

TREE-LIKE GRAPHINGS OF COUNTABLE BOREL EQUIVALENCE RELATIONS

ZHAOSHEN ZHAI

ABSTRACT. We present a streamlined exposition of a construction by R. Chen, A. Poulin, R. Tao, and A. Tserunyan, which proves the treeability of equivalence relations generated by any locally-finite Borel graph such that each component is a quasi-tree. More generally, we show that if each component of a locally-finite Borel graph admits a *finitely-separating Borel family of cuts*, then we may ‘canonically’ construct a forest of special ultrafilters; moreover, if the cuts are *dense towards ends*, then this forest is a Borel treeing.

The purpose of this note is to provide a streamlined proof of the main result in [CPTT23] and present the necessary constructions, in order to better understand the general formalism developed therein. We attempt to make this note self-contained, but nevertheless, we urge the reader to refer to the original paper for more detailed background/discussions and some generalizations of the results we have selected to include here.

Treeings of equivalence relations. A *countable Borel equivalence relation (CBER)* on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$ with each class countable. We are interested in special types of *graphings* of a CBER $E \subseteq X^2$, i.e. a Borel graph $G \subseteq X^2$ whose connectedness relation is precisely E . For instance, a graphing of E such that each component is a tree is called a *treeing* of E , and the CBERs that admit treeings are said to be *treeable*. The main results of [CPTT23] provide new sufficient criteria for treeability of certain classes of CBERs, and in particular, they prove the following

Theorem A (Section 4, [CPTT23, Theorem 1.1]). *If a CBER E admits a locally-finite graphing such that each component is a quasi-tree,¹ then E is treeable.*

Roughly speaking, the existence of a quasi-isometry $G|_C \rightarrow T_C$ to a simplicial tree T_C for each component $C \subseteq X$ induces a collection $\mathcal{H}(C) \subseteq 2^C$ of ‘cuts’ (subsets $H \subseteq C$ with finite boundary such that both H and $C \setminus H$ are connected), which are ‘tree-like’ in the sense that

1. $\mathcal{H}(C)$ is *finitely-separating*: each pair $x, y \in C$ is separated by finitely-many $H \in \mathcal{H}(C)$, and
2. $\mathcal{H}(C)$ is *dense towards ends*: each end in $G|_C$ has a neighborhood basis in $\mathcal{H}(C)$.

Condition (1) allows for an abstract construction of a tree whose vertices are special ‘ultrafilters’² on $\mathcal{H}(C)$, as outlined in the following diagram: starting from a finitely-separating family of cuts, one constructs a ‘dual median graph’ $\mathcal{M}(\mathcal{H}(C))$ with said ultrafilters; this median graph has finite ‘hyperplanes’, which allows one to apply a Borel cycle-cutting algorithm and obtain a ‘canonical’ spanning tree thereof.



Thus we have the following theorem, which can be viewed as a component-wise version of Theorem A.

Theorem B (Propositions 3.3, 3.5, 3.7). *For any finitely-separating family of cuts \mathcal{H} on a connected locally-finite graph, its dual median graph $\mathcal{M}(\mathcal{H})$ has finite hyperplanes, and fixing a Borel colouring on the intersection graph of those hyperplanes yields a canonical spanning tree of $\mathcal{M}(\mathcal{H})$.*

The additional condition (2) then shows that $\mathcal{M}(\mathcal{H})$ is locally-finite (Lemma 4.1), which ensures that such a spanning tree can be constructed uniformly for each component $C \subseteq G$; see Section 4 for details.

Date: August 16, 2024.

This work was partially supported by McGill University’s SURF (Science Undergraduate Research Award) grant.

¹Recall that metric spaces X and Y are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error, and X is a *quasi-tree* if it is quasi-isometric to a simplicial tree; see [Gro93] and [DK18].

²As in [CPTT23], we call them *orientations* instead, to avoid confusion with the more standard notion; see Definition 3.1.

Acknowledgements. I would like to thank Professor Anush Tserunyan for supervising me for this project, for her consistent support, patience, and feedback, and for guiding me through this fulfilling research experience. I also thank the original authors of [CPTT23] for their incredible work and detailed writing.

1. PRELIMINARIES ON POCSETS, ENDS OF GRAPHS, AND MEDIAN GRAPHS

Notation. A *graph* on a set X is a symmetric irreflexive binary relation $G \subseteq X^2$. For $A \subseteq X$, we say that A is *connected* if the induced subgraph $G[A]$ is. We always equip connected graphs with their *path metric* d , and let $\text{Ball}_r(x)$ be the closed ball of radius r around x ; more generally, we let $\text{Ball}_r(A) := \bigcup_{x \in A} \text{Ball}_r(x)$.

For a subset $A \subseteq X$, we let $\partial_v A := A \cap \text{Ball}_1(\neg A)$ be its *inner vertex boundary*, $\partial_{ov} A := \partial_v(\neg A)$ be its *outer vertex boundary*, and let $\partial_{ie} A := G \cap (\partial_{ov} A \times \partial_v A)$ and $\partial_{oe} A := \partial_{ie}(\neg A)$ respectively be its *inner* and *outer edge boundaries*. Let $\partial_v A := \partial_v A \sqcup \partial_{ov} A$ be the *(total) vertex boundary* of A .

Finally, for $x, y \in X$, the *interval* $[x, y]$ between x, y is the union of all geodesics between x, y , consisting of exactly those $z \in X$ with $d(x, z) + d(z, y) = d(x, y)$. We say that $A \subseteq X$ is *convex* if $[x, y] \subseteq A$ for all $x, y \in A$. For vertices $x, y, z \in X$, we write $x-y-z$ for $y \in [x, z]$. By the triangle inequality, we have for all $w, x, y, z \in X$ that

$$(w-x-y \text{ and } w-y-z) \Leftrightarrow (w-x-z \text{ and } x-y-z),$$

and both sides occur iff there is a geodesic from w to x to y to z , which we write as $w-x-y-z$.

1.1. Profinite pocsets of cuts. The construction starts by identifying a profinite pocset \mathcal{H} of ‘cuts’ in each component of the graphing, which we first study abstractly. The finitely-separating subpocsets of 2^X are well-known in metric geometry as *wallspaces*; see, e.g., [Nic04] and [CN05].

Definition 1.1. A *pocset* $(\mathcal{H}, \leq, \neg, 0)$ is a poset (\mathcal{H}, \leq) equipped with an order-reversing involution $\neg : \mathcal{H} \rightarrow \mathcal{H}$ and a least element $0 \neq \neg 0$ such that 0 is the only lower-bound of $H, \neg H$ for every $H \in \mathcal{H}$. We call the elements in \mathcal{H} *half-spaces*, and the edge boundaries $\partial_{ie} H$ for $H \in \mathcal{H}$ *hyperplanes*.

A *profinite pocset* is a pocset \mathcal{H} equipped with a compact topology making \neg continuous and is *totally order-disconnected*, in the sense that if $H \not\leq K$, then there is a clopen upward-closed $U \subseteq \mathcal{H}$ with $H \in U \not\leq K$.

We are primarily interested in subpocsets of $(2^X, \subseteq, \neg, \emptyset)$ for a fixed set X , which is profinite if equipped with the product topology of the discrete space 2 . Indeed, 2^X admits a base of *cylinder sets* – which are finite intersections of sets of the form $\pi_x^{-1}(i)$ where $x \in X$, $i \in \{0, 1\}$, and $\pi_x : 2^X \rightarrow 2$ is the projection – making \neg continuous since cylinders are clopen. Finally, for $H \not\leq K$, let U be the upward-closure of a clopen neighborhood $U_0 \ni H$ separating it from K , which is clopen since $\neg U_0$ is a finite union of cylinders.

Remark. We follow [CPTT23, Convention 2.7], where for a family $\mathcal{H} \subseteq 2^X$ of subsets of a fixed set X , we write $\mathcal{H}^* := \mathcal{H} \setminus \{\emptyset, X\}$ for the *non-trivial* elements of \mathcal{H} .

The following proposition gives a sufficient criteria for subpocsets of 2^X to be profinite. We also show in this case that every non-trivial element $H \in \mathcal{H}^*$ is isolated, which will be important in Section 3.1.

