TREE OF ORIENTATIONS ON A NESTED COLLECTION OF SETS

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Let $H \subseteq 2^X$ be a sub-pocset for some fixed set X (so that, in particular, H is closed under complements). With the definitions in Section 1, we prove the following

Theorem A (Propositions 2.5, 2.8, 2.9)**.** *If H is nested, then the graph $\mathcal{T}_H$, whose:*

- *Vertices are finitely-based orientations $p \subseteq H$;*
- *Edges are pairs $\{p, q\}$ such that $q = p \triangle \{h, \neg h\}$ for some $\subseteq$-minimal $h \in H$;*

is acyclic. Furthermore, if H is finitely-separating (or more generally, chain-vanishing), then $\mathcal{T}_H$ is a tree.

In particular, this applies to when (X, G) is a graph and H is a nested collection of cuts on X. No further assumptions on X (like local-finiteness) is needed.

1. Preliminaries

Let $H \subseteq 2^X$ be a sub-pocset for some fixed set X, whose elements $h \in H$ we call *half-spaces*.

Definition 1.1. Two elements $h, k \in H$ are *nested* if $h^i \cap k^j = \varnothing$ for some $i, j \in \{1, -1\}$, where $h^i \coloneqq h$ for $i = 1$ and $h^i \coloneqq h^c$ otherwise. We say that H is *nested* if every pair $h, k \in H$ are nested.

1.1. Orientations.

We give the standard definition of orientations on H and characterize them as 'consistent assignments of half-spaces to hyperplanes'.

Definition 1.2. An *orientation* on H is a subset $U \subseteq H$ such that

1. *(Upward-closure).* If $h \in U$ and $k \in H$ contains h, then $k \in U$.
2. *(Ultra).* For each $h \in H$, exactly one of h, h^c is contained in U.

Consider the equivalence relation $\sim$ on H generated by $h \sim h^c$ for all $h \in H$, whose classes are called *hyperplanes* $\partial h \coloneqq \{h, h^c\}$ where $\partial : H \to H/\sim$ is the projection. We show that an orientation on H is just a choice $\varphi : H/\sim \to H$ of a half-space for each hyperplane, that is consistent in the sense below.

Proposition 1.3. *An orientation $p \subseteq H$ is exactly the data of a function $\varphi : H/\sim \to H$ such that $\varphi(\partial h) \in \partial h$ and $\varphi(\partial h) \not\subseteq \varphi(\partial k)^c$ for every $h, k \in H$.*

Proof. Given an orientation $p \subseteq H$, let $\varphi_p(\partial h) \coloneqq h^i \in U$ for the unique $i \in \{1, -1\}$. That $\varphi_p(\partial h) \in \partial h$ is clear, and if $\varphi_p(\partial h) \subseteq \varphi_p(\partial k)^c$, then U contains both $\varphi_p(\partial k)$ and $\varphi_p(\partial k)^c$ by upward-closure, a contradiction.

Conversely, given such a function $\varphi : H/\sim \to H$, let $p_\varphi \coloneqq \operatorname{im} \varphi \subseteq H$. This is ultra since if $h \in H$ and $h^c \notin p_\varphi$, then $\varphi(\partial h) \in \partial h = \{h, h^c\}$ implies $h \in p_\varphi$. Furthermore, if $p_\varphi \ni h \subseteq k$, then $k^c \in p_\varphi$ implies $\varphi(\partial h) = h \subseteq k = \varphi(\partial k)^c$, a contradiction, so $k \in p_\varphi$ by the above.

Finally, given an orientation $p \subseteq H$, we have $h \in p$ iff $\varphi_p(\partial h) = h$, which occurs iff $h \in \operatorname{im} \varphi_p = p_{\varphi_p}$. Thus $p_{\varphi_p} = p$. Conversely, given $\varphi : H/\sim \to H$, and $h \in H$, we have $\varphi_{p_\varphi}(\partial h) = h^i$ iff $h^i \in p_\varphi = \operatorname{im} \varphi$, which occurs iff $\varphi(\partial h) = h^i$. Thus $\varphi_{p_\varphi} = \varphi$ too, as desired. ∎

Definition 1.4. A *base* for an orientation $p \subseteq H$ is a $\subseteq$-minimal subset $p_0 \subseteq p$ such that $p = \uparrow p_0$, where

$$\uparrow p_0 \coloneqq \bigcup_{h \in p_0} \uparrow h \coloneqq \bigcup_{h \in p_0} \{k \in H : k \supseteq h\}.$$

We say that p is *finitely-based* if it admits a finite basis.

