

# TREE-LIKE GRAPHINGS OF COUNTABLE BOREL EQUIVALENCE RELATIONS

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**ABSTRACT.** We present a streamlined exposition of a construction presented recently by R. Chen, A. Poulin, R. Tao, and A. Tserunyan, which establishes the treeability of the connectedness relation of every locally-finite Borel graph where each component is a quasi-tree. **Write some more details...**

The purpose of this note is to provide a streamlined proof of a particular case of a construction presented in [CPTT23], in order to better understand the general formalism developed therein. We attempt to make this note self-contained, but nevertheless urge the reader to refer to the original paper for more detailed discussions and some generalizations of the results we have selected to include here.

**Treeings of equivalence relations.** A *countable Borel equivalence relation (CBER)* on a standard Borel space  $X$  is a Borel equivalence relation  $E \subseteq X^2$  with each class countable. We are interested in special types of *graphings* of a CBER  $E \subseteq X^2$ , i.e. a Borel graph  $G \subseteq X^2$  whose connectedness relation is precisely  $E$ . For instance, a graphing of  $E$  such that each component is a tree is called a *treeing* of  $E$ , and the CBERs that admit treeings are said to be *treeable*. The main results of [CPTT23] provide new sufficient criteria for treeability of certain classes of CBERs, and in particular, they prove the following

**Theorem A** ([CPTT23, Theorem 1.1]). *If a CBER  $E$  admits a locally-finite graphing such that each component is a quasi-tree<sup>1</sup>, then  $E$  is treeable.*

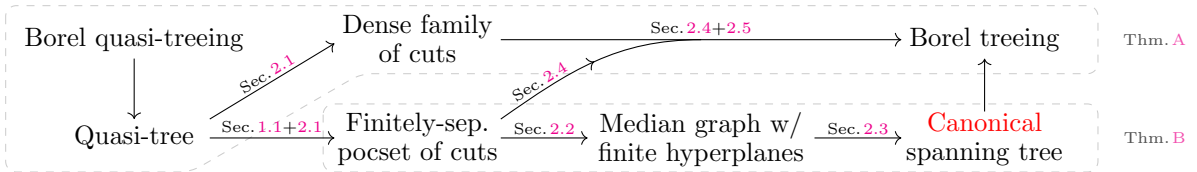
Roughly speaking, the existence of a quasi-isometry  $G|_C \rightarrow T_C$  to a simplicial tree  $T_C$  for each component  $C \subseteq X$  induces a collection  $\mathcal{H}(C) \subseteq 2^C$  of ‘cuts’ (subsets  $H \subseteq C$  with finite boundary such that both  $H$  and  $C \setminus H$  are connected), which are ‘tree-like’ in the sense that

1.  $\mathcal{H}(C)$  is *finitely-separating*: each pair  $x, y \in C$  is separated by finitely-many  $H \in \mathcal{H}(C)$ , and
2.  $\mathcal{H}(C)$  is *dense towards ends*: each end in  $G|_C$  has a neighborhood basis in  $\mathcal{H}(C)$ .

Condition (1) allows for an abstract (i.e., *not necessarily* Borel) but **canonical** construction of a tree from  $\mathcal{H}(C)$ . This construction starts by viewing  $\mathcal{H}(C)$  as a profinite pocset with non-trivial points isolated (i.e., a ‘walling’), which in turn provide exactly the data to construct a ‘median graph’  $\mathcal{M}(\mathcal{H}(C))$  whose vertices are special ‘ultrafilters’<sup>2</sup> on  $\mathcal{H}(C)$ . This median graph has finite ‘hyperplanes’, which allows us to apply a Borel ‘cycle-cutting’ algorithm and obtain a **canonical** spanning tree.

Thus we have the following theorem, which can be viewed as a component-wise version of Theorem A. In fact, the above construction can be done to *any* finitely-separating family of cuts on a connected locally-finite graph, not just a quasi-tree, but those cuts will not necessarily be dense towards ends.

**Theorem B.** *For any finitely-separating family of cuts on a connected locally-finite graph, its dual median graph admits a **canonical** spanning tree.*



The additional condition (2), which equivalently imposes certain finiteness conditions on the median graph  $\mathcal{M}(\mathcal{H}(C))$ , then ensures that such a spanning tree can be constructed in a uniform way to each component  $C \subseteq G$ . This gives us the desired treeing of the CBER, which we prove in Section 2.5.

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<sup>1</sup>Recall that metric spaces  $X$  and  $Y$  are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error, and  $X$  is a *quasi-tree* if it is quasi-isometric to a simplicial tree; see [Gro93] and [DK18].

<sup>2</sup>As in [CPTT23], we call them *orientations* instead, to avoid confusion with the more standard notion; see Definition 2.4.

## 1. PRELIMINARIES ON POCSSETS, ENDS OF GRAPHS, AND MEDIAN GRAPHS

**Notation.** A *graph* on a set  $X$  is a symmetric irreflexive binary relation  $G \subseteq X^2$ . For  $A \subseteq X$ , we say that  $A$  is *connected* if the induced subgraph  $G[A]$  is. We always equip connected graphs with their *path metric*  $d$ , and let  $\text{Ball}_r(x)$  be the closed ball of radius  $r$  around  $x$ ; more generally, we let  $\text{Ball}_r(A) := \bigcup_{x \in A} \text{Ball}_r(x)$ .

For a subset  $A \subseteq X$ , we let  $\partial_{iv}A := A \cap \text{Ball}_1(\neg A)$  be its *inner vertex boundary*,  $\partial_{ov}A := \partial_{iv}(\neg A)$  be its *outer vertex boundary*, and let  $\partial_{ie}A := G \cap (\partial_{ov}A \times \partial_{iv}A)$  and  $\partial_{oe}A := \partial_{ie}(\neg A)$  respectively be its *inner* and *outer edge boundaries*. Let  $\partial_vA := \partial_{iv}A \sqcup \partial_{ov}A$  be the *(total) vertex boundary* of  $A$ .

Finally, for  $x, y \in X$ , the *interval*  $[x, y]$  between  $x, y$  is the union of all geodesics between  $x, y$ , consisting of exactly those  $z \in X$  with  $d(x, z) + d(z, y) = d(x, y)$ . We say that  $A \subseteq X$  is *convex* if  $[x, y] \subseteq A$  for all  $x, y \in A$ . For vertices  $x, y, z \in X$ , we write  $x-y-z$  for  $y \in [x, z]$ . By the triangle inequality, we have for all  $w, x, y, z \in X$  that

$$(w-x-y \text{ and } w-y-z) \quad \Leftrightarrow \quad (w-x-z \text{ and } x-y-z),$$

and both sides occur iff there is a geodesic from  $w$  to  $x$  to  $y$  to  $z$ , which we write as  $w-x-y-z$ .

**1.1. Profinite pocsets of cuts.** The construction starts by identifying a profinite pocset  $\mathcal{H}$  of ‘cuts’ in each component of the graphing, which we first study abstractly. The finitely-separating subpocsets of  $2^X$  are well-known in metric geometry as *wallspaces*; see, e.g., [Nic04] and [CN05].

**Definition 1.1.** A *pocset*  $(\mathcal{H}, \leq, \neg, 0)$  is a poset  $(\mathcal{H}, \leq)$  equipped with an order-reversing involution  $\neg : \mathcal{H} \rightarrow \mathcal{H}$  and a least element  $0 \neq \neg 0$  such that  $0$  is the only lower-bound of  $H, \neg H$  for every  $H \in \mathcal{H}$ . We call the elements in  $\mathcal{H}$  *half-spaces*.

A *profinite pocset* is a pocset  $\mathcal{H}$  equipped with a compact topology making  $\neg$  continuous and is *totally order-disconnected*, in the sense that if  $H \not\leq K$ , then there is a clopen upward-closed  $U \subseteq \mathcal{H}$  with  $H \in U \not\leq K$ .

