

TREE-LIKE GRAPHINGS

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ABSTRACT. We present a streamlined exposition of a construction presented in [CPTT23], where it is proven that every locally-finite Borel graph with each component a quasi-tree induces a canonical treeable equivalence relation. *Write some more details...*

1. INTRODUCTION

The purpose of this note is to provide a streamlined proof of a particular case of a construction presented in [CPTT23], in order to better understand the general formalism developed therein. We attempt to make this note self-contained, but nevertheless urge the reader to refer to the original paper for more detailed discussion and some generalizations of the results we have selected to include here.

1.1. Treeings of equivalence relations. A *countable Borel equivalence relation (CBER)* on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$ with each class countable. We are interested in special types of *graphings* on a CBER $E \subseteq X^2$, i.e. a Borel graph $G \subseteq X^2$ whose connectedness relation is precisely E . For instance, a graphing of E such that each component is a tree is called a *treeing* of E , and the CBERs that admit treeings are said to be *treeable*. The main results of [CPTT23] provide new sufficient criteria for treeability of certain classes of CBERs, and in particular, they prove the following

Theorem A ([CPTT23, Theorem 1.1]). *If a CBER admits a locally-finite graphing whose components are quasi-trees, then it is treeable.*

Recall that metric spaces X and Y are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error, and X is a *quasi-tree* if it is quasi-isometric to a simplicial tree; see [Gro93] and [DK18].

1.2. Outline of the proof. Roughly speaking, the existence of a quasi-isometry $G|C \rightarrow T_C$ to a simplicial tree T_C for each component $C \subseteq X$ induces a collection $\mathcal{H}(C)$ of ‘cuts’ (subsets $H \subseteq C$ with finite boundary such that both H and $C \setminus H$ are connected), which are ‘tree-like’ in the sense that

1. $\mathcal{H}(C)$ is *finitely-separating*: each pair $x, y \in C$ is separated by finitely-many $H \in \mathcal{H}(C)$, and
2. $\mathcal{H}(C)$ is *dense towards ends*: each end in $G|C$ has a neighborhood basis in $\mathcal{H}(C)$.

By Condition (1), these cuts have the structure of a profinite pocset with non-trivial points isolated, which in turn provide exactly the data to construct a ‘median graph’ whose vertices are ‘ultrafilters’¹ thereof. Condition (2) then ensures that this graph has finite ‘hyperplanes’, which allows us to apply a Borel ‘cycle-cutting’ algorithm and obtain a canonical spanning tree thereof. Each step above can be done in a uniform way to each component $C \subseteq G$, giving us the desired treeing of the CBER.

$$\text{Quasi-tree} \xrightarrow{2} \begin{array}{c} \text{Dense family} \\ \text{of cuts} \end{array} \xrightarrow{3.1} \begin{array}{c} \text{Pocset w/} \\ \text{non-triv. pts.} \\ \text{isolated} \end{array} \xrightarrow{3.2+3.3} \begin{array}{c} \text{Median graph} \\ \text{w/ finite} \\ \text{hyperplanes} \end{array} \xrightarrow{4} \text{Canonical spanning tree}$$

Write some more stuff to tie things together...

Remark. We follow [CPTT23, Convention 2.7], where for a family $\mathcal{H} \subseteq 2^X$ of subsets of a fixed set X , we write $\mathcal{H}^* := \mathcal{H} \setminus \{\emptyset, X\}$ for the *non-trivial* elements of \mathcal{H} .

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¹As in [CPTT23], we call them *orientations* instead, to avoid confusion with the more standard notion; see Definition 3.4.

2. GRAPHS WITH DENSE FAMILIES OF CUTS

2.1. Ends of graphs. Let (X, G) be a connected locally-finite graph, which, in the context of Theorem A, will stand for a single component of the locally-finite graphing of a CBER.

Definition 2.1. For a subset $A \subseteq X$, we let $\partial_{iv}A := A \cap \text{Ball}_1(\neg A)$ be its *inner vertex boundary*, $\partial_{ov}A := \partial_{iv}(\neg A)$ be its *outer vertex boundary*, and let $\partial_{ie}A := G \cap (\partial_{ov}A \times \partial_{iv}A)$ and $\partial_{oe}A := \partial_{ie}(\neg A)$ respectively be its *inner* and *outer edge boundaries*. Let $\partial_vA := \partial_{iv}A \cup \partial_{ov}A$ be the *(total) vertex boundary* of A .

