TREE OF ORIENTATIONS ON A NESTED COLLECTION OF CUTS

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1. Introduction

With the definitions in Section 2, we prove the following

Theorem 1.1. *If (X, G) is a graph with a nested collection $\mathcal{C} \subseteq 2^X$ of cuts, then the graph $\mathcal{T}_\mathcal{C}$ whose:*

- *Vertices are finitely-based orientations on $\mathcal{C}$; and whose*
- *Neighbors of $\mathcal{U} \in \mathcal{T}_\mathcal{C}$ are $\mathcal{U} \triangle \{A, A^c\}$ for every minimal $A \in \mathcal{U}$;*

is acyclic. Furthermore, if $\mathcal{C}$ is closed under complements, then $\mathcal{T}_\mathcal{C}$ is a tree.

2. Preliminaries

2.1. **Orientations.** Let $\mathcal{C} \subseteq 2^X$ be a collection of non-empty subsets of a set X. Since we do not assume that $\mathcal{C}$ is closed under complements, we slightly modify the definition of orientations, as follows.

Definition 2.1. An *orientation* on $\mathcal{C}$ is a subset $\mathcal{U} \subseteq \mathcal{C}$ such that

1. *(Upward-closure).* If $A \in \mathcal{U}$ and $B \in \mathcal{C}$ contains A, then $B \in \mathcal{U}$.
2. *(Ultra).* If $A, A^c \in \mathcal{C}$, then either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$, but not both.

Lemma 2.2. *If $\mathcal{U} \subseteq \mathcal{C}$ is an orientation, then for any $\subseteq$-minimal $A \in \mathcal{U}$ with $A^c \in \mathcal{C}$, so is $\mathcal{U} \triangle \{A, A^c\}$.*

Proof. That $\mathcal{V} := \mathcal{U} \triangle \{A, A^c\} = \mathcal{U} \cup \{A^c\} \setminus \{A\}$ is upward-closed follows from $\subseteq$-minimality of A. Now, if $B, B^c \in \mathcal{C}$ and $B^c \notin \mathcal{V}$, we need to show that $B \in \mathcal{V}$.

To this end, note that $B^c \notin \mathcal{V}$ implies $A \neq B$ and either $B = A^c$ or $B^c \notin \mathcal{U}$. The former case follows from $A^c \in \mathcal{V}$, and for the latter, we have $B \in \mathcal{U} \setminus \{A\}$ since $\mathcal{U}$ is ultra. ■

Remark 2.3. In the above notations, clearly $\mathcal{U} \neq \mathcal{U} \triangle \{A, A^c\}$. Furthermore, for any other such orientation $\mathcal{U}'$ and $A' \in \mathcal{U}'$, that $\mathcal{U} = \mathcal{U}'$ and $\mathcal{U} \triangle \{A, A^c\} = \mathcal{U}' \triangle \{A', A'^c\}$ together imply $A = A'$.

Definition 2.4. A *base* for an orientation $\mathcal{U} \subseteq \mathcal{C}$ is a $\subseteq$-minimal subset $\mathcal{B} \subseteq \mathcal{U}$ such that $\mathcal{U} = \uparrow\mathcal{B}$, where

$$\uparrow\mathcal{B} := \bigcup_{B \in \mathcal{B}} \uparrow B := \bigcup_{B \in \mathcal{B}} \{A \in \mathcal{C} : A \supseteq B\}.$$

Definition 2.5. A collection $\mathcal{C}$ is said to be *nested* if every $C_1, C_2 \in \mathcal{C}$ has an empty corner, i.e., $C_1^i \cap C_2^j = \varnothing$ for some $i, j \in \{1, -1\}$, where $C^i := C$ if $i = 1$ and $C^i := C^c$ if $i = -1$.

Remark 2.6. If $\mathcal{C}$ is nested, then every $\subseteq$-minimal $B \in \mathcal{C}$ induces an orientation $\uparrow B := \{A \in \mathcal{C} : A \supseteq B\}$, called the *principal* orientation. Indeed, $\uparrow B$ is clearly upward-closed, and if $A, A^c \in \mathcal{C}$, then, by $\subseteq$-minimality of B and nestedness of $\mathcal{C}$, either $A \supseteq B$ or $A^c \supseteq B$ (but clearly not both).

This construction generalizes to any collection $\mathcal{B} \subseteq \mathcal{C}$ with each $B \in \mathcal{B}$ being $\subseteq$-minimal, in that $\uparrow\mathcal{B}$ is an orientation on $\mathcal{C}$.

Definition 2.7. An orientation $\mathcal{U} \subseteq \mathcal{C}$ is said to be *finitely-based* if it admits a finite base.

2.2. **Cuts in graphs.** Let (X, G) be a graph.

Definition 2.8. A *cut* in X is a subset $C \subseteq X$ contained in a single connected component $Y \subseteq X$ with $\partial_{\mathsf{v}} C$ finite such that both C and $Y \setminus C$ are infinite.

Definition 2.9.

3. The graph $\mathcal{T}_\mathcal{C}$