**Remark 1.2.** Such a topology is automatically Hausdorff and zero-dimensional, making it a Stone space.

We are primarily interested in subpocsets of  $(2^X, \subseteq, \neg, \emptyset)$  for a fixed set  $X$ , which is profinite if equipped with the product topology of the discrete space  $2$ . Indeed,  $2^X$  admits a base of *cylinder sets* – which are finite intersections of sets of the form  $\pi_x^{-1}(i)$  where  $x \in X$ ,  $i \in \{0, 1\}$ , and  $\pi_x : 2^X \rightarrow 2$  is the projection – making  $\neg$  continuous since cylinders are clopen. Finally, for  $H \not\leq K$ , let  $U$  be the upward-closure of a clopen neighborhood  $U_0 \ni H$  separating it from  $K$ , which is clopen since  $\neg U_0$  is a finite union of cylinders.

**Remark.** We follow [CPTT23, Convention 2.7], where for a family  $\mathcal{H} \subseteq 2^X$  of subsets of a fixed set  $X$ , we write  $\mathcal{H}^* := \mathcal{H} \setminus \{\emptyset, X\}$  for the *non-trivial* elements of  $\mathcal{H}$ .

The following proposition gives a sufficient criteria for subpocsets of  $2^X$  to be profinite. We also show in this case that every non-trivial element  $H \in \mathcal{H}^*$  is isolated, which will be important in Section 2.2.

**Lemma 1.3.** *If  $\mathcal{H} \subseteq 2^X$  is a finitely-separating pocset, then  $\mathcal{H}$  is closed and non-trivial elements are isolated.*

*Proof.* It suffices to show that the limit points of  $\mathcal{H}$  are trivial, so let  $A \in 2^X \setminus \{\emptyset, X\}$ . Fix  $x \in A \not\leq y$ . Since  $\mathcal{H}$  is finitely-separating, there are finitely-many  $H \in \mathcal{H}$  with  $x \in H \not\leq y$ , and for each such  $H \in \mathcal{H} \setminus \{A\}$ , we have either some  $x_H \in A \setminus H$  or  $y_H \in H \setminus A$ . Let  $U \subseteq 2^X$  be the family of all subsets  $B \subseteq X$  containing  $x$  and each  $x_H$ , but not  $y$  or any  $y_H$ .

This is the desired neighborhood isolating  $A \in U$ . Indeed, it is (cl)open since it is the *finite* intersection of cylinders prescribed by the  $x_H$ ’s and  $y_H$ ’s, and it is disjoint from  $\mathcal{H} \setminus \{A\}$  by construction. ■

Our main method of identifying the finitely-separating pocsets is as follows.

**Lemma 1.4.** *Let  $\mathcal{H} \subseteq 2^X$  be a pocset in a connected graph  $(X, G)$ . If each  $x \in X$  is on the vertex boundary of finitely-many half-spaces, then  $\mathcal{H}$  is finitely-separating. The converse holds too if  $(X, G)$  is locally-finite.*

*Proof.* Any  $H \in \mathcal{H}$  separating  $x, y \in X$  separates some edge on any fixed path between  $x$  and  $y$ , and there are only finitely-many such  $H$  for each edge. If  $(X, G)$  is locally-finite, then each  $x \in X$  is separated from each of its finitely-many neighbors by finitely-many  $H \in \mathcal{H}$ . ■

In the case that  $\mathcal{H}$  is a pocset consisting of connected co-connected half-spaces with finite vertex boundary, finite-separation also controls the degree of ‘non-nestedness’ of  $\mathcal{H}$ .

**Definition 1.5.** A *cut* in a connected locally-finite graph  $(X, G)$  is a half-space  $H \in \mathcal{H}_{\partial < \infty}(X) \cap \mathcal{H}_{\text{conn}}(X)$ .

**Definition 1.6.** Let  $\mathcal{H} \subseteq 2^X$  be a pocset. Two half-spaces  $H, K \in \mathcal{H}$  are *nested* if  $\neg^i H \cap \neg^j K = \emptyset$  for some  $i, j \in \{0, 1\}$ , where  $\neg^0 H := H$  and  $\neg^1 H := \neg H$ . We say that  $\mathcal{H}$  is *nested* if every pair  $H, K \in \mathcal{H}$  is nested.

**Lemma 1.7.** *In a pocset  $\mathcal{H}$  of finitely-separating cuts, each  $H \in \mathcal{H}$  is non-nested with finitely-many others.*

*Proof.* Fix  $H \in \mathcal{H}$  and let  $K \in \mathcal{H}$  be non-nested with  $H$ . By connectedness, the non-empty sets  $H \cap K$  and  $\neg H \cap K$  are joined by a path in  $K$ , so  $\partial_v H \cap K \neq \emptyset$ ; similarly,  $\partial_v H \cap \neg K \neq \emptyset$ . For each  $x \in \partial_v H \cap K$  and  $y \in \partial_v H \cap \neg K$ , any fixed path  $p_{xy}$  between them contains some  $z \in \partial_v K \cap p_{xy}$ ; thus, any  $K \in \mathcal{H}$  non-nested with  $H$  contains some  $z \in \partial_v K \cap p_{xy}$ .

Then, since there are finitely-many such  $x, y \in \partial_v H$ , for each of which there are finitely-many  $z \in p_{xy}$ , for each of which there are finitely-many  $K \in \mathcal{H}$  with  $z \in \partial_v K$  (by Lemma 1.4, since  $\mathcal{H}$  is finitely-separating), there can only be finitely-many  $K \in \mathcal{H}$  non-nested with  $H$ . ■

**1.2. Ends of graphs.** Let  $(X, G)$  be a connected locally-finite graph, and consider the Boolean algebra of finite vertex boundary half-spaces  $\mathcal{H}_{\partial < \infty}(X) \subseteq 2^X$ .

**Definition 1.8.** The *end compactification* of  $(X, G)$  is the Stone space  $\widehat{X}$  of  $\mathcal{H}_{\partial < \infty}(X)$ , whose non-principal ultrafilters are the *ends* of  $(X, G)$ .

We identify  $X \hookrightarrow \widehat{X}$  via principal ultrafilter map  $x \mapsto p_x$ , so  $\varepsilon(X) := \widehat{X} \setminus X$  is the set of ends of  $G$ . By definition,  $\widehat{X}$  admits a basis of clopen sets of the form  $\widehat{A} := \{p \in \widehat{X} : A \in p\}$  for each  $A \in \mathcal{H}_{\partial < \infty}(X)$ .

**Definition 1.9.** A pocset  $\mathcal{H} \subseteq \mathcal{H}_{\partial < \infty}(X)$  is *dense towards ends* of  $(X, G)$  if  $\mathcal{H}$  contains a neighborhood basis for every end in  $\varepsilon(X)$ .

In other words,  $\mathcal{H}$  is dense towards ends if for every  $p \in \varepsilon(X)$  and every (clopen) neighborhood  $\widehat{A} \ni p$ , where  $A \in \mathcal{H}_{\partial < \infty}(X)$ , there is some  $H \in \mathcal{H}$  with  $p \in \widehat{H} \subseteq \widehat{A}$ ; it is useful to note that  $\widehat{H} \subseteq \widehat{A}$  iff  $H \subseteq A$ , so we will abuse notation and write  $p \in H \subseteq A$  for the above condition.

We show in Section 2.1 that certain half-spaces  $\mathcal{H} \subseteq \mathcal{H}_{\partial < \infty}$  induced by a locally-finite quasi-tree is dense towards ends. It will also be important that these half-spaces be cuts, in that witnesses to density can also be found in  $\mathcal{H} \cap \mathcal{H}_{\text{conn}}$ . The following lemma takes care of this.

**Lemma 1.10.** *If a subpocset  $\mathcal{H} \subseteq \mathcal{H}_{\partial < \infty}$  is dense towards ends, then there is a subpocset  $\mathcal{H}' \subseteq \mathcal{H}_{\partial < \infty} \cap \mathcal{H}_{\text{conn}}$ , which is also dense towards ends, such that every  $H' \in \mathcal{H}'$  has  $\partial_{\text{ie}} H' \subseteq \partial_{\text{ie}} H$  for some  $H \in \mathcal{H}$ .*

*Proof.* A first attempt is to let  $\mathcal{H}'$  be the connected components  $H'_0$  of elements in  $\mathcal{H}$ , but this fails since  $\neg H'_0$  is not necessarily connected. Instead, we further take a component of  $\neg H'_0$ , whose complement clearly co-connected, and is connected since it consists of  $H'_0$  and the other components of  $\neg H'_0$ , each of which is connected to  $H'_0$  via  $\partial_{\text{ie}} H'_0$ . Formally, we let

$$\mathcal{H}' := \{H' \subseteq X : H \in \mathcal{H} \text{ and } H'_0 \in H/G \text{ and } \neg H' \in \neg H'_0/G\},$$

where  $H/G$  denotes the  $G$ -components of  $H$ . Clearly  $\partial_{\text{ie}} H' \subseteq \partial_{\text{ie}} H'_0 \subseteq \partial_{\text{ie}} H$ , and since  $H' \in \mathcal{H}_{\text{conn}}(X)$ , it remains to show that  $\mathcal{H}'$  is dense towards ends.