Lemma 1.2. *If $\mathcal{H} \subseteq 2^X$ is a finitely-separating pocset, then \mathcal{H} is closed and non-trivial elements are isolated.*

Proof. It suffices to show that the limit points of \mathcal{H} are trivial, so let $A \in 2^X \setminus \{\emptyset, X\}$. Fix $x \in A \not\leq y$. Since \mathcal{H} is finitely-separating, there are finitely-many $H \in \mathcal{H}$ with $x \in H \not\leq y$, and for each such $H \in \mathcal{H} \setminus \{A\}$, we have either some $x_H \in A \setminus H$ or $y_H \in H \setminus A$. Let $U \subseteq 2^X$ be the family of all subsets $B \subseteq X$ containing x and each x_H , but not y or any y_H .

This is the desired neighborhood isolating $A \in U$. Indeed, it is (cl)open since it is the *finite* intersection of cylinders prescribed by the x_H ’s and y_H ’s, and it is disjoint from $\mathcal{H} \setminus \{A\}$ by construction. ■

Our main method of identifying the finitely-separating pocsets is as follows.

Lemma 1.3. *Let $\mathcal{H} \subseteq 2^X$ be a pocset in a connected graph (X, G) . If each $x \in X$ is on the vertex boundary of finitely-many half-spaces, then \mathcal{H} is finitely-separating. The converse holds too if (X, G) is locally-finite.*

Proof. Any $H \in \mathcal{H}$ separating $x, y \in X$ separates some edge on any fixed path between x and y , and there are only finitely-many such H for each edge. If (X, G) is locally-finite, then each $x \in X$ is separated from each of its finitely-many neighbors by finitely-many $H \in \mathcal{H}$. ■

In the case that \mathcal{H} is a pocset consisting of connected co-connected half-spaces with finite vertex boundary, finite-separation also controls the degree of ‘non-nestedness’ of \mathcal{H} .

Definition 1.4. A *cut* in a connected locally-finite graph (X, G) is a half-space $H \in \mathcal{H}_{\partial < \infty}(X) \cap \mathcal{H}_{\text{conn}}(X)$.

Definition 1.5. Let $\mathcal{H} \subseteq 2^X$ be a pocset. Two half-spaces $H, K \in \mathcal{H}$ are *nested* if $\neg^i H \cap \neg^j K = \emptyset$ for some $i, j \in \{0, 1\}$, where $\neg^0 H := H$ and $\neg^1 H := \neg H$. We say that \mathcal{H} is *nested* if every pair $H, K \in \mathcal{H}$ is nested.

Lemma 1.6. In a pocset \mathcal{H} of finitely-separating cuts, each $H \in \mathcal{H}$ is non-nested with finitely-many others.

Proof. Fix $H \in \mathcal{H}$ and let $K \in \mathcal{H}$ be non-nested with H . By connectedness, the non-empty sets $H \cap K$ and $\neg H \cap K$ are joined by a path in K , so $\partial_v H \cap K \neq \emptyset$; similarly, $\partial_v H \cap \neg K \neq \emptyset$. For each $x \in \partial_v H \cap K$ and $y \in \partial_v H \cap \neg K$, any fixed path p_{xy} between them contains some $z \in \partial_v K \cap p_{xy}$; thus, any $K \in \mathcal{H}$ non-nested with H contains some $z \in \partial_v K \cap p_{xy}$.

Then, since there are finitely-many such $x, y \in \partial_v H$, for each of which there are finitely-many $z \in p_{xy}$, for each of which there are finitely-many $K \in \mathcal{H}$ with $z \in \partial_v K$ (by Lemma 1.3, since \mathcal{H} is finitely-separating), there can only be finitely-many $K \in \mathcal{H}$ non-nested with H . ■

1.2. Ends of graphs. Let (X, G) be a connected locally-finite graph, and consider the Boolean algebra of finite vertex boundary half-spaces $\mathcal{H}_{\partial < \infty}(X) \subseteq 2^X$.

Definition 1.7. The *end compactification* of (X, G) is the Stone space \widehat{X} of $\mathcal{H}_{\partial < \infty}(X)$, whose non-principal ultrafilters are the *ends* of (X, G) .

We identify $X \hookrightarrow \widehat{X}$ via principal ultrafilter map $x \mapsto p_x$, so $\varepsilon(X) := \widehat{X} \setminus X$ is the set of ends of G . By definition, \widehat{X} admits a basis of clopen sets of the form $\widehat{A} := \{p \in \widehat{X} : A \in p\}$ for each $A \in \mathcal{H}_{\partial < \infty}(X)$.

Lemma 1.8. A finite-boundary subset $A \in \mathcal{H}_{\partial < \infty}(X)$ is infinite iff it contains an end in (X, G) .

Proof. The converse direction follows since ends are non-principal. If A is infinite, then by local-finiteness of (X, G) , König's Lemma furnishes some infinite ray $(x_n) \subseteq A$. Then, A is contained in the filter

$$p := \{H \in \mathcal{H}_{\partial < \infty}(X) : \forall^\infty n (x_n \in H)\},$$

which is ultra since $H \in p$ are of finite-boundary, and is non-principal since it contains cofinite sets. ■

Definition 1.9. A pocset $\mathcal{H} \subseteq \mathcal{H}_{\partial < \infty}(X)$ is *dense towards ends* of (X, G) if \mathcal{H} contains a neighborhood basis for every end in $\varepsilon(X)$.

In other words, \mathcal{H} is dense towards ends if for every $p \in \varepsilon(X)$ and every (clopen) neighborhood $\widehat{A} \ni p$, where $A \in \mathcal{H}_{\partial < \infty}(X)$, there is some $H \in \mathcal{H}$ with $p \in \widehat{H} \subseteq \widehat{A}$; it is useful to note that $\widehat{H} \subseteq \widehat{A}$ iff $H \subseteq A$, so we will abuse notation and write $p \in H \subseteq A$ for the above condition.

We show in Section 2 that certain half-spaces $\mathcal{H} \subseteq \mathcal{H}_{\partial < \infty}$ induced by a locally-finite quasi-tree is dense towards ends. It will also be important that these half-spaces be cuts, in that witnesses to density can also be found in $\mathcal{H} \cap \mathcal{H}_{\text{conn}}$. The following lemma takes care of this.

Lemma 1.10. If a subpocset $\mathcal{H} \subseteq \mathcal{H}_{\partial < \infty}$ is dense towards ends, then there is a subpocset $\mathcal{H}' \subseteq \mathcal{H}_{\partial < \infty} \cap \mathcal{H}_{\text{conn}}$, which is also dense towards ends, such that every $H' \in \mathcal{H}'$ has $\partial_{\text{ie}} H' \subseteq \partial_{\text{ie}} H$ for some $H \in \mathcal{H}$.

Proof. A first attempt is to let \mathcal{H}' be the connected components H'_0 of elements in \mathcal{H} , but this fails since $\neg H'_0$ is not necessarily connected. Instead, we further take a component of $\neg H'_0$, whose complement clearly co-connected, and is connected since it consists of H'_0 and the other components of $\neg H'_0$, each of which is connected to H'_0 via $\partial_{\text{ie}} H'_0$. Formally, we let

$$\mathcal{H}' := \{H' \subseteq X : H \in \mathcal{H} \text{ and } H'_0 \in H/G \text{ and } \neg H' \in \neg H'_0/G\},$$

where H/G denotes the G -components of H . Clearly $\partial_{\text{ie}} H' \subseteq \partial_{\text{ie}} H'_0 \subseteq \partial_{\text{ie}} H$, and since $H' \in \mathcal{H}_{\text{conn}}(X)$, it remains to show that \mathcal{H}' is dense towards ends.

Fix an end $p \in \varepsilon(X)$ and a neighborhood $p \in \widehat{A} \in \mathcal{H}_{\partial < \infty}(X)$. Let $B \supseteq \partial_v A$ be finite connected, which can be obtained by adjoining paths between its components. Then $\neg B \in p$ since p is non-principal, so there is $H \in \mathcal{H}$ with $p \in H \subseteq \neg B$. We now use the above recipe to find the desired $H' \in \mathcal{H}'$:

- (1) Since $H \in \mathcal{H}_{\partial < \infty}(X)$, it has finitely-many connected components, so exactly one of them belongs to p , say $p \in H'_0 \subseteq H$. Note that $B \subseteq \neg H \subseteq \neg H'_0$.
- (2) Since B is connected, there is a unique component $\neg H' \subseteq \neg H'_0$ containing B .

Observe that $H' \in \mathcal{H}'$ and $p \in H'$. Lastly, since H' is connected and is disjoint from $\partial_v A \subseteq B$, and since $H' \subseteq \neg A$ would imply $\neg H' \in p$, this forces $H' \subseteq A$, and hence $p \in H' \subseteq A$ as desired. ■

1.3. Median graphs and projections. Starting from a profinite pocset \mathcal{H} with every non-trivial element isolated, we construct in Section 3.1 its dual median graph $\mathcal{M}(\mathcal{H})$.

