TREE-LIKE GRAPHINGS OF COUNTABLE BOREL EQUIVALENCE RELATIONS

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ABSTRACT. We present a streamlined exposition of a construction presented recently by R. Chen, A. Poulin, R. Tao, and A. Tserunyan, which proves the treeability of the equivalence relation generated by any locally-finite Borel graph such that each component is a quasi-tree. More generally, we show that if each component of a locally-finite Borel graph admits a *finitely-separating family of cuts*, then we may 'canonically' construct a forest of special ultrafilters; moreover, if the cuts are *dense towards ends*, then this forest is a Borel treeing.

The purpose of this note is to provide a streamlined proof of a particular case of a construction presented in [CPTT23], in order to better understand the general formalism developed therein. We attempt to make this note self-contained, but nevertheless urge the reader to refer to the original paper for more detailed discussions and some generalizations of the results we have selected to include here.

Treeings of equivalence relations. A *countable Borel equivalence relation (CBER)* on a standard Borel space X is a Borel equivalence relation $E \subseteq X^2$ with each class countable. We are interested in special types of *graphings* of a CBER $E \subseteq X^2$, i.e. a Borel graph $G \subseteq X^2$ whose connectedness relation is precisely E. For instance, a graphing of E such that each component is a tree is called a *treeing* of E, and the CBERs that admit treeings are said to be *treeable*. The main results of [CPTT23] provide new sufficient criteria for treeability of certain classes of CBERs, and in particular, they prove the following

Theorem A (Theorem 4.1, [CPTT23, Theorem 1.1])**.** *If a CBER E admits a locally-finite graphing such that each component is a quasi-tree*[1]*, then E is treeable.*

Roughly speaking, the existence of a quasi-isometry $G|C \to T_C$ to a simplicial tree T_C for each component $C \subseteq X$ induces a collection $\mathcal{H}(C) \subseteq 2^C$ of 'cuts' (subsets $H \subseteq C$ with finite boundary such that both H and $C \setminus H$ are connected), which are 'tree-like' in the sense that

1. $\mathcal{H}(C)$ is *finitely-separating*: each pair $x, y \in C$ is separated by finitely-many $H \in \mathcal{H}(C)$, and
2. $\mathcal{H}(C)$ is *dense towards ends*: each end in $G|C$ has a neighborhood basis in $\mathcal{H}(C)$.

Condition (1) allows for an abstract construction of a tree whose vertices are special 'ultrafilters'[2] on $\mathcal{H}(C)$, as outlined in the following diagram: starting from a finitely-separating family of cuts, one constructs a 'dual median graph' $\mathcal{M}(\mathcal{H}(C))$ with said ultrafilters; this median graph has finite 'hyperplanes', which allows one to apply a Borel cycle-cutting algorithm and obtain a 'canonical' spanning tree thereof.

$$\begin{tikzcd}
\text{Borel quasi-treeing} \arrow[d] & \text{Dense family of cuts} \arrow[rr, "\text{Lem. 4.2} \qquad \text{Theorem 4.1}"] & & \text{Borel treeing} & \text{Thm. A} \\
\text{Quasi-tree} \arrow[ur, "\text{Sec. 1.2+2}"] \arrow[r, "\text{Sec. 1.1+2}"] & \text{Finitely-sep. family of cuts} \arrow[urr, bend left, "\text{Lem. 4.2}"] \arrow[r, "\text{Sec. 3.1}"] & \text{Median graph w/ finite hyperplanes} \arrow[r, "\text{Sec. 3.2}"] & \text{`Canonical' spanning tree} \arrow[u] & \text{Thm. B}
\end{tikzcd}$$

Thus we have the following theorem, which can be viewed as a component-wise version of Theorem A.

Theorem B (Propositions 3.3, 3.5, 3.7)**.** *For any finitely-separating family of cuts $\mathcal{H}$ on a connected locally-finite graph, its dual median graph $\mathcal{M}(\mathcal{H})$ has finite hyperplanes, and fixing a Borel colouring on the intersection graph of those hyperplanes yields a canonical spanning tree of $\mathcal{M}(\mathcal{H})$.*

The additional condition (2), which equivalently imposes certain finiteness conditions on $\mathcal{H}(C)$, ensures that such a spanning tree can be constructed uniformly to each component $C \subseteq G$ (see Theorem 4.1), which gives us the desired treeing of the CBER.

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[1] Recall that metric spaces X and Y are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error, and X is a *quasi-tree* if it is quasi-isometric to a simplicial tree; see [Gro93] and [DK18].

[2] As in [CPTT23], we call them *orientations* instead, to avoid confusion with the more standard notion; see Definition 3.1.

1. Preliminaries on Pocsets, Ends of Graphs, and Median Graphs

Notation. A *graph* on a set X is a symmetric irreflexive binary relation $G \subseteq X^2$. For $A \subseteq X$, we say that A is *connected* if the induced subgraph $G[A]$ is. We always equip connected graphs with their *path metric* d, and let $\mathrm{Ball}_r(x)$ be the closed ball of radius r around x; more generally, we let $\mathrm{Ball}_r(A) \coloneqq \bigcup_{x\in A} \mathrm{Ball}_r(x)$.

For a subset $A \subseteq X$, we let $\partial_{\mathsf{iv}} A \coloneqq A \cap \mathrm{Ball}_1(\neg A)$ be its *inner vertex boundary*, $\partial_{\mathsf{ov}} A \coloneqq \partial_{\mathsf{iv}}(\neg A)$ be its *outer vertex boundary*, and let $\partial_{\mathsf{ie}} A \coloneqq G \cap (\partial_{\mathsf{ov}} A \times \partial_{\mathsf{iv}} A)$ and $\partial_{\mathsf{oe}} A \coloneqq \partial_{\mathsf{ie}}(\neg A)$ respectively be its *inner* and *outer edge boundaries*. Let $\partial_{\mathsf{v}} A \coloneqq \partial_{\mathsf{iv}} A \sqcup \partial_{\mathsf{ov}} A$ be the *(total) vertex boundary* of A.

Finally, for $x, y \in X$, the *interval* $[x, y]$ between x, y is the union of all geodesics between x, y, consisting of exactly those $z \in X$ with $d(x, z) + d(z, y) = d(x, y)$. We say that $A \subseteq X$ is *convex* if $[x, y] \subseteq A$ for all $x, y \in A$. For vertices $x, y, z \in X$, we write x—y—z for $y \in [x, z]$. By the triangle inequality, we have for all $w, x, y, z \in X$ that

$$(w—x—y \text{ and } w—y—z) \quad \Leftrightarrow \quad (w—x—z \text{ and } x—y—z),$$

and both sides occur iff there is a geodesic from w to x to y to z, which we write as w—x—y—z.

1.1. **Profinite pocsets of cuts.** The construction starts by identifying a profinite pocset $\mathcal{H}$ of 'cuts' in each component of the graphing, which we first study abstractly. The finitely-separating subpocsets of 2^X are well-known in metric geometry as *wallspaces*; see, e.g., [Nic04] and [CN05].

Definition 1.1. A *pocset* $(\mathcal{H}, \le, \neg, 0)$ is a poset $(\mathcal{H}, \le)$ equipped with an order-reversing involution $\neg : \mathcal{H} \to \mathcal{H}$ and a least element $0 \ne \neg 0$ such that 0 is the only lower-bound of $H, \neg H$ for every $H \in \mathcal{H}$. We call the elements in $\mathcal{H}$ *half-spaces*, and the edge boundaries $\partial_{\mathsf{ie}} H$ for $H \in \mathcal{H}$ *hyperplanes*.