Fix an end  $p \in \varepsilon(X)$  and a neighborhood  $p \in A \in \mathcal{H}_{\partial < \infty}(X)$ . Let  $B \supseteq \partial_v A$  be finite connected, which can be obtained by adjoining paths between its components. Then  $\neg B \in p$  since  $p$  is non-principal, so there is  $H \in \mathcal{H}$  with  $p \in H \subseteq \neg B$ . We now use the above recipe to find the desired  $H' \in \mathcal{H}'$ :

- (1) Since  $H \in \mathcal{H}_{\partial < \infty}(X)$ , it has finitely-many connected components, so exactly one of them belongs to  $p$ , say  $p \in H'_0 \subseteq H$ . Note that  $B \subseteq \neg H \subseteq \neg H'_0$ .
- (2) Since  $B$  is connected, there is a unique component  $\neg H' \subseteq \neg H'_0$  containing  $B$ .

Observe that  $H' \in \mathcal{H}'$  and  $p \in H'$ . Lastly, since  $H'$  is connected and is disjoint from  $\partial_v A \subseteq B$ , and since  $H' \subseteq \neg A$  would imply  $\neg H' \in p$ , this forces  $H' \subseteq A$ , and hence  $p \in H' \subseteq A$  as desired. ■

**1.3. Median graphs and projections.** Starting from a finitely-separating profinite pocset  $\mathcal{H}$ , we construct in Section 2.2 its dual median graph  $\mathcal{M}(\mathcal{H})$ . By Lemma 1.3, this applies to finitely-separating pocsets.

We devote this section and the next to study some basic properties of median graphs and their projections, which will be used in Section 2.3 to construct a spanning tree certain median graphs. For more comprehensive references of median graphs, and their general theory, see [Rol98] and [Bow22].

**Definition 1.11.** A *median graph* is a connected graph  $(X, G)$  such that for any  $x, y, z \in X$ , the intersection

$$[x, y] \cap [y, z] \cap [x, z]$$

is a singleton, whose element  $\langle x, y, z \rangle$  is called the *median* of  $x, y, z$ . Thus we have a ternary median operation  $\langle \cdot, \cdot, \cdot \rangle : X^3 \rightarrow X$ , and a *median homomorphism*  $f : (X, G) \rightarrow (Y, H)$  is a map preserving said operation.

**Lemma 1.12.** For any  $\emptyset \neq A \subseteq X$  and  $x \in X$ , there is a unique point in  $\text{cvx}(A)$  between  $x$  and every point in  $A$ , called the *projection of  $x$  towards  $A$* , denoted  $\text{proj}_A(x)$ .

Moreover, we have  $\bigcap_{a \in A} [x, a] = [x, \text{proj}_A(x)]$ , and for any  $y$  in this set, we have  $\text{proj}_A(y) = \text{proj}_A(x)$ .

*Proof.* To show existence, pick any  $a_0 \in A$ . Given  $a_n \in \text{cvx}(A)$ , if there exists  $a \in A$  with  $a_n \notin [x, a]$ , set  $a_{n+1} := \langle x, a, a_n \rangle \in \text{cvx}(A)$ . Then  $a_0 - a_1 - \dots - a_n - x$  for all  $n$ , so this sequence terminates in at most  $d(a_0, x)$  steps at a point in  $\text{cvx}(A)$  between  $x$  and every point in  $A$ . For uniqueness, if there exist two such points  $a, b \in \text{cvx}(A)$ , then  $x - a - b$  and  $x - b - a$ , forcing  $a = b$ .

Finally, if  $x - y - \text{proj}_A(x)$  and  $a \in A$ , then  $x - \text{proj}_A(x) - a$  and hence  $x - y - a$ . Conversely, let  $x - y - a$  for all  $a \in A$ . Since  $[y, a] \subseteq [x, a]$  for all  $a$ , we see that

$$\text{proj}_A(y) \in \text{cvx}(A) \cap \bigcap_{a \in A} [y, a] \subseteq \text{cvx}(A) \cap \bigcap_{a \in A} [x, a]$$

and hence  $\text{proj}_A(y) = \text{proj}_A(x)$  by uniqueness. But since  $y - \text{proj}_A(y) - a$ , we have  $x - y - \text{proj}_A(y)$ , and hence  $x - y - \text{proj}_A(x)$  as desired. ■

**Remark 1.13.** It follows from the proof above that for any median homomorphism  $f : (X, G) \rightarrow (Y, H)$ , we have  $f(\text{proj}_A(x)) = \text{proj}_{f(A)}(f(x))$  for any  $\emptyset \neq A \subseteq X$  and  $x \in X$ . Indeed, we have

$$\text{proj}_A(x) = \langle x, a_m, \dots, \langle x, a_2, \langle x, a_1, a_0 \rangle \rangle \dots \rangle$$

for some  $m \leq d(a_0, x)$  and  $a_0, \dots, a_m \in A$ , and this is preserved by  $f$ .

For  $A := \{a, b\}$ , we have  $\text{proj}_A(x) = \langle a, b, x \rangle$ , and hence  $\text{cvx}(A) = \text{proj}_A(X) = \langle a, b, X \rangle = [a, b]$ .

**Lemma 1.14.** For each  $x, y \in X$ ,  $\text{cone}_x(y)$  is convex, and if  $xGy$ , then  $\text{cone}_x(y) \sqcup \text{cone}_y(x) = X$ .

*Proof.* Fix  $a, b \in \text{cone}_x(y)$  and  $a - c - b$ . It suffices to show that  $x - y - \langle a, c, x \rangle$ , for then  $x - y - c$  since we have  $x - \langle a, c, x \rangle - c$ . Indeed, it follows from the following observations.

- $x - y - \langle a, b, x \rangle$ , since  $\langle a, b, x \rangle = \text{proj}_{\{a, b\}}(x)$  and so  $[x, \langle a, b, x \rangle] = [x, a] \cap [x, b] \ni y$  by Lemma 1.12.
- $x - \langle a, b, x \rangle - \langle a, c, x \rangle$ , which follows from  $\langle a, b, x \rangle - \langle a, c, x \rangle - a$ , since  $x - \langle a, b, x \rangle - a$  by definition. Indeed, we have  $\langle a, c, x \rangle$  is in both  $[a, x]$  and  $[a, c] \subseteq [a, b]$ , and since  $\text{proj}_{\{b, x\}}(a) = \langle a, b, x \rangle$ , we have again by Lemma 1.12 that  $[\langle a, b, x \rangle, a] = [a, x] \cap [a, b] \ni \langle a, c, x \rangle$ .

Finally, take  $z \in X$  and consider  $w := \langle x, y, z \rangle \subseteq [x, y]$ . Either  $w = x$  or  $w = y$  (but not both), giving us the desired partition. ■

**Remark 1.15.** In particular, this shows that if  $xGy$ , then  $\text{cone}_x(y) \in \mathcal{H}_{\text{cvx}}^*(X)$ . The convexity of cones also shows, in the situation of Lemma 1.12, that  $\text{proj}_A = \text{proj}_{\text{cvx}(A)}$ , i.e.,  $\text{proj}_A(x)$  is also between  $x$  and every point in  $\text{cvx}(A)$ : indeed, note that  $\text{cone}_x(\text{proj}_A(x))$  is convex and contains  $A$ , so it contains  $\text{cvx}(A)$  too.

**Lemma 1.16.**  $\text{proj}_A : X \rightarrow \text{cvx}(A)$  is a median homomorphism with  $\text{proj}_A \circ \text{cvx} = \text{cvx} \circ \text{proj}_A$ .

