TREE-LIKE GRAPHINGS

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ABSTRACT. We present a streamlined exposition of a construction presented in [CPTT23], where it is proven that every locally-finite Borel graph with each component a quasi-tree induces a canonical treeable equivalence relation. Write some more details...

1. INTRODUCTION

The purpose of this note is to provide a streamlined proof of a particular case of a construction presented in [CPTT23], in order to better understand the general formalism developed therein. We attempt to make this note self-contained, but nevertheless urge the reader to refer to the original paper for more detailed discussion and generalizations of the results we have selected to include here.

1.1. **Treeings of equivalence relations.** A *countable Borel equivalence relation (CBER)* on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$ with each class countable. We are interested in special types of *graphings* on a CBER $E \subseteq X^2$, i.e. a Borel graph $G \subseteq X^2$ whose connectedness relation is precisely E. For instance, a graphing of E such that each component is a tree is called a *treeing* of E, and the CBERs that admit a treeing are said to be *treeable*. The main results of [CPTT23] is to provide new sufficient criteria for treeability of CBERs, and in particular, they prove the following
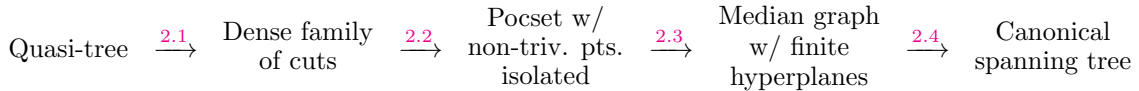
Theorem A ([CPTT23, Theorem 1.1]). *If a CBER admits a locally-finite graphing whose components are quasi-trees, then it is treeable.*

Recall that metric spaces X and Y are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error, and X is a *quasi-tree* if it is quasi-isometric to a simplicial tree; see [Gro93] and [DK18].

1.2. **Outline of the proof.** Roughly speaking, the existence of a quasi-isometry $G|C \to T_C$ to a simplicial tree T_C for each component $C \subseteq X$ induces a collection $\mathcal{H}(C)$ of 'cuts' (subsets $H \subseteq C$ with finite boundary such that both H and $C \setminus H$ are connected), which are 'tree-like' in the sense that

1. $\mathcal{H}(C)$ is *finitely-separating*: each pair $x, y \in C$ are separated by finitely-many $H \in \mathcal{H}(C)$, and
2. $\mathcal{H}(C)$ is *dense towards ends*: each end in $G|C$ has a neighborhood basis in $\mathcal{H}(C)$.

These cuts have the structure of a pocset with non-trivial points isolated, which in turn provide exactly the data to construct a 'median graph' whose vertices are 'ultrafilters'[1] thereof. Condition (2) then ensures that this graph has finite 'hyperplanes', which allows us to apply a Borel 'cycle-cutting' algorithm and obtain a canonical spanning tree thereof. Each step above can be done in a uniform way to each component $C \subseteq G$, giving us the desired treeing of the CBER.

$$\text{Quasi-tree} \xrightarrow{2.1} \begin{array}{c}\text{Dense family}\\\text{of cuts}\end{array} \xrightarrow{2.2} \begin{array}{c}\text{Pocset w/}\\\text{non-triv. pts.}\\\text{isolated}\end{array} \xrightarrow{2.3} \begin{array}{c}\text{Median graph}\\\text{w/ finite}\\\text{hyperplanes}\end{array} \xrightarrow{2.4} \begin{array}{c}\text{Canonical}\\\text{spanning tree}\end{array}$$

Remark. We follow [CPTT23, Convention 2.7], where for a family $\mathcal{H} \subseteq 2^X$ of subsets of a fixed set X, we write $\mathcal{H}^* := \mathcal{H} \setminus \{\varnothing, X\}$ for the *non-trivial* elements of $\mathcal{H}$.

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[1]As in [CPTT23], we call them *orientations* instead; see Definition 2.4.

2. Detailed Constructions

2.1. **Graphs with dense families of cuts.** Let (X, G) be a locally-finite graph.

2.1.1. *End compactification of graphs.*

2.2. **Pocsets of cuts.** The family $\mathcal{H}_{\mathrm{diam}\,\partial \leq R}(X)$ of cuts have the structure of a 'profinite pocset', which we first study abstractly. We then deduce some properties of the pocset induced by a *dense* family of cuts.

2.2.1. *Abstract profinite pocsets.*

Definition 2.1. A *pocset* $(P, \leq, \neg, 0)$ is a poset $(P, \leq)$ equipped with an order-reversing involution $\neg : P \to P$ and a least element $0 \neq \neg 0$ such that 0 is the only lower-bound of $p, \neg p$ for every $p \in P$.

A *profinite pocset* is a pocset P equipped with a compact topology making $\neg$ continuous and is *totally order-disconnected*, in the sense that if $p \not\leq q$, then there is a clopen upward-closed $U \subseteq P$ with $p \in U \not\ni q$.

Remark 2.2. Such a topology is automatically Hausdorff and zero-dimensional.

