

# TREE-LIKE GRAPHINGS

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ABSTRACT. We present a streamlined exposition of a construction presented in [CPTT23], where it is proven that every locally-finite Borel graph with each component a quasi-tree induces a canonical treeable equivalence relation. *Write some more details...*

## 1. INTRODUCTION

The purpose of this note is to provide a streamlined proof of a particular case of a construction presented in [CPTT23], in order to better understand the general formalism developed therein. We attempt to make this note self-contained, but nevertheless urge the reader to refer to the original paper for more detailed discussion and generalizations of the results we have selected to include here.

**1.1. Treeings of equivalence relations.** A *countable Borel equivalence relation* (CBER) on a standard Borel space  $X$  is a Borel equivalence relation  $E \subseteq X^2$  with each class countable. We are interested in special types of *graphings* on a CBER  $E \subseteq X^2$ , i.e. a Borel graph  $G \subseteq X^2$  whose connectedness relation is precisely  $E$ . For instance, a graphing of  $E$  such that each component is a tree is called a *treeing* of  $E$ , and the CBERs that admit a treeing are said to be *treeable*. The main results of [CPTT23] is to provide new sufficient criteria for treeability of CBERs, and in particular, they prove the following

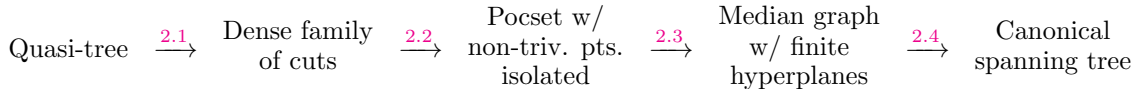
**Theorem A** ([CPTT23, Theorem 1.1]). *If a CBER admits a locally-finite graphing whose components are quasi-trees, then it is treeable.*

Recall that metric spaces  $X$  and  $Y$  are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error, and  $X$  is a *quasi-tree* if it is quasi-isometric to a simplicial tree; see [Gro93] and [DK18].

**1.2. Outline of the proof.** Roughly speaking, the existence of a quasi-isometry  $G|C \rightarrow T_C$  to a simplicial tree  $T_C$  for each component  $C \subseteq X$  induces a collection  $\mathcal{H}(C)$  of ‘cuts’ (subsets  $H \subseteq C$  with finite boundary such that both  $H$  and  $C \setminus H$  are connected), which are ‘tree-like’ in the sense that

1.  $\mathcal{H}(C)$  is *finitely-separating*: each pair  $x, y \in C$  are separated by finitely-many  $H \in \mathcal{H}(C)$ , and
2.  $\mathcal{H}(C)$  is *dense towards ends*: each end in  $G|C$  has a neighborhood basis in  $\mathcal{H}(C)$ .

These cuts have the structure of a pocset with non-trivial points isolated, which in turn provide exactly the data to construct a ‘median graph’ whose vertices are ‘ultrafilters’ thereof. Condition (2) then ensures that this graph has finite ‘hyperplanes’, which allows us to apply a Borel ‘cycle-cutting’ algorithm and obtain a canonical spanning tree thereof. Each step above can be done in a uniform way to each component  $C \subseteq G$ , giving us the desired treeing of the CBER.



**Remark.** We follow [CPTT23, Convention 2.7], where for a family  $\mathcal{H} \subseteq 2^X$  of subsets of a fixed set  $X$ , we write  $\mathcal{H}^* := \mathcal{H} \setminus \{\emptyset, X\}$  for the *non-trivial* elements of  $\mathcal{H}$ .

## 2. DETAILED CONSTRUCTIONS

**2.1. Graphs with dense families of cuts.** Let  $(X, G)$  be a locally-finite graph.

2.1.1. *End compactification of graphs.*

**2.2. Pocsets of cuts.** The family  $\mathcal{H}_{\text{diam } \partial \leq R}(X)$  of cuts have the structure of a ‘profinite pocset’, which we first study abstractly. We then deduce some properties of the pocset induced by a *dense* family of cuts.

2.2.1. *Abstract profinite pocsets.*

**Definition 2.1.** A *pocset*  $(P, \leq, \neg, 0)$  is a poset  $(P, \leq)$  equipped with an order-reversing involution  $\neg : P \rightarrow P$  and a least element  $0 \neq \neg 0$  such that  $0$  is the only lower-bound of  $p, \neg p$  for every  $p \in P$ .

A *profinite pocset* is a pocset  $P$  equipped with a compact topology making  $\neg$  continuous and is *totally order-disconnected*, in the sense that if  $p \not\leq q$ , then there is a clopen upward-closed  $U \subseteq P$  with  $p \in U \not\leq q$ .