*Proof.* The second claim follows from the first since, by Remark 1.13, we have

$$f(\text{cvx}(B)) = f(\text{proj}_B(X)) = \text{proj}_{f(B)}(f(X)) = \text{cvx}(f(B))$$

for all median homomorphisms  $f : X \rightarrow Y$  and  $B \subseteq X$ , so it in particular applies to  $f := \text{proj}_A$ .

To this end, let  $x - y - z \in X$  and set  $w := \langle \text{proj}_A(x), \text{proj}_A(y), \text{proj}_A(z) \rangle \in \text{cvx}(A)$ . It suffices to show that  $y - w - a$  for all  $a \in A$ , for then  $w = \text{proj}_A(y)$  and hence  $\text{proj}_A(x) - \text{proj}_A(y) - \text{proj}_A(z)$ . But we have  $y - \text{proj}_A(y) - a$  already, so it further suffices to show that  $y - w - \text{proj}_A(y)$ . For this, we note that

$$x - \text{proj}_A(x) - \text{proj}_A(y) \quad \text{and} \quad \text{proj}_A(x) - w - \text{proj}_A(y),$$

so  $x - w - \text{proj}_A(y)$ , and similarly  $z - w - \text{proj}_A(y)$ . Thus, it follows that

$$\begin{aligned} w \in [\text{proj}_A(y), x] \cap [\text{proj}_A(y), z] &= [\text{proj}_A(y), \text{proj}_{\{x, z\}}(\text{proj}_A(y))] && \text{Lemma 1.12} \\ &= [\text{proj}_A(y), \text{proj}_{[x, z]}(\text{proj}_A(y))] && \text{Remark 1.15} \\ &\subseteq [\text{proj}_A(y), y], && \text{Lemma 1.12} \end{aligned}$$

where the second equality follows from  $\text{cvx}(\{x, z\}) = [x, z]$ , and hence  $\text{proj}_{\{x, z\}} = \text{proj}_{[x, z]}$ . ■

**1.4. Convex half-spaces of median graphs.** We now use projections to explore the geometry of convex half-spaces in median graphs. For the axiomatics of convex structures, see [vdV93].

**Proposition 1.17.** *Each edge  $(x, y) \in G$  is on a unique hyperplane, namely the inward boundary of  $\text{cone}_x(y)$ , and conversely, each half-space  $H \in \mathcal{H}_{\text{cvx}}^*(X)$  is  $\text{cone}_x(y)$  for every  $(x, y) \in \partial_{\text{ie}}H$ .*

*Thus, hyperplanes are equivalence classes of edges. Furthermore, this equivalence relation is generated by parallel sides of squares (i.e., 4-cycles).*

*Proof.* We have  $\text{cone}_x(y) \in \mathcal{H}_{\text{cvx}}^*(X)$  by Lemma 1.14, and clearly  $(x, y) \in \partial_{\text{ie}} \text{cone}_x(y)$ . Conversely, take  $H \in \mathcal{H}_{\text{cvx}}^*(X)$  and any  $(x, y) \in \partial_{\text{ie}}H$ . Then  $H = \text{cone}_x(y)$ , for if  $z \in H \cap \neg \text{cone}_x(y)$ , then  $z \in \text{cone}_y(x)$ , and hence  $x \in [y, z] \subseteq H$  by convexity of  $H$ , a contradiction; if  $z \in \text{cone}_x(y) \cap \neg H$ , then  $[x, z] \subseteq \neg H$  by convexity of  $\neg H$ , and hence  $y \notin H$ , a contradiction.

Finally, parallel edges of a strip of squares generate the same hyperplane since, for a given square, each vertex is between its neighbors and hence any hyperplane containing an edge contains its opposite edge. On the other hand, let  $(a, b), (c, d) \in \partial_{\text{ie}}H$  for some  $H \in \mathcal{H}_{\text{cvx}}^*(X)$ . Note that  $\partial_{\text{ov}}H = \text{proj}_{\neg H}(H)$  is convex since  $H$  is, and since  $\text{proj}_{\neg H}$  preserves convexity by Lemma 1.16, any geodesic between  $a, c \in \partial_{\text{ov}}H$  lies in  $\partial_{\text{ov}}H$ . Matching this geodesic via  $\partial_{\text{ie}}H : \partial_{\text{ov}}H \rightarrow \partial_{\text{iv}}H$  gives us a geodesic between  $b, d$  in  $\partial_{\text{iv}}H$ , which together with the matching forms the desired strip of squares. ■

**Corollary 1.18.** *Two half-spaces  $H, K \in \mathcal{H}_{\text{cvx}}^*(X)$  are non-nested iff there is an embedding  $\{0, 1\}^2 \hookrightarrow X$  of the Hamming cube into the four corners  $\neg^i H \cap \neg^j K$ .*

*In particular, if  $H, K \in \mathcal{H}_{\text{cvx}}^*(X)$  are non-nested, then  $\partial_{\text{iv}}H \cap \partial_{\text{iv}}K \neq \emptyset$ .*

*Proof.* Let  $H, K$  be non-nested and take  $x_1 \in H \cap K$  and  $x_2 \in H \cap \neg K$ . Since  $H$  is connected, any geodesic between  $x_1, x_2$  crosses an edge  $(x'_1, x'_2) \in \partial_{\text{oe}}K$  in  $H$ . Similarly, there is an edge  $(y'_1, y'_2) \in \partial_{\text{oe}}K$  in  $\neg H$ , so we may slide both edges along  $\partial_{\text{oe}}K$  to obtain the desired square (see Proposition 1.17).

Conversely, the half-spaces cutting the square are clearly non-nested. ■

**Lemma 1.19 (Helly).** *Any finite intersection of pairwise-intersecting non-empty convex sets is non-empty.*

*Proof.* For pairwise-intersecting convex sets  $H_1, H_2, H_3$ , pick any  $x \in H_1 \cap H_2$ ,  $y \in H_1 \cap H_3$  and  $z \in H_2 \cap H_3$ ; their median  $\langle x, y, z \rangle$  then lies in  $H_1 \cap H_2 \cap H_3$ .

Suppose that it holds for some  $n \geq 3$  and let  $H_1, \dots, H_{n+1} \subseteq X$  pairwise-intersect. Then  $\{H_i \cap H_{n+1}\}_{i \leq n}$  is a family of  $n$  pairwise-intersecting convex sets, so  $\bigcap_{i \leq n+1} H_i = \bigcap_{i \leq n} (H_i \cap H_{n+1})$  is non-empty. ■

Lastly, we have some useful finiteness conditions on convex half-spaces; the former implies that  $\mathcal{H}_{\text{cvx}}(X)$  is finitely-separating, and the latter allows us to replace finite sets with their convex hulls.

**Lemma 1.20.** *Any two disjoint convex sets  $\emptyset \neq A, B \subseteq X$  can be separated by a half-space  $A \subseteq H \subseteq \neg B$ , and furthermore we have  $d(A, B) = |\{H \in \mathcal{H}_{\text{cvx}}(X) : A \subseteq H \subseteq \neg B\}|$ .*

*Proof.* Pick a geodesic  $A \ni x_0 G x_1 G \dots G x_n \in B$ , where  $n := d(A, B)$ . Then  $H := \text{cone}_{x_1}(x_0)$ , which is a half-space by Lemma 1.14, separates  $A, B$  since  $x_0 = \text{proj}_A(x_n)$ , and thus we have  $A \subseteq \text{cone}_{x_n}(x_0) \subseteq \text{cone}_{x_1}(x_0)$  and  $B \subseteq \text{cone}_{x_0}(x_n) \subseteq \text{cone}_{x_0}(x_1)$ .

Moreover, each such half-space  $A \subseteq H \subseteq \neg B$  satisfies  $x_i \in H \not\subseteq x_{i+1}$  for a unique  $i < n$ , and conversely each pair  $(x_i, x_{i+1})$  has a unique half-space separating them, so we have the desired bijection. ■

**Lemma 1.21.** *Every interval  $[x, y]$  is finite. More generally, if  $A \subseteq X$  is finite, then so is  $\text{cvx}(A)$ .*

*Proof.* The singletons  $\{x\}$  and  $\{y\}$  are convex, so there are finitely-many half-spaces  $H \subseteq [x, y]$ . But each  $z \in [x, y]$  is determined uniquely by those half-spaces containing it, so  $[x, y]$  is finite.

