

## QUESTIONS IN LEMMA 2.61

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### 1. RESULTS FROM THE PAPER

I'll first reference some previous results, and then reproduce Lemma 2.61 and its proof here.

**Corollary 2.34.** *Every interval  $[x, y]$  in a median graph is finite. More generally, a convex hull  $\text{cvx}(\{x_0, \dots, x_n\})$  of finitely many points is finite.*

**Lemma 2.58.** *For a median graph  $(X, G)$  with finite hyperplanes,  $\mathcal{H}_{\partial < \infty}(X) \subseteq 2^X$  is the Boolean subalgebra generated by  $\mathcal{H}_{\text{cvx}}(X)$ .*

**Lemma 2.61.** *For a locally-finite median graph  $(X, G)$  with finite hyperplanes, each end has a neighborhood basis of half-spaces.*

*Proof.* Let  $U \in \hat{X} \setminus X$  be an end,  $\hat{A} \ni U$  be a clopen neighborhood where  $A \in \mathcal{H}_{\partial < \infty}(X)$ . By Corollary 2.34,  $\text{cvx}(\partial_{\text{ov}} A) \subseteq X$  is finite. By Lemma 2.58 (applied to it and each point on its outer boundary), it is a finite intersection of half-spaces. One such half-space  $H$  must not contain  $U$ , since  $U \notin \text{cvx}(\partial_{\text{ov}} A)$ . Then  $U \in \neg \hat{H}$ , whence  $A \cap \neg H \neq \emptyset$ , whence  $\neg H \subseteq A$  since  $\partial_{\text{ov}} A \cap \neg H = \emptyset$ . ■

### 2. DISCUSSION AND QUESTIONS

I'll give some more details in the proof. Let  $U \in \hat{A}$  be as above.

*Proof.* ...  $Y := \text{cvx}(\partial_{\text{ov}} A)$  is finite, so there are  $H_0, \dots, H_n \in \mathcal{H}_{\text{cvx}}(X)$  such that  $Y = \bigcap_{i < n} H_i$ . Then there is some  $i < n$  such that  $H := H_i \notin U$ , since otherwise we have  $Y \in U$ , a contradiction since  $Y$  is finite and  $U$  is a *non-principal* ultrafilter. Thus  $\neg H \in U$ , so  $U \in \neg \hat{H}$ . Since  $A \in U$ , we have  $A \cap \neg H \neq \emptyset$ . We claim that  $\neg H \subseteq A$ , so that  $U \in \neg \hat{H} \subseteq \hat{A}$  as desired.

Indeed, fix some  $x_0 \in A \cap \neg H$ . If there is some  $x \in \neg A \cap \neg H$ , then the interval  $[x_0, x]$  contains some  $y \in \partial_{\text{ov}} A$ . Since  $\neg H$  is convex, we see that  $y \in [x_0, x] \subseteq \neg H$ , so that  $y \in \partial_{\text{ov}} A \cap \neg H$ . This is absurd, since  $\partial_{\text{ov}} A \subseteq Y \subseteq H$ . ■

**Question 2.1.** Lemma 2.58 only guarantees that every  $A \in \mathcal{H}_{\partial < \infty}(X)$  is a finite *boolean combination* of half-spaces in  $X$ , not finite *intersections* thereof.

Assuming that  $Y = \bigcap_{i < n} H_i$ , I think the rest of the proof applies also for non-locally-finite graphs, so somehow this should be justified by local-finiteness.

**Question 2.2.**