A *profinite pocset* is a pocset $\mathcal{H}$ equipped with a compact topology making $\neg$ continuous and is *totally order-disconnected*, in the sense that if $H \not\le K$, then there is a clopen upward-closed $U \subseteq \mathcal{H}$ with $H \in U \not\ni K$.

Remark 1.2. Such a topology is automatically Hausdorff and zero-dimensional, making it a Stone space.

We are primarily interested in subpocsets of $(2^X, \subseteq, \neg, \varnothing)$ for a fixed set X, which is profinite if equipped with the product topology of the discrete space 2. Indeed, 2^X admits a base of *cylinder sets* – which are finite intersections of sets of the form $\pi_x^{-1}(i)$ where $x \in X$, $i \in \{0, 1\}$, and $\pi_x : 2^X \to 2$ is the projection – making $\neg$ continuous since cylinders are clopen. Finally, for $H \not\le K$, let U be the upward-closure of a clopen neighborhood $U_0 \ni H$ separating it from K, which is clopen since $\neg U_0$ is a finite union of cylinders.

Remark. We follow [CPTT23, Convention 2.7], where for a family $\mathcal{H} \subseteq 2^X$ of subsets of a fixed set X, we write $\mathcal{H}^* \coloneqq \mathcal{H} \setminus \{\varnothing, X\}$ for the *non-trivial* elements of $\mathcal{H}$.

The following proposition gives a sufficient criteria for subpocsets of 2^X to be profinite. We also show in this case that every non-trivial element $H \in \mathcal{H}^*$ is isolated, which will be important in Section 3.1.

Lemma 1.3. *If $\mathcal{H} \subseteq 2^X$ is a finitely-separating pocset, then $\mathcal{H}$ is closed and non-trivial elements are isolated.*

Proof. It suffices to show that the limit points of $\mathcal{H}$ are trivial, so let $A \in 2^X \setminus \{\varnothing, X\}$. Fix $x \in A \not\ni y$. Since $\mathcal{H}$ is finitely-separating, there are finitely-many $H \in \mathcal{H}$ with $x \in H \not\ni y$, and for each such $H \in \mathcal{H} \setminus \{A\}$, we have either some $x_H \in A \setminus H$ or $y_H \in H \setminus A$. Let $U \subseteq 2^X$ be the family of all subsets $B \subseteq X$ containing x and each x_H, but not y or any y_H.

This is the desired neighborhood isolating $A \in U$. Indeed, it is (cl)open since it is the *finite* intersection of cylinders prescribed by the x_H's and y_H's, and it is disjoint from $\mathcal{H} \setminus \{A\}$ by construction. ∎

Our main method of identifying the finitely-separating pocsets is as follows.

Lemma 1.4. *Let $\mathcal{H} \subseteq 2^X$ be a pocset in a connected graph (X, G). If each $x \in X$ is on the vertex boundary of finitely-many half-spaces, then $\mathcal{H}$ is finitely-separating. The converse holds too if (X, G) is locally-finite.*

Proof. Any $H \in \mathcal{H}$ separating $x, y \in X$ separates some edge on any fixed path between x and y, and there are only finitely-many such H for each edge. If (X, G) is locally-finite, then each $x \in X$ is separated from each of its finitely-many neighbors by finitely-many $H \in \mathcal{H}$. ∎

In the case that $\mathcal{H}$ is a pocset consisting of connected co-connected half-spaces with finite vertex boundary, finite-separation also controls the degree of 'non-nestedness' of $\mathcal{H}$.

Definition 1.5. A *cut* in a connected locally-finite graph (X, G) is a half-space $H \in \mathcal{H}_{\partial<\infty}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$.

Definition 1.6. Let $\mathcal{H} \subseteq 2^X$ be a pocset. Two half-spaces $H, K \in \mathcal{H}$ are *nested* if $\neg^i H \cap \neg^j K = \varnothing$ for some $i, j \in \{0, 1\}$, where $\neg^0 H := H$ and $\neg^1 H := \neg H$. We say that $\mathcal{H}$ is *nested* if every pair $H, K \in \mathcal{H}$ is nested.

Lemma 1.7. *In a pocset $\mathcal{H}$ of finitely-separating cuts, each $H \in \mathcal{H}$ is non-nested with finitely-many others.*

Proof. Fix $H \in \mathcal{H}$ and let $K \in \mathcal{H}$ be non-nested with H. By connectedness, the non-empty sets $H \cap K$ and $\neg H \cap K$ are joined by a path in K, so $\partial_{\mathsf{v}} H \cap K \neq \varnothing$; similarly, $\partial_{\mathsf{v}} H \cap \neg K \neq \varnothing$. For each $x \in \partial_{\mathsf{v}} H \cap K$ and $y \in \partial_{\mathsf{v}} H \cap \neg K$, any fixed path p_{xy} between them contains some $z \in \partial_{\mathsf{v}} K \cap p_{xy}$; thus, any $K \in \mathcal{H}$ non-nested with H contains some $z \in \partial_{\mathsf{v}} K \cap p_{xy}$.

Then, since there are finitely-many such $x, y \in \partial_{\mathsf{v}} H$, for each of which there are finitely-many $z \in p_{xy}$, for each of which there are finitely-many $K \in \mathcal{H}$ with $z \in \partial_{\mathsf{v}} K$ (by Lemma 1.4, since $\mathcal{H}$ is finitely-separating), there can only be finitely-many $K \in \mathcal{H}$ non-nested with H. ∎

1.2. **Ends of graphs.** Let (X, G) be a connected locally-finite graph, and consider the Boolean algebra of finite vertex boundary half-spaces $\mathcal{H}_{\partial<\infty}(X) \subseteq 2^X$.

Definition 1.8. The *end compactification* of (X, G) is the Stone space $\widehat{X}$ of $\mathcal{H}_{\partial<\infty}(X)$, whose non-principal ultrafilters are the *ends* of (X, G).

We identify $X \hookrightarrow \widehat{X}$ via principal ultrafilter map $x \mapsto p_x$, so $\varepsilon(X) := \widehat{X} \setminus X$ is the set of ends of G. By definition, $\widehat{X}$ admits a basis of clopen sets of the form $\widehat{A} := \{p \in \widehat{X} : A \in p\}$ for each $A \in \mathcal{H}_{\partial<\infty}(X)$.

Definition 1.9. A pocset $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty}(X)$ is *dense towards ends* of (X, G) if $\mathcal{H}$ contains a neighborhood basis for every end in $\varepsilon(X)$.

In other words, $\mathcal{H}$ is dense towards ends if for every $p \in \varepsilon(X)$ and every (clopen) neighborhood $\widehat{A} \ni p$, where $A \in \mathcal{H}_{\partial<\infty}(X)$, there is some $H \in \mathcal{H}$ with $p \in \widehat{H} \subseteq \widehat{A}$; it is useful to note that $\widehat{H} \subseteq \widehat{A}$ iff $H \subseteq A$, so we will abuse notation and write $p \in H \subseteq A$ for the above condition.

Lemma 1.10. *A subset $A \in \mathcal{H}_{\partial<\infty}(X)$ is infinite iff it contains an end in (X, G).*

Proof. The converse direction follows since ends are non-principal. If A is infinite, then by local-finiteness of (X, G), Kőnig's Lemma furnishes some infinite ray $(x_n) \subseteq A$. Then, A is contained in the filter

$$p := \{H \in \mathcal{H}_{\partial<\infty}(X) : \forall^\infty n (x_n \in H)\},$$

which is ultra since $H \in p$ are of *finite*-boundary, and is non-principal since it contains cofinite sets. ∎

We show in Section 2 that certain half-spaces $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty}$ induced by a locally-finite quasi-tree is dense towards ends. It will also be important that these half-spaces be cuts, in that witnesses to density can also be found in $\mathcal{H} \cap \mathcal{H}_{\text{conn}}$. The following lemma takes care of this.

