TREE-LIKE GRAPHINGS OF COUNTABLE BOREL EQUIVALENCE RELATIONS

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Abstract. We present a streamlined exposition of a construction presented recently by R. Chen, A. Poulin, R. Tao, and A. Tserunyan, where it is proven that every locally-finite Borel graph with each component a quasi-tree induces a canonical treeable equivalence relation. Write some more details...

0. Introduction

The purpose of this note is to provide a streamlined proof of a particular case of a construction presented in [CPTT23], in order to better understand the general formalism developed therein. We attempt to make this note self-contained, but nevertheless urge the reader to refer to the original paper for more detailed discussions and some generalizations of the results we have selected to include here.

0.1. **Treeings of equivalence relations.** A *countable Borel equivalence relation (CBER)* on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$ with each class countable. We are interested in special types of *graphings* on a CBER $E \subseteq X^2$, i.e. a Borel graph $G \subseteq X^2$ whose connectedness relation is precisely E. For instance, a graphing of E such that each component is a tree is called a *treeing* of E, and the CBERs that admit treeings are said to be *treeable*. The main results of [CPTT23] provide new sufficient criteria for treeability of certain classes of CBERs, and in particular, they prove the following

Theorem A ([CPTT23, Theorem 1.1])**.** *If a CBER admits a locally-finite graphing whose components are quasi-trees, then it is treeable.*

Recall that metric spaces X and Y are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error, and X is a *quasi-tree* if it is quasi-isometric to a simplicial tree; see [Gro93] and [DK18].

0.2. **Outline of the proof.** Roughly speaking, the existence of a quasi-isometry $G|C \to T_C$ to a simplicial tree T_C for each component $C \subseteq X$ induces a collection $\mathcal{H}(C)$ of 'cuts' (subsets $H \subseteq C$ with finite boundary such that both H and $C \setminus H$ are connected), which are 'tree-like' in the sense that

1. $\mathcal{H}(C)$ is *finitely-separating*: each pair $x, y \in C$ is separated by finitely-many $H \in \mathcal{H}(C)$, and
2. $\mathcal{H}(C)$ is *dense towards ends*: each end in $G|C$ has a neighborhood basis in $\mathcal{H}(C)$.

By Condition (1), these cuts have the structure of a profinite pocset with non-trivial points isolated, which in turn provide exactly the data to construct a 'median graph' whose vertices are 'ultrafilters'[1] thereof. Condition (2) then ensures that this graph has finite 'hyperplanes', which allows us to apply a Borel 'cycle-cutting' algorithm and obtain a canonical spanning tree. Each step above can be done in a uniform way to each component $C \subseteq G$, giving us the desired treeing of the CBER.

$$\text{Quasi-tree} \xrightarrow{1} \begin{matrix}\text{Dense family}\\ \text{of cuts}\end{matrix} \xrightarrow{2} \begin{matrix}\text{Pocset w/}\\ \text{non-triv. pts.}\\ \text{isolated}\end{matrix} \xrightarrow{3} \begin{matrix}\text{Median graph}\\ \text{w/ finite}\\ \text{hyperplanes}\end{matrix} \xrightarrow{4} \begin{matrix}\text{Canonical}\\ \text{spanning tree}\end{matrix}$$

Write some more stuff to tie things together...

Remark. We follow [CPTT23, Convention 2.7], where for a family $\mathcal{H} \subseteq 2^X$ of subsets of a fixed set X, we write $\mathcal{H}^* := \mathcal{H} \setminus \{\varnothing, X\}$ for the *non-trivial* elements of $\mathcal{H}$.

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[1] As in [CPTT23], we call them *orientations* instead, to avoid confusion with the more standard notion; see Definition 3.1.

1. Graphs with Dense Families of Cuts

1.1. **Ends of graphs.** Let (X, G) be a connected locally-finite graph, which, in the context of Theorem A, will stand for a single component of the locally-finite graphing of a CBER.

Definition 1.1. For a subset $A \subseteq X$, we let $\partial_{\mathsf{iv}} A := A \cap \mathrm{Ball}_1(\neg A)$ be its *inner vertex boundary*, $\partial_{\mathsf{ov}} A := \partial_{\mathsf{iv}}(\neg A)$ be its *outer vertex boundary*, and let $\partial_{\mathsf{ie}} A := G \cap (\partial_{\mathsf{ov}} A \times \partial_{\mathsf{iv}} A)$ and $\partial_{\mathsf{oe}} A := \partial_{\mathsf{ie}}(\neg A)$ respectively be its *inner* and *outer edge boundaries*. Let $\partial_{\mathsf{v}} A := \partial_{\mathsf{iv}} A \cup \partial_{\mathsf{ov}} A$ be the *(total) vertex boundary* of A.

Let $\mathcal{H}_{\partial<\infty}(X) \subseteq 2^X$ be the Boolean algebra of all $A \subseteq X$ with finite vertex boundary, which we refer to as *cuts* in X. Later, we will restrict to the connected and co-connected cuts $\mathcal{H}_{\partial<\infty}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$.

Definition 1.2. The *end compactification* of (X, G) is the Stone space $\widehat{X}$ of $\mathcal{H}_{\partial<\infty}(X)$, whose non-principal ultrafilters are the *ends* of (X, G).

We identify $X \hookrightarrow \widehat{X}$ via principal ultrafilter map $x \mapsto p_x$, so $\widehat{X} \setminus X$ is the set of ends of G. By definition, $\widehat{X}$ admits a basis of clopen sets of the form $\widehat{A} := \{p \in \widehat{X} : A \in p\}$ for each $A \in \mathcal{H}_{\partial<\infty}(X)$.

Definition 1.3. A family $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty}(X)$ of cuts is *dense towards ends* of (X, G) if $\mathcal{H}$ contains a neighborhood basis for every end in $\widehat{X} \setminus X$.

In other words, $\mathcal{H}$ is dense towards ends if for every $p \in \widehat{X} \setminus X$ and every (clopen) neighborhood $\widehat{A} \ni p$, where $A \in \mathcal{H}_{\partial<\infty}(X)$, there is some $H \in \mathcal{H}$ with $p \in \widehat{H} \subseteq \widehat{A}$; it is useful to note that $\widehat{H} \subseteq \widehat{A}$ iff $H \subseteq A$.

The goal of this section is to show that certain cuts $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty}$ induced in a (locally-finite) quasi-tree is dense towards ends. It will also be important that these cuts be connected, in that witnesses to density can also be found in $\mathcal{H} \cap \mathcal{H}_{\mathrm{conn}}$, consisting of those cuts that are connected and co-connected.

Lemma 1.4. *If a subpocset $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty}$ is dense towards ends, then there is a subpocset $\mathcal{H}' \subseteq \mathcal{H}_{\partial<\infty} \cap \mathcal{H}_{\mathrm{conn}}$, which is also dense towards ends, such that every $H' \in \mathcal{H}'$ has $\partial_{\mathsf{ie}} H' \subseteq \partial_{\mathsf{ie}} H$ for some $H \in \mathcal{H}$.*

Proof. A first attempt is to let $\mathcal{H}'$ be the connected components H'_0 of elements in $\mathcal{H}$, but this fails since $\neg H'_0$ is not necessarily connected. Instead, we further take a component of $\neg H'_0$, whose complement clearly co-connected, and is connected since it consists of H'_0 and the other components of $\neg H'_0$, each of which is connected to H'_0 via $\partial_{\mathsf{ie}} H'_0$. Formally, we let

$$\mathcal{H}' := \{H' \subseteq X : H \in \mathcal{H} \text{ and } H'_0 \in H/G \text{ and } \neg H' \in \neg H'_0/G\},$$

where H/G denotes the G-components of H. Clearly $\partial_{\mathsf{ie}} H' \subseteq \partial_{\mathsf{ie}} H'_0 \subseteq \partial_{\mathsf{ie}} H$, so it remains to show that $\mathcal{H}'$ is dense towards ends.

