NESTED COLLECTION OF CUTS

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Notation 0.1. Let X be a set. For any family $\mathcal{C} \subseteq 2^X$ closed under the complement operation $\neg: 2^X \to 2^X$, we let $\mathcal{C}^* := \mathcal{C} \setminus \{\emptyset, X\}$ denote the *non-trivial* elements in \mathcal{C} .

1. Preliminaries

1.1. Ultrafilters and their bases. We recall some standard notions, mainly to establish notation.

Definition 1.1. A filter on X is a collection $\mathcal{F} \subseteq \mathcal{P}^*(X)$ that is closed-upwards and closed under pairwise intersections, i.e.

- (1) If $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$; and
- (2) If $A_1, A_2 \in \mathcal{F}$, then $A_1 \cap A_2 \in \mathcal{F}$.

Lemma 1.2. For any family $\mathcal{F}_0 \subseteq \mathcal{P}^*(X)$, the following are equivalent.

- (1) For all $F_1, F_2 \in \mathcal{F}_0$, there is some $F \in \mathcal{F}_0$ such that $F \subseteq F_1 \cap F_2$.
- (2) The collection $\mathcal{F} := \{A \subseteq X : A \supseteq F \text{ for some } F \in \mathcal{F}_0\} \subseteq \mathcal{P}^*(X) \text{ is a filter on } X.$

Proof. If (1) holds, then \mathcal{F} is upward closed since if $A \supseteq F$ for some $F \in \mathcal{F}_0$, then $B \supseteq F$ for any $B \supseteq A$. Moreover, if $A_i \supseteq F_i$ for i = 1, 2, then $A_1 \cap A_2 \supseteq F_1 \cap F_2 \supseteq F$ for some $F \in \mathcal{F}_0$ furnished by (1).

Conversely, let $F_1, F_2 \in \mathcal{F}_0$. Then $F_1, F_2 \in \mathcal{F}$, so $F_1 \cap F_2 \in \mathcal{F}$ by (2). Thus $F_1 \cap F_2 \supseteq F$ for some $F \in \mathcal{F}_0$, by definition of \mathcal{F} , as desired.

Definition 1.3. A basis for a filter on X is a family $\mathcal{F}_0 \subseteq \mathcal{P}^*(X)$ satisfying the equivalent definitions above, and we call the collection \mathcal{F} in (2) the filter generated by \mathcal{F}_0 .

Definition 1.4. An *ultrafilter* on X is a \subseteq -maximal filter on X.

1.2. **Nested cuts and** \mathcal{T} . Throughout this subsection, let (X,G) be a graph.

Definition 1.5. A *cut* in X is a subset $C \subseteq X$ contained in a single connected component $Y \subseteq X$ such that both C and $Y \setminus C$ are infinite but $\partial_{\nu}C$ is finite.

Definition 1.6. A family \mathcal{C} of cuts in X is *nested* if every $C_1, C_2 \in \mathcal{C}$ has an empty corner; i.e. $C_1^i \cap C_2^j = \emptyset$ for some $i, j = \pm 1$.

2. The graph $\mathcal{T}_{\mathcal{C}}$

Theorem 2.1. Let C be a family of nested cuts in X. The graph T_C , whose

- Vertices are finitely-based ultrafilters on C;
- \bullet Neighbors of

is acyclic. Furthermore, if something, then $\mathcal{T}_{\mathcal{C}}$ is a tree.

Proof. h

3. What goes wrong if C is non-nested?