Let  $A := \{x_0, \dots, x_n\}$ . Since  $\text{cvx}(A) = \text{proj}_A(X)$ , we have by Remark 1.13 that points in  $\text{cvx}(A)$  are of the form  $\langle x, x_n, \dots, \langle x, x_2, \langle x, x_1, x_0 \rangle \rangle \dots \rangle$ , which is finite by induction using that intervals are finite. ■

## 2. THE MAIN CONSTRUCTIONS

Let  $(X, G)$  be a connected locally-finite quasi-tree, which, in the context of Theorem A, stands for a single component of the locally-finite graphing of the CBER. For Theorem B to apply, we need to first identify a family of finitely-separating cuts therein, and we do so in such a way that the cuts are dense towards ends.



**2.1. Dense cuts induced by quasi-trees.** Since  $X$  is a quasi-tree, and thus does not have arbitrarily long cycles, we expect that there is some finite bound  $R < \infty$  such that the ends in  $\varepsilon(X)$  are ‘limits’ of cuts  $\mathcal{H}_{\text{diam}(\partial) \leq R}(X) \subseteq \mathcal{H}_{\partial < \infty}(X)$  with boundary diameter bounded by  $R$ . We show that this is indeed the case, in the sense that  $\mathcal{H}_{\text{diam}(\partial) \leq R}(X) \cap \mathcal{H}_{\text{conn}}(X)$  is dense towards ends of  $(X, G)$ .

**Lemma 2.1.** *Let  $f : (X, G) \rightarrow (Y, T)$  be a coarse-equivalence between connected graphs. For a fixed  $H \in \mathcal{H}_{\partial < \infty}(Y)$ ,  $\text{diam}(\partial_v f^{-1}(H))$  is uniformly bounded in terms of  $\text{diam}(\partial_v H)$ .*

*Proof.* Since  $f$  is bornologous, let  $S < \infty$  be such that  $xGx'$  implies  $d(f(x), f(x')) \leq S$ , so that for any  $(x, x') \in \partial_{\text{ie}} f^{-1}(H)$ , there is a path of length  $\leq S$  between  $f(x) \notin H$  and  $f(x') \in H$ . Thus both  $d(f(x), \partial_v H)$  and  $d(f(x'), \partial_v H)$  are bounded by  $S$ , so  $f(\partial_v f^{-1}(H)) \subseteq \text{Ball}_S(\partial_v H)$  and hence

$$\text{diam}(f(\partial_v f^{-1}(H))) \leq \text{diam}(\partial_v H) + 2S.$$

That  $f$  is a coarse-equivalence gives us a uniform bound of  $\text{diam}(\partial_v f^{-1}(H))$  in terms of  $\text{diam}(\partial_v H)$ .  $\blacksquare$

In particular, if  $\text{diam}(\partial_v H)$  is itself also uniformly bounded, then so is  $\text{diam}(\partial_v f^{-1}(H))$ .

**Proposition 2.2.** *The class of connected locally-finite graphs in which  $\mathcal{H}_{\text{diam}(\partial) \leq R}$  is dense towards ends for some  $R < \infty$  is invariant under coarse equivalence.*

*Proof.* Let  $(X, G), (Y, T)$  be connected locally-finite graphs,  $f : X \rightarrow Y$  be a coarse equivalence with quasi-inverse  $g : Y \rightarrow X$ , and suppose  $\mathcal{H}_{\text{diam}(\partial) \leq S}(Y)$  is dense towards ends for some  $S < \infty$ . By Lemma 2.1, pick some  $R < \infty$  so that for any  $H \in \mathcal{H}_{\text{diam}(\partial) \leq S}(Y)$ , we have  $f^{-1}(H) \in \mathcal{H}_{\text{diam}(\partial) \leq R}(X)$ .

Fix an end  $p \in \varepsilon(X)$  and a neighborhood  $p \in A \in \mathcal{H}_{\partial < \infty}(X)$ . We need to find some  $B \in \mathcal{H}_{\partial < \infty}(Y)$  such that  $f(p) \in B$  and  $f^{-1}(B) \subseteq A$ , for then  $f(p) \in H$  for some  $B \supseteq H \in \mathcal{H}_{\text{diam}(\partial) \leq S}(Y)$ , and hence we have

$$p \in f^{-1}(H) \subseteq f^{-1}(B) \subseteq A$$

with  $f^{-1}(H) \in \mathcal{H}_{\text{diam}(\partial) \leq R}(X)$ . For convenience, let  $D < \infty$  be the uniform distance  $d(1_X, g \circ f)$ .

To this end, note that  $f(p) \in B$  iff  $p \in f^{-1}(B)$ . Since  $p \in A$ , the latter can occur if  $|A \triangle f^{-1}(B)| < \infty$ , and so we need to find such a  $B \in \mathcal{H}_{\partial < \infty}(Y)$  with the additional property that  $f^{-1}(B) \subseteq A$ .

*Attempt 1.* Set  $B := g^{-1}(A) \in \mathcal{H}_{\partial < \infty}(Y)$ . Then  $f^{-1}(B) \subseteq \text{Ball}_D(A)$  since if  $(g \circ f)(x) \in A$ , then

$$d(x, A) \leq d(x, (g \circ f)(x)) \leq d(1_X, g \circ f) = D.$$

By local-finiteness of  $G$ , we see that  $A \triangle f^{-1}(B) = A \setminus f^{-1}(B)$  is finite, as desired.

However, it is *not* the case that  $f^{-1}(B) \subseteq A$ . To remedy this, we ‘shrink’  $A$  by  $D$  to  $A'$  so that  $\text{Ball}_D(A') \subseteq A$ , and take  $B := g^{-1}(A')$  instead. Indeed,  $A' := \neg \text{Ball}_D(\neg A) \subseteq A$  works, since  $f^{-1}(B) \subseteq \text{Ball}_D(A')$  as before, so  $A' \triangle f^{-1}(B) = A' \setminus f^{-1}(B)$  is finite. Also,  $A \triangle A'$  is finite since  $x \in A \triangle A'$  iff  $x \in A$  and  $d(x, \neg A) \leq D$ , so  $A \triangle f^{-1}(B)$  is finite too. It remains to show that  $\text{Ball}_D(A') \subseteq A$ , for then  $f^{-1}(B) \subseteq A$  as desired.

Indeed, if  $y \in \text{Ball}_D(A')$ , then by the (reverse) triangle-inequality we have  $d(y, \neg A) \geq d(x, \neg A) - d(x, y)$  for all  $x \in A'$ . But  $d(x, \neg A) > D$ , strictly, so  $d(y, \neg A) > D - D = 0$ , and hence  $y \in A$ .  $\blacksquare$

**Corollary 2.3.** *If  $(X, G)$  is a locally-finite quasi-tree, then the subpocset  $\mathcal{H}_{\text{diam}(\partial) \leq R}(X) \cap \mathcal{H}_{\text{conn}}(X)$  is dense towards ends for some  $R < \infty$ .*

*Proof.* Observe that  $\mathcal{H}_{\text{diam}(\partial) \leq 2}(T)$  is dense towards ends for any tree  $T$ , so Proposition 2.2 proves the density of  $\mathcal{H}_{\text{diam}(\partial) \leq R}(X)$  for some  $R < \infty$ . By Lemma 1.10, there is a subpocset  $\mathcal{H}' \subseteq \mathcal{H}_{\partial < \infty}(X) \cap \mathcal{H}_{\text{conn}}(X)$  dense towards ends such that for every  $H' \in \mathcal{H}'$ , we have  $\partial_{\text{ie}} H' \subseteq \partial_{\text{ie}} H$  for some  $H \in \mathcal{H}_{\text{diam}(\partial) \leq R}(X)$ . Hence we have  $H' \in \mathcal{H}_{\text{diam}(\partial) \leq R}(X) \cap \mathcal{H}_{\text{conn}}(X)$ , so the result follows.  $\blacksquare$

The cuts  $\mathcal{H} := \mathcal{H}_{\text{diam}(\partial) \leq R}(X) \cap \mathcal{H}_{\text{conn}}(X)$  obtained here will be our starting point for Theorem B, so we need to show that it is finitely-separating. Indeed, by local-finiteness of  $(X, G)$ , the  $R$ -ball around any fixed  $x \in X$  is finite, so since any cut  $H \in \mathcal{H}$  with  $x \in \partial_v H$  is contained in said  $R$ -ball, there are finitely-many such cuts. Thus, by Lemma 1.4,  $\mathcal{H}$  is finitely-separating.