Lemma 1.11. *If a subpocset $\mathcal{H} \subseteq \mathcal{H}_{\partial<\infty}$ is dense towards ends, then there is a subpocset $\mathcal{H}' \subseteq \mathcal{H}_{\partial<\infty} \cap \mathcal{H}_{\text{conn}}$, which is also dense towards ends, such that every $H' \in \mathcal{H}'$ has $\partial_{\mathsf{ie}} H' \subseteq \partial_{\mathsf{ie}} H$ for some $H \in \mathcal{H}$.*

Proof. A first attempt is to let $\mathcal{H}'$ be the connected components H'_0 of elements in $\mathcal{H}$, but this fails since $\neg H'_0$ is not necessarily connected. Instead, we further take a component of $\neg H'_0$, whose complement clearly co-connected, and is connected since it consists of H'_0 and the other components of $\neg H'_0$, each of which is connected to H'_0 via $\partial_{\mathsf{ie}} H'_0$. Formally, we let

$$\mathcal{H}' := \{H' \subseteq X : H \in \mathcal{H} \text{ and } H'_0 \in H/G \text{ and } \neg H' \in \neg H'_0/G\},$$

where H/G denotes the G-components of H. Clearly $\partial_{\mathsf{ie}} H' \subseteq \partial_{\mathsf{ie}} H'_0 \subseteq \partial_{\mathsf{ie}} H$, and since $H' \in \mathcal{H}_{\text{conn}}(X)$, it remains to show that $\mathcal{H}'$ is dense towards ends.

Fix an end $p \in \varepsilon(X)$ and a neighborhood $p \in A \in \mathcal{H}_{\partial<\infty}(X)$. Let $B \supseteq \partial_{\mathsf{v}} A$ be finite connected, which can be obtained by adjoining paths between its components. Then $\neg B \in p$ since p is non-principal, so there is $H \in \mathcal{H}$ with $p \in H \subseteq \neg B$. We now use the above recipe to find the desired $H' \in \mathcal{H}'$:

(1) Since $H \in \mathcal{H}_{\partial<\infty}(X)$, it has finitely-many connected components, so exactly one of them belongs to p, say $p \in H'_0 \subseteq H$. Note that $B \subseteq \neg H \subseteq \neg H'_0$.
(2) Since B is connected, there is a unique component $\neg H' \subseteq \neg H'_0$ containing B.

Observe that $H' \in \mathcal{H}'$ and $p \in H'$. Lastly, since H' is connected and is disjoint from $\partial_{\mathsf{v}} A \subseteq B$, and since $H' \subseteq \neg A$ would imply $\neg H' \in p$, this forces $H' \subseteq A$, and hence $p \in H' \subseteq A$ as desired. ∎

1.3. **Median graphs and projections.** Starting from a profinite pocset $\mathcal{H}$ with every non-trivial element isolated, we construct in Section 3.1 its dual median graph $\mathcal{M}(\mathcal{H})$.

We devote this section and the next to study some basic properties of median graphs and their projections, which will be used in Section 3.2 to construct a spanning tree certain median graphs. For more comprehensive references of median graphs, and their general theory, see [Rol98] and [Bow22].

Definition 1.12. A *median graph* is a connected graph (X, G) such that for any $x, y, z \in X$, the intersection

$$[x, y] \cap [y, z] \cap [x, z]$$

is a singleton, whose element $\langle x, y, z\rangle$ is called the *median* of x, y, z. Thus we have a ternary median operation $\langle \cdot, \cdot, \cdot\rangle : X^3 \to X$, and a *median homomorphism* $f : (X, G) \to (Y, H)$ is a map preserving said operation.

Lemma 1.13. *For any $\varnothing \neq A \subseteq X$ and $x \in X$, there is a unique point in* $\operatorname{cvx}(A)$ *between x and every point in A, called the* projection *of x towards A, denoted* $\operatorname{proj}_A(x)$.

Moreover, we have $\bigcap_{a\in A}[x, a] = [x, \operatorname{proj}_A(x)]$, and for any y in this set, we have $\operatorname{proj}_A(y) = \operatorname{proj}_A(x)$.

Proof. To show existence, pick any $a_0 \in A$. Given $a_n \in \operatorname{cvx}(A)$, if there exists $a \in A$ with $a_n \notin [x, a]$, set $a_{n+1} \coloneqq \langle x, a, a_n\rangle \in \operatorname{cvx}(A)$. Then $a_0—a_1— \cdots —a_n—x$ for all n, so this sequence terminates in at most $d(a_0, x)$ steps at a point in $\operatorname{cvx}(A)$ between x and every point in A. For uniqueness, if there exist two such points $a, b \in \operatorname{cvx}(A)$, then $x—a—b$ and $x—b—a$, forcing $a = b$.

Finally, if $x—y— \operatorname{proj}_A(x)$ and $a \in A$, then $x— \operatorname{proj}_A(x)—a$ and hence $x—y—a$. Conversely, let $x—y—a$ for all $a \in A$. Since $[y, a] \subseteq [x, a]$ for all a, we see that

$$\operatorname{proj}_A(y) \in \operatorname{cvx}(A) \cap \bigcap_{a\in A}[y, a] \subseteq \operatorname{cvx}(A) \cap \bigcap_{a\in A}[x, a]$$

and hence $\operatorname{proj}_A(y) = \operatorname{proj}_A(x)$ by uniqueness. But since $y— \operatorname{proj}_A(y)—a$, we have $x—y— \operatorname{proj}_A(y)$, and hence $x—y— \operatorname{proj}_A(x)$ as desired. ∎

Remark 1.14. It follows from the proof above that for any median homomorphism $f : (X, G) \to (Y, H)$, we have $f(\operatorname{proj}_A(x)) = \operatorname{proj}_{f(A)}(f(x))$ for any $\varnothing \neq A \subseteq X$ and $x \in X$. Indeed, we have

$$\operatorname{proj}_A(x) = \langle x, a_m, \dots, \langle x, a_2, \langle x, a_1, a_0\rangle\rangle \dots\rangle$$

for some $m \leq d(a_0, x)$ and $a_0, \dots, a_m \in A$, and this is preserved by f.

For $A \coloneqq \{a, b\}$, we have $\operatorname{proj}_A(x) = \langle a, b, x\rangle$, and hence $\operatorname{cvx}(A) = \operatorname{proj}_A(X) = \langle a, b, X\rangle = [a, b]$.

Lemma 1.15. *For each $x, y \in X$,* $\operatorname{cone}_x(y)$ *is convex, and if xGy, then* $\operatorname{cone}_x(y) \sqcup \operatorname{cone}_y(x) = X$.

Proof. Fix $a, b \in \operatorname{cone}_x(y)$ and $a—c—b$. It suffices to show that $x—y— \langle a, c, x\rangle$, for then $x—y—c$ since we have $x— \langle a, c, x\rangle —c$. Indeed, it follows from the following observations.

- $x—y— \langle a, b, x\rangle$, since $\langle a, b, x\rangle = \operatorname{proj}_{\{a,b\}}(x)$ and so $[x, \langle a, b, x\rangle] = [x, a] \cap [x, b] \ni y$ by Lemma 1.13.
- $x— \langle a, b, x\rangle — \langle a, c, x\rangle$, which follows from $\langle a, b, x\rangle — \langle a, c, x\rangle —a$, since $x— \langle a, b, x\rangle —a$ by definition. Indeed, we have $\langle a, c, x\rangle$ is in both $[a, x]$ and $[a, c] \subseteq [a, b]$, and since $\operatorname{proj}_{\{b,x\}}(a) = \langle a, b, x\rangle$, we have again by Lemma 1.13 that $[\langle a, b, x\rangle , a] = [a, x] \cap [a, b] \ni \langle a, c, x\rangle$.

