

# TREE-LIKE GRAPHINGS

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ABSTRACT. We present a streamlined exposition of a construction presented in [CPTT23], where it is proven that every locally-finite Borel graph with each component a quasi-tree induces a canonical treeable equivalence relation. Write some more details...

## 1. INTRODUCTION

The purpose of this note is to provide a streamlined proof of a particular case of a construction presented in [CPTT23], in order to better understand the general formalism developed therein. We attempt to make this note self-contained, but nevertheless urge the reader to refer to the original paper for more detailed discussion and some generalizations of the results we have selected to include here.

**1.1. Treeings of equivalence relations.** A *countable Borel equivalence relation (CBER)* on a standard Borel space  $X$  is a Borel equivalence relation  $E \subseteq X^2$  with each class countable. We are interested in special types of *graphings* on a CBER  $E \subseteq X^2$ , i.e. a Borel graph  $G \subseteq X^2$  whose connectedness relation is precisely  $E$ . For instance, a graphing of  $E$  such that each component is a tree is called a *treeing* of  $E$ , and the CBERs that admit treeings are said to be *treeable*. The main results of [CPTT23] provide new sufficient criteria for treeability of certain classes of CBERs, and in particular, they prove the following

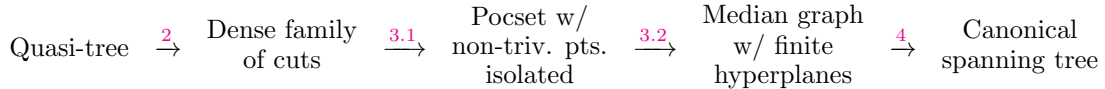
**Theorem A** ([CPTT23, Theorem 1.1]). *If a CBER admits a locally-finite graphing whose components are quasi-trees, then it is treeable.*

Recall that metric spaces  $X$  and  $Y$  are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error, and  $X$  is a *quasi-tree* if it is quasi-isometric to a simplicial tree; see [Gro93] and [DK18].

**1.2. Outline of the proof.** Roughly speaking, the existence of a quasi-isometry  $G|C \rightarrow T_C$  to a simplicial tree  $T_C$  for each component  $C \subseteq X$  induces a collection  $\mathcal{H}(C)$  of ‘cuts’ (subsets  $H \subseteq C$  with finite boundary such that both  $H$  and  $C \setminus H$  are connected), which are ‘tree-like’ in the sense that

1.  $\mathcal{H}(C)$  is *finitely-separating*: each pair  $x, y \in C$  is separated by finitely-many  $H \in \mathcal{H}(C)$ , and
2.  $\mathcal{H}(C)$  is *dense towards ends*: each end in  $G|C$  has a neighborhood basis in  $\mathcal{H}(C)$ .

By Condition (1), these cuts have the structure of a profinite pocset with non-trivial points isolated, which in turn provide exactly the data to construct a ‘median graph’ whose vertices are ‘ultrafilters’<sup>1</sup> thereof. Condition (2) then ensures that this graph has finite ‘hyperplanes’, which allows us to apply a Borel ‘cycle-cutting’ algorithm and obtain a canonical spanning tree thereof. Each step above can be done in a uniform way to each component  $C \subseteq G$ , giving us the desired treeing of the CBER.



Write some more stuff to tie things together...

**Remark.** We follow [CPTT23, Convention 2.7], where for a family  $\mathcal{H} \subseteq 2^X$  of subsets of a fixed set  $X$ , we write  $\mathcal{H}^* := \mathcal{H} \setminus \{\emptyset, X\}$  for the *non-trivial* elements of  $\mathcal{H}$ .

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<sup>1</sup>As in [CPTT23], we call them *orientations* instead, to avoid confusion with the more standard notion; see Definition 3.4.

