i^c$, and $A_i^c \subseteq A_{i-1}$, since then nestedness of P gives us $A_{i-1} \subsetneq A_i$, as desired.

- If $A_i \subseteq A_{i-1}$, then $A_{i-1} \in \mathcal{U}_i$, contradicting the definition of $\mathcal{U}_i$.
- If $A_{i-1} \subseteq A_i^c$, then $A_i^c \in \mathcal{U}_{i-1}$ by upward-closure of $\mathcal{U}_{i-1}$, a contradiction.
- If $A_i^c \subseteq A_{i-1}$, then $A_{i-1} \in \mathcal{U}_{i+1}$ by upward-closure of $\mathcal{U}_{i+1} \ni A_i^c$. But since $A_{i-1} \neq A_i^c$, we have by definition of $\mathcal{U}_{i+1}$ that $A_{i-1} \in \mathcal{U}_i$, a contradiction. ■

Proposition 2.5. *$\mathcal{T}_P$ is acyclic.*

Proof. Let $(\mathcal{U}_i)_{i<n}$ be a cycle in $\mathcal{T}_P$ induced by $(A_i)_{i<n}$. Since cycles are non-backtracking, we have $A_0 \subsetneq A_0$ by Lemma 2.4, a contradiction. ■

Proposition 2.6. *If P is chain-vanishing, then $\mathcal{T}_P$ is connected (and hence a tree).*

Proposition 2.7. *If P is closed under complements, then $\mathcal{T}_P$ is connected (and hence a tree).*

Proof. If $\mathcal{U}, \mathcal{U}' \in V(\mathcal{T}_P)$ are two finitely-based orientations on P, then swapping their basis elements one by one as in Remark 1.10 gives us a path between $\mathcal{U}$ and $\mathcal{U}'$; the closure of P is needed to ensure that those basis elements induce a path between the vertices, in that their complement lies in P. ■