Let $\mathcal{H}_{\partial < \infty}(X) \subseteq 2^X$ be the Boolean algebra of all $A \subseteq X$ with finite vertex boundary, called *cuts* in X .

Definition 2.2. The *end compactification* of (X, G) is the Stone space \widehat{X} of $\mathcal{H}_{\partial < \infty}(X)$, whose non-principal ultrafilters are the *ends* of (X, G) .

We identify $X \hookrightarrow \widehat{X}$ via principal ultrafilter map $x \mapsto p_x$, so $\widehat{X} \setminus X$ is the set of ends of G . By definition, \widehat{X} admits a basis of clopen sets of the form $\widehat{A} := \{p \in \widehat{X} : A \in p\}$ for each $A \in \mathcal{H}_{\partial < \infty}(X)$.

Definition 2.3. A family $\mathcal{H} \subseteq \mathcal{H}_{\partial < \infty}(X)$ of cuts is *dense towards ends* of (X, G) if \mathcal{H} contains a neighborhood basis for every end in $\widehat{X} \setminus X$.

In other words, \mathcal{H} is dense towards ends if for every $p \in \widehat{X} \setminus X$ and every (clopen) neighborhood $\widehat{A} \ni p$, where $A \in \mathcal{H}_{\partial < \infty}(X)$, there is some $H \in \mathcal{H}$ with $p \in \widehat{H} \subseteq \widehat{A}$; it is useful to note that $\widehat{H} \subseteq \widehat{A}$ iff $H \subseteq A$.

2.2. Dense cuts induced by quasi-trees. If X is a quasi-tree – and thus does not have arbitrary long cycles – we expect that there is some finite bound $R < \infty$ such that the ends in $\widehat{X} \setminus X$ are ‘limits’ of cuts $\mathcal{H}_{\text{diam}(\partial) \leq R}(X) \subseteq \mathcal{H}_{\partial < \infty}(X)$ with boundary diameter bounded by R . We show that this is indeed the case, in the sense that $\mathcal{H}_{\text{diam}(\partial) \leq R}(X)$ is dense towards ends of (X, G) .

Lemma 2.4. Let $f : (X, G) \rightarrow (Y, T)$ be a coarse-equivalence between connected graphs. For a fixed $H \in \mathcal{H}_{\partial < \infty}(Y)$, $\text{diam}(\partial_v f^{-1}(H))$ is uniformly bounded in terms of $\text{diam}(\partial_v H)$.

Proof. Since f is bornologous, let $S < \infty$ be such that xGx' implies $d(f(x), f(x')) \leq S$, so that for any $(x, x') \in \partial_{ie} f^{-1}(H)$, there is a path of length $\leq S$ between $f(x) \notin H$ and $f(x') \in H$. Thus both $d(f(x), \partial_v H)$ and $d(f(x'), \partial_v H)$ are bounded by S , so $f(\partial_v f^{-1}(H)) \subseteq \text{Ball}_S(\partial_v H)$ and hence

$$\text{diam}(f(\partial_v f^{-1}(H))) \leq \text{diam}(\partial_v H) + 2S.$$

That f is a coarse-equivalence gives us a uniform bound of $\text{diam}(\partial_v f^{-1}(H))$ in terms of $\text{diam}(\partial_v H)$. ■

In particular, if $\text{diam}(\partial_v H)$ is itself also uniformly bounded, then so is $\text{diam}(\partial_v f^{-1}(H))$.

Proposition 2.5. The class of connected locally-finite graphs in which $\mathcal{H}_{\text{diam}(\partial) \leq R}$ is dense towards ends for some $R < \infty$ is invariant under coarse equivalence.

Proof. Let $(X, G), (Y, T)$ be connected locally-finite graphs, $f : X \rightarrow Y$ be a coarse equivalence with quasi-inverse $g : Y \rightarrow X$, and suppose $\mathcal{H}_{\text{diam}(\partial) \leq S}(Y)$ is dense towards ends for some $S < \infty$. By Lemma 2.4, pick some $R < \infty$ so that for any $H \in \mathcal{H}_{\text{diam}(\partial) \leq S}(Y)$, we have $f^{-1}(H) \in \mathcal{H}_{\text{diam}(\partial) \leq R}(X)$.