Finally, take $z \in X$ and consider $w \coloneqq \langle x, y, z\rangle \subseteq [x, y]$. Either $w = x$ or $w = y$ (but not both), giving us the desired partition. ∎

Remark 1.16. In particular, this shows that if xGy, then $\operatorname{cone}_x(y) \in \mathcal{H}^*_{\operatorname{cvx}}(X)$. The convexity of cones also shows, in the situation of Lemma 1.13, that $\operatorname{proj}_A = \operatorname{proj}_{\operatorname{cvx}(A)}$, i.e., $\operatorname{proj}_A(x)$ is also between x and every point in $\operatorname{cvx}(A)$: indeed, note that $\operatorname{cone}_x(\operatorname{proj}_A(x))$ is convex and contains A, so it contains $\operatorname{cvx}(A)$ too.

Lemma 1.17. $\operatorname{proj}_A : X \twoheadrightarrow \operatorname{cvx}(A)$ *is a median homomorphism with* $\operatorname{proj}_A \circ \operatorname{cvx} = \operatorname{cvx} \circ \operatorname{proj}_A$.

Proof. The second claim follows from the first since, by Remark 1.14, we have

$$f(\operatorname{cvx}(B)) = f(\operatorname{proj}_B(X)) = \operatorname{proj}_{f(B)}(f(X)) = \operatorname{cvx}(f(B))$$

for all median homomorphisms $f : X \twoheadrightarrow Y$ and $B \subseteq X$, so it in particular applies to $f \coloneqq \operatorname{proj}_A$.

To this end, let $x—y—z \in X$ and set $w \coloneqq \langle \operatorname{proj}_A(x), \operatorname{proj}_A(y), \operatorname{proj}_A(z)\rangle \in \operatorname{cvx}(A)$. It suffices to show that $y—w—a$ for all $a \in A$, for then $w = \operatorname{proj}_A(y)$ and hence $\operatorname{proj}_A(x)— \operatorname{proj}_A(y)— \operatorname{proj}_A(z)$. But we have $y— \operatorname{proj}_A(y)—a$ already, so it further suffices to show that $y—w— \operatorname{proj}_A(y)$. For this, we note that

$$x— \operatorname{proj}_A(x)— \operatorname{proj}_A(y) \quad \text{and} \quad \operatorname{proj}_A(x)—w— \operatorname{proj}_A(y),$$

so x—w—$\mathrm{proj}_A(y)$, and similarly z—w—$\mathrm{proj}_A(y)$. Thus, it follows that

$$\begin{aligned} w \in [\mathrm{proj}_A(y), x] \cap [\mathrm{proj}_A(y), z] &= [\mathrm{proj}_A(y), \mathrm{proj}_{\{x,z\}}(\mathrm{proj}_A(y))] && \text{Lemma 1.13} \\ &= [\mathrm{proj}_A(y), \mathrm{proj}_{[x,z]}(\mathrm{proj}_A(y))] && \text{Remark 1.16} \\ &\subseteq [\mathrm{proj}_A(y), y], && \text{Lemma 1.13} \end{aligned}$$

where the second equality follows from $\mathrm{cvx}(\{x,z\}) = [x,z]$, and hence $\mathrm{proj}_{\{x,z\}} = \mathrm{proj}_{[x,z]}$. ■

1.4. **Convex half-spaces of median graphs.** We now use projections to explore the geometry of convex half-spaces in median graphs. For the axiomatics of convex structures, see [vdV93].

Proposition 1.18. *Each edge $(x,y) \in G$ is on a unique hyperplane, namely the inward boundary of $\mathrm{cone}_x(y)$, and conversely, each half-space $H \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ is $\mathrm{cone}_x(y)$ for every $(x,y) \in \partial_{\mathsf{ie}} H$.*

Thus, hyperplanes are equivalence classes of edges. Furthermore, this equivalence relation is generated by parallel sides of squares (i.e., 4-cycles).

Proof. We have $\mathrm{cone}_x(y) \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ by Lemma 1.15, and clearly $(x,y) \in \partial_{\mathsf{ie}} \mathrm{cone}_x(y)$. Conversely, take $H \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ and any $(x,y) \in \partial_{\mathsf{ie}} H$. Then $H = \mathrm{cone}_x(y)$, for if $z \in H \cap \neg\,\mathrm{cone}_x(y)$, then $z \in \mathrm{cone}_y(x)$, and hence $x \in [y,z] \subseteq H$ by convexity of H, a contradiction; if $z \in \mathrm{cone}_x(y) \cap \neg H$, then $[x,z] \subseteq \neg H$ by convexity of $\neg H$, and hence $y \notin H$, a contradiction.

Finally, parallel edges of a strip of squares generate the same hyperplane since, for a given square, each vertex is between its neighbors and hence any hyperplane containing an edge contains its opposite edge. On the other hand, let $(a,b), (c,d) \in \partial_{\mathsf{ie}} H$ for some $H \in \mathcal{H}^*_{\mathsf{cvx}}(X)$. Note that $\partial_{\mathsf{ov}} H = \mathrm{proj}_{\neg H}(H)$ is convex since H is, and since $\mathrm{proj}_{\neg H}$ preserves convexity by Lemma 1.17, any geodesic between $a, c \in \partial_{\mathsf{ov}} H$ lies in $\partial_{\mathsf{ov}} H$. Matching this geodesic via $\partial_{\mathsf{ie}} H : \partial_{\mathsf{ov}} H \to \partial_{\mathsf{iv}} H$ gives us a geodesic between b, d in $\partial_{\mathsf{iv}} H$, which together with the matching forms the desired strip of squares. ■

Corollary 1.19. *Two half-spaces $H, K \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ are non-nested iff there is an embedding $\{0,1\}^2 \hookrightarrow X$ of the Hamming cube into the four corners $\neg^i H \cap \neg^j K$.*

In particular, if $H, K \in \mathcal{H}^*_{\mathsf{cvx}}(X)$ are non-nested, then $\partial_{\mathsf{v}} H \cap \partial_{\mathsf{v}} K \neq \varnothing$.

Proof. Let H, K be non-nested and take $x_1 \in H \cap K$ and $x_2 \in H \cap \neg K$. Since H is connected, any geodesic between x_1, x_2 crosses an edge $(x_1', x_2') \in \partial_{\mathsf{oe}} K$ in H. Similarly, there is an edge $(y_1', y_2') \in \partial_{\mathsf{oe}} K$ in $\neg H$, so we may slide both edges along $\partial_{\mathsf{oe}} K$ to obtain the desired square (see Proposition 1.18).

Conversely, the half-spaces cutting the square are clearly non-nested. ■

Lemma 1.20 (Helly)**.** *Any finite intersection of pairwise-intersecting non-empty convex sets is non-empty.*

Proof. For pairwise-intersecting convex sets H_1, H_2, H_3, pick any $x \in H_1 \cap H_2$, $y \in H_1 \cap H_3$ and $z \in H_2 \cap H_3$; their median $\langle x, y, z\rangle$ then lies in $H_1 \cap H_2 \cap H_3$.