In the above correspondence, a base for $\varphi : H/\sim \to H$ is a function $\varphi_0 \subseteq \varphi$ where $\operatorname{dom} \varphi_0$ is a $\subseteq$-minimal subset of hyperplanes such that φ_0 extends uniquely to φ. Thus, the finitely-based orientations are the ones determined by a choice of half-spaces from finitely-many hyperplanes.

Date: June 26, 2024.

1.2. **Flipping basis elements.** We now investigate the behaviour of orientation when its choice on a single half-space is modified. Although the proofs work without the characterization in Proposition 1.3, it makes orientations a lot more intuitive to me, and so will be phrased this way.

Lemma 1.5. *Let $p \subseteq H$ be an orientation. Then $q := p \triangle \{h, h^c\}$ is an orientation iff $h \in p$ is $\subseteq$-minimal. Furthermore, if p is finitely-based, then so is q.*

Proof. First, note that $\varphi_q(\partial h) = \varphi_p(h)^c = h^c$ and $\varphi_q = \varphi_p$ away from ∂h.

($\Rightarrow$). If q is an orientation and $p \ni k \subset h$, then we have a contradiction since

$$\varphi_q(\partial k) = \varphi_p(\partial k) = k \subseteq h = \varphi_q(\partial h)^c.$$

($\Leftarrow$). Conversely, let $h \in p$ be $\subseteq$-minimal and suppose that q is not an orientation. Since only $\varphi_q(\partial h) = h^c$ differs from φ_p, this can only occur if $\varphi_q(\partial h) \subseteq \varphi_q(\partial k)^c$ for some $k \in H$. But then

$$h^c = \varphi_q(\partial h) \subseteq \varphi_q(\partial k)^c = k^i \notin p$$

for some unique $i \in \{1, -1\}$, so that $p \ni k^{-i} \subseteq h$ and contradicts that $h \in p$ is $\subseteq$-minimal.

Finally, suppose that p is finitely-based. ???. ∎

Remark 1.6. If $h \in p$ is $\subseteq$-minimal and $p_0 \subseteq p$ is any basis, then $h \in p_0$. Indeed, if $h \notin p_0$, then there is some $h_0 \in p_0 \subseteq p$ with $h \supset h_0$, contradicting the minimality of h.

Conversely, if $p_0 \subseteq p$ is a basis, then every $h \in p_0$ is $\subseteq$-minimal in p. Indeed, if $h \supset k$ for some $k \in p$, then there is some $k_0 \in p_0$ with $k \supseteq k_0$, so $h \supset k_0$. Thus $p = \uparrow(p_0 \setminus \{h\})$, contradicting the minimality of p_0.

2. The graph $\mathcal{T}_H$

Let $H \subseteq 2^X$ be a sub-pocset as before. Using Lemma 1.5, we construct a graph $\mathcal{T}_H$ whose:

- *Vertices* are finitely-based orientations $p \subseteq H$;
- *Edges* are pairs $\{p, q\}$ such that $q = p \triangle \{h, h^c\}$ for some $\subseteq$-minimal $h \in H$.

The goal of this section is to establish Theorem A, stating that if H is nested, then $\mathcal{T}_H$ is acyclic (Proposition 2.5). Furthermore, $\mathcal{T}_H$ is a tree when H is finitely-separating (Proposition 2.8), or more generally, when H is chain-vanishing (Proposition 2.9).

For the rest of this section, we let $H \subseteq 2^X$ be a *nested* sub-pocset.

2.1. **Acyclicity of $\mathcal{T}_H$.** Essentially, $\mathcal{T}_H$ is acyclic because any path in $\mathcal{T}_H$ (without backtracking) is induced by a sequence of the flipped half-spaces, and those form a *strictly*-increasing chain.

Example 2.1. This fails if H is non-nested. Indeed, let $X := 2 \times 2 := \{(x, y) : x, y \in 2\}$ and consider the set H of all fibers $h_y := \{(x, y) : x \in 2\}$ for each $y \in 2$ and $h^x := \{(x, y) : y \in 2\}$ for each $x \in 2$. All orientations on H are principal, of the form $\{h^x, h_y\}$ for $x, y \in 2$, and the resulting graph $\mathcal{T}_H$ is a square.