We devote this section and the next to study some basic properties of median graphs and their projections, which will be used in Section 3.2 to construct a spanning tree certain median graphs. For more comprehensive references of median graphs, and their general theory, see [Rol98] and [Bow22].

Definition 1.11. A *median graph* is a connected graph (X, G) such that for any $x, y, z \in X$, the intersection

$$[x, y] \cap [y, z] \cap [x, z]$$

is a singleton, whose element $\langle x, y, z \rangle$ is called the *median* of x, y, z . Thus we have a ternary median operation $\langle \cdot, \cdot, \cdot \rangle : X^3 \rightarrow X$, and a *median homomorphism* $f : (X, G) \rightarrow (Y, H)$ is a map preserving said operation.

Lemma 1.12. For any $\emptyset \neq A \subseteq X$ and $x \in X$, there is a unique point in $\text{cvx}(A)$ between x and every point in A , called the *projection of x towards A* , denoted $\text{proj}_A(x)$.

Moreover, we have $\bigcap_{a \in A} [x, a] = [x, \text{proj}_A(x)]$, and for any y in this set, we have $\text{proj}_A(y) = \text{proj}_A(x)$.

Proof. To show existence, pick any $a_0 \in A$. Given $a_n \in \text{cvx}(A)$, if there exists $a \in A$ with $a_n \notin [x, a]$, set $a_{n+1} := \langle x, a, a_n \rangle \in \text{cvx}(A)$. Then $a_0 - a_1 - \dots - a_n - x$ for all n , so this sequence terminates in at most $d(a_0, x)$ steps at a point in $\text{cvx}(A)$ between x and every point in A . For uniqueness, if there exist two such points $a, b \in \text{cvx}(A)$, then $x - a - b$ and $x - b - a$, forcing $a = b$.

Finally, if $x - y - \text{proj}_A(x)$ and $a \in A$, then $x - \text{proj}_A(x) - a$ and hence $x - y - a$. Conversely, let $x - y - a$ for all $a \in A$. Since $[y, a] \subseteq [x, a]$ for all a , we see that

$$\text{proj}_A(y) \in \text{cvx}(A) \cap \bigcap_{a \in A} [y, a] \subseteq \text{cvx}(A) \cap \bigcap_{a \in A} [x, a]$$

and hence $\text{proj}_A(y) = \text{proj}_A(x)$ by uniqueness. But since $y - \text{proj}_A(y) - a$, we have $x - y - \text{proj}_A(y)$, and hence $x - y - \text{proj}_A(x)$ as desired. ■

Remark 1.13. It follows from the proof above that for any median homomorphism $f : (X, G) \rightarrow (Y, H)$, we have $f(\text{proj}_A(x)) = \text{proj}_{f(A)}(f(x))$ for any $\emptyset \neq A \subseteq X$ and $x \in X$. Indeed, we have

$$\text{proj}_A(x) = \langle x, a_m, \dots, \langle x, a_2, \langle x, a_1, a_0 \rangle \rangle \dots \rangle$$

for some $m \leq d(a_0, x)$ and $a_0, \dots, a_m \in A$, and this is preserved by f .

For $A := \{a, b\}$, we have $\text{proj}_A(x) = \langle a, b, x \rangle$, and hence $\text{cvx}(A) = \text{proj}_A(X) = \langle a, b, X \rangle = [a, b]$.

Lemma 1.14. For each $x, y \in X$, $\text{cone}_x(y)$ is convex, and if xGy , then $\text{cone}_x(y) \sqcup \text{cone}_y(x) = X$.

Proof. Fix $a, b \in \text{cone}_x(y)$ and $a - c - b$. It suffices to show that $x - y - \langle a, c, x \rangle$, for then $x - y - c$ since we have $x - \langle a, c, x \rangle - c$. Indeed, it follows from the following observations.

- $x - y - \langle a, b, x \rangle$, since $\langle a, b, x \rangle = \text{proj}_{\{a, b\}}(x)$ and so $[x, \langle a, b, x \rangle] = [x, a] \cap [x, b] \ni y$ by Lemma 1.12.
- $x - \langle a, b, x \rangle - \langle a, c, x \rangle$, which follows from $\langle a, b, x \rangle - \langle a, c, x \rangle - a$, since $x - \langle a, b, x \rangle - a$ by definition. Indeed, we have $\langle a, c, x \rangle$ is in both $[a, x]$ and $[a, c] \subseteq [a, b]$, and since $\text{proj}_{\{b, x\}}(a) = \langle a, b, x \rangle$, we have again by Lemma 1.12 that $[\langle a, b, x \rangle, a] = [a, x] \cap [a, b] \ni \langle a, c, x \rangle$.

Finally, take $z \in X$ and consider $w := \langle x, y, z \rangle \subseteq [x, y]$. Either $w = x$ or $w = y$ (but not both), giving us the desired partition. ■

Remark 1.15. In particular, this shows that if xGy , then $\text{cone}_x(y) \in \mathcal{H}_{\text{cvx}}^*(X)$. The convexity of cones also shows, in the situation of Lemma 1.12, that $\text{proj}_A = \text{proj}_{\text{cvx}(A)}$, i.e., $\text{proj}_A(x)$ is also between x and every point in $\text{cvx}(A)$: indeed, note that $\text{cone}_x(\text{proj}_A(x))$ is convex and contains A , so it contains $\text{cvx}(A)$ too.

Lemma 1.16. $\text{proj}_A : X \rightarrow \text{cvx}(A)$ is a median homomorphism with $\text{proj}_A \circ \text{cvx} = \text{cvx} \circ \text{proj}_A$.

Proof. The second claim follows from the first since, by Remark 1.13, we have

$$f(\text{cvx}(B)) = f(\text{proj}_B(X)) = \text{proj}_{f(B)}(f(X)) = \text{cvx}(f(B))$$

for all median homomorphisms $f : X \rightarrow Y$ and $B \subseteq X$, so it in particular applies to $f := \text{proj}_A$.

To this end, let $x - y - z \in X$ and set $w := \langle \text{proj}_A(x), \text{proj}_A(y), \text{proj}_A(z) \rangle \in \text{cvx}(A)$. It suffices to show that $y - w - a$ for all $a \in A$, for then $w = \text{proj}_A(y)$ and hence $\text{proj}_A(x) - \text{proj}_A(y) - \text{proj}_A(z)$. But we have $y - \text{proj}_A(y) - a$ already, so it further suffices to show that $y - w - \text{proj}_A(y)$. For this, we note that

$$x - \text{proj}_A(x) - \text{proj}_A(y) \quad \text{and} \quad \text{proj}_A(x) - w - \text{proj}_A(y),$$

so $x \text{---} w \text{---} \text{proj}_A(y)$, and similarly $z \text{---} w \text{---} \text{proj}_A(y)$. Thus, it follows that

$$\begin{aligned} w \in [\text{proj}_A(y), x] \cap [\text{proj}_A(y), z] &= [\text{proj}_A(y), \text{proj}_{\{x,z\}}(\text{proj}_A(y))] && \text{Lemma 1.12} \\ &= [\text{proj}_A(y), \text{proj}_{[x,z]}(\text{proj}_A(y))] && \text{Remark 1.15} \\ &\subseteq [\text{proj}_A(y), y], && \text{Lemma 1.12} \end{aligned}$$

where the second equality follows from $\text{cvx}(\{x, z\}) = [x, z]$, and hence $\text{proj}_{\{x,z\}} = \text{proj}_{[x,z]}$. \blacksquare

1.4. Convex half-spaces of median graphs. We now use projections to explore the geometry of convex half-spaces in median graphs. For the axiomatics of convex structures, see [vdV93].

Proposition 1.17. *Each edge $(x, y) \in G$ is on a unique hyperplane, namely the inward boundary of $\text{cone}_x(y)$, and conversely, each half-space $H \in \mathcal{H}_{\text{cvx}}^*(X)$ is $\text{cone}_x(y)$ for every $(x, y) \in \partial_{\text{ie}}H$.*

Thus, hyperplanes are equivalence classes of edges. Furthermore, this equivalence relation is generated by parallel sides of squares (i.e., 4-cycles).