## 2. GRAPHS WITH DENSE FAMILIES OF CUTS

**2.1. Ends of graphs.** Let  $(X, G)$  be a locally-finite graph.

**Definition 2.1.** For a subset  $A \subseteq X$ , we let  $\partial_v A := A \cap \text{Ball}_1(\neg A)$  be its *inner vertex boundary*,  $\partial_{ov} A := \partial_v(\neg A)$  be its *outer vertex boundary*, and let  $\partial_e A := G \cap (\partial_{ov} A \times \partial_v A)$  and  $\partial_{oe} A := \partial_e(\neg A)$  respectively be its *inner* and *outer edge boundaries*. Let  $\partial_v A := \partial_v A \cup \partial_{ov} A$  be the *(total) vertex boundary* of  $A$ .

Let  $\mathcal{H}_{\partial < \infty}(X) \subseteq 2^X$  be the Boolean algebra of all  $A \subseteq X$  with finite vertex boundary, called *cuts* in  $X$ .

**Definition 2.2.** The *end compactification* of  $(X, G)$  is the Stone space  $\widehat{X}$  of  $\mathcal{H}_{\partial < \infty}(X)$ , whose non-principal ultrafilters are the *ends* of  $(X, G)$ .

We identify  $X \hookrightarrow \widehat{X}$  via principal ultrafilter map  $x \mapsto p_x$ , so  $\widehat{X} \setminus X$  is the set of ends of  $G$ . By definition,  $\widehat{X}$  admits a basis of clopen sets of the form  $\widehat{A} := \{p \in \widehat{X} : A \in p\}$  for each  $A \in \mathcal{H}_{\partial < \infty}(X)$ .

**Definition 2.3.** A family  $\mathcal{H} \subseteq \mathcal{H}_{\partial < \infty}(X)$  of cuts is *dense towards ends* of  $(X, G)$  if  $\mathcal{H}$  contains a neighborhood basis for every end in  $\widehat{X} \setminus X$ .

In other words,  $\mathcal{H}$  is dense towards ends if for every  $p \in \widehat{X} \setminus X$  and every (clopen) neighborhood  $\widehat{A} \ni p$ , where  $A \in \mathcal{H}_{\partial < \infty}(X)$ , there is some  $H \in \mathcal{H}$  with  $p \in \widehat{H} \subseteq \widehat{A}$ ; it is useful to note that  $\widehat{H} \subseteq \widehat{A}$  iff  $H \subseteq A$ .

**2.2. Dense cuts induced by quasi-isometries.** If  $X$  is a quasi-tree – and thus does not have arbitrary long cycles – we expect that there is some finite bound  $R < \infty$  such that the ends in  $\widehat{X} \setminus X$  are ‘limits’ of cuts  $\mathcal{H}_{\text{diam}(\partial) \leq R}(X) \subseteq \mathcal{H}_{\partial < \infty}(X)$  with boundary diameter bounded by  $R$ . We show that this is indeed the case, in the sense that  $\mathcal{H}_{\text{diam}(\partial) \leq R}(X)$  is dense towards ends of  $(X, G)$ .

**Lemma 2.4.** Let  $f : (X, G) \rightarrow (Y, T)$  be a coarse-equivalence between connected locally-finite graphs. For a fixed  $H \in \mathcal{H}_{\partial < \infty}(Y)$ ,  $\text{diam}(\partial_v f^{-1}(H))$  is uniformly bounded in terms of  $\text{diam}(\partial_v H)$ .

*Proof.* Since  $f$  is bornologous, let  $S < \infty$  be such that  $xGx'$  implies  $d(f(x), f(x')) \leq S$ , so that for any  $(x, x') \in \partial_e f^{-1}(H)$ , there is a path of length  $\leq S$  between  $f(x) \notin H$  and  $f(x') \in H$ . Thus both  $d(f(x), \partial_v H)$  and  $d(f(x'), \partial_v H)$  are bounded by  $S$ , so  $f(\partial_v f^{-1}(H)) \subseteq \text{Ball}_S(\partial_v H)$  and hence

$$\text{diam}(f(\partial_v f^{-1}(H))) \leq \text{diam}(\partial_v H) + 2S.$$

That  $f$  is a coarse-equivalence gives us a uniform bound of  $\text{diam}(\partial_v f^{-1}(H))$  in terms of  $\text{diam}(\partial_v H)$ .  $\blacksquare$

In particular, if  $\text{diam}(\partial_v H)$  is itself also uniformly bounded, then so is  $\text{diam}(\partial_v f^{-1}(H))$ .