Fix an end $p \in \widehat{X} \setminus X$ with $p \in \widehat{A}$ for some $A \in \mathcal{H}_{\partial<\infty}(X)$. Let $B \supseteq \partial_{\mathsf{v}} A$ be finite connected, which can be obtained by adjoining paths between its components. Then $\neg B \in p$ since p is non-principal, so there is $H \in \mathcal{H}$ with $p \in \widehat{H} \subseteq \neg\widehat{B}$. We now use the above recipe to find the desired $H' \in \mathcal{H}'$:

(1) Since $H \in \mathcal{H}_{\partial<\infty}(X)$, it has finitely-many connected components, so exactly one of them belongs to p, say $p \in \widehat{H}'_0 \subseteq \widehat{H}$. Note that $B \subseteq \neg H \subseteq \neg H'_0$.
(2) Since B is connected, there is a unique component $\neg H' \subseteq \neg H'_0$ containing B.

Observe that $H' \in \mathcal{H}'$ and $p \in \widehat{H}'$. Lastly, since H' is connected and is disjoint from $\partial_{\mathsf{v}} A \subseteq B$, and since $H' \subseteq \neg A$ would imply $\neg H' \in p$, this forces $H' \subseteq A$, and hence $p \in \widehat{H}' \subseteq \widehat{A}$ as desired. ∎

1.2. **Dense cuts induced by quasi-trees.** If X is a quasi-tree – and thus does not have arbitrarily long cycles – we expect that there is some finite bound $R < \infty$ such that the ends in $\widehat{X} \setminus X$ are 'limits' of cuts $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \subseteq \mathcal{H}_{\partial<\infty}(X)$ with boundary diameter bounded by R. We show that this is indeed the case, in the sense that $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ is dense towards ends of (X, G).

Lemma 1.5. *Let $f : (X, G) \to (Y, T)$ be a coarse-equivalence between connected graphs. For a fixed $H \in \mathcal{H}_{\partial<\infty}(Y)$, $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$ is uniformly bounded in terms of $\mathrm{diam}(\partial_{\mathsf{v}} H)$.*

Proof. Since f is bornologous, let $S < \infty$ be such that xGx' implies $d(f(x), f(x')) \leq S$, so that for any $(x, x') \in \partial_{\mathsf{ie}} f^{-1}(H)$, there is a path of length $\leq S$ between $f(x) \notin H$ and $f(x') \in H$. Thus both $d(f(x), \partial_{\mathsf{v}} H)$ and $d(f(x'), \partial_{\mathsf{v}} H)$ are bounded by S, so $f(\partial_{\mathsf{v}} f^{-1}(H)) \subseteq \mathrm{Ball}_S(\partial_{\mathsf{v}} H)$ and hence

$$\mathrm{diam}(f(\partial_{\mathsf{v}} f^{-1}(H))) \leq \mathrm{diam}(\partial_{\mathsf{v}} H) + 2S.$$

That f is a coarse-*equivalence* gives us a uniform bound of $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$ in terms of $\mathrm{diam}(\partial_{\mathsf{v}} H)$. ∎

In particular, if $\operatorname{diam}(\partial_{\mathsf{v}} H)$ is itself also uniformly bounded, then so is $\operatorname{diam}(\partial_{\mathsf{v}} f^{-1}(H))$.

Proposition 1.6. *The class of connected locally-finite graphs in which $\mathcal{H}_{\operatorname{diam}(\partial)\leq R}$ is dense towards ends for some $R < \infty$ is invariant under coarse equivalence.*

Proof. Let (X, G), (Y, T) be connected locally-finite graphs, $f : X \to Y$ be a coarse equivalence with quasi-inverse $g : Y \to X$, and suppose $\mathcal{H}_{\operatorname{diam}(\partial)\leq S}(Y)$ is dense towards ends for some $S < \infty$. By Lemma 1.5, pick some $R < \infty$ so that for any $H \in \mathcal{H}_{\operatorname{diam}(\partial)\leq S}(Y)$, we have $f^{-1}(H) \in \mathcal{H}_{\operatorname{diam}(\partial)\leq R}(X)$.

Fix an end $p \in \widehat{X} \setminus X$ with $p \in \widehat{A}$ for some $A \in \mathcal{H}_{\partial<\infty}(X)$. We need to find some $B \in \mathcal{H}_{\partial<\infty}(Y)$ such that $\widehat{f}(p) \in \widehat{B}$ and $f^{-1}(B) \subseteq A$, for then $\widehat{f}(p) \in \widehat{H}$ for some $B \supseteq H \in \mathcal{H}_{\operatorname{diam}(\partial)\leq S}(Y)$, and hence we have

$$p \in \widehat{f^{-1}(H)} \subseteq \widehat{f^{-1}(B)} \subseteq \widehat{A}$$

with $f^{-1}(H) \in \mathcal{H}_{\operatorname{diam}(\partial)\leq R}(X)$. For convenience, let $D < \infty$ be the uniform distance $d(1_X, g \circ f)$.

To this end, note that $\widehat{f}(p) \in \widehat{B}$ iff $p \in \widehat{f^{-1}(B)}$. Since $p \in \widehat{A}$, the latter can occur if $|A \triangle f^{-1}(B)| < \infty$, and so we need to find such a $B \in \mathcal{H}_{\partial<\infty}(Y)$ with the additional property that $f^{-1}(B) \subseteq A$.

> *Attempt 1.* Set $B \coloneqq g^{-1}(A) \in \mathcal{H}_{\partial<\infty}(Y)$. Then $f^{-1}(B) \subseteq \operatorname{Ball}_D(A)$ since if $(g \circ f)(x) \in A$, then
> $$d(x, A) \leq d(x, (g \circ f)(x)) \leq d(1_X, g \circ f) = D.$$
> By local-finiteness of G, we see that $A \triangle f^{-1}(B) = A \setminus f^{-1}(B)$ is finite, as desired.

However, it is *not* the case that $f^{-1}(B) \subseteq A$. To remedy this, we 'shrink' A by D to A' so that $\operatorname{Ball}_D(A') \subseteq A$, and take $B \coloneqq g^{-1}(A')$ instead. Indeed, $A' \coloneqq \neg \operatorname{Ball}_D(\neg A) \subseteq A$ works, since $f^{-1}(B) \subseteq \operatorname{Ball}_D(A')$ as before, so $A' \triangle f^{-1}(B) = A' \setminus f^{-1}(B)$ is finite. Also, $A \triangle A'$ is finite since $x \in A \triangle A'$ iff $x \in A$ and $d(x, \neg A) \leq D$, so $A \triangle f^{-1}(B)$ is finite too. It remains to show that $\operatorname{Ball}_D(A') \subseteq A$, for then $f^{-1}(B) \subseteq A$ as desired.