Proof. We have $\text{cone}_x(y) \in \mathcal{H}_{\text{cvx}}^*(X)$ by Lemma 1.14, and clearly $(x, y) \in \partial_{\text{ie}}\text{cone}_x(y)$. Conversely, take $H \in \mathcal{H}_{\text{cvx}}^*(X)$ and any $(x, y) \in \partial_{\text{ie}}H$. Then $H = \text{cone}_x(y)$, for if $z \in H \cap \neg\text{cone}_x(y)$, then $z \in \text{cone}_y(x)$, and hence $x \in [y, z] \subseteq H$ by convexity of H , a contradiction; if $z \in \text{cone}_x(y) \cap \neg H$, then $[x, z] \subseteq \neg H$ by convexity of $\neg H$, and hence $y \notin H$, a contradiction.

Finally, parallel edges of a strip of squares generate the same hyperplane since, for a given square, each vertex is between its neighbors and hence any hyperplane containing an edge contains its opposite edge. On the other hand, let $(a, b), (c, d) \in \partial_{\text{ie}}H$ for some $H \in \mathcal{H}_{\text{cvx}}^*(X)$. Note that $\partial_{\text{ov}}H = \text{proj}_{\neg H}(H)$ is convex since H is, and since $\text{proj}_{\neg H}$ preserves convexity by Lemma 1.16, any geodesic between $a, c \in \partial_{\text{ov}}H$ lies in $\partial_{\text{ov}}H$. Matching this geodesic via $\partial_{\text{ie}}H : \partial_{\text{ov}}H \rightarrow \partial_{\text{iv}}H$ gives us a geodesic between b, d in $\partial_{\text{iv}}H$, which together with the matching forms the desired strip of squares. \blacksquare

Corollary 1.18. *Two half-spaces $H, K \in \mathcal{H}_{\text{cvx}}^*(X)$ are non-nested iff there is an embedding $\{0, 1\}^2 \hookrightarrow X$ of the Hamming cube into the four corners $\neg^i H \cap \neg^j K$.*

In particular, if $H, K \in \mathcal{H}_{\text{cvx}}^(X)$ are non-nested, then $\partial_{\text{v}}H \cap \partial_{\text{v}}K \neq \emptyset$.*

Proof. Let H, K be non-nested and take $x_1 \in H \cap K$ and $x_2 \in H \cap \neg K$. Since H is connected, any geodesic between x_1, x_2 crosses an edge $(x'_1, x'_2) \in \partial_{\text{oe}}K$ in H . Similarly, there is an edge $(y'_1, y'_2) \in \partial_{\text{oe}}K$ in $\neg H$, so we may slide both edges along $\partial_{\text{oe}}K$ to obtain the desired square (see Proposition 1.17).

Conversely, the half-spaces cutting the square are clearly non-nested. \blacksquare

Lemma 1.19 (Helly). *Any finite intersection of pairwise-intersecting non-empty convex sets is non-empty.*

Proof. For pairwise-intersecting convex sets H_1, H_2, H_3 , pick any $x \in H_1 \cap H_2$, $y \in H_1 \cap H_3$ and $z \in H_2 \cap H_3$; their median $\langle x, y, z \rangle$ then lies in $H_1 \cap H_2 \cap H_3$.

Suppose that it holds for some $n \geq 3$ and let $H_1, \dots, H_{n+1} \subseteq X$ pairwise-intersect. Then $\{H_i \cap H_{n+1}\}_{i \leq n}$ is a family of n pairwise-intersecting convex sets, so $\bigcap_{i \leq n+1} H_i = \bigcap_{i \leq n} (H_i \cap H_{n+1})$ is non-empty. \blacksquare

Lastly, we have some useful finiteness conditions on convex half-spaces; the former implies that $\mathcal{H}_{\text{cvx}}(X)$ is finitely-separating, and the latter allows us to replace finite sets with their convex hulls.

Lemma 1.20. *Any two disjoint convex sets $\emptyset \neq A, B \subseteq X$ can be separated by a half-space $A \subseteq H \subseteq \neg B$, and furthermore we have $d(A, B) = |\{H \in \mathcal{H}_{\text{cvx}}(X) : A \subseteq H \subseteq \neg B\}|$.*

Proof. Pick a geodesic $A \ni x_0 G x_1 G \dots G x_n \in B$, where $n := d(A, B)$. Then $H := \text{cone}_{x_1}(x_0)$, which is a half-space by Lemma 1.14, separates A, B since $x_0 = \text{proj}_A(x_n)$, and thus we have $A \subseteq \text{cone}_{x_n}(x_0) \subseteq \text{cone}_{x_1}(x_0)$ and $B \subseteq \text{cone}_{x_0}(x_n) \subseteq \text{cone}_{x_0}(x_1)$.

Moreover, each such half-space $A \subseteq H \subseteq \neg B$ satisfies $x_i \in H \not\subseteq x_{i+1}$ for a unique $i < n$, and conversely each pair (x_i, x_{i+1}) has a unique half-space separating them, so we have the desired bijection. \blacksquare

Lemma 1.21. *Every interval $[x, y]$ is finite. More generally, if $A \subseteq X$ is finite, then so is $\text{cvx}(A)$.*

Proof. The singletons $\{x\}$ and $\{y\}$ are convex, so there are finitely-many half-spaces $H \subseteq [x, y]$. But each $z \in [x, y]$ is determined uniquely by those half-spaces containing it, so $[x, y]$ is finite.

Let $A := \{x_0, \dots, x_n\}$. Since $\text{cvx}(A) = \text{proj}_A(X)$, we have by Remark 1.13 that points in $\text{cvx}(A)$ are of the form $\langle x, x_n, \dots, \langle x, x_2, \langle x, x_1, x_0 \rangle \rangle \dots \rangle$, which is finite by induction using that intervals are finite. \blacksquare

2. GRAPHS WITH DENSE FAMILIES OF CUTS

Let (X, G) be a connected locally-finite quasi-tree, which, in the context of Theorem A, stands for a single component of the locally-finite graphing of the CBER. For Theorem B to apply, we need to first identify a family of finitely-separating cuts therein, and we do so in such a way that the cuts are dense towards ends.

Since X is a quasi-tree, and thus does not have arbitrarily long cycles, we expect that there is some finite bound $R < \infty$ such that the ends in $\varepsilon(X)$ are ‘limits’ of cuts $\mathcal{H} := \mathcal{H}_{\text{diam}(\partial) \leq R}(X) \cap \mathcal{H}_{\text{conn}}(X)$ with boundary diameter bounded by R . We show that this is indeed the case, in the sense that \mathcal{H} is dense towards ends.

Lemma 2.1. *Let $f : (X, G) \rightarrow (Y, T)$ be a coarse-equivalence between connected graphs. For a fixed $H \in \mathcal{H}_{\partial < \infty}(Y)$, $\text{diam}(\partial_v f^{-1}(H))$ is uniformly bounded in terms of $\text{diam}(\partial_v H)$.*

Proof. Since f is bornologous, let $S < \infty$ be such that xGx' implies $d(f(x), f(x')) \leq S$, so that for any $(x, x') \in \partial_v f^{-1}(H)$, there is a path of length $\leq S$ between $f(x) \notin H$ and $f(x') \in H$. Thus both $d(f(x), \partial_v H)$ and $d(f(x'), \partial_v H)$ are bounded by S , so $f(\partial_v f^{-1}(H)) \subseteq \text{Ball}_S(\partial_v H)$ and hence

$$\text{diam}(f(\partial_v f^{-1}(H))) \leq \text{diam}(\partial_v H) + 2S.$$

That f is a coarse-equivalence gives us a uniform bound of $\text{diam}(\partial_v f^{-1}(H))$ in terms of $\text{diam}(\partial_v H)$. \blacksquare

In particular, if $\text{diam}(\partial_v H)$ is itself also uniformly bounded, then so is $\text{diam}(\partial_v f^{-1}(H))$.

Proposition 2.2. *The class of connected locally-finite graphs in which $\mathcal{H}_{\text{diam}(\partial) \leq R}$ is dense towards ends for some $R < \infty$ is invariant under coarse equivalence.*

Proof. Let $(X, G), (Y, T)$ be connected locally-finite graphs, $f : X \rightarrow Y$ be a coarse equivalence with quasi-inverse $g : Y \rightarrow X$, and suppose $\mathcal{H}_{\text{diam}(\partial) \leq S}(Y)$ is dense towards ends for some $S < \infty$. By Lemma 2.1, pick some $R < \infty$ so that for any $H \in \mathcal{H}_{\text{diam}(\partial) \leq S}(Y)$, we have $f^{-1}(H) \in \mathcal{H}_{\text{diam}(\partial) \leq R}(X)$.