**Proposition 2.5.** The class of connected locally-finite graphs in which  $\mathcal{H}_{\text{diam}(\partial) \leq R}$  is dense towards ends for some  $R < \infty$  is invariant under coarse equivalence.

*Proof.* Let  $(X, G), (Y, T)$  be connected locally-finite graphs,  $f : X \rightarrow Y$  be a coarse equivalence with quasi-inverse  $g : Y \rightarrow X$ , and suppose  $\mathcal{H}_{\text{diam}(\partial) \leq S}(Y)$  is dense towards ends for some  $S < \infty$ . By Lemma 2.4, pick some  $R < \infty$  so that for any  $H \in \mathcal{H}_{\text{diam}(\partial) \leq S}(Y)$ , we have  $f^{-1}(H) \in \mathcal{H}_{\text{diam}(\partial) \leq R}(X)$ .

Fix an end  $U \in \widehat{X} \setminus X$  with  $U \in \widehat{A}$  for some  $A \in \mathcal{H}_{\partial < \infty}(X)$ . We need to find some  $B \in \mathcal{H}_{\partial < \infty}(Y)$  such that  $\widehat{f}(U) \in \widehat{B}$  and  $f^{-1}(B) \subseteq A$ , for then  $\widehat{f}(U) \in \widehat{B}$  for some  $B \supseteq H \in \mathcal{H}_{\text{diam}(\partial) \leq S}(Y)$ , and hence we have

$$U \in \widehat{f^{-1}(H)} \subseteq \widehat{f^{-1}(B)} \subseteq \widehat{A}$$

with  $f^{-1}(H) \in \mathcal{H}_{\text{diam}(\partial) \leq R}(X)$ . For convenience, let  $D < \infty$  be the uniform distance  $d(1_X, g \circ f)$ .

To this end, note that  $\widehat{f}(U) \in \widehat{B}$  iff  $U \in \widehat{f^{-1}(B)}$ . Since  $U \in \widehat{A}$ , the latter can occur if  $|A \triangle f^{-1}(B)| < \infty$ , and so we need to find such a  $B \in \mathcal{H}_{\partial < \infty}(Y)$  with the additional property that  $f^{-1}(B) \subseteq A$ .

*Attempt 1.* Set  $B := g^{-1}(A) \in \mathcal{H}_{\partial < \infty}(Y)$ . Then  $f^{-1}(B) \subseteq \text{Ball}_D(A)$  since if  $(g \circ f)(x) \in A$ , then

$$d(x, A) \leq d(x, (g \circ f)(x)) \leq d(1_X, g \circ f) = D.$$

By local-finiteness of  $G$ , we see that  $A \triangle f^{-1}(B) = A \setminus f^{-1}(B)$  is finite, as desired.

However, it is *not* the case that  $f^{-1}(B) \subseteq A$ . To remedy this, we ‘shrink’  $A$  by  $D$  to  $A'$  so that  $\text{Ball}_D(A') \subseteq A$ , and take  $B := g^{-1}(A')$  instead. Indeed,  $A' := \neg \text{Ball}_D(\neg A) \subseteq A$  works, since  $f^{-1}(B) \subseteq \text{Ball}_D(A')$  as before,

so  $A' \triangle f^{-1}(B) = A' \setminus f^{-1}(B)$  is finite. Also,  $A \triangle A'$  is finite since  $x \in A \triangle A'$  iff  $x \in A$  and  $d(x, \neg A) \leq D$ , so  $A \triangle f^{-1}(B)$  is finite too. It remains to show that  $\text{Ball}_D(A') \subseteq A$ , for then  $f^{-1}(B) \subseteq A$  as desired.