Suppose that it holds for some $n \geq 3$ and let $H_1, \ldots, H_{n+1} \subseteq X$ pairwise-intersect. Then $\{H_i \cap H_{n+1}\}_{i \leq n}$ is a family of n pairwise-intersecting convex sets, so $\bigcap_{i \leq n+1} H_i = \bigcap_{i \leq n} (H_i \cap H_{n+1})$ is non-empty. ■

Lastly, we have some useful finiteness conditions on convex half-spaces; the former implies that $\mathcal{H}_{\mathsf{cvx}}(X)$ is finitely-separating, and the latter allows us to replace finite sets with their convex hulls.

Lemma 1.21. *Any two disjoint convex sets $\varnothing \neq A, B \subseteq X$ can be separated by a half-space $A \subseteq H \subseteq \neg B$, and furthermore we have $d(A,B) = |\{H \in \mathcal{H}_{\mathsf{cvx}}(X) : A \subseteq H \subseteq \neg B\}|$.*

Proof. Pick a geodesic $A \ni x_0 G x_1 G \cdots G x_n \in B$, where $n \coloneqq d(A,B)$. Then $H \coloneqq \mathrm{cone}_{x_1}(x_0)$, which is a half-space by Lemma 1.15, separates A, B since $x_0 = \mathrm{proj}_A(x_n)$, and thus we have $A \subseteq \mathrm{cone}_{x_n}(x_0) \subseteq \mathrm{cone}_{x_1}(x_0)$ and $B \subseteq \mathrm{cone}_{x_0}(x_n) \subseteq \mathrm{cone}_{x_0}(x_1)$.

Moreover, each such half-space $A \subseteq H \subseteq \neg B$ satisfies $x_i \in H \not\ni x_{i+1}$ for a unique $i < n$, and conversely each pair (x_i, x_{i+1}) has a unique half-space separating them, so we have the desired bijection. ■

Lemma 1.22. *Every interval $[x,y]$ is finite. More generally, if $A \subseteq X$ is finite, then so is $\mathrm{cvx}(A)$.*

Proof. The singletons $\{x\}$ and $\{y\}$ are convex, so there are finitely-many half-spaces $H \subseteq [x,y]$. But each $z \in [x,y]$ is determined uniquely by those half-spaces containing it, so $[x,y]$ is finite.

Let $A \coloneqq \{x_0, \ldots, x_n\}$. Since $\mathrm{cvx}(A) = \mathrm{proj}_A(X)$, we have by Remark 1.14 that points in $\mathrm{cvx}(A)$ are of the form $\langle x, x_n, \ldots, \langle x, x_2, \langle x, x_1, x_0\rangle\rangle \ldots\rangle$, which is finite by induction using that intervals are finite. ■

2. Graphs with Dense Families of Cuts

Let (X, G) be a connected locally-finite quasi-tree, which, in the context of Theorem A, stands for a single component of the locally-finite graphing of the CBER. For Theorem B to apply, we need to first identify a family of finitely-separating cuts therein, and we do so in such a way that the cuts are dense towards ends.

Since X is a quasi-tree, and thus does not have arbitrarily long cycles, we expect that there is some finite bound $R < \infty$ such that the ends in $\varepsilon(X)$ are 'limits' of cuts $\mathcal{H} \coloneqq \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ with boundary diameter bounded by R. We show that this is indeed the case, in the sense that $\mathcal{H}$ is dense towards ends.

Lemma 2.1. *Let $f : (X, G) \to (Y, T)$ be a coarse-equivalence between connected graphs. For a fixed $H \in \mathcal{H}_{\partial<\infty}(Y)$, $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$ is uniformly bounded in terms of $\mathrm{diam}(\partial_{\mathsf{v}} H)$.*

Proof. Since f is bornologous, let $S < \infty$ be such that xGx' implies $d(f(x), f(x')) \leq S$, so that for any $(x, x') \in \partial_{\mathsf{ie}} f^{-1}(H)$, there is a path of length $\leq S$ between $f(x) \notin H$ and $f(x') \in H$. Thus both $d(f(x), \partial_{\mathsf{v}} H)$ and $d(f(x'), \partial_{\mathsf{v}} H)$ are bounded by S, so $f(\partial_{\mathsf{v}} f^{-1}(H)) \subseteq \mathrm{Ball}_S(\partial_{\mathsf{v}} H)$ and hence

$$\mathrm{diam}(f(\partial_{\mathsf{v}} f^{-1}(H))) \leq \mathrm{diam}(\partial_{\mathsf{v}} H) + 2S.$$

That f is a coarse-*equivalence* gives us a uniform bound of $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$ in terms of $\mathrm{diam}(\partial_{\mathsf{v}} H)$. ∎

In particular, if $\mathrm{diam}(\partial_{\mathsf{v}} H)$ is itself also uniformly bounded, then so is $\mathrm{diam}(\partial_{\mathsf{v}} f^{-1}(H))$.

Proposition 2.2. *The class of connected locally-finite graphs in which $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}$ is dense towards ends for some $R < \infty$ is invariant under coarse equivalence.*

Proof. Let (X, G), (Y, T) be connected locally-finite graphs, $f : X \to Y$ be a coarse equivalence with quasi-inverse $g : Y \to X$, and suppose $\mathcal{H}_{\mathrm{diam}(\partial)\leq S}(Y)$ is dense towards ends for some $S < \infty$. By Lemma 2.1, pick some $R < \infty$ so that for any $H \in \mathcal{H}_{\mathrm{diam}(\partial)\leq S}(Y)$, we have $f^{-1}(H) \in \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$.

Fix an end $p \in \varepsilon(X)$ and a neighborhood $p \in A \in \mathcal{H}_{\partial<\infty}(X)$. We need to find some $B \in \mathcal{H}_{\partial<\infty}(Y)$ such that $f(p) \in B$ and $f^{-1}(B) \subseteq A$, for then $f(p) \in H$ for some $B \supseteq H \in \mathcal{H}_{\mathrm{diam}(\partial)\leq S}(Y)$, and hence we have

$$p \in f^{-1}(H) \subseteq f^{-1}(B) \subseteq A$$

with $f^{-1}(H) \in \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$. For convenience, let $D < \infty$ be the uniform distance $d(1_X, g \circ f)$.

To this end, note that $f(p) \in B$ iff $p \in f^{-1}(B)$. Since $p \in A$, the latter can occur if $|A \triangle f^{-1}(B)| < \infty$, and so we need to find such a $B \in \mathcal{H}_{\partial<\infty}(Y)$ with the additional property that $f^{-1}(B) \subseteq A$.

> *Attempt 1.* Set $B \coloneqq g^{-1}(A) \in \mathcal{H}_{\partial<\infty}(Y)$. Then $f^{-1}(B) \subseteq \mathrm{Ball}_D(A)$ since if $(g \circ f)(x) \in A$, then
>
> $$d(x, A) \leq d(x, (g \circ f)(x)) \leq d(1_X, g \circ f) = D.$$
>
> By local-finiteness of G, we see that $A \triangle f^{-1}(B) = A \setminus f^{-1}(B)$ is finite, as desired.

However, it is *not* the case that $f^{-1}(B) \subseteq A$. To remedy this, we 'shrink' A by D to A' so that $\mathrm{Ball}_D(A') \subseteq A$, and take $B \coloneqq g^{-1}(A')$ instead. Indeed, $A' \coloneqq \neg\,\mathrm{Ball}_D(\neg A) \subseteq A$ works, since $f^{-1}(B) \subseteq \mathrm{Ball}_D(A')$ as before, so $A' \triangle f^{-1}(B) = A' \setminus f^{-1}(B)$ is finite. Also, $A \triangle A'$ is finite since $x \in A \triangle A'$ iff $x \in A$ and $d(x, \neg A) \leq D$, so $A \triangle f^{-1}(B)$ is finite too. It remains to show that $\mathrm{Ball}_D(A') \subseteq A$, for then $f^{-1}(B) \subseteq A$ as desired.