Indeed, if $y \in \operatorname{Ball}_D(A')$, then by the (reverse) triangle-inequality we have $d(y, \neg A) \geq d(x, \neg A) - d(x, y)$ for all $x \in A'$. But $d(x, \neg A) > D$, strictly, so $d(y, \neg A) > D - D = 0$, and hence $y \in A$. ∎

Corollary 1.7. *If (X, G) is a locally-finite quasi-tree, then the subpocset $\mathcal{H}_{\operatorname{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\operatorname{conn}}(X)$ is dense towards ends for some $R < \infty$.*

Proof. Observe that $\mathcal{H}_{\operatorname{diam}(\partial)\leq 2}(T)$ is dense towards ends for any tree T, so Proposition 1.6 proves the density of $\mathcal{H}_{\operatorname{diam}(\partial)\leq R}(X)$ for some $R < \infty$. By Lemma 1.4, there is a subpocset $\mathcal{H}' \subseteq \mathcal{H}_{\partial<\infty}(X) \cap \mathcal{H}_{\operatorname{conn}}(X)$ dense towards ends such that for every $H' \in \mathcal{H}'$, we have $\partial_{\mathsf{ie}} H' \subseteq \partial_{\mathsf{ie}} H$ for some $H \in \mathcal{H}_{\operatorname{diam}(\partial)\leq R}(X)$. Hence we have $H' \in \mathcal{H}_{\operatorname{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\operatorname{conn}}(X)$, so the result follows. ∎

2. Pocsets of Dense Families of Cuts

2.1. **Pocsets of cuts.** The family $\mathcal{H}_{\operatorname{diam}\partial\leq R}(X)$ of cuts have the structure of a 'profinite pocset', which we first study abstractly. We then deduce some properties of the pocset induced by a *dense* family of cuts.

Definition 2.1. A *pocset* $(\mathcal{H}, \leq, \neg, 0)$ is a poset $(\mathcal{H}, \leq)$ equipped with an order-reversing involution $\neg : \mathcal{H} \to \mathcal{H}$ and a least element $0 \neq \neg 0$ such that 0 is the only lower-bound of $H, \neg H$ for every $H \in \mathcal{H}$. We call the elements in $\mathcal{H}$ *half-spaces*.

A *profinite pocset* is a pocset $\mathcal{H}$ equipped with a compact topology making $\neg$ continuous and is *totally order-disconnected*, in the sense that if $H \not\leq K$, then there is a clopen upward-closed $U \subseteq \mathcal{H}$ with $H \in U \not\ni K$.

Remark 2.2. Such a topology is automatically Hausdorff and zero-dimensional, making it a Stone space.

We are primarily interested in subpocsets of $(2^X, \subseteq, \neg, \varnothing)$, which is profinite if equipped with the product topology of the discrete space 2. Indeed, 2^X admits a base of *cylinder sets* $-$ which are finite intersections of sets of the form $\pi_x^{-1}(i)$ where $x \in X$, $i \in \{0, 1\}$, and $\pi_x : 2^X \to 2$ is the projection $-$ making $\neg$ continuous since cylinders are clopen. Finally, for $H \not\leq K$, let U be the upward-closure of a clopen neighborhood $U_0 \ni H$ separating it from K, which is clopen since $\neg U_0$ is a finite union of cylinders.

The following proposition gives a sufficient criteria for subpocsets of 2^X to be profinite. We also show in this case that every non-trivial element $H \in \mathcal{H}^*$ is isolated, which will important in Section 3.

Lemma 2.3. *Let X be a set and $\mathcal{H} \subseteq 2^X$ be a subpocset. If $\mathcal{H}$ is finitely-separating, then $\mathcal{H} \subseteq 2^X$ is closed and every non-trivial element is isolated.*

Proof. It suffices to show that the limit points of $\mathcal{H}$ are trivial, so let $A \in 2^X \setminus \{\varnothing, X\}$. Fix $x \in A \not\ni y$. Since $\mathcal{H}$ is finitely-separating, there are finitely-many $H \in \mathcal{H}$ with $x \in H \not\ni y$, and for each such $H \in \mathcal{H} \setminus \{A\}$, we have either some $x_H \in A \setminus H$ or $y_H \in H \setminus A$. Let $U \subseteq 2^X$ be the family of all subsets $B \subseteq X$ containing x and each x_H but not y or any y_H.

This is the desired neighborhood isolating $A \in U$. Indeed, it is (cl)open since it is the *finite* intersection of cylinders prescribed by the x_H's and y_H's, and it is disjoint from $\mathcal{H} \setminus \{A\}$ by construction. ∎

In the case when $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty}$ is a subpocset of cuts in a graph (X, G), we can deduce that $\mathcal{H} \cap \mathcal{H}_{\text{conn}} \subseteq 2^X$ is closed. Hence, the above lemma refines to the following

Lemma 2.4. *Let $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty}$ be a subpocset of cuts. If $\mathcal{H}$ is finitely-separating, then $\mathcal{H} \cap \mathcal{H}_{\text{conn}}$ is closed and every non-trivial element is isolated.*

Proof. It suffices to show that no $H \in \mathcal{H}^*$ is a limit point of $\mathcal{H}_{\text{conn}}(X)$, for then $\mathcal{H} \cap \mathcal{H}_{\text{conn}}(X)$ is closed, so fix $H \in \mathcal{H}^*$. Let $U \subseteq 2^X$ be the family of all subsets $B \subseteq X$ containing $\partial_{\mathsf{iv}} H$ and disjoint from $\partial_{\mathsf{ov}} H$, which clearly contains H, and is clopen since $\partial_{\mathsf{v}} H$ is finite.

Suppose that there is some $B \in U \cap \mathcal{H}_{\text{conn}}(X)$. Then $H \subseteq B$, since if $x \in H \cap \neg B$, then there is a path in $\neg B$ to some $x' \in \partial_{\mathsf{ov}} H \subseteq \neg B$, which passes through $\partial_{\mathsf{iv}} H \subseteq B$, a contradiction. The converse is similar. ∎

Lemma 2.5. *Let $\mathcal{H} \subseteq 2^X$ be a subpocset in a connected graph (X, G). If each $x \in X$ is on the boundary of finitely-many half-spaces, then $\mathcal{H}$ is finitely-separating. The converse holds too if (X, G) is locally-finite.*

Proof. Any $H \in \mathcal{H}$ separating $x, y \in X$ separates some edge on any fixed path between x and y, and there are only finitely-many such H for each edge. If (X, G) is locally-finite, then each $x \in X$ is separated from each of its finitely-many neighbors by finitely-many $H \in \mathcal{H}$. ∎

In particular, if (X, G) is locally-finite, then $\mathcal{H}_{\text{diam}(\partial)\leq R}(X)$ for any fixed $R < \infty$ (see Section 1.2) is finitely-separating. Indeed, since the R-ball around any fixed $x \in X$ is finite, any $H \in \mathcal{H}_{\text{diam}(\partial)\leq R}(X)$ with $x \in \partial_{\mathsf{v}} H$ is contained in said R-ball, so there are finitely-many such half-spaces.