Fix an end $p \in \varepsilon(X)$ and a neighborhood $p \in A \in \mathcal{H}_{\partial < \infty}(X)$. We need to find some $B \in \mathcal{H}_{\partial < \infty}(Y)$ such that $f(p) \in B$ and $f^{-1}(B) \subseteq A$, for then $f(p) \in B$ for some $B \supseteq H \in \mathcal{H}_{\text{diam}(\partial) \leq S}(Y)$, and hence we have

$$p \in f^{-1}(H) \subseteq f^{-1}(B) \subseteq A$$

with $f^{-1}(H) \in \mathcal{H}_{\text{diam}(\partial) \leq R}(X)$. For convenience, let $D < \infty$ be the uniform distance $d(1_X, g \circ f)$.

To this end, note that $f(p) \in B$ iff $p \in f^{-1}(B)$. Since $p \in A$, the latter can occur if $|A \triangle f^{-1}(B)| < \infty$, and so we need to find such a $B \in \mathcal{H}_{\partial < \infty}(Y)$ with the additional property that $f^{-1}(B) \subseteq A$.

Attempt 1. Set $B := g^{-1}(A) \in \mathcal{H}_{\partial < \infty}(Y)$. Then $f^{-1}(B) \subseteq \text{Ball}_D(A)$ since if $(g \circ f)(x) \in A$, then

$$d(x, A) \leq d(x, (g \circ f)(x)) \leq d(1_X, g \circ f) = D.$$

By local-finiteness of G , we see that $A \triangle f^{-1}(B) = A \setminus f^{-1}(B)$ is finite, as desired.

However, it is *not* the case that $f^{-1}(B) \subseteq A$. To remedy this, we ‘shrink’ A by D to A' so that $\text{Ball}_D(A') \subseteq A$, and take $B := g^{-1}(A')$ instead. Indeed, $A' := \neg \text{Ball}_D(\neg A) \subseteq A$ works, since $f^{-1}(B) \subseteq \text{Ball}_D(A')$ as before, so $A' \triangle f^{-1}(B) = A' \setminus f^{-1}(B)$ is finite. Also, $A \triangle A'$ is finite since $x \in A \triangle A'$ iff $x \in A$ and $d(x, \neg A) \leq D$, so $A \triangle f^{-1}(B)$ is finite too. It remains to show that $\text{Ball}_D(A') \subseteq A$, for then $f^{-1}(B) \subseteq A$ as desired.

Indeed, if $y \in \text{Ball}_D(A')$, then by the (reverse) triangle-inequality we have $d(y, \neg A) \geq d(x, \neg A) - d(x, y)$ for all $x \in A'$. But $d(x, \neg A) > D$, strictly, so $d(y, \neg A) > D - D = 0$, and hence $y \in A$. \blacksquare

Corollary 2.3. *If (X, G) is a locally-finite quasi-tree, then the subpocset $\mathcal{H}_{\text{diam}(\partial) \leq R}(X) \cap \mathcal{H}_{\text{conn}}(X)$ is dense towards ends for some $R < \infty$.*

Proof. Observe that $\mathcal{H}_{\text{diam}(\partial) \leq 2}(T)$ is dense towards ends for any tree T , so Proposition 2.2 proves the density of $\mathcal{H}_{\text{diam}(\partial) \leq R}(X)$ for some $R < \infty$. By Lemma 1.10, there is a subpocset $\mathcal{H}' \subseteq \mathcal{H}_{\partial < \infty}(X) \cap \mathcal{H}_{\text{conn}}(X)$ dense towards ends such that for every $H' \in \mathcal{H}'$, we have $\partial_e H' \subseteq \partial_e H$ for some $H \in \mathcal{H}_{\text{diam}(\partial) \leq R}(X)$. Hence we have $H' \in \mathcal{H}_{\text{diam}(\partial) \leq R}(X) \cap \mathcal{H}_{\text{conn}}(X)$, so the result follows. \blacksquare

The cuts $\mathcal{H} := \mathcal{H}_{\text{diam}(\partial) \leq R}(X) \cap \mathcal{H}_{\text{conn}}(X)$ obtained here will be our starting point for Theorem B, so we need to show that it is finitely-separating. Indeed, by local-finiteness of (X, G) , the R -ball around any fixed $x \in X$ is finite, so since any cut $H \in \mathcal{H}$ with $x \in \partial_v H$ is contained in said R -ball, there are finitely-many such cuts. Thus, by Lemma 1.3, \mathcal{H} is finitely-separating.

Furthermore, we have by Lemma 1.2 that $\mathcal{H} \subseteq 2^X$ is closed and every non-trivial element is isolated, and those conditions allow for the construction in Theorem B to continue.

Finally, we will need to represent ‘density towards ends’ in a Borel manner, which ultimately is to ensure that the bounds $R < \infty$ can be obtained uniformly across all components; see Section 4 for details.

Proposition 2.4. *If \mathcal{H} is a finitely-separating pocset of cuts on a connected locally-finite graph, then \mathcal{H} is dense towards ends iff (1) each \mathcal{H} -block is finite, and (2) each $H \in \mathcal{H}^*$ has finitely-many successors.*

Proof. If \mathcal{H} is dense towards ends, we will cover (certain closed sets in) $\varepsilon(X)$ by cuts in \mathcal{H} , which cuts down to a finite subcover. We will show that a certain Boolean combination thereof, which has finite-boundary, is finite using Lemma 1.8, giving us the desired finiteness claim. Below are the details.

1. Fix an \mathcal{H} -block $[x]_{\mathcal{H}}$ and let $A := \neg\{x\}$. By density, we can cover each end $p \in \varepsilon(X)$ with some $H_p \subseteq A$, which gives us a finite subcover $\bigcup_{i < n} H_i$ of $\varepsilon(X)$. Note that $\bigcap_{i < n} \neg H_i \in \mathcal{H}_{\partial < \infty}(X)$ contains $[x]_{\mathcal{H}}$, and is finite since it contains no ends in $\varepsilon(X)$.
2. Fix $H \in \mathcal{H}^*$ and let $K_\alpha \in \mathcal{H}^*$ be the successors of H . By density, each end $p \in \varepsilon(\neg H)$ is contained in some $H_p \subseteq \neg H$, which in turn is contained in some $\neg K_\alpha$; this gives us a finite subcover $\bigcup_{i < n} \neg K_i$ of $\varepsilon(\neg H)$. Any other successor $H \subset K \in \mathcal{H}^*$ not in this subcover may be assumed to be nested with each K_i , since there are finitely-many non-nested ones by Lemma 1.6. But then $K \not\subseteq K_i \not\subseteq K$ and $K \cap K_i \neq \emptyset$ for all $i < n$, which forces $\neg K \subseteq \neg H \cap \bigcap_{i < n} K_i \in \mathcal{H}_{\partial < \infty}(X)$; the latter contains no ends in $\varepsilon(X)$, so it is finite, and hence there are finitely-many possibilities for such K .

Conversely, fix an end $p \in \varepsilon(X)$ and a neighborhood $p \in A \in \mathcal{H}_{\partial < \infty}(X)$. Since \mathcal{H} consists of connected sets, it suffices to find some $H \in \mathcal{H}$ containing $\partial_v A$ but not p , for then $p \in \neg H \subseteq A$ as desired.

Observation. Finitely-many $H, \neg H \in \mathcal{H}$ may be removed and still satisfy conditions (1) and (2) above.

That (2) still holds is obvious; for (1), we may remove a single pair, since with $\mathcal{H}' := \mathcal{H} \setminus \{H, \neg H\}$, the map $X/\mathcal{H} \rightarrow X/\mathcal{H}'$ sending $[x]_{\mathcal{H}} \rightarrow [x]_{\mathcal{H}'}$ is surjective and at-most 2-to-1.

Thus, for each of the finitely-many $x, y \in \partial_v A$, we may remove the finitely-many half-spaces separating them, so we may assume that $\mathcal{H} = \mathcal{H}_A \sqcup \neg \mathcal{H}_A$ is partitioned into the half-spaces in \mathcal{H}_A containing $\partial_v A$, and its complements which are disjoint from $\partial_v A$. By contradiction, if each $H \in \mathcal{H}_A$ contains p , we will construct a strictly decreasing chain $(H_n) \subseteq \mathcal{H}_A^*$, which gives an infinite family separating $\partial_v A$ from $\neg H_0$.

