

TREE OF ORIENTATIONS ON A NESTED COLLECTION OF SETS

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Let $H \subseteq 2^X$ be a sub-pocset for some fixed set X (so that, in particular, H is closed under complements). With the definitions in Section 1, we prove the following

Theorem A (Corollary 2.5, Proposition 2.8). *If H is nested, then the graph \mathcal{T}_H , whose:*

- *Vertices are based orientations $p \subseteq H$;*
- *Edges are pairs $\{p, q\}$ such that $q = p \triangle \{h, h^c\}$ for some \subseteq -minimal $h \in H$;*

is acyclic. Furthermore, if H is interval-finite, then \mathcal{T}_H is a tree.

In particular, this applies to when (X, G) is a graph and H is a nested collection of cuts on X . No further assumptions on X (like local-finiteness) is needed.

1. PRELIMINARIES

Let $H \subseteq 2^X$ be a sub-pocset for some fixed set X , whose elements $h \in H$ we call *half-spaces*.

Definition 1.1. Two elements $h, k \in H$ are *nested* if $h^i \cap k^j = \emptyset$ for some $i, j \in \{1, -1\}$, where $h^i := h$ for $i = 1$ and $h^i := h^c$ otherwise. We say that H is *nested* if every pair $h, k \in H$ are nested.

1.1. Orientations. We give the standard definition of orientations on H and characterize them as ‘consistent assignments of half-spaces to hyperplanes’.

Definition 1.2. An *orientation* on H is a subset $U \subseteq H$ such that

1. (*Upward-closure*). If $h \in U$ and $k \in H$ contains h , then $k \in U$.
2. (*Ultra*). For each $h \in H$, exactly one of h, h^c is contained in U .

Consider the equivalence relation \sim on H generated by $h \sim h^c$ for all $h \in H$, whose classes are called *hyperplanes* $\partial h := \{h, h^c\}$ where $\partial : H \rightarrow H/\sim$ is the projection. We show that an orientation on H is just a choice $\varphi : H/\sim \rightarrow H$ of a half-space for each hyperplane, that is consistent in the sense below.

Proposition 1.3. *An orientation $p \subseteq H$ is exactly the data of a function $\varphi : H/\sim \rightarrow H$ such that $\varphi(\partial h) \in \partial h$ and $\varphi(\partial h) \not\subseteq \varphi(\partial k)^c$ for every $h, k \in H$.*

Proof. Given an orientation $p \subseteq H$, let $\varphi_p(\partial h) := h^i \in U$ for the unique $i \in \{1, -1\}$. That $\varphi_p(\partial h) \in \partial h$ is clear, and if $\varphi_p(\partial h) \subseteq \varphi_p(\partial k)^c$, then U contains both $\varphi_p(\partial k)$ and $\varphi_p(\partial k)^c$ by upward-closure, a contradiction.

Conversely, given such a function $\varphi : H/\sim \rightarrow H$, let $p_\varphi := \text{im } \varphi \subseteq H$. This is ultra since if $h \in H$ and $h^c \notin p_\varphi$, then $\varphi(\partial h) \in \partial h = \{h, h^c\}$ implies $h \in p_\varphi$. Furthermore, if $p_\varphi \ni h \subseteq k$, then $k^c \in p_\varphi$ implies $\varphi(\partial h) = h \subseteq k = \varphi(\partial k)^c$, a contradiction, so $k \in p_\varphi$ by the above.

Finally, given an orientation $p \subseteq H$, we have $h \in p$ iff $\varphi_p(\partial h) = h$, which occurs iff $h \in \text{im } \varphi_p = p_{\varphi_p}$. Thus $p_{\varphi_p} = p$. Conversely, given $\varphi : H/\sim \rightarrow H$, and $h \in H$, we have $\varphi_{p_\varphi}(\partial h) = h^i$ iff $h^i \in p_\varphi = \text{im } \varphi$, which occurs iff $\varphi(\partial h) = h^i$. Thus $\varphi_{p_\varphi} = \varphi$ too, as desired. \blacksquare

Definition 1.4. A *base* for an orientation $p \subseteq H$ is a \subseteq -minimal subset $p_0 \subseteq p$ such that $p = \uparrow p_0$, where

$$\uparrow p_0 := \bigcup_{h \in p_0} \uparrow h := \bigcup_{h \in p_0} \{k \in H : k \supseteq h\}.$$

We say that p is *based* if it admits a basis.