This example, in tandem with Corollary 1.7, gives us, for each component $C \subseteq X$ of a quasi-treeing G of a CBER, a family of cuts $\mathcal{H}(C) \coloneqq \mathcal{H}_{\text{diam}(\partial)\leq R_C}(C) \cap \mathcal{H}_{\text{conn}}(C)$ which is finitely-separating and dense towards ends of $G|C$ for some $R_C < \infty$. We use the former condition in Section 3 to construct a 'median graph' $\mathcal{M}(\mathcal{H}(C))$, and use the latter to provide restrictions on $\mathcal{H}(C)$, and hence on $\mathcal{M}(\mathcal{H}(C))$ too, in Section 2.2. The conditions on $\mathcal{M}(\mathcal{H}(C))$ then allows us to *canonically* construct a spanning tree thereof; we do so in Section 4.3, and prove in Section ??? that this is the desired treeing of the CBER.

2.2. Finiteness conditions on $\mathcal{H}$ induced by dense cuts.

Let (X, G) be a connected locally-finite graph and fix a finitely-separating subpocset $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty}(X) \cap \mathcal{H}_{\text{conn}}(X)$ of cuts.

Definition 2.6. Let $\mathcal{H} \subseteq 2^X$ be a pocset. Two half-spaces $H, K \in \mathcal{H}$ are *nested* if $\neg^i H \cap \neg^j K = \varnothing$ for some $i, j \in \{0, 1\}$, where $\neg^0 H \coloneqq H$ and $\neg^1 H \coloneqq \neg H$. We say that $\mathcal{H}$ is *nested* if every pair $H, K \in \mathcal{H}$ is nested.

Lemma 2.7. *Each $H \in \mathcal{H}$ is non-nested with finitely-many $K \in \mathcal{H}$.*

Proof. Fix $H \in \mathcal{H}$ and let $K \in \mathcal{H}$ be non-nested with H. By connectedness, the non-empty sets $H \cap K$ and $\neg H \cap K$ are joined by a path in K, so $\partial_{\mathsf{v}} H \cap K \neq \varnothing$; similarly, $\partial_{\mathsf{v}} H \cap \neg K \neq \varnothing$. For each $x \in \partial_{\mathsf{v}} H \cap K$ and $y \in \partial_{\mathsf{v}} H \cap \neg K$, any fixed path p_{xy} between them contains some $z \in \partial_{\mathsf{v}} K \cap p_{xy}$; thus, any $K \in \mathcal{H}$ non-nested with H contains some $z \in \partial_{\mathsf{v}} K \cap p_{xy}$.

Then, since there are finitely-many such $x, y \in \partial_{\mathsf{v}} H$, for each of which there are finitely-many $z \in p_{xy}$, for each of which there are finitely-many $K \in \mathcal{H}$ with $z \in \partial_{\mathsf{v}} K$ (by Lemma 2.5, since $\mathcal{H}$ is finitely-separating), there can only be finitely-many $K \in \mathcal{H}$ non-nested with H. ∎

Proposition 2.8. *If $\mathcal{H}$ is dense towards ends, then each $\mathcal{H}$-block is finite and each $H \in \mathcal{H}^*$ has finitely-many successors $K \in \mathcal{H}^*$.*

Proof. ∎

3. The Dual Median Graph of a Profinite Pocset

Let $\mathcal{H}$ be a profinite subpocset with every non-trivial element isolated; say, if $\mathcal{H}$ is finitely-separating, and in particular the cuts $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$ for some locally-finite connected graph (X, G). We present a classical construction in geometric group theory (see [Dun79], [Rol98], [Sag95], and [NR03] for other applications) of a 'tree-like' graph associated to such a pocset.

Definition 3.1. An *orientation* on $\mathcal{H}$ is an upward-closed subset $U \subseteq \mathcal{H}$ containing exactly one of $H, \neg H$ for each $H \in \mathcal{H}$. We let $\mathcal{U}(\mathcal{H})$ denote the set of all orientations on $\mathcal{H}$ and let $\mathcal{U}^\circ(\mathcal{H})$ denote the clopen ones.

Intuitively, an orientation is a 'maximally consistent' choice of half-spaces[2].

Example 3.2. Each $x \in X$ induces its *principal orientation* $\widehat{x} \coloneqq \{H \in \mathcal{H} : x \in H\} = \mathcal{H} \cap \pi_x^{-1}(1)$ – which is clearly clopen in $\mathcal{H}$ – and gives us a canonical map $X \to \mathcal{U}^\circ(\mathcal{H})$. However, this map is *not necessarily* injective, and we call a fiber $[x]_{\mathcal{H}} \coloneqq \{y \in X : \widehat{x} = \widehat{y}\}$ thereof an *$\mathcal{H}$-block*.

The goal of this section is to canonically construct a graph whose vertices are clopen orientations on $\mathcal{H}$.

Definition 3.3. A *median graph* is a connected graph (X, G) such that for any $x, y, z \in X$, the intersection

$$[x, y] \cap [y, z] \cap [x, z]$$

is a singleton, whose element $\langle x, y, z\rangle$ is called the *median* of x, y, z. Thus we have a ternary median operation $\langle \cdot, \cdot, \cdot\rangle : X^3 \to X$, and a *median homomorphism* $f : (X, G) \to (Y, H)$ is a map preserving said operation.

Some basic properties of median graphs are given in Section 4.2. For more comprehensive references and their general theory, see [Rol98] and [Bow22].

Proposition 3.4. *Let $\mathcal{H}$ be a profinite pocset with every non-trivial element isolated. Then the graph $\mathcal{M}(\mathcal{H})$, whose vertices are clopen orientations $\mathcal{U}^\circ(\mathcal{H})$ and whose edges are pairs $\{U, V\}$ with $V = U \triangle \{H, \neg H\}$ for some $\subseteq$-minimal $H \in U \setminus \{\varnothing, X\}$, is a median graph with path metric $d(U, V) = |U \triangle V|/2$ and medians*

$$\begin{aligned}\langle U, V, W\rangle &\coloneqq \{H \in \mathcal{H} : H \text{ belongs to at least two of } U, V, W\} \\ &= (U \cap V) \cup (V \cap W) \cup (U \cap W).\end{aligned}$$

Proof. First, $V \coloneqq U \triangle \{H, \neg H\}$ as above is clopen since $H, \neg H \in \mathcal{H}^*$ are isolated (whence $\{H\}, \{\neg H\}$ are clopen), and it is an orientation by $\subseteq$-minimality of H. That $\mathcal{M}(\mathcal{H})$ is connected follows from the following claim by noting that $U \triangle V \not\ni \varnothing, X$ is clopen, so it is a compact set of isolated points, whence finite.

Claim ([Sag95, Theorem 3.3])**.** There is a path between U, V iff $U \triangle V$ is finite, in which case

$$d(U, V) = |U \triangle V|/2 = |U \setminus V| = |V \setminus U|.$$

Proof. If $(U_i)_{i<n}$ is a path from $U =: U_0$ to $V =: U_{n-1}$, then, letting $\{H_i, \neg H_i\} \coloneqq U_i \triangle U_{i-1}$ for all $1 \leq i < n$ gives us a sequence $(H_i)_{i<n}$ inducing[a] this path, whence $U \triangle V$ consists of $\{H_i\}_{i<n}$ and their complements. Thus $U \triangle V = 2n = 2d(U, V)$, as desired.