**Remark 2.2.** Such a topology is automatically Hausdorff and zero-dimensional.

We are primarily interested in subpocsets of  $(2^X, \subseteq, \neg, \emptyset)$ , which is profinite if equipped with the product topology of the discrete space  $2$ . Indeed,  $2^X$  admits a base of *cylinder sets* – which are finite intersections of sets of the form  $\pi_x^{-1}(i)$  where  $x \in X$ ,  $i \in \{0, 1\}$ , and  $\pi_x : 2^X \rightarrow 2$  is the projection – making  $\neg$  continuous since cylinders are clopen. **Show that it is totally order-disconnected.**

The following proposition gives a sufficient criteria for subpocsets of  $2^X$  to be profinite. We also show in this case that every non-trivial element  $H \in \mathcal{H}^*$  is isolated, which will important in Section 2.3.

**Proposition 2.3.** *Let  $X$  be a set and  $\mathcal{H} \subseteq 2^X$  be a subpocset. If  $\mathcal{H}$  is finitely-separating, then  $\mathcal{H} \subseteq 2^X$  is closed and every non-trivial element is isolated.*

*Proof.* It suffices to show that the limit points of  $\mathcal{H}$  are trivial, so let  $A \in 2^X \setminus \{\emptyset, X\}$ . Fix  $x \in A \not\leq y$ . Since  $\mathcal{H}$  is finitely-separating, there are finitely-many  $H \in \mathcal{H}$  with  $x \in H \not\leq y$ , and for each such  $H \in \mathcal{H} \setminus \{A\}$ , we have either some  $x_H \in A \setminus H$  or  $y_H \in H \setminus A$ . Let  $U \subseteq 2^X$  be the family of all subsets  $B \subseteq X$  containing  $x$  and each  $x_H$  but not  $y$  or any  $y_H$ .

This is the desired neighborhood isolating  $A \in U$ . Indeed, it is (cl)open since it is the *finite* intersection of cylinders prescribed by the  $x_H$ ’s and  $y_H$ ’s, and it is disjoint from  $\mathcal{H} \setminus \{A\}$  by construction. ■

2.2.2. *The pocset induced by a dense family cuts.*

**2.3. The dual median graph of a pocset.** Let  $\mathcal{H} \subseteq 2^X$  be a finitely-separating pocset, whose elements we call *half-spaces*.

**Definition 2.4.** An *orientation* on  $\mathcal{H}$  is an upward-closed subset  $U \subseteq \mathcal{H}$  containing exactly one of  $H, \neg H$  for each  $H \in \mathcal{H}$ . We let  $\mathcal{U}(\mathcal{H})$  denote the set of all orientations on  $\mathcal{H}$ , and let  $\mathcal{U}^\circ(\mathcal{H}) \subseteq \mathcal{U}(\mathcal{H})$  denote the clopen orientations on  $\mathcal{H}$ .

**Remark 2.5.** Intuitively, an orientation is a ‘maximally consistent’ choice of a half-space from each hyperplane. To formalize this, let  $\sim$  be the equivalence relation on  $\mathcal{H}$  given by  $H \sim \neg H$ , and let  $\partial : \mathcal{H} \rightarrow \mathcal{H}/\sim$  be the quotient map. Orientations  $U \subseteq \mathcal{H}$  then correspond precisely to sections  $\varphi : \mathcal{H}/\sim \rightarrow \mathcal{H}$  of  $\partial$  such that  $\varphi(\partial H) \not\leq \neg \varphi(\partial K)$  for every  $H, K \in \mathcal{H}$ ; the latter condition rules out ‘orientations’ of the form  $\leftarrow \mid \rightarrow$ .

The *clopen* orientations  $U \in \mathcal{U}^\circ(\mathcal{H})$  are then those

**Theorem 2.6.**

**2.4. Spanning trees of median graphs with finite hyperplanes.**

### A. TREE OF ORIENTATIONS ON A NESTED POCKET OF SETS

Let  $\mathcal{H} \subseteq 2^X$  be a finitely-separating subpocset (see Definition 2.2.1), so every non-trivial element is isolated by Proposition 2.3. As shown in Theorem 2.6, the clopen orientations  $\mathcal{U}^\circ(\mathcal{H})$  form the vertices of a median graph, whose edges are given by ‘minimal base-flipping’.

We devote this appendix to show that if  $\mathcal{H}$  is *nested*, then  $\mathcal{U}^\circ(\mathcal{H})$  is in fact a tree, and so we may bypass the Borel cycle-cutting algorithm in Section 2.4 and obtain a treeing directly.

## REFERENCES

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- [Gro93] Mikhail Gromov, *Asymptotic invariants of infinite groups*, Geometric Group Theory, Vol. 2 (Sussex, 1991), London Mathematical Society Lecture Note Series, vol. 182, Cambridge University Press, 1993.
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