Indeed, if $y \in \mathrm{Ball}_D(A')$, then by the (reverse) triangle-inequality we have $d(y, \neg A) \geq d(x, \neg A) - d(x, y)$ for all $x \in A'$. But $d(x, \neg A) > D$, strictly, so $d(y, \neg A) > D - D = 0$, and hence $y \in A$. ∎

Corollary 2.3. *If (X, G) is a locally-finite quasi-tree, then the subpocset $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ is dense towards ends for some $R < \infty$.*

Proof. Observe that $\mathcal{H}_{\mathrm{diam}(\partial)\leq 2}(T)$ is dense towards ends for any tree T, so Proposition 2.2 proves the density of $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$ for some $R < \infty$. By Lemma 1.11, there is a subpocset $\mathcal{H}' \subseteq \mathcal{H}_{\partial<\infty}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ dense towards ends such that for every $H' \in \mathcal{H}'$, we have $\partial_{\mathsf{ie}} H' \subseteq \partial_{\mathsf{ie}} H$ for some $H \in \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X)$. Hence we have $H' \in \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$, so the result follows. ∎

The cuts $\mathcal{H} \coloneqq \mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ obtained here will be our starting point for Theorem B, so we need to show that it is finitely-separating. Indeed, by local-finiteness of (X, G), the R-ball around any fixed $x \in X$ is finite, so since any cut $H \in \mathcal{H}$ with $x \in \partial_{\mathsf{v}} H$ is contained in said R-ball, there are finitely-many such cuts. Thus, by Lemma 1.4, $\mathcal{H}$ is finitely-separating.

Furthermore, we have by Lemma 1.3 that $\mathcal{H} \subseteq 2^X$ is closed and every non-trivial element is isolated, and those conditions allow for the construction in Theorem B to continue.

3. The Dual Median Graph of a Profinite Pocset and its Spanning Trees

3.1. **Construction of the dual median graph.** We present a classical construction in geometric group theory of a median graph associated to a profinite pocset with every non-trivial element isolated; see [Dun79], [Rol98], [Sag95], and [NR03] for other applications.

In the context of Theorem A, this will be applied to the pocset $\mathcal{H}_{\mathrm{diam}(\partial)\leq R}(X) \cap \mathcal{H}_{\mathrm{conn}}(X)$ of cuts in a locally-finite graph (X, G), and is also the first step in the construction in Theorem B.

Definition 3.1. An *orientation* on $\mathcal{H}$ is an upward-closed subset $U \subseteq \mathcal{H}$ containing exactly one of $H, \neg H$ for each $H \in \mathcal{H}$. We let $\mathcal{U}(\mathcal{H})$ denote the set of all orientations on $\mathcal{H}$ and let $\mathcal{U}^\circ(\mathcal{H})$ denote the clopen ones.

Intuitively, an orientation is a 'maximally consistent' choice of half-spaces[3].

Example 3.2. Each $x \in X$ induces its *principal orientation* $\widehat{x} := \{H \in \mathcal{H} : x \in H\} = \mathcal{H} \cap \pi_x^{-1}(1)$ – which is clearly clopen in $\mathcal{H}$ – and gives us a canonical map $X \to \mathcal{U}^\circ(\mathcal{H})$. However, this map is *not necessarily* injective, and we call a fiber $[x]_{\mathcal{H}} := \{y \in X : \widehat{x} = \widehat{y}\}$ thereof an *$\mathcal{H}$-block*. This induces an equivalence relation on X by declaring $x \sim_{\mathcal{H}} y$ iff x and y are contained in exactly the same half-spaces in $\mathcal{H}$.

Proposition 3.3. *Let $\mathcal{H}$ be a profinite pocset with every non-trivial element isolated. Then the graph $\mathcal{M}(\mathcal{H})$, whose vertices are clopen orientations $\mathcal{U}^\circ(\mathcal{H})$ and whose edges are pairs $\{U, V\}$ with $V = U \triangle \{H, \neg H\}$ for some minimal $H \in U \setminus \{\varnothing, X\}$, is a median graph with path metric $d(U, V) = |U \triangle V|/2$ and medians*

$$\begin{aligned}\langle U, V, W\rangle &:= \{H \in \mathcal{H} : H \text{ belongs to at least two of } U, V, W\} \\ &= (U \cap V) \cup (V \cap W) \cup (U \cap W).\end{aligned}$$

Proof. First, $V := U \triangle \{H, \neg H\}$ as above is clopen since $H, \neg H \in \mathcal{H}^*$ are isolated (whence $\{H\}, \{\neg H\}$ are clopen), and it is an orientation by minimality of H. That $\mathcal{M}(\mathcal{H})$ is connected follows from the following claim by noting that $U \triangle V \not\ni \varnothing, X$ is clopen, so it is a compact set of isolated points, whence finite.

Claim ([Sag95, Theorem 3.3])**.** There is a path between U, V iff $U \triangle V$ is finite, in which case

$$d(U, V) = |U \triangle V|/2 = |U \setminus V| = |V \setminus U|.$$

Proof. If $(U_i)_{i<n}$ is a path from $U =: U_0$ to $V =: U_{n-1}$, then, letting $\{H_i, \neg H_i\} := U_i \triangle U_{i-1}$ for all $1 \leq i < n$ gives us a sequence $(H_i)_{i<n}$ inducing[a] this path, whence $U \triangle V$ consists of $\{H_i\}_{i<n}$ and their complements. Thus $U \triangle V = 2n = 2d(U, V)$, as desired.

Conversely, if $U \triangle V = \{H_1, \ldots, H_n\} \sqcup \{K_1, \ldots, K_m\}$ with $U \setminus V = \{H_i\}$ and $V \setminus U = \{K_j\}$, then $\neg H_i \in V \setminus U$ and $\neg K_j \in U \setminus V$ for all $i < n$ and $j < m$, so $n = m$ and $V = U \cup \{\neg H_i\} \setminus \{H_i\}$. We claim that there is a permutation $\sigma \in S_n$ such that $(H_{\sigma(i)})$ induces a path from $U =: U_0$, which is the desired path from U to V. Choose a minimal $H \in \{H_i\}$, which is also minimal in U: if $K \subseteq H$ for some $K \in U$, then $\neg H \subseteq \neg K$, and hence $\neg K \in V$, so $K = H_i \subseteq H$ for some i, forcing $K = H$. Set $U_1 := U \triangle \{H, \neg H\}$, which is a clopen orientation. Continuing in this manner by choosing a minimal element in $\{H_i\} \setminus \{H\}$ – and so on – gives us the desired path with $d(U, V) = n$. □

[a] In the sense that $U_i = U_{i-1} \triangle \{H_i, \neg H_i\}$ and $H_i \in U_i$ for each $1 \leq i < n$; see [Tse20, Definition 2.20].

Finally, we show that $\mathcal{M}(\mathcal{H})$ is a median graph. Fix $U, V, W \in \mathcal{U}^\circ(\mathcal{H})$, and note that for any $M \in \mathcal{U}^\circ(\mathcal{H})$, we have by the triangle inequality that $M \in [U, V]$ iff $(U \setminus M) \cup (M \setminus V) \subseteq U \setminus V$, which clearly occurs iff $U \cap V \subseteq M \subseteq U \cup V$. Thus, a vertex M lies in the triple intersection $[U, V] \cap [V, W] \cap [U, W]$ iff

$$(U \cap V) \cup (V \cap W) \cup (U \cap W) \subseteq M \subseteq (U \cup V) \cap (V \cup W) \cap (U \cup W).$$

Note that the two sides coincide, so $M = \langle U, V, W\rangle$ – which is clopen if U, V, W are – is as claimed. ■

Given such a pocset $\mathcal{H}$, the graph $\mathcal{M}(\mathcal{H})$ constructed above is called the *dual*[4] median graph of $\mathcal{H}$. An important special case of this construction is when $\mathcal{H}$ is *nested*, in which case $\mathcal{M}(\mathcal{H})$ is a tree.