Conversely, if $U \triangle V = \{H_1, \ldots, H_n\} \sqcup \{K_1, \ldots, K_m\}$ with $U \setminus V = \{H_i\}$ and $V \setminus U = \{K_j\}$, then $\neg H_i \in V \setminus U$ and $\neg K_j \in U \setminus V$ for all $i < n$ and $j < m$, so $n = m$ and $V = U \cup \{\neg H_i\} \setminus \{H_i\}$. We claim that there is a permutation $\sigma \in S_n$ such that $(H_{\sigma(i)})$ induces a path from $U =: U_0$, which is the desired path from U to V. Choose a minimal $H \in \{H_i\}$, which is also minimal in U: if $K \subseteq H$ for some $K \in U$, then $\neg H \subseteq \neg K$, and hence $\neg K \in V$, so $K = H_i \subseteq H$ for some i, forcing $K = H$. Set $U_1 \coloneqq U \triangle \{H, \neg H\}$, which is a clopen orientation. Continuing in this manner by choosing a minimal element in $\{H_i\} \setminus \{H\}$ – and so on – gives us the desired path with $d(U, V) = n$. ☐

[a] In the sense that $U_i = U_{i-1} \triangle \{H_i, \neg H_i\}$ and $H_i \in U_i$ for each $1 \leq i < n$; see [Tse20, Definition 2.20].

Finally, we show that $\mathcal{M}(\mathcal{H})$ is a median graph. Fix $U, V, W \in \mathcal{U}^\circ(\mathcal{H})$, and note that for any $M \in \mathcal{U}^\circ(\mathcal{H})$, we have by the triangle inequality that $M \in [U, V]$ iff $(U \setminus M) \cup (M \setminus V) \subseteq U \setminus V$, which clearly occurs iff $U \cap V \subseteq M \subseteq U \cup V$. Thus, a vertex M lies in the triple intersection $[U, V] \cap [V, W] \cap [U, W]$ iff

$$(U \cap V) \cup (V \cap W) \cup (U \cap W) \subseteq M \subseteq (U \cup V) \cap (V \cup W) \cap (U \cup W).$$

[2] This can be formalized by letting $\sim$ be the equivalence relation on $\mathcal{H}$ given by $H \sim \neg H$. Letting $\partial : \mathcal{H} \to \mathcal{H}/{\sim}$ denote the quotient map, orientations $U \subseteq \mathcal{H}$ then correspond precisely to sections $\varphi : \mathcal{H}/{\sim} \to \mathcal{H}$ of ∂ such that $\varphi(\partial H) \not\subseteq \neg\varphi(\partial K)$ for every $H, K \in \mathcal{H}$; the latter condition rules out 'orientations' of the form $\leftarrow\!\mid\;\mid\!\rightarrow$.

Note that the two sides coincide, so $M = \langle U, V, W\rangle$ − which is clopen if U, V, W are − is as claimed. ■

Given such a pocset $\mathcal{H}$, the graph $\mathcal{M}(\mathcal{H})$ constructed above is called the *dual*[3] median graph of $\mathcal{H}$. An important special case of this construction is when $\mathcal{H}$ is *nested*, in which case $\mathcal{M}(\mathcal{H})$ is a tree.

Lemma 3.5. *Let $\mathcal{H}$ be a profinite pocset with non-trivial points isolated. If $\mathcal{H}$ is nested and $(H_i)_{i<n} \subseteq \mathcal{H}$ induces a path in $\mathcal{M}(\mathcal{H})$ with no backtracking, then (H_i) is strictly increasing.*

Proof. Let $(U_i)_{i<n}$ be the induced path. By definition, we have $U_{i+1} = U_{i-1} \cup \{\neg H_i, \neg H_{i-1}\} \setminus \{H_i, H_{i-1}\}$, so $H_i \neq \neg H_{i-1}$, lest $U_{i+1} = U_{i-1}$. Thus, since $H_i \in U_i = U_{i-1} \cup \{\neg H_{i-1}\} \setminus \{H_{i-1}\}$, we see that $H_i \in U_{i-1}$. Clearly $H_i \neq H_{i-1}$. It suffices to remove the three cases when $\neg H_i \subseteq H_{i-1}$, $H_{i-1} \subseteq \neg H_i$, and $H_i \subseteq H_{i-1}$, since then nestedness of $\mathcal{H}$ gives us $H_{i-1} \subsetneq H_i$, as desired.

Indeed, if $\neg H_i \subseteq H_{i-1}$, then $H_{i-1} \in U_{i+1}$ by upward-closure of $U_{i+1} \ni \neg H_i$. But since $H_{i-1} \neq \neg H_i$, we have by definition of U_{i+1} that $H_{i-1} \in U_i$, a contradiction. The other cases are similar. ■

Corollary 3.6. *If $\mathcal{H}$ is nested, then $\mathcal{M}(\mathcal{H})$ is a tree.*

Proof. A cycle in $\mathcal{M}(\mathcal{H})$ is induced by a *strictly* increasing chain $H_0 \subset \cdots \subset H_n \subset H_0$, which is absurd. ■

Nonetheless, in the general non-nested case, the condition that $\mathcal{H}$ is dense towards ends (and in particular, the finiteness conditions on $\mathcal{H}$ that follows; see Section 2.2) gives us a sufficient criteria for the existence of a *canonical* spanning tree of $\mathcal{M}(\mathcal{H})$, which we prove in the following section.

4. Median Graphs, Convex Half-spaces, and Canonical Spanning Trees

Throughout this section, let (X, G) be a median graph, which, in the context of Theorem A, is the dual median graph of the pocset $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}$ given by the quasi-isometry.

The presence of the median operation gives rise to a canonical finitely-separating pocset of *convex* (and co-convex) half-spaces $\mathcal{H}_{\mathrm{cvx}}(X) \subseteq 2^X$ with a very rigid structure. To study them, we need to first understand the geometry of *projections* therein.

4.1. Projections in median graphs.

For vertices $x, y, z \in X$, we write x—y—z for $y \in [x, z]$. By the triangle inequality, we have for all $w, x, y, z \in X$ that

$$(w\text{—}x\text{—}y \text{ and } w\text{—}y\text{—}z) \quad \Leftrightarrow \quad (w\text{—}x\text{—}z \text{ and } x\text{—}y\text{—}z),$$

and both sides occur iff there is a geodesic from w to x to y to z, which we write as w—x—y—z.

Lemma 4.1. *For any $\varnothing \neq A \subseteq X$ and $x \in X$, there is a unique point in $\mathrm{cvx}(A)$ between x and every point in A, called the* projection *of x towards A, denoted $\mathrm{proj}_A(x)$.*

Moreover, we have $\bigcap_{a\in A}[x, a] = [x, \mathrm{proj}_A(x)]$, and for any y in this set, we have $\mathrm{proj}_A(y) = \mathrm{proj}_A(x)$.

Proof. To show existence, pick any $a_0 \in A$. Given $a_n \in \mathrm{cvx}(A)$, if there exists $a \in A$ with $a_n \notin [x, a]$, set $a_{n+1} := \langle x, a, a_n\rangle \in \mathrm{cvx}(A)$. Then a_0—a_1—$\cdots$—a_n—x for all n, so this sequence terminates in at most $d(a_0, x)$ steps at a point in $\mathrm{cvx}(A)$ between x and every point in A. For uniqueness, if there exist two such points $a, b \in \mathrm{cvx}(A)$, then x—a—b and x—b—a, forcing $a = b$.

