# TREE-LIKE GRAPHINGS

ZHAOSHEN ZHAI

Abstract. We present a streamlined exposition of a construction presented in [CPTT23], where it is proven that every locally-finite Borel graph with each component a quasi-tree induces a canonical treeable equivalence relation. Write some more details...

## 1. Introduction

The purpose of this note is to provide a streamlined proof of a particular case of a construction presented in [CPTT23], in order to better understand the general formalism developed therein. We attempt to make this note self-contained, but nevertheless urge the reader to refer to the original paper for more detailed discussion and generalizations of the results we have selected to include here.

1.1. **Treeings of equivalence relations.** A *countable Borel equivalence relation (CBER)* on a standard Borel space $X$ is a Borel equivalence relation $E \subseteq X^2$ with each class countable. We are interested in special types of *graphings* on a CBER $E \subseteq X^2$, i.e. a Borel graph $G \subseteq X^2$ whose connectedness relation is precisely $E$. For instance, a graphing of $E$ such that each component is a tree is called a *treeing* of $E$, and the CBERs that admit a treeing are said to be *treeable*. The main results of [CPTT23] is to provide new sufficient criteria for treeability of CBERs, and in particular, they prove the following

**Theorem A** ([CPTT23, Theorem 1.1])**.** *If a CBER admits a locally-finite graphing whose components are quasi-trees, then it is treeable.*

Recall that metric spaces $X$ and $Y$ are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error, and $X$ is a *quasi-tree* if it is quasi-isometric to a simplicial tree; see [Gro93] and [DK18].

1.2. **Outline of the proof.** Roughly speaking, the existence of a quasi-isometry $G|C \to T_C$ to a simplicial tree $T_C$ for each component $C \subseteq X$ induces a collection $\mathcal{H}(C)$ of 'cuts' (subsets $H \subseteq C$ with finite boundary such that both $H$ and $C \setminus H$ are connected), which are 'tree-like' in the sense that

1. $\mathcal{H}(C)$ is *finitely-separating*: each pair $x, y \in C$ are separated by finitely-many $H \in \mathcal{H}(C)$, and
2. $\mathcal{H}(C)$ is *dense towards ends*: each end in $G|C$ has a neighborhood basis in $\mathcal{H}(C)$.

These cuts have the structure of a pocset with non-trivial points isolated, which in turn provide exactly the data to construct a 'median graph' whose vertices are 'ultrafilters' thereof. Condition (2) then ensures that this graph has finite 'hyperplanes', which allows us to apply a Borel 'cycle-cutting' algorithm and obtain a canonical spanning tree thereof. Each step above can be done in a uniform way to each component $C \subseteq G$, giving us the desired treeing of the CBER.

$$\text{Quasi-tree} \xrightarrow{2.1} \begin{array}{c}\text{Dense family}\\\text{of cuts}\end{array} \xrightarrow{2.2} \begin{array}{c}\text{Pocset w/}\\\text{non-triv. pts.}\\\text{isolated}\end{array} \xrightarrow{2.3} \begin{array}{c}\text{Median graph}\\\text{w/ finite}\\\text{hyperplanes}\end{array} \xrightarrow{2.4} \begin{array}{c}\text{Canonical}\\\text{spanning tree}\end{array}$$

**Remark.** We follow [CPTT23, Convention 2.7], where for a family $\mathcal{H} \subseteq 2^X$ of subsets of a fixed set $X$, we write $\mathcal{H}^* := \mathcal{H} \setminus \{\varnothing, X\}$ for the *non-trivial* elements of $\mathcal{H}$.

## 2. Detailed Constructions

2.1. **Graphs with dense families of cuts.** Let $(X, G)$ be a locally-finite graph.

2.1.1. *End compactification of graphs.*

*Date*: July 24, 2024.

2.2. **Pocsets of cuts.** The family $\mathcal{H}_{\operatorname{diam}\partial\le R}(X)$ of cuts have the following structure.

**Definition 2.1.** A *pocset* $(P, \le, \neg, 0)$ is a poset $(P, \le)$ equipped with an order-reversing involution $\neg : P \to P$ and a least element $0 \ne \neg 0$ such that $0$ is the only lower-bound of $p, \neg p$ for every $p \in P$.

A *profinite pocset* is a pocset $P$ equipped with a compact topology making $\neg$ continuous and is *totally order-disconnected*, in the sense that if $p \not\le q$, then there is a clopen upward-closed $U \subseteq P$ with $p \in U \not\ni q$.

**Remark 2.2.** Such a topology is automatically Hausdorff (if $p \ne q$, then either $p \not\le q$ or $q \not\le p$)

We are primarily interested in subpocsets of $2^X$, which has a canonical pocset structure $(2^X, \subseteq, \neg, \varnothing)$.

**Lemma 2.3.** *The pocset $2^X$ equipped with the product topology of the discrete space $2 \coloneqq \{0, 1\}$ is profinite, and admits a basis of 'cylinders': for each finite $A \subseteq X$ and $W \in 2^A$, the* cylinder *at $W$ is*

$$[W] \coloneqq \left\{ H \in 2^X : H \cap A = W \right\}.$$

*Proof.* ■

**Proposition 2.4.** *Let $X$ be a set and $\mathcal{H} \subseteq 2^X$ be a subpocset. If $\mathcal{H}$ is finitely-separating, then $\mathcal{H} \subseteq 2^X$ is closed and every non-trivial element is isolated.*

*Proof.* ■

2.3. **The dual median graph of a pocset.**

2.4. **Spanning trees of median graphs with finite hyperplanes.**

## References


[CPTT23] Ruiyuan Chen, Antoine Poulin, Ran Tao, and Anush Tserunyan, *Tree-like graphings, wallings, and median graphings of equivalence relations* (2023), available at https://arxiv.org/abs/2308.13010.

[Gro93] Mikhail Gromov, *Asymptotic invariants of infinite groups*, Geometric Group Theory, Vol. 2 (Sussex, 1991), London Mathematical Society Lecture Note Series, vol. 182, Cambridge University Press, 1993.

[DK18] Cornelia Druţu and Michael Kapovich, *Geometric Group Theory*, with appendix by Bogdan Nica, Vol. 63, American Mathematical Society Colloquium Publications, 2018.