

TREE OF ORIENTATIONS ON A NESTED COLLECTION OF CUTS

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1. INTRODUCTION

Let $\mathcal{C} \subseteq 2^X$ be a collection of non-empty subsets of a set X . With the definitions in Section 2, we prove the following

Theorem 1.1. *If \mathcal{C} is nested, then the graph $\mathcal{T}_{\mathcal{C}}$, whose:*

- *Vertices are finitely-based orientations on \mathcal{C} ; and whose*
- *Neighbors of $\mathcal{U} \in V(\mathcal{T}_{\mathcal{C}})$ are $\mathcal{U} \triangle \{A, A^c\}$ for every minimal $A \in \mathcal{U}$ with $A^c \in \mathcal{C}$;*

is acyclic. Furthermore, $\mathcal{T}_{\mathcal{C}}$ is a tree iff \mathcal{C} is closed under complements.

In particular, this applies to when (X, G) is a graph and \mathcal{C} is a nested collection of cuts on X .

2. PRELIMINARIES

2.1. Orientations. Since we do not assume that \mathcal{C} is closed under complements, we slightly modify the definition of orientations, as follows.

Definition 2.1. An *orientation* on \mathcal{C} is a subset $\mathcal{U} \subseteq \mathcal{C}$ such that

1. (*Upward-closure*). If $A \in \mathcal{U}$ and $B \in \mathcal{C}$ contains A , then $B \in \mathcal{U}$.
2. (*Ultra*). If $A, A^c \in \mathcal{C}$, then either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$, but not both.

Remark 2.2. This coincides with the standard definition when \mathcal{C} is a subposet of 2^X .

Lemma 2.3. *If $\mathcal{U} \subseteq \mathcal{C}$ is an orientation, then for any \subseteq -minimal $A \in \mathcal{U}$ with $A^c \in \mathcal{C}$, so is $\mathcal{U} \triangle \{A, A^c\}$.*

Proof. That $\mathcal{V} := \mathcal{U} \triangle \{A, A^c\} = \mathcal{U} \cup \{A^c\} \setminus \{A\}$ is upward-closed follows from \subseteq -minimality of A . Now, if $B, B^c \in \mathcal{C}$ and $B^c \notin \mathcal{V}$, we need to show that $B \in \mathcal{V}$.

To this end, note that $B^c \notin \mathcal{V}$ implies $A \neq B$ and either $B = A^c$ or $B^c \notin \mathcal{U}$. The former case follows from $A^c \in \mathcal{V}$, and for the latter, we have $B \in \mathcal{U} \setminus \{A\}$ since \mathcal{U} is ultra. ■

Remark 2.4. In the above notations, clearly $\mathcal{U} \neq \mathcal{U} \triangle \{A, A^c\}$. Furthermore, for any other such orientation \mathcal{U}' and $A' \in \mathcal{U}'$, that $\mathcal{U} = \mathcal{U}'$ and $\mathcal{U} \triangle \{A, A^c\} = \mathcal{U}' \triangle \{A', A'^c\}$ together imply $A = A'$.

Definition 2.5. A *base* for an orientation $\mathcal{U} \subseteq \mathcal{C}$ is a \subseteq -minimal subset $\mathcal{B} \subseteq \mathcal{U}$ such that $\mathcal{U} = \uparrow \mathcal{B}$, where

$$\uparrow \mathcal{B} := \bigcup_{B \in \mathcal{B}} \uparrow B := \bigcup_{B \in \mathcal{B}} \{A \in \mathcal{C} : A \supseteq B\}.$$

Definition 2.6. A collection \mathcal{C} is said to be *nested* if every $C_1, C_2 \in \mathcal{C}$ has an empty corner, i.e., $C_1^i \cap C_2^j = \emptyset$ for some $i, j \in \{1, -1\}$, where $C^i := C$ if $i = 1$ and $C^i := C^c$ if $i = -1$.