Finally, if x—y—$\mathrm{proj}_A(x)$ and $a \in A$, then x—$\mathrm{proj}_A(x)$—a and hence x—y—a. Conversely, let x—y—a for all $a \in A$. Since $[y, a] \subseteq [x, a]$ for all a, we see that

$$\mathrm{proj}_A(y) \in \mathrm{cvx}(A) \cap \bigcap_{a\in A}[y, a] \subseteq \mathrm{cvx}(A) \cap \bigcap_{a\in A}[x, a]$$

and hence $\mathrm{proj}_A(y) = \mathrm{proj}_A(x)$ by uniqueness. But since y—$\mathrm{proj}_A(y)$—a, we have x—y—$\mathrm{proj}_A(y)$, and hence x—y—$\mathrm{proj}_A(x)$ as desired. ■

[3]The name is justified by a Stone-type duality between median graphs with median homomorphisms and profinite pocsets whose non-trivial points are isolated with continuous maps, where from a median graph one can construct a canonical pocset of 'convex' half-spaces (see [CPTT23, Section 2.D] for details).

Remark 4.2. It follows from the proof above that for any median homomorphism $f : (X, G) \to (Y, H)$, we have $f(\mathrm{proj}_A(x)) = \mathrm{proj}_{f(A)}(f(x))$ for any $\varnothing \neq A \subseteq X$ and $x \in X$. Indeed, we have

$$\mathrm{proj}_A(x) = \langle x, a_m, \dots, \langle x, a_2, \langle x, a_1, a_0\rangle\rangle \dots\rangle$$

for some $m \leq d(a_0, x)$ and $a_0, \dots, a_m \in A$, and this is preserved by f.

For $A \coloneqq \{a, b\}$, we have $\mathrm{proj}_A(x) = \langle a, b, x\rangle$, and hence $\mathrm{cvx}(A) = \mathrm{proj}_A(X) = \langle a, b, X\rangle = [a, b]$.

Lemma 4.3. *For each $x, y \in X$, $\mathrm{cone}_x(y)$ is convex, and if xGy, then $\mathrm{cone}_x(y) \sqcup \mathrm{cone}_y(x) = X$.*

Proof. Fix $a, b \in \mathrm{cone}_x(y)$ and a—c—b. It suffices to show that x—y—$\langle a, c, x\rangle$, for then x—y—c since we have x—$\langle a, c, x\rangle$—c. Indeed, it follows from the following observations.

- x—y—$\langle a, b, x\rangle$, since $\langle a, b, x\rangle = \mathrm{proj}_{\{a,b\}}(x)$ and so $[x, \langle a, b, x\rangle] = [x, a] \cap [x, b] \ni y$ by Lemma 4.1.
- x—$\langle a, b, x\rangle$—$\langle a, c, x\rangle$, which follows from $\langle a, b, x\rangle$—$\langle a, c, x\rangle$—a, since x—$\langle a, b, x\rangle$—a by definition. Indeed, we have $\langle a, c, x\rangle$ is in both $[a, x]$ and $[a, c] \subseteq [a, b]$, and since $\mathrm{proj}_{\{b,x\}}(a) = \langle a, b, x\rangle$, we have again by Lemma 4.1 that $[\langle a, b, x\rangle, a] = [a, x] \cap [a, b] \ni \langle a, c, x\rangle$.

Finally, take $z \in X$ and consider $w \coloneqq \langle x, y, z\rangle \subseteq [x, y]$. Either $w = x$ or $w = y$ (but not both), giving us the desired partition. ■

In particular, this shows that if xGy, then $\mathrm{cone}_x(y) \in \mathcal{H}^*_{\mathsf{cvx}}(X)$. The convexity of cones also shows, in the situation of Lemma 4.1, that $\mathrm{proj}_A = \mathrm{proj}_{\mathrm{cvx}(A)}$, i.e., $\mathrm{proj}_A(x)$ is also between x and every point in $\mathrm{cvx}(A)$. Indeed, note that $\mathrm{cone}_x(\mathrm{proj}_A(x))$ is convex and contains A, so it contains $\mathrm{cvx}(A)$ too.

Lemma 4.4. $\mathrm{proj}_A : X \twoheadrightarrow \mathrm{cvx}(A)$ *is a median homomorphism with* $\mathrm{proj}_A \circ \mathrm{cvx} = \mathrm{cvx} \circ \mathrm{proj}_A$.

Proof. The second claim follows from the first since, by Remark 4.2, we have

$$f(\mathrm{cvx}(B)) = f(\mathrm{proj}_B(X)) = \mathrm{proj}_{f(B)}(f(X)) = \mathrm{cvx}(f(B))$$

for all median homomorphisms $f : X \twoheadrightarrow Y$ and $B \subseteq X$, so it in particular applies to $f \coloneqq \mathrm{proj}_A$.

To this end, let x—y—$z \in X$ and set $w \coloneqq \langle \mathrm{proj}_A(x), \mathrm{proj}_A(y), \mathrm{proj}_A(z)\rangle \in \mathrm{cvx}(A)$. It suffices to show that y—w—a for all $a \in A$, for then $w = \mathrm{proj}_A(y)$ and hence $\mathrm{proj}_A(x)$—$\mathrm{proj}_A(y)$—$\mathrm{proj}_A(z)$. But we have y—$\mathrm{proj}_A(y)$—a already, so it further suffices to show that y—w—$\mathrm{proj}_A(y)$. For this, we note that

$$x\text{—}\mathrm{proj}_A(x)\text{—}\mathrm{proj}_A(y) \quad \text{and} \quad \mathrm{proj}_A(x)\text{—}w\text{—}\mathrm{proj}_A(y),$$

so x—w—$\mathrm{proj}_A(y)$, and similarly z—w—$\mathrm{proj}_A(y)$. Thus, it follows that

$$\begin{aligned} w \in [\mathrm{proj}_A(y), x] \cap [\mathrm{proj}_A(y), z] &= [\mathrm{proj}_A(y), \mathrm{proj}_{\{x,z\}}(\mathrm{proj}_A(y))] && \text{Lemma 4.1} \\ &= [\mathrm{proj}_A(y), \mathrm{proj}_{[x,z]}(\mathrm{proj}_A(y))] \\ &\subseteq [\mathrm{proj}_A(y), y], && \text{Lemma 4.1} \end{aligned}$$

where the second equality follows from $\mathrm{cvx}(\{x, z\}) = [x, z]$, and hence $\mathrm{proj}_{\{x,z\}} = \mathrm{proj}_{[x,z]}$. ■

4.2. **Convex half-spaces of median graphs.** We now use projections to explore the geometry of convex half-spaces in median graphs. In particular, they are very rigid:

Proposition 4.5. *Each edge $(x, y) \in G$ is on a unique hyperplane, namely the inward boundary of $\mathrm{cone}_x(y)$, and conversely, each half-space $H \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ is $\mathrm{cone}_x(y)$ for every $(x, y) \in \partial_{\mathsf{ie}} H$.*

Thus, hyperplanes are equivalence classes of edges. Furthermore, this equivalence relation is generated by parallel sides of squares (i.e., 4-cycles).