In the above correspondence, a base for $\varphi : H/\sim \rightarrow H$ is a function $\varphi_0 \subseteq \varphi$ where $\text{dom } \varphi_0$ is a minimal subset of hyperplanes such that φ_0 extends uniquely to φ .

1.2. Flipping basis elements. We now investigate the behaviour of an orientation when its choice on a single half-space is modified. Although the proofs work without the characterization in Proposition 1.3, it makes orientations more intuitive to me, and so will be phrased this way.

Lemma 1.5. *Let $p \subseteq H$ be an orientation. Then $q := p \triangle \{h, h^c\}$ is an orientation iff $h \in p$ is \subseteq -minimal. Furthermore, if p is based, then so is q .*

Proof. First, note that $\varphi_q(\partial h) = \varphi_p(\partial h)^c = h^c$ and $\varphi_q = \varphi_p$ away from ∂h .

(\Rightarrow). If q is an orientation and $p \ni k \subsetneq h$, then we have a contradiction since $\partial k \neq \partial h$ and hence

$$\varphi_q(\partial k) = \varphi_p(\partial k) = k \subseteq h = \varphi_q(\partial h)^c.$$

(\Leftarrow). Conversely, let $h \in p$ be \subseteq -minimal and suppose that q is not an orientation. Since only $\varphi_q(\partial h) = h^c$ differs from φ_p , this can only occur if $\varphi_q(\partial h) \subseteq \varphi_q(\partial k)^c$ for some $k \in H$. But then

$$h^c = \varphi_q(\partial h) \subseteq \varphi_q(\partial k)^c = k^i \notin p$$

for some unique $i \in \{1, -1\}$, so that $p \ni k^{-i} \subseteq h$ and contradicts that $h \in p$ is \subseteq -minimal.

Finally, suppose that p is based. ???.

Remark 1.6. If $h \in p$ is \subseteq -minimal and $p_0 \subseteq p$ is any basis, then $h \in p_0$. Indeed, if $h \notin p_0$, then there is some $h_0 \in p_0 \subseteq p$ with $h \supset h_0$, contradicting the minimality of h .

Conversely, if $p_0 \subseteq p$ is a basis, then every $h \in p_0$ is \subseteq -minimal in p . Indeed, if $h \supset k$ for some $k \in p$, then there is some $k_0 \in p_0$ with $k \supseteq k_0$, so $h \supset k_0$. Thus $p = \uparrow(p_0 \setminus \{h\})$, contradicting the minimality of p_0 .

2. THE GRAPH \mathcal{T}_H

Let $H \subseteq 2^X$ be a sub-pocset as before. Using Lemma 1.5, we construct a graph \mathcal{T}_H whose:

- *Vertices* are based orientations $p \subseteq H$;
- *Edges* are pairs $\{p, q\}$ such that $q = p \triangle \{h, h^c\}$ for some \subseteq -minimal $h \in H$.

The goal of this section is to establish Theorem A, stating that if H is nested, then \mathcal{T}_H is acyclic (Corollary 2.5), and furthermore, \mathcal{T}_H is a tree when H is interval-finite (Proposition 2.8).

For the rest of this section, we let $H \subseteq 2^X$ be a *nested* sub-pocset.

2.1. Acyclicity of \mathcal{T}_H . Essentially, \mathcal{T}_H is acyclic because any path in \mathcal{T}_H (without backtracking) is induced by a sequence of the flipped half-spaces, and those form a *strictly-increasing* chain.

Example 2.1. This fails if H is non-nested. Indeed, let $X := 2 \times 2 := \{(x, y) : x, y \in 2\}$ and consider the set H of all fibers $h_y := \{(x, y) : x \in 2\}$ for each $y \in 2$ and $h^x := \{(x, y) : y \in 2\}$ for each $x \in 2$. All orientations on H are principal – of the form $\{h^x, h_y\}$ for $x, y \in 2$ – and the resulting graph \mathcal{T}_H is a square.