Let $H_0 \in \mathcal{H}_A^*$ be arbitrary, which exists since X is infinite, so \mathcal{H} -blocks being finite forces $\mathcal{H}^* = \mathcal{H}_A^* \sqcup \neg \mathcal{H}_A^*$ to be infinite. There are finitely-many half-spaces non-nested with H_0 by Lemma 1.6, so we may assume that there is none. We may further assume that all successors H of $\neg H_0$ lie in \mathcal{H}_A^* , for otherwise $\mathcal{H}_A^* \ni \neg H \subset H_0$ and we may take $H_1 := \neg H$. Thus, $B := H_0 \cap \bigcap_{H \supset \neg H_0} H$ is a *finite* intersection of the minimal ones in \mathcal{H}_A^* , and since $p \in H$ for each such H by assumption, B contains p and is infinite. Since \mathcal{H} -blocks are finite and B is \mathcal{H} -invariant, it contains infinitely-many \mathcal{H} -blocks; any two such \mathcal{H} -blocks is separated by some $H \in \mathcal{H}_A^*$ nested with H_0 , and since $H \not\subseteq H_0^c$ and H contains neither H_0 nor $\neg H_0$, we can set $H_1 := H \subset H_0$. ■

3. THE DUAL MEDIAN GRAPH OF A PROFINITE POCSET AND ITS SPANNING TREES

3.1. Construction of the dual median graph. We present a classical construction in geometric group theory of a median graph associated to a profinite pocset with every non-trivial element isolated; see [Dun79], [Rol98], [Sag95], and [NR03] for other applications.

In the context of Theorem A, this will be applied to the pocset $\mathcal{H}_{\text{diam}(\partial) \leq R}(X) \cap \mathcal{H}_{\text{conn}}(X)$ of cuts in a locally-finite graph (X, G) , and is also the first step in the construction in Theorem B.

Definition 3.1. An *orientation* on \mathcal{H} is an upward-closed subset $U \subseteq \mathcal{H}$ containing exactly one of $H, \neg H$ for each $H \in \mathcal{H}$. We let $\mathcal{U}(\mathcal{H})$ denote the set of all orientations on \mathcal{H} and let $\mathcal{U}^\circ(\mathcal{H})$ denote the clopen ones.

Intuitively, an orientation is a ‘maximally consistent’ choice of half-spaces³.

Example 3.2. Each $x \in X$ induces its *principal orientation* $\hat{x} := \{H \in \mathcal{H} : x \in H\} = \mathcal{H} \cap \pi_x^{-1}(1)$ – which is clearly clopen in \mathcal{H} – and gives us a canonical map $X \rightarrow \mathcal{U}^\circ(\mathcal{H})$. However, this map is *not necessarily* injective, and we call a fiber $[x]_{\mathcal{H}} := \{y \in X : \hat{x} = \hat{y}\}$ thereof an \mathcal{H} -block. This induces an equivalence relation on X by declaring $x \sim_{\mathcal{H}} y$ iff x and y are contained in exactly the same half-spaces in \mathcal{H} .

³This can be formalized by letting \sim be the equivalence relation on \mathcal{H} given by $H \sim \neg H$. Letting $\partial : \mathcal{H} \rightarrow \mathcal{H}/\sim$ denote the quotient map, orientations $U \subseteq \mathcal{H}$ then correspond precisely to sections $\varphi : \mathcal{H}/\sim \rightarrow \mathcal{H}$ of ∂ such that $\varphi(\partial H) \not\subseteq \neg \varphi(\partial K)$ for every $H, K \in \mathcal{H}$; the latter condition rules out ‘orientations’ of the form $\leftarrow | \rightarrow$.

Proposition 3.3. *Let \mathcal{H} be a profinite pocset with every non-trivial element isolated. Then the graph $\mathcal{M}(\mathcal{H})$, whose vertices are clopen orientations $\mathcal{U}^\circ(\mathcal{H})$ and whose edges are pairs $\{U, V\}$ with $V = U \triangle \{H, \neg H\}$ for some minimal $H \in U \setminus \{\emptyset, X\}$, is a median graph with path metric $d(U, V) = |U \triangle V|/2$ and medians*

$$\begin{aligned} \langle U, V, W \rangle &:= \{H \in \mathcal{H} : H \text{ belongs to at least two of } U, V, W\} \\ &= (U \cap V) \cup (V \cap W) \cup (U \cap W). \end{aligned}$$

Proof. First, $V := U \triangle \{H, \neg H\}$ as above is clopen since $H, \neg H \in \mathcal{H}^*$ are isolated (whence $\{H\}, \{\neg H\}$ are clopen), and it is an orientation by minimality of H . That $\mathcal{M}(\mathcal{H})$ is connected follows from the following claim by noting that $U \triangle V \not\equiv \emptyset, X$ is clopen, so it is a compact set of isolated points, whence finite.

Claim ([Sag95, Theorem 3.3]). *There is a path between U, V iff $U \triangle V$ is finite, in which case*

$$d(U, V) = |U \triangle V|/2 = |U \setminus V| = |V \setminus U|.$$

Proof. If $(U_i)_{i < n}$ is a path from $U =: U_0$ to $V =: U_{n-1}$, then, letting $\{H_i, \neg H_i\} := U_i \triangle U_{i-1}$ for all $1 \leq i < n$ gives us a sequence $(H_i)_{i < n}$ inducing^a this path, whence $U \triangle V$ consists of $\{H_i\}_{i < n}$ and their complements. Thus $U \triangle V = 2n = 2d(U, V)$, as desired.

Conversely, if $U \triangle V = \{H_1, \dots, H_n\} \sqcup \{K_1, \dots, K_m\}$ with $U \setminus V = \{H_i\}$ and $V \setminus U = \{K_j\}$, then $\neg H_i \in V \setminus U$ and $\neg K_j \in U \setminus V$ for all $i < n$ and $j < m$, so $n = m$ and $V = U \cup \{\neg H_i\} \setminus \{H_i\}$. We claim that there is a permutation $\sigma \in S_n$ such that $(H_{\sigma(i)})$ induces a path from $U =: U_0$, which is the desired path from U to V . Choose a minimal $H \in \{H_i\}$, which is also minimal in U : if $K \subseteq H$ for some $K \in U$, then $\neg H \subseteq \neg K$, and hence $\neg K \in V$, so $K = H_i \subseteq H$ for some i , forcing $K = H$. Set $U_1 := U \triangle \{H, \neg H\}$, which is a clopen orientation. Continuing in this manner by choosing a minimal element in $\{H_i\} \setminus \{H\}$ – and so on – gives us the desired path with $d(U, V) = n$. \square

^aIn the sense that $U_i = U_{i-1} \triangle \{H_i, \neg H_i\}$ and $H_i \in U_i$ for each $1 \leq i < n$; see [Tse20, Definition 2.20].

Finally, we show that $\mathcal{M}(\mathcal{H})$ is a median graph. Fix $U, V, W \in \mathcal{U}^\circ(\mathcal{H})$, and note that for any $M \in \mathcal{U}^\circ(\mathcal{H})$, we have by the triangle inequality that $M \in [U, V]$ iff $(U \setminus M) \cup (M \setminus V) \subseteq U \setminus V$, which clearly occurs iff $U \cap V \subseteq M \subseteq U \cup V$. Thus, a vertex M lies in the triple intersection $[U, V] \cap [V, W] \cap [U, W]$ iff

$$(U \cap V) \cup (V \cap W) \cup (U \cap W) \subseteq M \subseteq (U \cup V) \cap (V \cup W) \cap (U \cup W).$$

Note that the two sides coincide, so $M = \langle U, V, W \rangle$ – which is clopen if U, V, W are – is as claimed. \blacksquare

Given such a pocset \mathcal{H} , the graph $\mathcal{M}(\mathcal{H})$ constructed above is called the *dual*⁴ median graph of \mathcal{H} . An important special case of this construction is when \mathcal{H} is *nested*, in which case $\mathcal{M}(\mathcal{H})$ is a tree.

Corollary 3.4. *Let \mathcal{H} be a profinite pocset with non-trivial points isolated. If \mathcal{H} is nested, then the median graph $\mathcal{M}(\mathcal{H})$ is acyclic, and hence $\mathcal{M}(\mathcal{H})$ is a tree.*

Proof. Let $(U_i)_{i < n}$ be a (non-backtracking) cycle in $\mathcal{M}(\mathcal{H})$, say induced by some sequence $(H_i)_{i < n} \subseteq \mathcal{H}$ of half-spaces. We show that (H_i) is *strictly* increasing, so that $H_0 \subset \dots \subset H_n \subset H_0$, which is absurd.