Indeed, if  $y \in \text{Ball}_D(A')$ , then by the (reverse) triangle-inequality we have  $d(y, \neg A) \geq d(x, \neg A) - d(x, y)$  for all  $x \in A'$ . But  $d(x, \neg A) > D$ , strictly, so  $d(y, \neg A) > D - D = 0$ , and hence  $y \in A$ . ■

**Corollary 2.6.** *If  $(X, G)$  is a locally-finite quasi-tree, then  $\mathcal{H}_{\text{diam}(\partial) \leq R}$  is dense towards ends of  $G$  for some  $R < \infty$ .*

*Proof.* Observe that  $\mathcal{H}_{\text{diam}(\partial) \leq 2}(T)$  is dense towards ends for any tree  $T$ , and invoke Proposition 2.5. ■

### 3. POCSSETS AND ORIENTATIONS

**3.1. Pocsets of cuts.** The family  $\mathcal{H}_{\text{diam} \partial \leq R}(X)$  of cuts have the structure of a ‘profinite pocset’, which we first study abstractly. We then deduce some properties of the pocset induced by a *dense* family of cuts.

**3.1.1. Abstract profinite pocsets.**

**Definition 3.1.** A *pocset*  $(\mathcal{H}, \leq, \neg, 0)$  is a poset  $(\mathcal{H}, \leq)$  equipped with an order-reversing involution  $\neg : \mathcal{H} \rightarrow \mathcal{H}$  and a least element  $0 \neq \neg 0$  such that  $0$  is the only lower-bound of  $H, \neg H$  for every  $H \in \mathcal{H}$ .

A *profinite pocset* is a pocset  $\mathcal{H}$  equipped with a compact topology making  $\neg$  continuous and is *totally order-disconnected*, in the sense that if  $H \not\leq K$ , then there is a clopen upward-closed  $U \subseteq \mathcal{H}$  with  $H \in U \not\leq K$ .

**Remark 3.2.** Such a topology is automatically Hausdorff and zero-dimensional.

We are primarily interested in subpocsets of  $(2^X, \subseteq, \neg, \emptyset)$ , which is profinite if equipped with the product topology of the discrete space  $2$ . Indeed,  $2^X$  admits a base of *cylinder sets* – which are finite intersections of sets of the form  $\pi_x^{-1}(i)$  where  $x \in X$ ,  $i \in \{0, 1\}$ , and  $\pi_x : 2^X \rightarrow 2$  is the projection – making  $\neg$  continuous since cylinders are clopen. **Show that it is totally order-disconnected.**

The following proposition gives a sufficient criteria for subpocsets of  $2^X$  to be profinite. We also show in this case that every non-trivial element  $H \in \mathcal{H}^*$  is isolated, which will be important in Section 3.2.

**Proposition 3.3.** *Let  $X$  be a set and  $\mathcal{H} \subseteq 2^X$  be a subpocset. If  $\mathcal{H}$  is finitely-separating, then  $\mathcal{H} \subseteq 2^X$  is closed and every non-trivial element is isolated.*

*Proof.* It suffices to show that the limit points of  $\mathcal{H}$  are trivial, so let  $A \in 2^X \setminus \{\emptyset, X\}$ . Fix  $x \in A \not\leq y$ . Since  $\mathcal{H}$  is finitely-separating, there are finitely-many  $H \in \mathcal{H}$  with  $x \in H \not\leq y$ , and for each such  $H \in \mathcal{H} \setminus \{A\}$ , we have either some  $x_H \in A \setminus H$  or  $y_H \in H \setminus A$ . Let  $U \subseteq 2^X$  be the family of all subsets  $B \subseteq X$  containing  $x$  and each  $x_H$  but not  $y$  or any  $y_H$ .

This is the desired neighborhood isolating  $A \in U$ . Indeed, it is (cl)open since it is the *finite* intersection of cylinders prescribed by the  $x_H$ ’s and  $y_H$ ’s, and it is disjoint from  $\mathcal{H} \setminus \{A\}$  by construction. ■

**3.1.2. Finiteness conditions on the pocset of a dense family of cuts.**

**3.2. The dual median graph of a pocset.** Let  $\mathcal{H} \subseteq 2^X$  be a finitely-separating subpocset, whose elements we call *half-spaces*. By Proposition 3.3,  $\mathcal{H}$  is profinite and every non-trivial element thereof is isolated.