Definition 2.2. Fix $p_0 \in V(\mathcal{T}_H)$ and $n \in \mathbb{N}$. A sequence $(h_i)_{i < n} \subseteq H$ is said to *induce a path in \mathcal{T}_H from p_0* if $(p_i)_{i < n}$, defined by $p_i := p_{i-1} \triangle \{h_{i-1}, h_{i-1}^c\}$ for every $1 \leq i < n$, is a path in \mathcal{T}_H with each $h_i \in p_i$.

Lemma 2.3. *A path from p_0 induced by $(h_i)_{i < n}$, $n \geq 3$, has no backtracking iff $h_i \neq h_{i-1}^c$ for every $1 \leq i < n$.*

Proof. Take $2 \leq i \leq n$. It suffices to show that $p_{i-2} = p_i$ iff $h_{i-1} = h_{i-2}^c$.

(\Rightarrow). We have by definition that $p_i = p_{i-2} \cup \{h_{i-1}^c, h_{i-2}^c\} \setminus \{h_{i-1}, h_{i-2}\}$, so since $h_{i-2} \in p_{i-2} = p_i$, we have $h_{i-2} = h_{i-1}^c$ as desired.

(\Leftarrow). Again by definition, by noting that the basis-flipping cancels out.

Proposition 2.4. *If $(h_i)_{i < n}$ induces a path in \mathcal{T}_H with no backtracking, then $(h_i)_{i < n}$ is strictly increasing.*

Proof. By Lemma 2.3, we have $h_i \neq h_{i-1}^c$ for every $1 \leq i < n$. Thus, since $h_i \in p_i = p_{i-1} \cup \{h_{i-1}^c\} \setminus \{h_{i-1}\}$, we see that $h_i \in p_{i-1}$. Clearly $h_i \neq h_{i-1}$. It suffices to remove the three cases when $h_i \subseteq h_{i-1}$, $h_{i-1} \subseteq h_i^c$, and $h_i^c \subseteq h_{i-1}$, since then nestedness of H gives us $h_{i-1} \subsetneq h_i$, as desired.

- If $h_i \subseteq h_{i-1}$, then $h_{i-1} \in p_i$, contradicting the definition of p_i .
- If $h_{i-1} \subseteq h_i^c$, then $h_i^c \in p_{i-1}$ by upward-closure of p_{i-1} , a contradiction.
- If $h_i^c \subseteq h_{i-1}$, then $h_{i-1} \in p_{i+1}$ by upward-closure of $p_{i+1} \ni h_i^c$. But since $h_{i-1} \neq h_i^c$, we have by definition of p_{i+1} that $h_{i-1} \in p_i$, a contradiction.

Corollary 2.5. \mathcal{T}_H is acyclic.

2.2. Connectedness of \mathcal{T}_H . In general, \mathcal{T}_H need not be connected, as shown by the following

Example 2.6 (Tserunyan). Let $X := \omega_1 := \omega + 1 := \omega \sqcup \{\omega\}$ and let

$$H := \{[0, n] : n < \omega\} \cup \{(n, \omega] : n < \omega\} \cup \{\omega, \{\omega\}\}.$$

We claim that every based orientation $p \subseteq H$ is principal (i.e., $p = \hat{x} := \{h \in H : h \ni x\}$ for some $x \in X$).

Proof. For each $h \in H$, the hyperplane ∂h is either $\{[0, n], (n, \omega]\}$ for some $n < \omega$, or $\{\omega, \{\omega\}\}$. Consider the set $N_p := \{n < \omega : [0, n] \in p\}$, for which we have two cases.

