NESTED COLLECTION OF CUTS

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Notation 0.1. Let X be a set. For any family $\mathcal{C} \subseteq 2^X$ closed under the complement operation $\neg : 2^X \to 2^X$, we let $\mathcal{C}^* \coloneqq \mathcal{C} \setminus \{\varnothing, X\}$ denote the *non-trivial* elements in $\mathcal{C}$.

1. Preliminaries

1.1. Ultrafilters and their bases.

We recall some standard notions, mainly to establish notation.

Definition 1.1. A *filter* on X is a collection $\mathcal{F} \subseteq \mathcal{P}^*(X)$ that is closed-upwards and closed under pairwise intersections, i.e.

(1) If $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$; and
(2) If $A_1, A_2 \in \mathcal{F}$, then $A_1 \cap A_2 \in \mathcal{F}$.

Lemma 1.2. *For any family $\mathcal{F}_0 \subseteq \mathcal{P}^*(X)$, the following are equivalent.*

(1) For all $F_1, F_2 \in \mathcal{F}_0$, there is some $F \in \mathcal{F}_0$ such that $F \subseteq F_1 \cap F_2$.
(2) The collection $\mathcal{F} \coloneqq \{A \subseteq X : A \supseteq F \text{ for some } F \in \mathcal{F}_0\} \subseteq \mathcal{P}^*(X)$ is a filter on X.

Proof. If (1) holds, then $\mathcal{F}$ is upward closed since if $A \supseteq F$ for some $F \in \mathcal{F}_0$, then $B \supseteq F$ for any $B \supseteq A$. Moreover, if $A_i \supseteq F_i$ for $i = 1, 2$, then $A_1 \cap A_2 \supseteq F_1 \cap F_2 \supseteq F$ for some $F \in \mathcal{F}_0$ furnished by (1).

Conversely, let $F_1, F_2 \in \mathcal{F}_0$. Then $F_1, F_2 \in \mathcal{F}$, so $F_1 \cap F_2 \in \mathcal{F}$ by (2). Thus $F_1 \cap F_2 \supseteq F$ for some $F \in \mathcal{F}_0$, by definition of $\mathcal{F}$, as desired. ∎

Definition 1.3. A *basis for a filter* on X is a family $\mathcal{F}_0 \subseteq \mathcal{P}^*(X)$ satisfying the equivalent definitions above, and we call the collection $\mathcal{F}$ in (2) the *filter generated by* $\mathcal{F}_0$.

Definition 1.4. An *ultrafilter* on X is a $\subseteq$-maximal filter on X.

1.2. Nested cuts and $\mathcal{T}$.

Throughout this subsection, let (X, G) be a graph.

Definition 1.5. A *cut* in X is a subset $C \subseteq X$ contained in a single connected component $Y \subseteq X$ such that both C and $Y \setminus C$ are infinite but $\partial_{\mathsf{v}} C$ is finite.

Definition 1.6. A family $\mathcal{C}$ of cuts in X is *nested* if every $C_1, C_2 \in \mathcal{C}$ has an empty corner; i.e. $C_1^i \cap C_2^j = \varnothing$ for some $i, j = \pm 1$

2. The graph $\mathcal{T}_\mathcal{C}$

Theorem 2.1. *Let $\mathcal{C}$ be a family of nested cuts in X. The graph $\mathcal{T}_\mathcal{C}$, whose*

- *Vertices are finitely-based ultrafilters on $\mathcal{C}$;*
- *Neighbors of*

is acyclic. Furthermore, if something, then $\mathcal{T}_\mathcal{C}$ is a tree.

Proof. h ∎

3. What goes wrong if $\mathcal{C}$ is non-nested?