**3.2.1. Median graph of orientations.**

**Definition 3.4.** An *orientation* on  $\mathcal{H}$  is an upward-closed subset  $U \subseteq \mathcal{H}$  containing exactly one of  $H, \neg H$  for each  $H \in \mathcal{H}$ . We let  $\mathcal{U}(\mathcal{H})$  denote the set of all orientations on  $\mathcal{H}$  and let  $\mathcal{U}^\circ(\mathcal{H})$  denote the clopen ones.

The *clopen* orientations are exactly those which are *finitely-based*, in the sense that there is a  $\subseteq$ -minimal finite subset  $U_0 \subseteq U$  such that  $U = \uparrow U_0$ , where

$$\uparrow U_0 := \bigcup_{H \in U_0} \uparrow H := \bigcup_{H \in U_0} \{K \in \mathcal{H} : K \supseteq H\}.$$

Indeed, if  $U$  is clopen, then there is a finite subset  $A \subseteq X$  such that for all  $H \in \mathcal{H}$ , we have  $H \in U$  iff there is some  $K \in U$  with  $H \cap A = K \cap A$ .

**Example 3.5.** Every  $x \in X$  induces its *principal orientation*  $\hat{x} := \{H \in \mathcal{H} : x \in H\} = \mathcal{H} \cap \pi_x^{-1}(1)$ , which is clearly clopen in  $\mathcal{H}$ , and gives us a canonical map  $\mathbf{p} : X \rightarrow \mathcal{U}^\circ(\mathcal{H})$ . Note, however, that  $\mathbf{p}$  is *not necessarily* injective; we call a fiber  $[x]_{\mathcal{H}} := \mathbf{p}^{-1}(\hat{x}) = \{y \in X : \hat{x} = \hat{y}\}$  an  *$\mathcal{H}$ -block*.

**Remark 3.6.** Intuitively, an orientation is a ‘maximally consistent’ choice of half-spaces. To formalize this, let  $\sim$  be the equivalence relation on  $\mathcal{H}$  given by  $H \sim \neg H$ , and let  $\partial : \mathcal{H} \rightarrow \mathcal{H}/\sim$  be the quotient map. Orientations  $U \subseteq \mathcal{H}$  then correspond precisely to sections  $\varphi : \mathcal{H}/\sim \rightarrow \mathcal{H}$  of  $\partial$  such that  $\varphi(\partial H) \not\subseteq \neg\varphi(\partial K)$  for every  $H, K \in \mathcal{H}$ ; the latter condition rules out ‘orientations’ of the form  $\leftarrow \mid \rightarrow$ .

The goal of this section is to canonically construct a graph whose vertices are clopen orientations on  $\mathcal{H}$ .

**Theorem 3.7.** *Let  $\mathcal{H} \subseteq 2^X$  be a finitely-separating subpocset. Then ...*

3.2.2. *Finiteness conditions on  $\mathcal{U}^\circ(\mathcal{H})$ .*

#### 4. MEDIAN GRAPHS AND THEIR CANONICAL SPANNING TREES

##### A. TREE OF ORIENTATIONS ON A NESTED POCSET OF SETS

Let  $\mathcal{H} \subseteq 2^X$  be a finitely-separating subpocset (see Definition 3.1.1), so every non-trivial element is isolated by Proposition 3.3. As shown in Theorem 3.7, the clopen orientations  $\mathcal{U}^\circ(\mathcal{H})$  form the vertices of a median graph, whose edges are given by ‘minimal base-flipping’.

We devote this appendix to show that if  $\mathcal{H}$  is *nested*, then  $\mathcal{U}^\circ(\mathcal{H})$  is in fact a tree, and so we may bypass the Borel cycle-cutting algorithm in Section 4 and obtain a treeing directly.

#### REFERENCES

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