Definition 2.2. Fix $p_0 \in V(\mathcal{T}_H)$ and $n \in \mathbb{N}$. A sequence $(h_i)_{i<n} \subseteq H$ is said to *induce a path in $\mathcal{T}_H$ from* p_0 if $(p_i)_{i<n}$, defined by $p_i := p_{i-1} \triangle \{h_{i-1}, h_{i-1}^c\}$ for every $1 \leq i < n$, is a path in $\mathcal{T}_H$ with each $h_i \in p_i$.

Lemma 2.3. *A path from p_0 induced by $(h_i)_{i<n}$, $n \geq 3$, has no backtracking iff $h_i \neq h_{i-1}^c$ for every $1 \leq i < n$.*

Proof. Take $2 \leq i \leq n$. It suffices to show that $p_{i-2} = p_i$ iff $h_{i-1} = h_{i-2}^c$.

($\Rightarrow$). We have by definition that $p_i = p_{i-2} \cup \{h_{i-1}^c, h_{i-2}^c\} \setminus \{h_{i-1}, h_{i-2}\}$, so since $h_{i-2} \in p_{i-2} = p_i$, we have $h_{i-2} = h_{i-1}^c$ as desired.

($\Leftarrow$). Again by definition, by noting that the basis-flipping cancels out. ∎

Lemma 2.4. *If $(h_i)_{i<n}$ induces a path in $\mathcal{T}_H$ with no backtracking, then $(h_i)_{i<n}$ is strictly increasing.*

Proof. By Lemma 2.3, we have $h_i \neq h_{i-1}^c$ for every $1 \leq i < n$. Thus, since $h_i \in p_i = p_{i-1} \cup \{h_{i-1}^c\} \setminus \{h_{i-1}\}$, we see that $h_i \in p_{i-1}$. Clearly $h_i \neq h_{i-1}$. It suffices to remove the three cases when $h_i \subseteq h_{i-1}$, $h_{i-1} \subseteq h_i^c$, and $h_i^c \subseteq h_{i-1}$, since then nestedness of H gives us $h_{i-1} \subsetneq h_i$, as desired.

- If $h_i \subseteq h_{i-1}$, then $h_{i-1} \in p_i$, contradicting the definition of p_i.
- If $h_{i-1} \subseteq h_i^c$, then $h_i^c \in p_{i-1}$ by upward-closure of p_{i-1}, a contradiction.
- If $h_i^c \subseteq h_{i-1}$, then $h_{i-1} \in p_{i+1}$ by upward-closure of $p_{i+1} \ni h_i^c$. But since $h_{i-1} \neq h_i^c$, we have by definition of p_{i+1} that $h_{i-1} \in p_i$, a contradiction. ∎

Proposition 2.5. *$\mathcal{T}_H$ is acyclic.*

Proof. Let $(p_i)_{i<n}$ be a cycle in $\mathcal{T}_H$ induced by $(h_i)_{i<n}$. Since cycles are non-backtracking, we have $h_0 \subsetneq h_0$ by Lemma 2.4, a contradiction. ∎

2.2. **Connectedness of $\mathcal{T}_H$.** In general, $\mathcal{T}_H$ need not be connected, as shown by the following

Example 2.6 (Tserunyan)**.** Let $X \coloneqq \omega_1 \coloneqq \omega + 1 \coloneqq \omega \sqcup \{\omega\}$ and let

$$H \coloneqq \{[0,n] : n < \omega\} \cup \{(n,\omega] : n < \omega\} \cup \{\omega, \{\omega\}\}.$$

We claim that every finitely-based orientation $p \subseteq H$ is principal (i.e., of the form $\widehat{x} \coloneqq \{h \in H : h \ni x\}$ for some $x \in X$).

Proof. For each $h \in H$, the hyperplane ∂h is either $\{[0,n],(n,\omega]\}$ for some $n < \omega$, or $\{\omega, \{\omega\}\}$. Consider the set $N_p \coloneqq \{n < \omega : [0,n] \in p\}$, for which we have two cases.

