

TREE OF ORIENTATIONS ON A NESTED COLLECTION OF SETS

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Let $H \subseteq 2^X$ be a sub-pocset for some fixed set X (so that, in particular, H is closed under complements). With the definitions in Section 1, we prove the following

Theorem A (Propositions 2.5, 2.6). *If H is nested, then the graph \mathcal{T}_H , whose:*

- *Vertices are finitely-based orientations on H ;*
- *Edges are pairs $\{p, q\}$ such that $q = p \triangle \{h, -h\}$ for some minimal $h \in H$;*

is acyclic. Furthermore, if H is finitely-separating (or more generally, chain-vanishing), then \mathcal{T}_H is a tree.

In particular, this applies to when (X, G) is a graph and H is a nested collection of cuts on X . No further assumptions on X (like local-finiteness) is needed.

1. PRELIMINARIES

Let $H \subseteq 2^X$ be a sub-pocset for some fixed set X , whose elements $h \in H$ are called *half-spaces*.

Definition 1.1. Two elements $h, k \in H$ are *nested* if $h^i \cap k^j = \emptyset$ for some $i, j \in \{1, -1\}$, where $h^i := h$ for $i = 1$ and $h^i := h^c$ otherwise. We say that H is *nested* if every pair $h, k \in H$ are nested.

1.1. Orientations. We give the standard definition of an orientation on h and characterize them as ‘consistent assignments of half-spaces to hyperplanes’.

Definition 1.2. An *orientation* on H is a subset $U \subseteq H$ such that

1. (*Upward-closure*). If $h \in U$ and $k \in H$ contains h , then $k \in U$.
2. (*Ultra*). For each $h \in H$, exactly one of h, h^c is contained in U .

Lemma 1.3. *If $\mathcal{U} \subseteq P$ is an orientation, then for any \subseteq -minimal $A \in \mathcal{U}$ with $A^c \in P$, so is $\mathcal{U} \triangle \{A, A^c\}$.*

Proof. That $\mathcal{V} := \mathcal{U} \triangle \{A, A^c\} = \mathcal{U} \cup \{A^c\} \setminus \{A\}$ is upward-closed follows from \subseteq -minimality of A . Now, if $B, B^c \in P$ and $B^c \notin \mathcal{V}$, we need to show that $B \in \mathcal{V}$.

To this end, note that $B^c \notin \mathcal{V}$ implies $A \neq B$ and either $B = A^c$ or $B^c \notin \mathcal{U}$. The former case follows since $A^c \in \mathcal{V}$, and for the latter, we have $B \in \mathcal{U} \setminus \{A\}$ since \mathcal{U} is ultra. ■

Remark 1.4. In the above notations, clearly $\mathcal{U} \neq \mathcal{U} \triangle \{A, A^c\}$. Furthermore, for any other such orientation \mathcal{U}' and $A' \in \mathcal{U}'$, that $\mathcal{U} = \mathcal{U}'$ and $\mathcal{U} \triangle \{A, A^c\} = \mathcal{U}' \triangle \{A', A'^c\}$ together imply $A = A'$.

1.2. Finitely-based orientations.

Definition 1.5. A *base* for an orientation $\mathcal{U} \subseteq P$ is a \subseteq -minimal subset $\mathcal{B} \subseteq \mathcal{U}$ such that $\mathcal{U} = \uparrow \mathcal{B}$, where

$$\uparrow \mathcal{B} := \bigcup_{B \in \mathcal{B}} \uparrow B := \bigcup_{B \in \mathcal{B}} \{A \in P : A \supseteq B\}.$$

Remark 1.6. If P is nested, then every \subseteq -minimal $B \in P$ induces an orientation $\uparrow B := \{A \in P : A \supseteq B\}$, called a *principal* orientation. Indeed, $\uparrow B$ is clearly upward-closed, and if $A, A^c \in P$, then, by \subseteq -minimality of B and nestedness of P , either $A \supseteq B$ or $A^c \supseteq B$ (but clearly not both).

This construction generalizes to any collection $\mathcal{B} \subseteq P$ with each $B \in \mathcal{B}$ being \subseteq -minimal in P , so that $\uparrow \mathcal{B}$ is an orientation on P .

Definition 1.7. An orientation $\mathcal{U} \subseteq P$ is said to be *finitely-based* if it admits a finite base.

Remark 1.8. If $\mathcal{U} = \uparrow \{B_1, \dots, B_n\}$ is finitely-based, then for any \subseteq -minimal $A \in \mathcal{U}$ with $A^c \in P$, so is the orientation $\mathcal{V} := \mathcal{U} \triangle \{A, A^c\}$. Indeed, we have $A = B_i$ for some $1 \leq i \leq n$, and $\mathcal{V} = \uparrow(\{A^c\} \cup \{B_j\}_{j \neq i})$.