Furthermore, we have by Lemma 1.3 that  $\mathcal{H} \subseteq 2^X$  is closed and every non-trivial element is isolated, and those conditions allow for the construction in Theorem B to continue.

**2.2. The dual median graph of a profinite pocset.** We now present a classical construction in geometric group theory of a median graph associated to a profinite pocset with every non-trivial element isolated; see [Dun79], [Rol98], [Sag95], and [NR03] for other applications.

In the context of Theorem A, this will be applied to the pocset  $\mathcal{H}_{\text{diam}(\partial) \leq R}(X) \cap \mathcal{H}_{\text{conn}}(X)$  of cuts in a locally-finite graph  $(X, G)$ , and is also the first step in the construction in Theorem B.

**Definition 2.4.** An *orientation* on  $\mathcal{H}$  is an upward-closed subset  $U \subseteq \mathcal{H}$  containing exactly one of  $H, \neg H$  for each  $H \in \mathcal{H}$ . We let  $\mathcal{U}(\mathcal{H})$  denote the set of all orientations on  $\mathcal{H}$  and let  $\mathcal{U}^\circ(\mathcal{H})$  denote the clopen ones.

Intuitively, an orientation is a ‘maximally consistent’ choice of half-spaces<sup>3</sup>.

**Example 2.5.** Each  $x \in X$  induces its *principal orientation*  $\hat{x} := \{H \in \mathcal{H} : x \in H\} = \mathcal{H} \cap \pi_x^{-1}(1)$  – which is clearly clopen in  $\mathcal{H}$  – and gives us a canonical map  $X \rightarrow \mathcal{U}^\circ(\mathcal{H})$ . However, this map is *not necessarily* injective, and we call a fiber  $[x]_{\mathcal{H}} := \{y \in X : \hat{x} = \hat{y}\}$  thereof an  $\mathcal{H}$ -block. This induces an equivalence relation on  $X$  by declaring  $x \sim_{\mathcal{H}} y$  iff  $x$  and  $y$  are contained in exactly the same half-spaces in  $\mathcal{H}$ .

**Proposition 2.6.** Let  $\mathcal{H}$  be a profinite pocset with every non-trivial element isolated. Then the graph  $\mathcal{M}(\mathcal{H})$ , whose vertices are clopen orientations  $\mathcal{U}^\circ(\mathcal{H})$  and whose edges are pairs  $\{U, V\}$  with  $V = U \triangle \{H, \neg H\}$  for some  $\subseteq$ -minimal  $H \in U \setminus \{\emptyset, X\}$ , is a median graph with path metric  $d(U, V) = |U \triangle V|/2$  and medians

$$\begin{aligned} \langle U, V, W \rangle &:= \{H \in \mathcal{H} : H \text{ belongs to at least two of } U, V, W\} \\ &= (U \cap V) \cup (V \cap W) \cup (U \cap W). \end{aligned}$$

*Proof.* First,  $V := U \triangle \{H, \neg H\}$  as above is clopen since  $H, \neg H \in \mathcal{H}^*$  are isolated (whence  $\{H\}, \{\neg H\}$  are clopen), and it is an orientation by  $\subseteq$ -minimality of  $H$ . That  $\mathcal{M}(\mathcal{H})$  is connected follows from the following claim by noting that  $U \triangle V \not\subseteq \emptyset, X$  is clopen, so it is a compact set of isolated points, whence finite.

**Claim** ([Sag95, Theorem 3.3]). There is a path between  $U, V$  iff  $U \triangle V$  is finite, in which case

$$d(U, V) = |U \triangle V|/2 = |U \setminus V| = |V \setminus U|.$$

*Proof.* If  $(U_i)_{i < n}$  is a path from  $U := U_0$  to  $V := U_{n-1}$ , then, letting  $\{H_i, \neg H_i\} := U_i \triangle U_{i-1}$  for all  $1 \leq i < n$  gives us a sequence  $(H_i)_{i < n}$  inducing<sup>a</sup> this path, whence  $U \triangle V$  consists of  $\{H_i\}_{i < n}$  and their complements. Thus  $U \triangle V = 2n = 2d(U, V)$ , as desired.

Conversely, if  $U \triangle V = \{H_1, \dots, H_n\} \sqcup \{K_1, \dots, K_m\}$  with  $U \setminus V = \{H_i\}$  and  $V \setminus U = \{K_j\}$ , then  $\neg H_i \in V \setminus U$  and  $\neg K_j \in U \setminus V$  for all  $i < n$  and  $j < m$ , so  $n = m$  and  $V = U \cup \{\neg H_i\} \setminus \{H_i\}$ . We claim that there is a permutation  $\sigma \in S_n$  such that  $(H_{\sigma(i)})$  induces a path from  $U := U_0$ , which is the desired path from  $U$  to  $V$ . Choose a minimal  $H \in \{H_i\}$ , which is also minimal in  $U$ : if  $K \subseteq H$  for some  $K \in U$ , then  $\neg H \subseteq \neg K$ , and hence  $\neg K \in V$ , so  $K = H_i \subseteq H$  for some  $i$ , forcing  $K = H$ . Set  $U_1 := U \triangle \{H, \neg H\}$ , which is a clopen orientation. Continuing in this manner by choosing a minimal element in  $\{H_i\} \setminus \{H\}$  – and so on – gives us the desired path with  $d(U, V) = n$ .  $\square$

<sup>a</sup>In the sense that  $U_i = U_{i-1} \triangle \{H_i, \neg H_i\}$  and  $H_i \in U_i$  for each  $1 \leq i < n$ ; see [Tse20, Definition 2.20].

Finally, we show that  $\mathcal{M}(\mathcal{H})$  is a median graph. Fix  $U, V, W \in \mathcal{U}^\circ(\mathcal{H})$ , and note that for any  $M \in \mathcal{U}^\circ(\mathcal{H})$ , we have by the triangle inequality that  $M \in [U, V]$  iff  $(U \setminus M) \cup (M \setminus V) \subseteq U \setminus V$ , which clearly occurs iff  $U \cap V \subseteq M \subseteq U \cup V$ . Thus, a vertex  $M$  lies in the triple intersection  $[U, V] \cap [V, W] \cap [U, W]$  iff

$$(U \cap V) \cup (V \cap W) \cup (U \cap W) \subseteq M \subseteq (U \cup V) \cap (V \cup W) \cap (U \cup W).$$

Note that the two sides coincide, so  $M = \langle U, V, W \rangle$  – which is clopen if  $U, V, W$  are – is as claimed.  $\blacksquare$

Given such a pocset  $\mathcal{H}$ , the graph  $\mathcal{M}(\mathcal{H})$  constructed above is called the *dual*<sup>4</sup> median graph of  $\mathcal{H}$ . An important special case of this construction is when  $\mathcal{H}$  is *nested*, in which case  $\mathcal{M}(\mathcal{H})$  is a tree.

**Lemma 2.7.** Let  $\mathcal{H}$  be a profinite pocset with non-trivial points isolated. If  $\mathcal{H}$  is nested and  $(H_i)_{i < n} \subseteq \mathcal{H}$  induces a path in  $\mathcal{M}(\mathcal{H})$  with no backtracking, then  $(H_i)$  is strictly increasing.

<sup>3</sup>This can be formalized by letting  $\sim$  be the equivalence relation on  $\mathcal{H}$  given by  $H \sim \neg H$ . Letting  $\partial : \mathcal{H} \rightarrow \mathcal{H}/\sim$  denote the quotient map, orientations  $U \subseteq \mathcal{H}$  then correspond precisely to sections  $\varphi : \mathcal{H}/\sim \rightarrow \mathcal{H}$  of  $\partial$  such that  $\varphi(\partial H) \not\subseteq \neg\varphi(\partial K)$  for every  $H, K \in \mathcal{H}$ ; the latter condition rules out ‘orientations’ of the form  $\leftarrow | \rightarrow$ .