- If $N_p = \varnothing$, then $(n,\omega] \in p$ for all p. We claim that $p = \widehat{\omega}$. If $\omega \in p$, then p is not (finitely-)based since $(n,\omega]$, for n large enough, does not contain any basis element. Thus $\{\omega\} \in p$. Both types of half-spaces contain ω, and those are the only half-spaces in p.
- If $N_p \neq \varnothing$, then there is a minimal $n < \omega$ with $[0,n] \in p$; we claim that $p = \widehat{n}$. Indeed, we have $(m,\omega] \in p$ if $m < n$, $[0,m] \in p$ if $m \geq n$, and $\omega \in p$ by upward-closure. All three types of half-spaces contain n, and those are the only half-spaces in p. ∎

Now, each orientation $\widehat{n}$ is generated by the basis $\{[0,n],(n-1,\omega]\}$, and $\widehat{\omega}$ is generated by $\{\{\omega\}\}$. Thus, we see that $\widehat{\alpha}, \widehat{\beta}$, where $\alpha < \beta \leq \omega$, is joined by an edge iff $\beta = \alpha + 1$, where the $\subseteq$-minimal element $[0,\alpha] \in \widehat{\alpha}$ is flipped to $(\alpha,\omega] = (\beta - 1, \omega] \in \widehat{\beta}$. This shows that $\mathcal{T}_H$ is disconnected, with ω an isolated point.

Sufficient criteria ruling out Example 2.6 are given in Proposition 2.8 and 2.9. We need the following

Lemma 2.7. *There is a path between $p, q \in V(\mathcal{T}_H)$ iff $p \triangle q$ is finite, in which case $d(p,q) = (p \triangle q)/2$.*

Proof. If $p \triangle q = \{h_1, \ldots, h_n\} \cup \{k_1, \ldots, k_m\}$ with $p \setminus q = \{h_i\}$ and $q \setminus q = \{k_j\}$, then $h_i^c \in q \setminus p$ and $k_j^c \in p \setminus q$ for all $i < n$ and $j < m$, so $n = m$ and $q = p \cup \{h_i^c\} \setminus \{h_i\}$. We claim that there is a permutation $\sigma \in S_n$ such that $(h_{\sigma(i)})$ induces a path from $p =: p_0$, which is the desired path from p to q. Choose a minimal $h \in \{h_i\}$, which is also minimal in p: if $k \subseteq h$ for some $k \in p$, then $h^c \subseteq k^c$, and hence $k^c \in q$, so $k = h_i \subseteq h$ for some i, forcing $k = h$. Set $p_1 \coloneqq p \triangle \{h, h^c\}$, which is an orientation by Lemma 1.5. Continuing in this manner by choosing a minimal element in $\{h_i\} \setminus \{h\}$ gives us the desired path.

Conversely, suppose that $\mathcal{T}_H$ is connected and let $(p_i)_{i<n}$ be a path from $p = p_0$ to $q = p_{n-1}$. Then, letting $h_i \coloneqq p_i \triangle p_{i-1}$ for all $1 \leq i < n$ gives us a sequence $(h_i)_{i<n}$ inducing such a path, whence $p \triangle q$ consists of $\{h_i\}$ and their complements; hence, $p \triangle q$ is finite. ∎

Proposition 2.8. *If H is finitely-separating (i.e., for every $h, h' \in H$, there are finitely-many $k \in H$ with $h \subseteq k \subseteq h'$), then $\mathcal{T}_H$ is connected (and hence a tree).*

Proof. Fix $p, q \in V(\mathcal{T}_H)$. Since they are finitely-based, we can use Remark 1.6 to choose minimal elements $h_1 \in p \setminus q$ and $h_2 \in q \setminus p$, and by assumption there are finitely-many $k \in H$ with $h_1 \subseteq k \subseteq h_2^c$, say $k_1, \ldots, k_n$. We claim that $p \triangle q \subseteq \{k_i\} \cup \{k_i^c\}$, so the result follows from Lemma 2.7.

Indeed, it suffices to show that $k \in \{k_i\}$ for every $k \in p \setminus q$, for then any $k' \in q \setminus p$ is in $k' \in \{k_i^c\}$. Let $k \in p \setminus q$, which by minimality of h_1, implies $k \not\subseteq h_1$. Thus, either $k^c \subseteq h_1$, $k \subseteq h_1^c$, or $h_1 \subseteq k$ by nestedness of H, and the first two cases are impossible since either case forces $h_1 \in q$. A similar argument using the observation that $k^c \in q \setminus p$, and hence $k^c \not\subseteq h_2$, shows that $k \subseteq h_2^c$. Thus $h_1 \subseteq k \subseteq h_2^c$, as desired. ∎

Proposition 2.9.