Fix an end $U \in \widehat{X} \setminus X$ with $U \in \widehat{A}$ for some $A \in \mathcal{H}_{\partial < \infty}(X)$. We need to find some $B \in \mathcal{H}_{\partial < \infty}(Y)$ such that $\widehat{f}(U) \in \widehat{B}$ and $f^{-1}(B) \subseteq A$, for then $\widehat{f}(U) \in \widehat{B}$ for some $B \supseteq H \in \mathcal{H}_{\text{diam}(\partial) \leq S}(Y)$, and hence we have

$$U \in \widehat{f^{-1}(H)} \subseteq \widehat{f^{-1}(B)} \subseteq \widehat{A}$$

with $f^{-1}(H) \in \mathcal{H}_{\text{diam}(\partial) \leq R}(X)$. For convenience, let $D < \infty$ be the uniform distance $d(1_X, g \circ f)$.

To this end, note that $\widehat{f}(U) \in \widehat{B}$ iff $U \in \widehat{f^{-1}(B)}$. Since $U \in \widehat{A}$, the latter can occur if $|A \triangle f^{-1}(B)| < \infty$, and so we need to find such a $B \in \mathcal{H}_{\partial < \infty}(Y)$ with the additional property that $f^{-1}(B) \subseteq A$.

Attempt 1. Set $B := g^{-1}(A) \in \mathcal{H}_{\partial < \infty}(Y)$. Then $f^{-1}(B) \subseteq \text{Ball}_D(A)$ since if $(g \circ f)(x) \in A$, then

$$d(x, A) \leq d(x, (g \circ f)(x)) \leq d(1_X, g \circ f) = D.$$

By local-finiteness of G , we see that $A \triangle f^{-1}(B) = A \setminus f^{-1}(B)$ is finite, as desired.

However, it is *not* the case that $f^{-1}(B) \subseteq A$. To remedy this, we ‘shrink’ A by D to A' so that $\text{Ball}_D(A') \subseteq A$, and take $B := g^{-1}(A')$ instead. Indeed, $A' := \neg \text{Ball}_D(\neg A) \subseteq A$ works, since $f^{-1}(B) \subseteq \text{Ball}_D(A')$ as before, so $A' \triangle f^{-1}(B) = A' \setminus f^{-1}(B)$ is finite. Also, $A \triangle A'$ is finite since $x \in A \triangle A'$ iff $x \in A$ and $d(x, \neg A) \leq D$, so $A \triangle f^{-1}(B)$ is finite too. It remains to show that $\text{Ball}_D(A') \subseteq A$, for then $f^{-1}(B) \subseteq A$ as desired.

Indeed, if $y \in \text{Ball}_D(A')$, then by the (reverse) triangle-inequality we have $d(y, \neg A) \geq d(x, \neg A) - d(x, y)$ for all $x \in A'$. But $d(x, \neg A) > D$, strictly, so $d(y, \neg A) > D - D = 0$, and hence $y \in A$. ■

Corollary 2.6. *If (X, G) is a locally-finite quasi-tree, then $\mathcal{H}_{\text{diam}(\partial) \leq R}(X)$ is dense towards ends for some $R < \infty$.*

Proof. Observe that $\mathcal{H}_{\text{diam}(\partial) \leq 2}(T)$ is dense towards ends for any tree T , and invoke Proposition 2.5. ■

3. POCSSETS AND ORIENTATIONS

3.1. Pocsets of cuts. The family $\mathcal{H}_{\text{diam} \partial \leq R}(X)$ of cuts have the structure of a ‘profinite pocset’, which we first study abstractly. We then deduce some properties of the pocset induced by a *dense* family of cuts.

Definition 3.1. A *pocset* $(\mathcal{H}, \leq, \neg, 0)$ is a poset (\mathcal{H}, \leq) equipped with an order-reversing involution $\neg : \mathcal{H} \rightarrow \mathcal{H}$ and a least element $0 \neq \neg 0$ such that 0 is the only lower-bound of $H, \neg H$ for every $H \in \mathcal{H}$. We call the elements in \mathcal{H} *half-spaces*.

A *profinite pocset* is a pocset \mathcal{H} equipped with a compact topology making \neg continuous and is *totally order-disconnected*, in the sense that if $H \not\leq K$, then there is a clopen upward-closed $U \subseteq \mathcal{H}$ with $H \in U \not\leq K$.

Remark 3.2. Such a topology is automatically Hausdorff and zero-dimensional.