We have $U_{i+1} = U_{i-1} \cup \{\neg H_i, \neg H_{i-1}\} \setminus \{H_i, H_{i-1}\}$, so $H_i \neq \neg H_{i-1}$ (for otherwise $U_{i+1} = U_{i-1}$). Since $H_i \in U_i = U_{i-1} \cup \{\neg H_{i-1}\} \setminus \{H_{i-1}\}$, we see that $H_i \in U_{i-1}$, and since $H_i \neq H_{i-1}$, it suffices by nestedness of \mathcal{H} to remove the three cases when $\neg H_i \subseteq H_{i-1}$, $H_{i-1} \subseteq \neg H_i$, and $H_i \subseteq H_{i-1}$.

Indeed, if $\neg H_i \subseteq H_{i-1}$, then $H_{i-1} \in U_{i+1}$ by upward-closure of $U_{i+1} \ni \neg H_i$. But since $H_{i-1} \neq \neg H_i$, we have by definition of U_{i+1} that $H_{i-1} \in U_i$, a contradiction. The other cases are similar. \blacksquare

Nonetheless, in the general non-nested case, $\mathcal{M}(\mathcal{H})$ still admits a *canonical* spanning tree if we fix a Borel colouring of $\mathcal{H}_{\text{cvx}}^*(\mathcal{M}(\mathcal{H}))$ into its nested sub-pocsets, the existence of which follows from the following

Proposition 3.5. *The dual median graph $\mathcal{M}(\mathcal{H})$ of a pocset of finitely-separating cuts has finite hyperplanes.*

Proof. Fix $K \in \mathcal{H}_{\text{cvx}}^*(\mathcal{M}(\mathcal{H}))$, which by Proposition 1.17 is of the form $K = \text{cone}_V(U)$ for some (and hence any) $(U, V) \in \partial_{\text{ie}} K$, and we have by Proposition 3.3 that $V = U \triangle \{H, \neg H\}$ for some (non-trivial) minimal $H \in U$. We claim that any other edge $(U', V') \in \partial_{\text{ie}} K$ can be reached from (U, V) by simultaneously flipping only the half-spaces $H'_0, \dots, H'_n \in \mathcal{H}^*$ non-nested with H , of which there are finitely-many by Lemma 1.6.

⁴The name is justified by a Stone-type duality between $\{\text{median graphs, median homomorphisms}\}$ and $\{\text{profinite pocsets with non-trivial points isolated, continuous maps}\}$, where from a median graph X one can construct a canonical pocset $\mathcal{H}_{\text{cvx}}(X)$ of convex half-spaces (see [CPTT23, Section 2.D] for details).

Since the edges $(U, V), (U', V')$ induce the same hyperplane $\partial_e K$, it suffices by Proposition 1.17 to prove this for when (U', V') is an edge of a square parallel to (U, V) , in which case there is some minimal $H' \in \mathcal{H}^*$ flipping both U to U' and V to V' . Note that H, H' are non-nested since $H' \not\subseteq H$ and $H \not\subseteq H'$ by minimality; if $H \subseteq \neg H'$, then $\neg H' \in U$; and if $\neg H \subseteq H'$, then H' is not minimal in U . Moreover, $H \in U'$ is still minimal since $U' = U \triangle \{H', \neg H'\}$, and the only way this can fail is if $\neg H' \subseteq H$, contradicting minimality of H . Thus we have $V' = U' \triangle \{H, \neg H\}$, so the induction continues with (U', V') in place of (U, V) . ■

3.2. Canonical spanning trees. We now present the Borel cycle-cutting algorithm that can be preformed on any countable median graph with finite hyperplanes. Applying this algorithm to the dual median graph of a finitely-separating family of cuts, which has finite hyperplanes by Proposition 3.5, proves Theorem B.

Lemma 3.6. *For any subpocset $\mathcal{H} \subseteq \mathcal{H}_{\text{cvx}}(X)$ on a median graph (X, G) , the map $X \rightarrow \mathcal{U}^o(\mathcal{H})$ is surjective.*

Proof. Let $U \in \mathcal{U}^o(\mathcal{H})$, we need to find some $x \in X$ with $U = \hat{x}$. Since $U \subseteq \mathcal{H}$ is clopen, there is a finite set $A \subseteq X$ – which we may assume to be convex by Lemma 1.21 – such that for all $H \in \mathcal{H}$, we have $H \in U$ iff there is $H' \in U$ with $H \cap A = H' \cap A$. Note that $H \cap A \neq \emptyset$ for every $H \in U$, since otherwise $\emptyset \in U$. Furthermore, $H \cap H' \neq \emptyset$ for every $H, H' \in U$, since otherwise we have $H \subseteq \neg H'$, and so $\neg H' \in U$.

By Lemma 1.19, the intersection $(H \cap A) \cap (H' \cap A) = H \cap H' \cap A$ is non-empty, and applying it again furnishes some $x \in \bigcap_{H \in U} H \cap A$ in X . Thus $U \subseteq \hat{x}$, so $U = \hat{x}$ since both are orientations. ■

This induces a G -adjacency graph $X/\mathcal{H} \cong \mathcal{M}(\mathcal{H})$; explicitly, two \mathcal{H} -blocks $([x]_{\mathcal{H}}, [y]_{\mathcal{H}})$ are G -adjacent if $(\hat{x}, \hat{y}) \in \mathcal{M}(\mathcal{H})$. Note that $\mathcal{M}(\mathcal{H})$ may be constructed as in Proposition 3.3 since $\mathcal{H} \subseteq \mathcal{H}_{\text{cvx}}(X)$ is finitely-separating by Lemma 1.20. In particular, if \mathcal{H} is nested, then X/\mathcal{H} is a tree by Corollary 3.4.

Proposition 3.7. *If (X, G) is a countable median graph with finite hyperplanes, then fixing any colouring of $\mathcal{H}_{\text{cvx}}^*(X)$ into nested sub-pocsets yields a canonical spanning tree thereof.*

Proof. Such a colouring exists, since, by Corollary 1.18, if two half-spaces $H, K \in \mathcal{H}_{\text{cvx}}^*(X)$ are non-nested, then $\partial_v H \cap \partial_v K \neq \emptyset$. Thus, the intersection graph of the boundaries admits a countable colouring, which descends into a colouring $\mathcal{H}_{\text{cvx}}^*(X) = \bigsqcup_{n \in \mathbb{N}} \mathcal{H}_n^*$ such that each $H, \neg H$ receive the same colour and that each $\mathcal{H}_n := \mathcal{H}_n^* \cup \{\emptyset, X\}$ is a nested subpocset. For each $n \in \mathbb{N}$, let $\mathcal{K}_n := \bigcup_{m \geq n} \mathcal{H}_m$.

We shall inductively construct an increasing chain of subforests $T_n \subseteq \bar{G}$ such that the components of T_n are exactly the \mathcal{K}_n -blocks. Then, the increasing union $T := \bigcup_n T_n$ is a spanning tree, since each $(x, y) \in G$ lies in a \mathcal{K}_n -block for sufficiently large n (namely, the n such that $\text{cone}_x(y) \in \mathcal{H}_{n-1}^*$, since $\text{cone}_x(y)$ and its complement are the only half-spaces separating x and y by Proposition 1.17).

Since each pair of distinct points is separated by a half-space, the \mathcal{K}_0 -blocks are singletons, so put $T_0 := \emptyset$. Suppose that a forest T_n is constructed as required. Note that each \mathcal{K}_{n+1} -block $Y \in X/\mathcal{K}_{n+1}$ is not separated by any half-spaces in \mathcal{H}_m for $m > n$, but is separated by \mathcal{H}_n into the \mathcal{K}_n -blocks contained in Y , which are precisely the \mathcal{H}_n -blocks in Y/\mathcal{H}_n . Pick an edge from the finite hyperplane $\partial_e H$ for each $H \in \mathcal{H}_n$, which connects a unique pair of G -adjacent blocks in Y/\mathcal{H}_n . Since each Y/\mathcal{H}_n is a tree by Corollary 3.4, and each pair of G -adjacent blocks in Y/\mathcal{H}_n is connected by a single picked edge, the graph T_{n+1} obtained from T_n by adding all such edges is a forest whose components are exactly the \mathcal{K}_{n+1} -blocks. ■

4. BOREL TREEINGS OF GRAPHINGS WITH DENSE CUTS

We finally prove Theorem A, stating that if a CBER (X, E) admits a locally-finite graphing G such that each component is a quasi-tree, then E is treeable. The first step is to identify, for each component $G|C$, a family $\mathcal{H}(C)$ of finitely-separating cuts that is dense towards ends of $G|C$; since each $G|C$ is a quasi-tree, the cuts $\mathcal{H}(C) := \mathcal{H}_{\text{diam}(\partial) \leq R_C}(C) \cap \mathcal{H}_{\text{conn}}(C)$ for some $R_C < \infty$ from Section 2 will do. Applying Theorem B then gives us, for each component $G|C$, a median graph $\mathcal{M}(\mathcal{H}(C))$ on $\mathcal{U}^o(\mathcal{H}(C))$ with finite hyperplanes.