We are primarily interested in subpocsets of $(2^X, \subseteq, \neg, \varnothing)$, which is profinite if equipped with the product topology of the discrete space 2. Indeed, 2^X admits a base of *cylinder sets* − which are finite intersections of sets of the form $\pi_x^{-1}(i)$ where $x \in X$, $i \in \{0, 1\}$, and $\pi_x : 2^X \to 2$ is the projection − making $\neg$ continuous since cylinders are clopen. Show that it is totally order-disconnected.

The following proposition gives a sufficient criteria for subpocsets of 2^X to be profinite. We also show in this case that every non-trivial element $H \in \mathcal{H}^*$ is isolated, which will important in Section 2.3.

Proposition 2.3. *Let X be a set and $\mathcal{H} \subseteq 2^X$ be a subpocset. If $\mathcal{H}$ is finitely-separating, then $\mathcal{H} \subseteq 2^X$ is closed and every non-trivial element is isolated.*

Proof. It suffices to show that the limit points of $\mathcal{H}$ are trivial, so let $A \in 2^X \setminus \{\varnothing, X\}$. Fix $x \in A \not\ni y$. Since $\mathcal{H}$ is finitely-separating, there are finitely-many $H \in \mathcal{H}$ with $x \in H \not\ni y$, and for each such $H \in \mathcal{H} \setminus \{A\}$, we have either some $x_H \in A \setminus H$ or $y_H \in H \setminus A$. Let $U \subseteq 2^X$ be the family of all subsets $B \subseteq X$ containing x and each x_H but not y or any y_H.

This is the desired neighborhood isolating $A \in U$. Indeed, it is (cl)open since it is the *finite* intersection of cylinders prescribed by the x_H's and y_H's, and it is disjoint from $\mathcal{H} \setminus \{A\}$ by construction. ∎

2.2.2. *The pocset induced by a dense family cuts.*

2.3. **The dual median graph of a pocset.** Let $\mathcal{H} \subseteq 2^X$ be a finitely-separating pocset, whose elements we call *half-spaces*. By Proposition 2.3, $\mathcal{H}$ is profinite and every non-trivial element thereof is isolated.

Definition 2.4. An *orientation* on $\mathcal{H}$ is an upward-closed subset $U \subseteq \mathcal{H}$ containing exactly one of $H, \neg H$ for each $H \in \mathcal{H}$. We let $\mathcal{U}(\mathcal{H})$ denote the set of all orientations on $\mathcal{H}$ and let $\mathcal{U}^\circ(\mathcal{H})$ denote the clopen ones.

Remark 2.5. Intuitively, an orientation is a 'maximally consistent' choice of a half-space from each hyper-plane. To formalize this, let $\sim$ be the equivalence relation on $\mathcal{H}$ given by $H \sim \neg H$, and let $\partial : \mathcal{H} \to \mathcal{H}/\!\!\sim$ be the quotient map. Orientations $U \subseteq \mathcal{H}$ then correspond precisely to sections $\varphi : \mathcal{H}/\!\!\sim \to \mathcal{H}$ of ∂ such that $\varphi(\partial H) \not\subseteq \neg\varphi(\partial K)$ for every $H, K \in \mathcal{H}$; the latter condition rules out 'orientations' of the form $\leftarrow\!\!\mid \;\mid\!\!\mapsto$.

Example 2.6. Every $x \in X$ induces its *principal orientation* $\widehat{x} := \{H \in \mathcal{H} : x \in H\} = \mathcal{H} \cap \pi_x^{-1}(1)$, which is clearly clopen in $\mathcal{H}$, and gives us a canonical map $\mathfrak{p} : X \to \mathcal{U}^\circ(\mathcal{H})$. Note, however, that $\mathfrak{p}$ is *not necessarily* injective; we call a fiber $[x]_{\mathcal{H}} := \mathfrak{p}^{-1}(\widehat{x}) = \{y \in X : \widehat{x} = \widehat{y}\}$ an $\mathcal{H}$-*block*.

The goal of this section is to canonically construct a graph whose vertices are clopen orientations on $\mathcal{H}$.

Theorem 2.7.

2.4. **Spanning trees of median graphs with finite hyperplanes.**

A. Tree of orientations on a nested pocset of sets

Let $\mathcal{H} \subseteq 2^X$ be a finitely-separating subpocset (see Definition 2.2.1), so every non-trivial element is isolated by Proposition 2.3. As shown in Theorem 2.7, the clopen orientations $\mathcal{U}^\circ(\mathcal{H})$ form the vertices of a median graph, whose edges are given by 'minimal base-flipping'.

We devote this appendix to show that if $\mathcal{H}$ is *nested*, then $\mathcal{U}^\circ(\mathcal{H})$ is in fact a tree, and so we may bypass the Borel cycle-cutting algorithm in Section 2.4 and obtain a treeing directly.

References

[CPTT23] Ruiyuan Chen, Antoine Poulin, Ran Tao, and Anush Tserunyan, *Tree-like graphings, wallings, and median graphings of equivalence relations* (2023), available at https://arxiv.org/abs/2308.13010.

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