We are primarily interested in subpocsets of $(2^X, \subseteq, \neg, \emptyset)$, which is profinite if equipped with the product topology of the discrete space 2 . Indeed, 2^X admits a base of *cylinder sets* – which are finite intersections of sets of the form $\pi_x^{-1}(i)$ where $x \in X$, $i \in \{0, 1\}$, and $\pi_x : 2^X \rightarrow 2$ is the projection – making \neg continuous since cylinders are clopen. **Show that it is totally order-disconnected.**

The following proposition gives a sufficient criteria for subpocsets of 2^X to be profinite. We also show in this case that every non-trivial element $H \in \mathcal{H}^*$ is isolated, which will important in Section 3.2.

Proposition 3.3. *Let X be a set and $\mathcal{H} \subseteq 2^X$ be a subpocset. If \mathcal{H} is finitely-separating, then $\mathcal{H} \subseteq 2^X$ is closed and every non-trivial element is isolated.*

Proof. It suffices to show that the limit points of \mathcal{H} are trivial, so let $A \in 2^X \setminus \{\emptyset, X\}$. Fix $x \in A \not\leq y$. Since \mathcal{H} is finitely-separating, there are finitely-many $H \in \mathcal{H}$ with $x \in H \not\leq y$, and for each such $H \in \mathcal{H} \setminus \{A\}$, we have either some $x_H \in A \setminus H$ or $y_H \in H \setminus A$. Let $U \subseteq 2^X$ be the family of all subsets $B \subseteq X$ containing x and each x_H but not y or any y_H .

This is the desired neighborhood isolating $A \in U$. Indeed, it is (cl)open since it is the *finite* intersection of cylinders prescribed by the x_H ’s and y_H ’s, and it is disjoint from $\mathcal{H} \setminus \{A\}$ by construction. ■

For a locally-finite connected graph (X, G) , a subpocset $\mathcal{H} \subseteq \mathcal{H}_{\partial < \infty}(X)$ of cuts is finitely-separating iff each $x \in X$ is on the boundary of finitely-many $H \in \mathcal{H}$. Indeed, if \mathcal{H} is finitely-separating, then each $x \in X$ is separated from each of its finitely-many neighbors by finitely-many $H \in \mathcal{H}$. Conversely, any $H \in \mathcal{H}$ separating $x, y \in X$ separates some edge on any fixed path between x and y , and there are only finitely-many such H for each edge.

In particular, the subpocset $\mathcal{H}_{\text{diam}(\partial) \leq R}(X)$ for any fixed $R < \infty$ (see Section 2.2) is finitely-separating.

3.2. The dual median graph of a pocset. Let \mathcal{H} be a profinite subpocset with every non-trivial element isolated; for instance, if \mathcal{H} is finitely-separating, and in particular the cuts $\mathcal{H}_{\text{diam}(\partial) \leq R}(X)$ for some locally-finite connected graph (X, G) .

Definition 3.4. An *orientation* on \mathcal{H} is an upward-closed subset $U \subseteq \mathcal{H}$ containing exactly one of $H, \neg H$ for each $H \in \mathcal{H}$. We let $\mathcal{U}(\mathcal{H})$ denote the set of all orientations on \mathcal{H} and let $\mathcal{U}^\circ(\mathcal{H})$ denote the clopen ones.

Intuitively, an orientation is a ‘maximally consistent’ choice of half-spaces².

²This can be formalized by letting \sim be the equivalence relation on \mathcal{H} given by $H \sim \neg H$. Letting $\partial : \mathcal{H} \rightarrow \mathcal{H}/\sim$ denote the quotient map, orientations $U \subseteq \mathcal{H}$ then correspond precisely to sections $\varphi : \mathcal{H}/\sim \rightarrow \mathcal{H}$ of ∂ such that $\varphi(\partial H) \not\leq \neg \varphi(\partial K)$ for every $H, K \in \mathcal{H}$; the latter condition rules out ‘orientations’ of the form $\leftarrow | \rightarrow$.

Example 3.5. Each $x \in X$ induces its *principal orientation* $\hat{x} := \{H \in \mathcal{H} : x \in H\} = \mathcal{H} \cap \pi_x^{-1}(1)$ – which is clearly clopen in \mathcal{H} – and gives us a canonical map $X \rightarrow \mathcal{U}^\circ(\mathcal{H})$. However, this map is *not necessarily* injective, and we call a fiber $[x]_{\mathcal{H}} := \{y \in X : \hat{x} = \hat{y}\}$ thereof an \mathcal{H} -*block*.