[3] This can be formalized by letting $\sim$ be the equivalence relation on $\mathcal{H}$ given by $H \sim \neg H$. Letting $\partial : \mathcal{H} \to \mathcal{H}/\!\sim$ denote the quotient map, orientations $U \subseteq \mathcal{H}$ then correspond precisely to sections $\varphi : \mathcal{H}/\!\sim \to \mathcal{H}$ of ∂ such that $\varphi(\partial H) \not\subseteq \neg\varphi(\partial K)$ for every $H, K \in \mathcal{H}$; the latter condition rules out 'orientations' of the form $\leftarrow\!\mid\;\mid\!\rightarrow$.

[4] The name is justified by a Stone-type duality between {median graphs, median homomorphisms} and {profinite pocsets with non-trivial points isolated, continuous maps}, where from a median graph X one can construct a canonical pocset $\mathcal{H}_{\mathrm{cvx}}(X)$ of convex half-spaces (see [CPTT23, Section 2.D] for details).

Corollary 3.4. *Let $\mathcal{H}$ be a profinite pocset with non-trivial points isolated. If $\mathcal{H}$ is nested, then the median graph $\mathcal{M}(\mathcal{H})$ is acyclic, and hence $\mathcal{M}(\mathcal{H})$ is a tree.*

Proof. Let $(U_i)_{i<n}$ be a (non-backtracking) cycle in $\mathcal{M}(\mathcal{H})$, say induced by some sequence $(H_i)_{i<n} \subseteq \mathcal{H}$ of half-spaces. We show that (H_i) is *strictly* increasing, so that $H_0 \subset \cdots \subset H_n \subset H_0$, which is absurd.

We have $U_{i+1} = U_{i-1} \cup \{\neg H_i, \neg H_{i-1}\} \setminus \{H_i, H_{i-1}\}$, so $H_i \neq \neg H_{i-1}$ (for otherwise $U_{i+1} = U_{i-1}$). Since $H_i \in U_i = U_{i-1} \cup \{\neg H_{i-1}\} \setminus \{H_{i-1}\}$, we see that $H_i \in U_{i-1}$, and since $H_i \neq H_{i-1}$, it suffices by nestedness of $\mathcal{H}$ to remove the three cases when $\neg H_i \subseteq H_{i-1}$, $H_{i-1} \subseteq \neg H_i$, and $H_i \subseteq H_{i-1}$.

Indeed, if $\neg H_i \subseteq H_{i-1}$, then $H_{i-1} \in U_{i+1}$ by upward-closure of $U_{i+1} \ni \neg H_i$. But since $H_{i-1} \neq \neg H_i$, we have by definition of U_{i+1} that $H_{i-1} \in U_i$, a contradiction. The other cases are similar. ■

Nonetheless, in the general non-nested case, $\mathcal{M}(\mathcal{H})$ still admits a *canonical* spanning tree if we fix a Borel colouring of $\mathcal{H}^*_{\mathrm{cvx}}(\mathcal{M}(\mathcal{H}))$ into its nested sub-pocsets, the existence of which follows from the following

Proposition 3.5. *The dual median graph $\mathcal{M}(\mathcal{H})$ of a pocset of finitely-separating cuts has finite hyperplanes.*

Proof. Fix $K \in \mathcal{H}^*_{\mathrm{cvx}}(\mathcal{M}(\mathcal{H}))$, which by Proposition 1.18 is of the form $K = \mathrm{cone}_V(U)$ for some (and hence any) $(U, V) \in \partial_{\mathsf{ie}} K$, and we have by Proposition 3.3 that $V = U \triangle \{H, \neg H\}$ for some (non-trivial) minimal $H \in U$. We claim that any other edge $(U', V') \in \partial_{\mathsf{ie}} K$ can be reached from (U, V) by simultaneously flipping only the half-spaces $H'_0, \ldots, H'_n \in \mathcal{H}^*$ non-nested with H, of which there are finitely-many by Lemma 1.7.

Since the edges $(U, V), (U', V')$ induce the same hyperplane $\partial_{\mathsf{ie}} K$, it suffices by Proposition 1.18 to prove this for when (U', V') is an edge of a square parallel to (U, V), in which case there is some minimal $H' \in \mathcal{H}^*$ flipping both U to U' and V to V'. We have some easy observations.

1. *H, H' are non-nested.* Indeed, $H' \not\subseteq H$ and $H \not\subseteq H'$ by minimality; if $H \subseteq \neg H'$, then $\neg H' \in U$; and if $\neg H \subseteq H'$, then H' is not minimal in U.
2. *$H \in U'$ is still minimal.* Indeed, since $U' = U \triangle \{H', \neg H'\}$, the only way this can fail is if $\neg H' \subseteq H$, but this contradicts minimality of H.

Thus, we have $V' = U' \triangle \{H, \neg H\}$, so the induction continues with (U', V') in place of (U, V). ■

3.2. **Canonical spanning trees.** We now present the Borel cycle-cutting algorithm that can be preformed on any countable median graph with finite hyperplanes. We do so by colouring the half-spaces $\mathcal{H}^*_{\mathrm{cvx}}(X)$ into certain nested sub-pocsets $\mathcal{H}_n \subseteq \mathcal{H}_{\mathrm{cvx}}(X)$, from which we inductively build a spanning forest by leveraging a tree structure on the $\mathcal{H}_n$-blocks $X/\mathcal{H}_n$. This tree is constructed as follows.

Lemma 3.6. *For any subpocset $\mathcal{H} \subseteq \mathcal{H}_{\mathrm{cvx}}(X)$ on a median graph (X, G), the map $X \to \mathcal{U}^\circ(\mathcal{H})$ is surjective.*

Proof. Let $U \in \mathcal{U}^\circ(\mathcal{H})$, we need to find some $x \in X$ with $U = \widehat{x}$. Since $U \subseteq \mathcal{H}$ is clopen, there is a finite set $A \subseteq X$ – which we may assume to be convex by Lemma 1.22 – such that for all $H \in \mathcal{H}$, we have $H \in U$ iff there is $K \in U$ with $H \cap A = K \cap A$. Note that $K \cap A \neq \varnothing$ for every $K \in U$, since otherwise $\varnothing \in U$. Furthermore, $H \cap K \neq \varnothing$ for every $H, K \in U$, since otherwise we have $H \subseteq \neg K$, and so $\neg K \in U$.

By Lemma 1.20, the intersection $(H \cap A) \cap (K \cap A) = H \cap K \cap A$ is non-empty, and applying it again furnishes some $x \in \bigcap_{H \in U} H \cap A$ in X. Thus $U \subseteq \widehat{x}$, so $U = \widehat{x}$ since both are orientations. ■

This induces a *G-adjacency* graph $X/\mathcal{H} \cong \mathcal{M}(\mathcal{H})$; explicitly, two $\mathcal{H}$-blocks $([x]_{\mathcal{H}}, [y]_{\mathcal{H}})$ are G-adjacent if $(\widehat{x}, \widehat{y}) \in \mathcal{M}(\mathcal{H})$. Note that $\mathcal{M}(\mathcal{H})$ may be constructed as in Proposition 3.3 since $\mathcal{H} \subseteq \mathcal{H}_{\mathrm{cvx}}(X)$ is finitely-separating by Lemma 1.21, and in particular, if $\mathcal{H}$ is nested, then $X/\mathcal{H}$ is a tree by Corollary 3.4.