<sup>4</sup>The name is justified by a Stone-type duality between  $\{\text{median graphs, median homomorphisms}\}$  and  $\{\text{profinite pocsets with non-trivial points isolated, continuous maps}\}$ , where from a median graph  $X$  one can construct a canonical pocset  $\mathcal{H}_{\text{cvx}}(X)$  of convex half-spaces (see [CPTT23, Section 2.D] for details).

*Proof.* Let  $(U_i)_{i < n}$  be the induced path. By definition, we have  $U_{i+1} = U_{i-1} \cup \{\neg H_i, \neg H_{i-1}\} \setminus \{H_i, H_{i-1}\}$ , so  $H_i \neq \neg H_{i-1}$ , lest  $U_{i+1} = U_{i-1}$ . Thus, since  $H_i \in U_i = U_{i-1} \cup \{\neg H_{i-1}\} \setminus \{H_{i-1}\}$ , we see that  $H_i \in U_{i-1}$ . Clearly  $H_i \neq H_{i-1}$ . It suffices to remove the three cases when  $\neg H_i \subseteq H_{i-1}$ ,  $H_{i-1} \subseteq \neg H_i$ , and  $H_i \subseteq H_{i-1}$ , since then nestedness of  $\mathcal{H}$  gives us  $H_{i-1} \subsetneq H_i$ , as desired.

Indeed, if  $\neg H_i \subseteq H_{i-1}$ , then  $H_{i-1} \in U_{i+1}$  by upward-closure of  $U_{i+1} \ni \neg H_i$ . But since  $H_{i-1} \neq \neg H_i$ , we have by definition of  $U_{i+1}$  that  $H_{i-1} \in U_i$ , a contradiction. The other cases are similar. ■

**Corollary 2.8.** *If  $\mathcal{H}$  is nested, then  $\mathcal{M}(\mathcal{H})$  is a tree.*

*Proof.* A cycle in  $\mathcal{M}(\mathcal{H})$  is induced by a *strictly* increasing chain  $H_0 \subset \cdots \subset H_n \subset H_0$ , which is absurd. ■

Nonetheless, in the general non-nested case,  $\mathcal{M}(\mathcal{H})$  still admits a **canonical** spanning tree, which we prove in the following section.

**2.3. Canonical spanning trees.** We now present the Borel cycle-cutting algorithm that can be preformed **canonically** on any countable median graph with finite hyperplanes. We do so by colouring the half-spaces  $\mathcal{H}_{\text{cvx}}^*(X)$  into certain nested half-spaces  $\mathcal{H}_n \subseteq \mathcal{H}_{\text{cvx}}(X)$ , from which we inductively build a spanning forest by leveraging a tree structure on the  $\mathcal{H}_n$ -blocks  $X/\mathcal{H}_n$ . This tree is constructed as follows.

**Lemma 2.9.** *For any subpocset  $\mathcal{H} \subseteq \mathcal{H}_{\text{cvx}}(X)$ , the principal orientations map  $X \rightarrow \mathcal{U}^\circ(\mathcal{H})$  is surjective.*

*Proof.* Let  $U \in \mathcal{U}^\circ(\mathcal{H})$ , we need to find some  $x \in X$  with  $U = \hat{x}$ . Since  $U \subseteq \mathcal{H}$  is clopen, there is a finite set  $A \subseteq X$  – which we may assume to be convex by Lemma 1.21 – such that for all  $H \in \mathcal{H}$ , we have  $H \in U$  iff there is  $K \in U$  with  $H \cap A = K \cap A$ . Note that  $K \cap A \neq \emptyset$  for every  $K \in U$ , since otherwise  $\emptyset \in U$ . Furthermore,  $H \cap K \neq \emptyset$  for every  $H, K \in U$ , since otherwise we have  $H \subseteq \neg K$ , and so  $\neg K \in U$ .

By Lemma 1.19, the intersection  $(H \cap A) \cap (K \cap A) = H \cap K \cap A$  is non-empty, and applying it again furnishes some  $x \in \bigcap_{H \in U} H \cap A$  in  $X$ . Thus  $U \subseteq \hat{x}$ , so  $U = \hat{x}$  since both are orientations. ■

Thus, we have a graph isomorphism  $X/\mathcal{H} \cong \mathcal{M}(\mathcal{H})$ ; explicitly, two  $\mathcal{H}$ -blocks  $([x]_{\mathcal{H}}, [y]_{\mathcal{H}})$  are said to be *G-adjacent* if  $(\hat{x}, \hat{y}) \in \mathcal{M}(\mathcal{H})$ . Note that  $\mathcal{M}(\mathcal{H})$  may be constructed as in Proposition 2.6 since  $\mathcal{H} \subseteq \mathcal{H}_{\text{cvx}}(X)$  is finitely-separating by Lemma 1.20, and in particular, if  $\mathcal{H}$  is nested, then  $X/\mathcal{H}$  is a tree by Corollary 2.8.

**Lemma 2.10.** *The dual median graph  $\mathcal{M}(\mathcal{H})$  of a pocset  $\mathcal{H}$  of finitely-separating cuts has finite hyperplanes.*

*Proof.* Fix  $K \in \mathcal{H}_{\text{cvx}}^*(\mathcal{M}(\mathcal{H}))$ , which by Proposition 1.17 is of the form  $K = \text{cone}_V(U)$  for some (and hence any)  $(U, V) \in \partial_{\text{ie}} K$ , and we have by Proposition 2.6 that  $V = U \triangle \{H, \neg H\}$  for some (non-trivial) minimal  $H \in U$ . We claim that any other edge  $(U', V') \in \partial_{\text{ie}} K$  can be reached from  $(U, V)$  by simultaneously flipping only the half-spaces  $H'_0, \dots, H'_n \in \mathcal{H}^*$  non-nested with  $H$ , of which there are finitely-many by Lemma 1.7.

Since the edges  $(U, V), (U', V')$  induce the same hyperplane  $\partial_{\text{ie}} K$ , it suffices by Proposition 1.17 to prove this for when  $(U', V')$  is an edge of a square parallel to  $(U, V)$ , in which case there is some minimal  $H' \in \mathcal{H}^*$  flipping both  $U$  to  $U'$  and  $V$  to  $V'$ . We have some easy observations.

1.  $H, H'$  are non-nested. Indeed,  $H' \not\subseteq H$  and  $H \not\subseteq H'$  by minimality; if  $H \subseteq \neg H'$ , then  $\neg H' \in U$ ; and if  $\neg H \subseteq H'$ , then  $H'$  is not minimal in  $U$ .
2.  $H \in U'$  is still minimal. Indeed, since  $U' = U \triangle \{H', \neg H'\}$ , the only way this can fail is if  $\neg H' \subseteq H$ , but this contradicts minimality of  $H$ .

Thus, we have  $V' = U' \triangle \{H, \neg H\}$ , so the induction continues with  $(U', V')$  in place of  $(U, V)$ . ■

**Proposition 2.11.** *Every countable median graph with finite hyperplanes admits a **canonical** spanning tree.*

*Proof.* Let  $(X, G)$  be a countable median graph with finite hyperplanes. By Corollary 1.18, if two half-spaces  $H, K \in \mathcal{H}_{\text{cvx}}^*(X)$  are non-nested, then  $\partial_v H \cap \partial_v K \neq \emptyset$ , so looking at the intersection graph of the boundaries furnishes a countable colouring  $\mathcal{H}_{\text{cvx}}^*(X) = \bigsqcup_{n \in \mathbb{N}} \mathcal{H}_n^*$  such that each  $H, \neg H$  receive the same color and that each  $\mathcal{H}_n := \mathcal{H}_n^* \cup \{\emptyset, X\}$  is a *nested* subpocset. For each  $n \in \mathbb{N}$ , let  $\mathcal{K}_n := \bigcup_{m \geq n} \mathcal{H}_m$ .