The issue lies in making the family $\mathcal{U}^o(\mathcal{H}) := \bigsqcup_C \mathcal{U}^o(\mathcal{H}(C))$ of all clopen orientations on $\mathcal{H} := \bigsqcup_C \mathcal{H}(C)$ into a standard Borel space. If $\mathcal{U}^o(\mathcal{H})$ is standard Borel, the above partition induces a CBER \mathcal{E} admitting a median graphing $\mathcal{M}(\mathcal{H}) := \bigsqcup_C \mathcal{M}(\mathcal{H}(C))$ with finite hyperplanes, from which one can implement the proof of Proposition 3.7 in a Borel manner (using [KM04, Lemma 7.3] for a countable colouring of the intersection graph of finite hyperplanes therein) to obtain a treeing of \mathcal{E} . Finally, E is Borel bireducible with \mathcal{E} via the principal orientations map $X \ni x \mapsto \hat{x} \in \mathcal{U}^o(\mathcal{H})$, so E is also treeable by [JKL02, Proposition 3.3 (ii)].

We will remedy this issue using the fact that the cuts $\mathcal{H}(C)$ are dense towards ends of $G|C$. In particular, we have the following crucial lemma, which, by Proposition 3.3, shows that $\mathcal{M}(\mathcal{H}(C))$ is locally-finite.

Lemma 4.1. *Let \mathcal{H} be a finitely-separating pocset of cuts on a connected locally-finite graph (X, G) . If \mathcal{H} is dense towards ends, then each clopen orientation $U \in \mathcal{U}^\circ(\mathcal{H})$ contains finitely-many minimal cuts $H \in \mathcal{H}$.*

Proof. Fix a vertex $U \in \mathcal{U}^\circ(\mathcal{H})$ and let $K_\alpha \in U$ be its minimal elements. Since $U \subseteq \mathcal{H}$ is clopen, there is a finite set $A \subseteq X$ such that for all $H \in \mathcal{H}$, we have $H \in U$ iff there is $H' \in U$ with $H \cap A = H' \cap A$. Note that $H \cap A \neq \emptyset$ for every $H \in U$, for otherwise $\emptyset \in U$; in particular, we have $\neg H \in U$ for every $H \subseteq \neg A$.

Each end $p \in \varepsilon(X)$ lies in $\neg A$, so density of \mathcal{H} furnishes $H_p \in \mathcal{H}$ with $p \in H_p \subseteq \neg A$, and thus $\neg H_p \in U$. Since U is clopen, we have $K_\alpha \subseteq \neg H_p$ for some α . Thus $\varepsilon(X) \subseteq \bigcup_\alpha \neg K_\alpha$, which by compactness cuts down to a finite subcover $\varepsilon(X) \subseteq \bigcup_{i < n} \neg K_i$. We show that there are at-most finitely-many more minimal $K \in U$.

Let $K \neq K_i$ be any other minimal element in U . By Lemma 1.6, each K_i is non-nested with finitely-many other half-spaces, so we may assume that K is nested with every K_i . But then $K \not\subseteq K_i \not\subseteq K$ and $K \cap K_i \neq \emptyset$ for all $i < n$, which forces $\neg K \subseteq \bigcap_{i < n} K_i \in \mathcal{H}_{\partial < \infty}(X)$; the latter contains no ends in $\varepsilon(X)$, so it is finite by Lemma 1.8, and hence there are at-most finitely-many more minimal $K \in U$. ■

We now describe the encoding of $\mathcal{U}^\circ(\mathcal{H})$ into a standard Borel space. Since cuts have finite edge boundary, we may first represent the space \mathcal{K} of all non-trivial cuts of G as a Borel subset of $[G]^{<\infty}$. The subcollection $\mathcal{H} \subseteq \mathcal{K}$ consisting of those cuts with component-wise bounded boundary diameter is also Borel, since each $R_C < \infty$ can be witnessed as the minimal number making $\mathcal{H}_{\text{diam}(\partial) \leq R_C}(C)$ dense towards ends of $G|C$, and the latter is a Borel condition as characterized in Proposition 2.4. Finally, $\mathcal{U}^\circ(\mathcal{H})$ is a Borel subset of $[\mathcal{H}]^{<\infty}$, since we may encode each clopen orientation $U \in \mathcal{U}^\circ(\mathcal{H}(C))$ by its set of minimal elements in $\mathcal{H}(C)$, which is finite by Lemma 4.1. This makes $\mathcal{U}^\circ(\mathcal{H})$ a standard Borel space, and finishes the proof of Theorem A.

The above discussion actually proves the following generalization of Theorem A, which is no longer about quasi-trees; rather, we only require that the (locally-finite) graphing admits a Borel family of ‘tree-like’ cuts.

Theorem 4.2. *If a CBER (X, E) admits a locally-finite graphing G such that each component $G|C$ admits a family $\mathcal{H}(C)$ of finitely-separating cuts that is dense towards ends of $G|C$, and if $\mathcal{H} := \bigsqcup_C \mathcal{H}(C)$ is a Borel subset of the standard Borel space of all cuts of G , then E is treeable.*

REFERENCES

- [CPTT23] Ruiyuan Chen, Antoine Poulin, Ran Tao, and Anush Tserunyan, *Tree-like graphings, wallings, and median graphings of equivalence relations* (2023), available at <https://arxiv.org/abs/2308.13010>.
- [Gro93] Mikhail Gromov, *Asymptotic invariants of infinite groups*, Geometric Group Theory, vol. 2 (Sussex, 1991), London Mathematical Society Lecture Note Series, vol. 182, Cambridge University Press, 1993.
- [DK18] Cornelia Druţu and Michael Kapovich, *Geometric Group Theory*, with appendix by Bogdan Nica, Vol. 63, American Mathematical Society Colloquium Publications, 2018.
- [Nic04] B. Nica, *Cubulating Spaces with Walls*, Algebraic & Geometric Topology **4** (2004), 297-309.
- [CN05] I. Chatterji and G.A. Niblo, *From Wall Spaces to CAT(0) Cube Complexes*, Internat. J. Algebra Comput. **15** (2005), no. 5-6, 875-885.
- [Rol98] Martin Roller, *Poc sets, median algebras and group actions*, Universität Regensburg (1998), available at <https://arxiv.org/abs/1607.07747>. Ph.D. thesis.
- [Bow22] Brian H. Bowditch, *Median Algebras* (2022), available at <https://homepages.warwick.ac.uk/~masgak/papers/median-algebras.pdf>.
- [vdV93] M. L. J. van de Vel, *Theory of Convex Structures*, North-Holland Mathematical Library, vol. 50, North-Holland Publishing Co., Amsterdam, 1993.
- [Dun79] M. J. Dunwoody, *Accessibility and Groups of Cohomological Dimension One*, Proceedings of the London Mathematical Society (3) **38** (1979), no. 2, 193-215.
- [Sag95] Michah Sageev, *Ends of Group Pairs and Non-Positively Curved Cube Complexes*, Proceedings of the London Mathematical Society (3) **71** (1995), no. 3, 585-617.
- [NR03] G.A. Niblo and L.D. Reeves, *Coxeter Groups act on CAT(0) cube complexes*, Journal of Group Theory **6** (2003), no. 3, 399-413.
- [Tse20] Anush Tserunyan, *A descriptive construction of trees and Stallings’ theorem*, Trends in Set Theory, Contemporary Mathematics **752** (2020), 191-207.
- [KM04] Alexander S. Kechris and Benjamin D. Miller, *Topics in Orbit Equivalence*, Lecture Notes in Mathematics, vol. 1852, Springer-Verlag Berlin, Heidelberg, 2004.
- [JKL02] S. Jackson, Alexander S. Kechris, and A. Louveau, *Countable Borel Equivalence Relations*, Journal of Mathematical Logic **2** (2002), no. 1, 1-80.

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY, 805 SHERBROOKE STREET WEST, MONTREAL, QC, H3A 0B9, CANADA

Email address: zhaoshen.zhai@mail.mcgill.ca