Proof. We have $\mathrm{cone}_x(y) \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ by the above lemma, and clearly $(x, y) \in \partial_{\mathsf{ie}} \mathrm{cone}_x(y)$. Conversely, take $H \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ and any $(x, y) \in \partial_{\mathsf{ie}} H$. Then $H = \mathrm{cone}_x(y)$, for if $z \in H \cap \neg\,\mathrm{cone}_x(y)$, then $z \in \mathrm{cone}_y(x)$, and hence $x \in [y, z] \subseteq H$ by convexity of H, a contradiction; if $z \in \mathrm{cone}_x(y) \cap \neg H$, then $[x, z] \subseteq \neg H$ by convexity of $\neg H$, and hence $y \notin H$, a contradiction.

Finally, parallel edges of a strip of squares generate the same hyperplane since, for a given square, each vertex is between its neighbors and hence any hyperplane containing an edge contains its opposite edge. On the other hand, let $(a, b), (c, d) \in \partial_{\mathsf{ie}} H$ for some $H \in \mathcal{H}^*_{\mathsf{cvx}}(X)$. Note that $\partial_{\mathsf{ov}} H = \mathrm{proj}_{\neg H}(H)$ is convex since H is, and $\mathrm{proj}_{\neg H}$ preserves convexity by Proposition 4.4, so any geodesic between $a, c \in \partial_{\mathsf{ov}} H$ lies in $\partial_{\mathsf{ov}} H$. Matching this geodesic via $\partial_{\mathsf{ie}} H : \partial_{\mathsf{ov}} H \to \partial_{\mathsf{iv}} H$ gives us a geodesic between b, d in $\partial_{\mathsf{iv}} H$, which together with the matching forms the desired strip of squares. ■

Corollary 4.6. *Two half-spaces $H, K \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ are non-nested iff there is an embedding $\{0,1\}^2 \hookrightarrow X$ of the Hamming cube into the four corners $\neg^i H \cap \neg^j K$.*

In particular, if $H, K \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ are non-nested, then $\partial_{\mathsf{v}} H \cap \partial_{\mathsf{v}} K \neq \varnothing$.

Proof. Let H, K be non-nested and take $x_1 \in H \cap K$ and $x_2 \in H \cap \neg K$. Since H is connected, any geodesic between x_1, x_2 crosses an edge $(x_1', x_2') \in \partial_{\mathsf{oe}} K$ in H. Similarly, there is an edge $(y_1', y_2') \in \partial_{\mathsf{oe}} K$ in $\neg H$, so we may slide both edges along $\partial_{\mathsf{oe}} K$ to obtain the desired square (see Proposition 4.5).

Conversely, the half-spaces cutting the square are clearly non-nested. ∎

Lemma 4.7. *A median graph (X, G) has finite hyperplanes iff each half-space $H \in \mathcal{H}_{\mathsf{cvx}}(X)$ is non-nested with finitely-many others.*

Proof. By Corollary 4.6, a half-space K non-nested with H corresponds uniquely to a pair $(x, y) \in \partial_{\mathsf{ie}} H$ with $x, y \in \partial_{\mathsf{iv}} K$, so there are finitely-many such half-spaces K iff $\partial_{\mathsf{ie}} H$ is finite. ∎

Lemma 4.8 (Helly)**.** *Any finite intersection of pairwise-intersecting non-empty convex sets is non-empty.*

Proof. For pairwise-intersecting convex sets H_1, H_2, H_3, pick any $x \in H_1 \cap H_2$, $y \in H_1 \cap H_3$ and $z \in H_2 \cap H_3$; their median $\langle x, y, z \rangle$ then lies in $H_1 \cap H_2 \cap H_3$.

Suppose that it holds for some $n \geq 3$ and let $H_1, \ldots, H_{n+1} \subseteq X$ pairwise-intersect. Then $\{H_i \cap H_{n+1}\}_{i \leq n}$ is a family of n pairwise-intersecting convex sets, so $\bigcap_{i \leq n+1} H_i = \bigcap_{i \leq n} (H_i \cap H_{n+1})$ is non-empty. ∎

Lastly, we have some useful finiteness conditions on convex half-spaces; the former implies that $\mathcal{H}_{\mathsf{cvx}}(X)$ is finitely-separating, and the latter allows us to replace finite sets with their convex hulls.

Lemma 4.9. *Any two disjoint convex sets $\varnothing \neq A, B \subseteq X$ can be separated by a half-space $A \subseteq H \subseteq \neg B$, and furthermore we have $d(A, B) = |\{H \in \mathcal{H}_{\mathsf{cvx}}(X) : A \subseteq H \subseteq \neg B\}|$.*

Proof. Pick a geodesic $A \ni x_0 G x_1 G \cdots G x_n \in B$, where $n \coloneqq d(A, B)$. Then $H \coloneqq \mathrm{cone}_{x_1}(x_0)$, which is a half-space by Lemma 4.3, separates A, B since $x_0 = \mathrm{proj}_A(x_n)$, and thus we have $A \subseteq \mathrm{cone}_{x_n}(x_0) \subseteq \mathrm{cone}_{x_1}(x_0)$ and $B \subseteq \mathrm{cone}_{x_0}(x_n) \subseteq \mathrm{cone}_{x_0}(x_1)$.

Moreover, each such half-space $A \subseteq H \subseteq \neg B$ satisfies $x_i \in H \not\ni x_{i+1}$ for a unique $i < n$, and conversely each pair (x_i, x_{i+1}) has a unique half-space separating them, so we have the desired bijection. ∎

Lemma 4.10. *Every interval $[x, y]$ is finite. More generally, if $A \subseteq X$ is finite, then so is* $\mathrm{cvx}(A)$*.*

Proof. The singletons $\{x\}$ and $\{y\}$ are convex, so there are finitely-many half-spaces $H \subsetneq [x, y]$. But each $z \in [x, y]$ is determined uniquely by those half-spaces containing it, so $[x, y]$ is finite.

Let $A \coloneqq \{x_0, \ldots, x_n\}$. Since $\mathrm{cvx}(A) = \mathrm{proj}_A(X)$, we have by Remark 4.2 that points in $\mathrm{cvx}(A)$ are of the form $\langle x, x_n, \ldots, \langle x, x_2, \langle x, x_1, x_0 \rangle \rangle \rangle$, which is finite by induction using that intervals are finite. ∎

4.3. **Canonical spanning trees.** We now present the Borel cycle-cutting algorithm that can be preformed canonically on any countable median graph with finite hyperplanes. We do so by colouring the half-spaces $\mathcal{H}^*_{\mathsf{cvx}}(X)$ into certain nested half-spaces $\mathcal{H}_n \subseteq \mathcal{H}_{\mathsf{cvx}}(X)$, from which we inductively build a spanning forest by leveraging a tree structure on the $\mathcal{H}_n$-blocks $X/\mathcal{H}_n$. This tree is constructed as follows.

