

NESTED COLLECTION OF CUTS

ZHAOSHEN ZHAI

1. INTRODUCTION

With the definitions in Section 2, we prove the following

Theorem 1.1. *If (X, G) is a graph with a nested collection $\mathcal{C} \subseteq 2^X$ of cuts, then the graph $\mathcal{T}_{\mathcal{C}}$ whose:*

- *Vertices are finitely-based orientations on \mathcal{C} ; and whose*
- *Neighbors of $\mathcal{U} \in \mathcal{T}_{\mathcal{C}}$ are $\mathcal{U} \triangle \{A, A^c\}$ for every minimal $A \in \mathcal{U}$;*

is acyclic. Furthermore, if \mathcal{C} is closed under complements, then $\mathcal{T}_{\mathcal{C}}$ is a tree.

2. PRELIMINARIES

2.1. Orientations. Let $\mathcal{C} \subseteq 2^X$ be a collection of non-empty subsets of a set X . Since we do not assume that \mathcal{C} is closed under complements, we slightly modify the definition of orientations, as follows.

Definition 2.1. An *orientation* on \mathcal{C} is a subset $\mathcal{U} \subseteq \mathcal{C}$ such that

1. (*Upward-closure*). If $A \in \mathcal{U}$ and $B \in \mathcal{C}$ contains A , then $B \in \mathcal{U}$.
2. (*Ultra*). If $A, A^c \in \mathcal{C}$, then either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$, but not both.

Definition 2.2. A *base* for an orientation $\mathcal{U} \subseteq \mathcal{C}$ is a \subseteq -minimal subset $\mathcal{B} \subseteq \mathcal{U}$ such that $\mathcal{U} = \uparrow \mathcal{B}$, where

$$\uparrow \mathcal{B} := \bigcup_{B \in \mathcal{B}} \uparrow B := \bigcup_{B \in \mathcal{B}} \{A \in \mathcal{C} : A \supseteq B\}.$$

Definition 2.3. A collection \mathcal{C} is said to be *nested* if every $C_1, C_2 \in \mathcal{C}$ has an empty corner, i.e., $C_1^i \cap C_2^j = \emptyset$ for some $i, j = \pm 1$, where $C^i := C$ if $i = 1$ and $C^i := C^c$ if $i = -1$.

Remark 2.4. If \mathcal{C} is nested, then every \subseteq -minimal $B \in \mathcal{C}$ induces an orientation $\uparrow B := \{A \in \mathcal{C} : A \supseteq B\}$, called the *principal* orientation. Indeed, $\uparrow B$ is clearly upward-closed, and if $A, A^c \in \mathcal{C}$, then, by \subseteq -minimality of B and nestedness of \mathcal{C} , either $A \supseteq B$ or $A^c \supseteq B$ (but clearly not both).

This construction generalizes to any collection $\mathcal{B} \subseteq \mathcal{C}$, in that $\uparrow \mathcal{B}$ is always an orientation on \mathcal{C} .

Definition 2.5. An orientation $\mathcal{U} \subseteq \mathcal{C}$ is said to be *finitely-based* if it admits a finite base.

2.2. Cuts in graphs. Let (X, G) be a graph.

Definition 2.6. A *cut* in X is a subset $C \subseteq X$ contained in a single connected component $Y \subseteq X$ such that both C and $Y \setminus C$ are infinite but $\partial_v C$ is finite.

3. THE GRAPH $\mathcal{T}_{\mathcal{C}}$