2. THE GRAPH \mathcal{T}_P

Fix a nested collection of non-empty subsets of a set X . Using Lemma 1.3 and Remarks 1.4 and 1.8, we construct a graph \mathcal{T}_P whose:

- *Vertices* of \mathcal{T}_P are finitely-based orientations on P .
- *Neighbors* of $\mathcal{U} \in V(\mathcal{T}_P)$ are $\mathcal{U} \triangle \{A, A^c\}$ for every minimal $A \in \mathcal{U}$ with $A^c \in P$.

The goal of this section is to establish Theorem A, stating that \mathcal{T}_P is acyclic (Proposition 2.5), and furthermore, \mathcal{T}_P is a tree when P is closed under complements (Proposition 2.6).

Definition 2.1. Fix $\mathcal{U}_0 \in V(\mathcal{T}_P)$ and $n \in \mathbb{N}$. A sequence $(A_i)_{i < n} \subseteq P$ is said to *induce a path from \mathcal{U}_0* if $(\mathcal{U}_i)_{i < n}$, defined by $\mathcal{U}_i := \mathcal{U}_{i-1} \triangle \{A_{i-1}, A_{i-1}^c\}$ for every $1 \leq i < n$, is a path in \mathcal{T}_P with each $A_i \in \mathcal{U}_i$.

Remark 2.2. Any path in \mathcal{T}_P is induced by its sequence of flipped basis elements.

Lemma 2.3. Let $n \geq 3$. A path in \mathcal{T}_P from \mathcal{U}_0 induced by $(A_i)_{i < n}$ has no backtracking iff $A_i \neq A_{i-1}^c$ for every $1 \leq i < n$.

Proof. Take $2 \leq i \leq n$. It suffices to show that $\mathcal{U}_{i-2} = \mathcal{U}_i$ iff $A_{i-1} = A_{i-2}^c$.

(\Rightarrow). We have by definition that $\mathcal{U}_i = \mathcal{U}_{i-2} \cup \{A_{i-1}^c, A_{i-2}^c\} \setminus \{A_{i-1}, A_{i-2}\}$, so since $A_{i-2} \in \mathcal{U}_{i-2} = \mathcal{U}_i$, we have $A_{i-2} = A_{i-1}^c$ as desired.

(\Leftarrow). Again by definition, by noting that the basis-flipping cancels out. ■

Lemma 2.4. If $(A_i)_{i < n}$ induces a path in \mathcal{T}_P with no backtracking, then $(A_i)_{i < n}$ is strictly increasing.

Proof. By Lemma 2.3, we have $A_i \neq A_{i-1}^c$ for every $1 \leq i < n$. Thus, since $A_i \in \mathcal{U}_i = \mathcal{U}_{i-1} \cup \{A_{i-1}^c\} \setminus \{A_{i-1}\}$, we see that $A_i \in \mathcal{U}_{i-1}$. Clearly $A_i \neq A_{i-1}$. It suffices to remove the three cases when $A_i \subseteq A_{i-1}$, $A_{i-1} \subseteq A_i^c$, and $A_i^c \subseteq A_{i-1}$, since then nestedness of P gives us $A_{i-1} \subsetneq A_i$, as desired.

- If $A_i \subseteq A_{i-1}$, then $A_{i-1} \in \mathcal{U}_i$, contradicting the definition of \mathcal{U}_i .
- If $A_{i-1} \subseteq A_i^c$, then $A_i^c \in \mathcal{U}_{i-1}$ by upward-closure of \mathcal{U}_{i-1} , a contradiction.
- If $A_i^c \subseteq A_{i-1}$, then $A_{i-1} \in \mathcal{U}_{i+1}$ by upward-closure of $\mathcal{U}_{i+1} \ni A_i^c$. But since $A_{i-1} \neq A_i^c$, we have by definition of \mathcal{U}_{i+1} that $A_{i-1} \in \mathcal{U}_i$, a contradiction. ■

Proposition 2.5. \mathcal{T}_P is acyclic.

Proof. Let $(\mathcal{U}_i)_{i < n}$ be a cycle in \mathcal{T}_P induced by $(A_i)_{i < n}$. Since cycles are non-backtracking, we have $A_0 \subsetneq A_0$ by Lemma 2.4, a contradiction. ■

Proposition 2.6. If P is chain-vanishing, then \mathcal{T}_P is connected (and hence a tree).

Proposition 2.7. If P is closed under complements, then \mathcal{T}_P is connected (and hence a tree).

Proof. If $\mathcal{U}, \mathcal{U}' \in V(\mathcal{T}_P)$ are two finitely-based orientations on P , then swapping their basis elements one by one as in Remark 1.8 gives us a path between \mathcal{U} and \mathcal{U}' ; the closure of P is needed to ensure that those basis elements induce a path between the vertices, in that their complement lies in P . ■