Lemma 4.11. *For any subpocset $\mathcal{H} \subseteq \mathcal{H}_{\mathsf{cvx}}(X)$, the principal orientations map $X \to \mathcal{U}^\circ(\mathcal{H})$ is surjective.*

Proof. Let $U \in \mathcal{U}^\circ(\mathcal{H})$. Since $U \subseteq \mathcal{H}$ is clopen, there is a finite set $A \subseteq X$ − which we may assume to be convex by Corollary 4.10 − such that for all $H \in \mathcal{H}$, we have $H \in U$ iff there is $K \in U$ with $H \cap A = K \cap A$. Note that $K \cap A \neq \varnothing$ for every $K \in U$, since otherwise $\varnothing \in U$. Furthermore, $H \cap K \neq \varnothing$ for every $H, K \in U$, since otherwise we have $H \subseteq \neg K$, and so $\neg K \in U$. By Lemma 4.8, we have $(H \cap A) \cap (K \cap A) = H \cap K \cap A \neq \varnothing$, and applying it again furnishes some $x \in \bigcap_{H \in U} H \cap A$ in X; note that the latter intersection is finite. Thus $U \subseteq \widehat{x}$, so $U = \widehat{x}$ since both are orientations. ∎

This lemma is crucial, in that it induces an isomorphism $X/\mathcal{H} \cong \mathcal{M}(\mathcal{H})$. Indeed, $\mathcal{H}$ is finitely-separating by Lemma 2.5 and Proposition 4.5, and so $\mathcal{U}^\circ(\mathcal{H})$ is the vertices of the median graph $\mathcal{M}(\mathcal{H})$ by Lemma 2.3 and Proposition 3.4. The fibers of $X \to \mathcal{U}^\circ(\mathcal{H})$ are the $\mathcal{H}$-blocks, so we have the desired bijection.

Explicitly, two $\mathcal{H}$-blocks $([x]_{\mathcal{H}}, [y]_{\mathcal{H}})$ are said to be *G-adjacent* if $(\widehat{x}, \widehat{y}) \in \mathcal{M}(\mathcal{H})$.

Proposition 4.12. *Every countable median graph with finite hyperplanes admits a canonical spanning tree.*

Proof. Let (X, G) be a countable median graph with finite hyperplanes. By Corollary 4.6, if two half-spaces $H, K \in \mathcal{H}^*_{\mathrm{cvx}}(X)$ are non-nested, then $\partial_{\mathsf{v}} H \cap \partial_{\mathsf{v}} K \neq \varnothing$, so looking at the intersection graph of the boundaries furnishes a countable colouring $\mathcal{H}^*_{\mathrm{cvx}}(X) = \bigsqcup_{n\in\mathbb{N}} \mathcal{H}^*_n$ such that each $H, \neg H$ receive the same color and that each $\mathcal{H}_n \coloneqq \mathcal{H}^*_n \cup \{\varnothing, X\}$ is a *nested* subpocset. For each $n \in \mathbb{N}$, let $\mathcal{K}_n \coloneqq \bigcup_{m\geq n} \mathcal{H}_m$.

We shall inductively construct an increasing chain of subforests $T_n \subseteq G$ such that the components of T_n are exactly the $\mathcal{K}_n$-blocks. Then, the increasing union $T \coloneqq \bigcup_n T_n$ is a spanning tree, since each $(x, y) \in G$ lies in a $\mathcal{K}_n$-block for sufficiently large n (namely, the n such that $\mathrm{cone}_x(y) \in \mathcal{H}^*_{n-1}$, since $\mathrm{cone}_x(y)$ and its complement are the only half-spaces separating x and y by Proposition 4.5).

Since each pair of distinct points is separated by a half-space, the $\mathcal{K}_0 = \mathcal{H}_{\mathrm{cvx}}(X)$-blocks are singletons, so put $T_0 \coloneqq \varnothing$. Suppose that a forest T_n is constructed as required. Note that each $\mathcal{K}_{n+1}$-block $Y \in X/\mathcal{K}_{n+1}$ is not separated by any half-spaces in $\mathcal{H}_m$ for $m > n$, but is separated (by Proposition 4.5) by $\mathcal{H}_n$ into the $\mathcal{K}_n$-blocks contained in Y, and those correspond precisely to the $\mathcal{H}_n$-blocks in $Y/\mathcal{H}_n$. For each G-adjacent pair $A, B \in Y/\mathcal{H}_n$, which is separated by a unique half-space $H \in \mathcal{H}_n$ by Lemma 4.9, we may pick an edge from the *finite* hyperplane $\partial_{\mathsf{ie}} H$. Since each $Y/\mathcal{H}_n$ is a tree by Corollary 3.6, and a single edge is picked between every pair of G-adjacent blocks therein, the graph T_{n+1} obtained from T_n by adding all such edges is a forest whose components are exactly the $\mathcal{K}_{n+1}$-blocks. ■

To apply this proposition to the dual median graph $\mathcal{M}(\mathcal{H})$ of a subpocset $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty} \cap \mathcal{H}_{\mathrm{conn}}$ of finitely-separating cuts, by Lemma 4.7, it suffices to show that each half-space in $\mathcal{H}_{\mathrm{cvx}}(\mathcal{M}(\mathcal{H}))$ is non-nested with finitely-many others. However, given that $\mathcal{H}$ is finitely-separating, so that each $H \in \mathcal{H}$ is non-nested with finitely-others by Lemma 2.7, we need to transfer this result from $\mathcal{H}$ to $\mathcal{H}_{\mathrm{cvx}}(\mathcal{M}(\mathcal{H}))$.

Lemma 4.13. *Fix a half-space $K \in \mathcal{H}^*_{\mathrm{cvx}}(\mathcal{M}(\mathcal{H}))$. Then there is a unique $H \in \mathcal{H}^*$ such that*

$$K = \widehat{H} \coloneqq \{W \in \mathcal{U}^\circ(\mathcal{H}) : H \in W\},$$

and if $K' \in \mathcal{H}^*_{\mathrm{cvx}}(\mathcal{M}(\mathcal{H}))$ is non-nested with K, say with $K' = \widehat{H}'$, then H' is non-nested with H.

Proof. We claim that for any $(U, V) \in \partial_{\mathsf{ie}} K$, we have $V \setminus U = \{H\}$ iff $K = \widehat{H}$.

> Indeed, we have $H = \mathrm{cone}_U(V)$ by Proposition 4.5, so for all $W \in \mathcal{U}^\circ(\mathcal{H})$, we have $W \in K$ iff $V \in [W, U]$, which by (the proof of) Proposition 3.4, occurs iff $(W \setminus V) \cup (V \setminus U) \subseteq W \setminus U$. If $V \setminus U = \{H\}$, then the latter is equivalent to $H \in W$, so we have $K = \widehat{H}$. Conversely, if $K = \widehat{H}$, then we have $V \setminus U \in K$ by the above, so $H \in V \setminus U$. Since U, V are adjacent, we have $V \setminus U = \{H\}$ as desired.

This gives existence, since $V \setminus U$ is a singleton for any such (U, V) by Proposition 3.4, and uniqueness follows too since if $\widehat{H} = K = \widehat{H}'$, then any $(U, V) \in \partial_{\mathsf{ie}} K$ has $V \setminus U = \{H\} = \{H'\}$.

Finally, let $K' = \widehat{H}' \in \mathcal{H}^*_{\mathrm{cvx}}(\mathcal{M}(\mathcal{H}))$ be non-nested with $K = \widehat{H}$, so there are edges $(U, V), (U', V') \in \partial_{\mathsf{ie}} K$ forming parallel sides of a square, by Corollary 4.6, say with $U, V \in \partial_{\mathsf{iv}} K'$. Since $V \in K \cap K'$ is an orientation containing H, H', which are respectively flipped to U, V', we see that both H, H' are $\subseteq$-minimal in V, and that $H \cap H' \neq \varnothing$. Finally, note that $\neg H, \neg H' \in U'$ since U' is opposite to V, so $\neg H \cap \neg H'$ too, and hence H' is non-nested with H as desired. ■

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