Remark 2.7. If \mathcal{C} is nested, then every \subseteq -minimal $B \in \mathcal{C}$ induces an orientation $\uparrow B := \{A \in \mathcal{C} : A \supseteq B\}$, called the *principal* orientation. Indeed, $\uparrow B$ is clearly upward-closed, and if $A, A^c \in \mathcal{C}$, then, by \subseteq -minimality of B and nestedness of \mathcal{C} , either $A \supseteq B$ or $A^c \supseteq B$ (but clearly not both).

This construction generalizes to any collection $\mathcal{B} \subseteq \mathcal{C}$ with each $B \in \mathcal{B}$ being \subseteq -minimal, in that $\uparrow \mathcal{B}$ is an orientation on \mathcal{C} .

Definition 2.8. An orientation $\mathcal{U} \subseteq \mathcal{C}$ is said to be *finitely-based* if it admits a finite base.

Remark 2.9. If $\mathcal{U} = \uparrow \{B_1, \dots, B_n\}$ is finitely-based, then for any \subseteq -minimal $A \in \mathcal{U}$ with $A^c \in \mathcal{C}$, so is the orientation $\mathcal{V} := \mathcal{U} \triangle \{A, A^c\}$. Indeed, $A = B_i$ for some $1 \leq i \leq n$, and $\mathcal{V} = \uparrow(\{A\} \cup \{B_j\}_{j \neq i})$.

3. THE GRAPH $\mathcal{T}_{\mathcal{C}}$

Fix a nested collection of non-empty subsets of a set X . Using Lemma 2.3 and Remarks 2.4 and 2.9, we construct a graph $\mathcal{T}_{\mathcal{C}}$ whose:

- *Vertices* of $\mathcal{T}_{\mathcal{C}}$ are finitely-based orientations on \mathcal{C} .
- *Neighbors* of $\mathcal{U} \in V(\mathcal{T}_{\mathcal{C}})$ are the finitely-based orientations $\mathcal{U} \triangle \{A, A^c\}$ for every minimal $A \in \mathcal{U}$.

The goal of this section is to establish Theorem 1.1, stating that $\mathcal{T}_{\mathcal{C}}$ is acyclic (Proposition 3.4), and furthermore, $\mathcal{T}_{\mathcal{C}}$ is a tree precisely when \mathcal{C} is closed under complements (Proposition 3.5).

3.1. Paths in $\mathcal{T}_{\mathcal{C}}$. To show that $\mathcal{T}_{\mathcal{C}}$ is acyclic, we characterize backtracking paths in $\mathcal{T}_{\mathcal{C}}$ as follows.

Definition 3.1. Fix $\mathcal{U}_0 \in V(\mathcal{T}_{\mathcal{C}})$ and $\alpha \leq \omega$. A sequence $(A_n)_{n < \alpha}$ is said to *represent a path from \mathcal{U}_0* if $(\mathcal{U}_n)_{n \leq \alpha}$, defined by $\mathcal{U}_n := \mathcal{U}_{n-1} \triangle \{A_{n-1}\}$ for every $1 \leq n \leq \alpha$, is a path in $\mathcal{T}_{\mathcal{C}}$.

Remark 3.2. Any path in $\mathcal{T}_{\mathcal{C}}$ is represented by its sequence of flipped basis elements.

Lemma 3.3. Let $\alpha \geq 3$. A path in $\mathcal{T}_{\mathcal{C}}$ from \mathcal{U}_0 represented by $(A_n)_{n \leq \alpha}$ has no backtracking iff $A_n \neq A_{n-1}^c$ for every $1 \leq n \leq \alpha$.

Proof. Take $2 \leq n \leq \alpha$. It suffices to show that $\mathcal{U}_{n-2} = \mathcal{U}_n$ iff $A_{n-1} = A_{n-2}^c$.

- (\Rightarrow) . We have by definition that $\mathcal{U}_n = \mathcal{U}_{n-2} \cup \{A_{n-1}^c, A_{n-2}^c\} \setminus \{A_{n-1}, A_{n-2}\}$

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Proposition 3.4.

Proof.

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Proposition 3.5.

Proof.

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