Proposition 3.7. *If (X, G) is a countable median graph with finite hyperplanes, then fixing any colouring of $\mathcal{H}^*_{\mathrm{cvx}}(X)$ into nested sub-pocsets yields a canonical spanning tree thereof.*

Proof. Such a colouring exists, since, by Corollary 1.19, if two half-spaces $H, K \in \mathcal{H}^*_{\mathrm{cvx}}(X)$ are non-nested, then $\partial_{\mathsf{v}} H \cap \partial_{\mathsf{v}} K \neq \varnothing$. Thus, the intersection graph of the boundaries admits a countable colouring, which descends into a colouring $\mathcal{H}^*_{\mathrm{cvx}}(X) = \bigsqcup_{n \in \mathbb{N}} \mathcal{H}^*_n$ such that each $H, \neg H$ receive the same colour and that each $\mathcal{H}_n \coloneqq \mathcal{H}^*_n \cup \{\varnothing, X\}$ is a *nested* subpocset. For each $n \in \mathbb{N}$, let $\mathcal{K}_n \coloneqq \bigcup_{m \geq n} \mathcal{H}_m$.

We shall inductively construct an increasing chain of subforests $T_n \subseteq G$ such that the components of T_n are exactly the $\mathcal{K}_n$-blocks. Then, the increasing union $T \coloneqq \bigcup_n T_n$ is a spanning tree, since each $(x, y) \in G$ lies in a $\mathcal{K}_n$-block for sufficiently large n (namely, the n such that $\mathrm{cone}_x(y) \in \mathcal{H}^*_{n-1}$, since $\mathrm{cone}_x(y)$ and its complement are the only half-spaces separating x and y by Proposition 1.18).

Since each pair of distinct points is separated by a half-space, the $\mathcal{K}_0 = \mathcal{H}_{\mathrm{cvx}}(X)$-blocks are singletons, so put $T_0 \coloneqq \varnothing$. Suppose that a forest T_n is constructed as required. Note that each $\mathcal{K}_{n+1}$-block $Y \in X/\mathcal{K}_{n+1}$ is not separated by any half-spaces in $\mathcal{H}_m$ for $m > n$, but is separated (by Proposition 1.18) by $\mathcal{H}_n$ into the $\mathcal{K}_n$-blocks contained in Y, and those correspond precisely to the $\mathcal{H}_n$-blocks in $Y/\mathcal{H}_n$. Pick an edge from the *finite* hyperplane $\partial_{\mathsf{ie}} H$ for each $H \in \mathcal{H}_n$, which connects a unique pair of G-adjacent blocks in $Y/\mathcal{H}_n$. Since each $Y/\mathcal{H}_n$ is a tree by Corollary 3.4, and each pair of G-adjacent blocks in $Y/\mathcal{H}_n$ is connected by a single picked edge, the graph T_{n+1} obtained from T_n by adding all such edges is a forest whose components are exactly the $\mathcal{K}_{n+1}$-blocks. ■

Applying this to the dual median graph $\mathcal{M}(\mathcal{H})$ of a finitely-separating family of cuts $\mathcal{H}$ on a connected locally-finite graph, which has finite hyperplanes by Lemma 3.5, proves Theorem B.

4. Borel Treeings of Graphings with Dense Cuts

In this last section, we prove Theorem A, stating that if a CBER E admits a locally-finite graphing such that each component is a quasi-tree, then E is treeable. In fact, we prove the following more general result.

Theorem 4.1. *If a CBER E admits a locally-finite graphing (X, G) such that each component $G|C$ admits a family $\mathcal{H}(C)$ of finitely-separating cuts that is dense towards ends of $G|C$, then E is treeable.*

This implies Theorem A, since each quasi-tree $G|C$ admits a family $\mathcal{H}(C) \coloneqq \mathcal{H}_{\mathrm{diam}(\partial) \le R_C}(C) \cap \mathcal{H}_{\mathrm{conn}}(C)$ of finitely-separating cuts, which is dense towards ends of $G|C$ for some $R_C < \infty$; see Section 2.

The proof of this theorem relies on representing the family $\mathcal{U}^\circ(\mathcal{K}) \coloneqq \bigsqcup_{G|C} \mathcal{U}^\circ(\mathcal{H}(C))$ of *all* clopen orientations as a standard Borel space, which is possible due to the following

Lemma 4.2. *Let $\mathcal{H}$ be a finitely-separating pocset of cuts on a connected locally-finite graph (X, G). If $\mathcal{H}$ is dense towards ends, then the dual median graph $\mathcal{M}(\mathcal{H})$ is locally-finite.*

Proof. Fix a vertex $U \in \mathcal{U}^\circ(\mathcal{H})$ and let $H_\alpha \in U$ be its minimal elements. Since $U \subseteq \mathcal{H}$ is clopen, there is a finite set $A \subseteq X$ such that for all $H \in \mathcal{H}$, we have $H \in U$ iff there is $K \in U$ with $H \cap A = K \cap A$.

We have $p \in \neg A$ for each end $p \in \varepsilon(X)$, so density of $\mathcal{H}$ furnishes some $H \in \mathcal{H}$ with $p \in H \subseteq \neg A$; in particular, $H \notin U$. Since U is clopen, $H_\alpha \subseteq \neg H$ for some α. Thus we have a cover $\varepsilon(X) \subseteq \bigcup_\alpha \neg H_\alpha$, which by compactness cuts down to a finite subcover $\varepsilon(X) \subseteq \bigcup_{i<n} \neg H_i$.

Any other minimal $H \in U$ not in this subcover may be assumed to be nested with each H_i, since there are finitely-many non-nested ones by Lemma 1.7. But then $H \not\subseteq H_i \not\subseteq H$ and $H \cap H_i \neq \varnothing$ for all $i < n$, which forces $\neg H \subseteq \bigcap_{i<n} H_i \in \mathcal{H}_{\partial<\infty}(X)$; the latter contains no ends in $\varepsilon(X)$, so it is finite by Lemma 1.10, and hence there are at-most finitely-many more minimal $H \in U$. ■

Proof of Theorem 4.1. We may represent each cut by its finite hyperplane, which makes the collection $\mathcal{K}^* \coloneqq \bigsqcup_{G|C} \mathcal{H}^*(C)$ of all non-trivial cuts into a standard Borel space. Thus, the set $\mathcal{U}^\circ(\mathcal{K})$ of clopen all orientations on $\mathcal{K}$ is also a standard Borel space, since we may encode each $U \in \mathcal{U}^\circ(\mathcal{H}(C))$ by its set of minimal elements, which is finite by Proposition 3.3 and Lemma 4.2.

This allows us to collect the dual median graphs into a median *graphing* $\mathcal{M}(\mathcal{K})$ of $\mathcal{U}^\circ(\mathcal{K})$, and we can implement the proof of Proposition 3.7 in a Borel manner (using [KM04, Lemma 7.3] for a countable colouring of the intersection graph of finite hyperplanes therein) to obtain a treeing of the relation generated by $\mathcal{M}(\mathcal{K})$. But this relation is Borel bireducible with (X, E) via $X \ni x \mapsto \widehat{x} \in \mathcal{U}^\circ(\mathcal{H}([x]_E))$, so (X, E) is also treeable by [JKL02, Proposition 3.3 (ii)], as desired. ■

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