We shall inductively construct an increasing chain of subforests  $T_n \subseteq G$  such that the components of  $T_n$  are exactly the  $\mathcal{K}_n$ -blocks. Then, the increasing union  $T := \bigcup_n T_n$  is a spanning tree, since each  $(x, y) \in G$  lies in a  $\mathcal{K}_n$ -block for sufficiently large  $n$  (namely, the  $n$  such that  $\text{cone}_x(y) \in \mathcal{H}_{n-1}^*$ , since  $\text{cone}_x(y)$  and its complement are the only half-spaces separating  $x$  and  $y$  by Proposition 1.17).

Since each pair of distinct points is separated by a half-space, the  $\mathcal{K}_0 = \mathcal{H}_{\text{cvx}}(X)$ -blocks are singletons, so put  $T_0 := \emptyset$ . Suppose that a forest  $T_n$  is constructed as required. Note that each  $\mathcal{K}_{n+1}$ -block  $Y \in X/\mathcal{K}_{n+1}$



is not separated by any half-spaces in  $\mathcal{H}_m$  for  $m > n$ , but is separated (by Proposition 1.17) by  $\mathcal{H}_n$  into the  $\mathcal{K}_n$ -blocks contained in  $Y$ , and those correspond precisely to the  $\mathcal{H}_n$ -blocks in  $Y/\mathcal{H}_n$ . For each  $G$ -adjacent pair  $A, B \in Y/\mathcal{H}_n$ , which is separated by a unique half-space  $H \in \mathcal{H}_n$  by Lemma 1.20, we may pick an edge from the *finite* hyperplane  $\partial_e H$ . Since each  $Y/\mathcal{H}_n$  is a tree by Corollary 2.8, and a single edge is picked between every pair of  $G$ -adjacent blocks therein, the graph  $T_{n+1}$  obtained from  $T_n$  by adding all such edges is a forest whose components are exactly the  $\mathcal{K}_{n+1}$ -blocks. ■

**2.4. Finiteness conditions on pocsets.** Let  $(X, G)$  be a connected locally-finite graph and fix a finitely-separating pocset  $\mathcal{H}$  of cuts. We now investigate certain finiteness conditions on  $\mathcal{H}$  when it is dense towards ends. These conditions will be leveraged in Section 2.5, in which we prove Theorem A.

**Lemma 2.12.** *A subset  $A \in \mathcal{H}_{\partial < \infty}(X)$  is infinite iff it contains an end in  $(X, G)$ .*

*Proof.* The converse direction follows since ends are non-principal. If  $A$  is infinite, then by local-finiteness of  $(X, G)$ , König's Lemma furnishes some infinite ray  $(x_n) \subseteq A$ . Then,  $A$  is contained in the filter

$$p := \{H \in \mathcal{H}_{\partial < \infty}(X) : \forall^\infty n (x_n \in H)\},$$

which is ultra since  $H \in p$  are of *finite*-boundary, and is non-principal since it contains cofinite sets. ■

**Proposition 2.13.** *If  $\mathcal{H}$  is a finitely-separating pocset of cuts on a connected locally-finite graph, then  $\mathcal{H}$  is dense towards ends iff the following two conditions hold.*

1. *Each  $\mathcal{H}$ -block is finite.*
2. *Each  $H \in \mathcal{H}^*$  has finitely-many successors.*

*Proof.* If  $\mathcal{H}$  is dense towards ends, we will cover (certain closed sets in)  $\varepsilon(X)$  by cuts in  $\mathcal{H}$ , which cuts down to a finite subcover. We will show that a certain Boolean combination thereof, which has finite-boundary, is finite using Lemma 2.12, giving us the desired finiteness claim. Below are the details.

1. Fix an  $\mathcal{H}$ -block  $[x]_{\mathcal{H}}$  and let  $A := \neg\{x\}$ . By density, we can cover each end  $p \in \varepsilon(X)$  with some  $H_p \subseteq A$ , which gives us a finite subcover  $\bigcup_{i < n} H_i$  of  $\varepsilon(X)$ . Note that  $\bigcap_{i < n} \neg H_i \in \mathcal{H}_{\partial < \infty}(X)$  contains  $[x]_{\mathcal{H}}$ , and is finite since it contains no ends in  $\varepsilon(X)$ .
2. Fix  $H \in \mathcal{H}^*$  and let  $K_\alpha \in \mathcal{H}^*$  be the successors of  $H$ . By density, each end  $p \in \varepsilon(\neg H)$  is contained in some  $H_p \subseteq \neg H$ , which in turn is contained in some  $\neg K_\alpha$ ; this gives us a finite subcover  $\bigcup_{i < n} \neg K_i$  of  $\varepsilon(\neg H)$ . Any other successor  $H \subset K \in \mathcal{H}^*$  not in this subcover may be assumed to be nested with each  $K_i$ , since there are finitely-many non-nested ones by Lemma 1.7. But then  $K \not\subseteq K_i \not\subseteq K$  and  $K \cap K_i \neq \emptyset$  for all  $i < n$ , which forces  $\neg K \subseteq \neg H \cap \bigcap_{i < n} K_i \in \mathcal{H}_{\partial < \infty}(X)$ ; the latter contains no ends in  $\varepsilon(X)$ , so it is finite, and hence there are finitely-many possibilities for such  $K$ .

Conversely, fix an end  $p \in \varepsilon(X)$  and a neighborhood  $p \in A \in \mathcal{H}_{\partial < \infty}(X)$ . Since  $\mathcal{H}$  consists of connected sets, it suffices to find some  $H \in \mathcal{H}$  containing  $\partial_v A$  but not  $p$ , for then  $p \in \neg H \subseteq A$  as desired.

**Observation.** Finitely-many  $H, \neg H \in \mathcal{H}$  may be removed and still satisfy conditions (1) and (2) above.

That (2) still holds is obvious; for (1), we may remove a single pair, since with  $\mathcal{H}' := \mathcal{H} \setminus \{H, \neg H\}$ , the map  $X/\mathcal{H} \rightarrow X/\mathcal{H}'$  sending  $[x]_{\mathcal{H}} \rightarrow [x]_{\mathcal{H}'}$  is surjective and at-most 2-to-1.

Thus, for each of the finitely-many  $x, y \in \partial_v A$ , we may remove the finitely-many half-spaces separating them, so we may assume that  $\mathcal{H} = \mathcal{H}_A \sqcup \neg \mathcal{H}_A$  is partitioned into the half-spaces in  $\mathcal{H}_A$  containing  $\partial_v A$ , and its complements which are disjoint from  $\partial_v A$ . By contradiction, if each  $H \in \mathcal{H}_A$  contains  $p$ , we will construct a strictly decreasing chain  $(H_n) \subseteq \mathcal{H}_A^*$ , which gives an infinite family separating  $\partial_v A$  from  $\neg H_0$ .

Let  $H_0 \in \mathcal{H}_A^*$  be arbitrary, which exists since  $X$  is infinite, so  $\mathcal{H}$ -blocks being finite forces  $\mathcal{H}^* = \mathcal{H}_A^* \sqcup \neg \mathcal{H}_A^*$  to be infinite. There are finitely-many half-spaces non-nested with  $H_0$  by Lemma 1.7, so we may assume that there is none. We may further assume that all successors  $H$  of  $\neg H_0$  lie in  $\mathcal{H}_A^*$ , for otherwise  $\mathcal{H}_A^* \ni \neg H \subset H_0$  and we may take  $H_1 := \neg H$ . Thus,  $B := H_0 \cap \bigcap_{H \supset \neg H_0} H$  is a *finite* intersection of the minimal ones in  $\mathcal{H}_A^*$ , and since  $p \in H$  for each such  $H$  by assumption,  $B$  contains  $p$  and is infinite. Since  $\mathcal{H}$ -blocks are finite and  $B$  is  $\mathcal{H}$ -invariant, it contains infinitely-many  $\mathcal{H}$ -blocks; any two such  $\mathcal{H}$ -blocks is separated by some  $H \in \mathcal{H}_A^*$  nested with  $H_0$ , and since  $H \not\subseteq H_0^c$  and  $H$  contains neither  $H_0$  nor  $\neg H_0$ , we can set  $H_1 := H \subset H_0$ . ■

## 2.5. Borel subtreeing of quasi-trees. TODO

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