The *clopen* orientations are exactly those which are *finitely-based*, in the sense that there is a \subseteq -minimal finite subset $U_0 \subseteq U$ such that $U = \uparrow U_0$, where

$$\uparrow U_0 := \bigcup_{H \in U_0} \uparrow H := \bigcup_{H \in U_0} \{K \in \mathcal{H} : K \supseteq H\}.$$

Indeed, if U is clopen, then there is a finite subset $A \subseteq X$ such that for all $H \in \mathcal{H}$, we have $H \in U$ iff there is some $K \in U$ with $H \cap A = K \cap A$.

The goal of this section is to canonically construct a graph whose vertices are clopen orientations on \mathcal{H} .

Definition 3.6. A *median graph* is a connected graph (X, G) such that for any $x, y, z \in X$, the intersection

$$[x, y] \cap [y, z] \cap [x, z]$$

is a singleton. Its unique element $\langle x, y, z \rangle$ is called the *median* of x, y, z .

An exposition on some generalities of median graphs can be found in [CPTT23, Section 2.B/2.C], from which we refer to as needed in Section 4. For more comprehensive references, see [Rol98] and [Bow22].

Theorem 3.7. Let \mathcal{H} be a profinite pocset with every non-trivial element isolated. Then the graph $\mathcal{M}(\mathcal{H})$, whose vertices are clopen orientations $\mathcal{U}^\circ(\mathcal{H})$ and whose edges are pairs $\{U, V\}$ with $V = U \triangle \{H, \neg H\}$ for some \subseteq -minimal $H \in \mathcal{H}^*$, is a median graph with path metric $d(U, V) = (U \triangle V)/2$ and medians

$$\langle U, V, W \rangle := \{H \in \mathcal{H} : H \text{ belongs to at least two of } U, V, W\}.$$

Proof. First, $V := U \triangle \{H, \neg H\}$ as above is clopen since $H, \neg H \in \mathcal{H}^*$ is isolated (whence $\{H\}$ and $\{\neg H\}$ are clopen), and it is an orientation by \subseteq -minimality of H .

Claim. There is a path between U, V iff $U \triangle V$ is finite, in which case $d(U, V) = (U \triangle V)/2$.

Proof. If $(U_i)_{i < n}$ is a path from $U = U_0$ to $V = U_{n-1}$, then letting $\{H_i, \neg H_i\} := U_i \triangle U_{i-1}$ for all $1 \leq i < n$ gives us a sequence $(H_i)_{i < n}$ inducing this path, whence $U \triangle V$ consists of $\{H_i\}_{i < n}$ and their complements. Thus $U \triangle V = 2n = 2d(U, V)$, as desired.

Conversely, if $U \triangle V = \{H_1, \dots, H_n\} \sqcup \{K_1, \dots, K_m\}$ with $U \setminus V = \{H_i\}$ and $V \setminus U = \{K_j\}$, then $\neg H_i \in V \setminus U$ and $\neg K_j \in U \setminus V$ for all $i < n$ and $j < m$, so $n = m$ and $V = U \cup \{\neg H_i\} \setminus \{H_i\}$. We claim that there is a permutation $\sigma \in S_n$ such that $(H_{\sigma(i)})$ induces a path from $U =: U_0$, which is the desired path from U to V . Choose a minimal $H \in \{H_i\}$, which is also minimal in U : if $K \subseteq H$ for some $K \in U$, then $\neg H \subseteq \neg K$, and hence $\neg K \in V$, so $K = H_i \subseteq H$ for some i , forcing $K = H$. Set $U_1 := U \triangle \{H, \neg H\}$, which is a clopen orientation. Continuing in this manner by choosing a minimal element in $\{H_i\} \setminus \{H\}$ – and so on – gives us the desired path with $d(U, V) = n$. \square (claim)

■

3.3. Finiteness conditions on $\mathcal{U}^\circ(\mathcal{H})$ induced by dense cuts. Hi

4. MEDIAN GRAPHS AND THEIR CANONICAL SPANNING TREES

TODO

A. TREE OF ORIENTATIONS ON A NESTED POCSET OF SETS

Let $\mathcal{H} \subseteq 2^X$ be a finitely-separating subpocset (see Definition 3.1), so every non-trivial element is isolated by Proposition 3.3. As shown in Theorem 3.7, the clopen orientations $\mathcal{U}^\circ(\mathcal{H})$ form the vertices of a median graph, whose edges are given by ‘minimal base-flipping’.

We devote this appendix to show that if \mathcal{H} is *nested*, then $\mathcal{U}^\circ(\mathcal{H})$ is in fact a tree, and so we may bypass the Borel cycle-cutting algorithm in Section 4 and obtain a treeing directly.

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