

TREE-LIKE GRAPHINGS

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ABSTRACT. We present a streamlined exposition of a construction presented in [CPTT23], where it is proven that every locally-finite Borel graph with each component a quasi-tree induces a canonical treeable equivalence relation.

1. INTRODUCTION

The purpose of this note is to provide a streamlined proof of a particular case of a construction presented in [CPTT23], in order to better understand the general formalism developed therein. We attempt to make this note self-contained, but nevertheless urge the reader to refer to the original paper for more detailed discussion and generalizations of the results we have selected to include here.

1.1. Treeings of Equivalence Relations. A *countable Borel equivalence relation* (CBER) on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$, such that each class is countable. We are interested in special types of *graphings* on a CBER $E \subseteq X^2$, i.e. a Borel graph $G \subseteq X^2$ whose connectedness relation is precisely E . For instance, a graphing of E such that each component is a tree is called a *treeing* of E , and the CBERs that admit a treeing are said to be *treeable*. The main results of [CPTT23] is to provide new sufficient criteria for treeability of CBERs, and in particular, they prove the following

Theorem A ([CPTT23, Theorem 1.1], Corollary 2.1). *If a CBER admits a locally-finite graphing whose components are quasi-trees, then it is treeable.*

Recall that two metric spaces X and Y are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error, and X is a *quasi-tree* if it is quasi-isometric to a simplicial tree; see [DK18].

1.2. Outline of the Proof. Roughly speaking, the existence of a quasi-isometry $G|_C \rightarrow T_C$ to a simplicial tree T_C for each component $C \subseteq G$ induces a collection $\mathcal{H}(C)$ of ‘cuts’ (subsets $H \subseteq C$ with finite boundary such that both H and $C \setminus H$ are connected), which are ‘tree-like’ in the sense that

1. $\mathcal{H}(C)$ is a *walling*: each vertex $x \in C$ is on the boundary of finitely-many $H \in \mathcal{H}(C)$, and
2. $\mathcal{H}(C)$ is *dense towards ends*: each end in $G|_C$ has a neighborhood basis in $\mathcal{H}(C)$.

These cuts provide exactly the data to construct a ‘median graph’ whose vertices are ‘ultrafilters’ thereof, and condition (2) ensures that this graph has finite ‘hyperplanes’. This finiteness condition allows us to apply a Borel ‘cycle-cutting’ algorithm to obtain a subtree thereof. Each step above can be done in a uniform way to each component $C \subseteq G$, giving us the desired treeing of the CBER.

2. MAIN CONSTRUCTIONS

Corollary 2.1. *If a CBER admits a locally-finite graphing whose components are quasi-trees, then it is treeable.*

2.1. Wallings of Cuts.

2.2. Median Graph of Orientations.

2.3. Canonical Construction of a Sub-treeing.

REFERENCES

- [CPTT23] Ruiyuan Chen, Antoine Poulin, Ran Tao, and Anush Tserunyan, *Tree-like graphings, wallings, and median graphings of equivalence relations* (2023), available at <https://arxiv.org/abs/2308.13010>.
- [DK18] Cornelia Druţu and Michael Kapovich, *Geometric Group Theory*, with appendix by Bogdan Nica, Vol. 63, American Mathematical Society Colloquium Publications, 2018.