- If $N_p = \emptyset$, then $(n, \omega] \in p$ for all p . We claim that $p = \hat{\omega}$. If $\omega \in p$, then p is not based since $(n, \omega]$, for n large enough, does not contain any basis element. Thus $\{\omega\} \in p$. Both types of half-spaces contain ω , and those are the only half-spaces in p .
- If $N_p \neq \emptyset$, then there is a minimal $n < \omega$ with $[0, n] \in p$; we claim that $p = \hat{n}$. Indeed, we have $(m, \omega] \in p$ if $m < n$, $[0, m] \in p$ if $m \geq n$, and $\omega \in p$ by upward-closure. All three types of half-spaces contain n , and those are the only half-spaces in p . ■

Now, each orientation \hat{n} is generated by the basis $\{[0, n], (n-1, \omega]\}$, and $\hat{\omega}$ is generated by $\{\{\omega\}\}$. Thus, we see that $\hat{\alpha}, \hat{\beta}$, where $\alpha < \beta \leq \omega$, is joined by an edge iff $\beta = \alpha + 1$, where the \subseteq -minimal element $[0, \alpha] \in \hat{\alpha}$ is flipped to $(\alpha, \omega] = (\beta-1, \omega] \in \hat{\beta}$. This shows that \mathcal{T}_H is disconnected, with ω an isolated point.

A sufficient criterion ruling out Example 2.6 is given in Proposition 2.8, which needs the following

Lemma 2.7. *There is a path between $p, q \in V(\mathcal{T}_H)$ iff $p \triangle q$ is finite, in which case $d(p, q) = (p \triangle q)/2$.*

Proof. If $p \triangle q = \{h_1, \dots, h_n\} \cup \{k_1, \dots, k_m\}$ with $p \setminus q = \{h_i\}$ and $q \setminus p = \{k_j\}$, then $h_i^c \in q \setminus p$ and $k_j^c \in p \setminus q$ for all $i < n$ and $j < m$, so $n = m$ and $q = p \cup \{h_i^c\} \setminus \{h_i\}$. We claim that there is a permutation $\sigma \in S_n$ such that $(h_{\sigma(i)})$ induces a path from $p =: p_0$, which is the desired path from p to q . Choose a minimal $h \in \{h_i\}$, which is also minimal in p : if $k \subseteq h$ for some $k \in p$, then $h^c \subseteq k^c$, and hence $k^c \in q$, so $k = h_i \subseteq h$ for some i , forcing $k = h$. Set $p_1 := p \triangle \{h, h^c\}$, which is an orientation by Lemma 1.5. Continuing in this manner by choosing a minimal element in $\{h_i\} \setminus \{h\}$ gives us the desired path.

Conversely, suppose that \mathcal{T}_H is connected and let $(p_i)_{i < n}$ be a path from $p = p_0$ to $q = p_{n-1}$. Then letting $h_i := p_i \triangle p_{i-1}$ for all $1 \leq i < n$ gives us a sequence $(h_i)_{i < n}$ inducing such a path, whence $p \triangle q$ consists of $\{h_i\}$ and their complements; hence, $p \triangle q$ is finite with $p \triangle q = 2d(p, q)$. ■

Proposition 2.8. *If H is interval-finite (i.e., for every $h, h' \in H$, the interval $[h, h'] := \{k \in H : h \subseteq k \subseteq h'\}$ is finite), then \mathcal{T}_H is connected.*

Proof. Fix $p, q \in V(\mathcal{T}_H)$. Since they are based, we can use Remark 1.6 to choose minimal elements $h_1 \in p \setminus q$ and $h_2 \in q \setminus p$, and by assumption there are finitely-many $k \in H$ with $h_1 \subseteq k \subseteq h_2^c$, say k_1, \dots, k_n . We claim that $p \triangle q \subseteq \{k_i\} \cup \{k_i^c\}$, so the result follows from Lemma 2.7.

Indeed, it suffices to show that $k \in \{k_i\}$ for every $k \in p \setminus q$, for then any $k' \in q \setminus p$ is in $k' \in \{k_i^c\}$. Let $k \in p \setminus q$, which by minimality of h_1 , implies $k \not\subseteq h_1$. Thus, either $k^c \subseteq h_1$, $k \subseteq h_1^c$, or $h_1 \subseteq k$ by nestedness of H , and the first two cases are impossible since either case forces $h_1 \in q$. A similar argument using the observation that $k^c \in q \setminus p$, and hence $k^c \not\subseteq h_2$, shows that $k \subseteq h_2^c$. Thus $h_1 \subseteq k \subseteq h_2^c$, as desired. ■