Tree-like graphings of countable Borel equivalence relations

An exposition to

Tree-like graphings, wallings, and median graphings of equivalence relations

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October 1, 2024

Countable Borel Equivalence Relations

Definition

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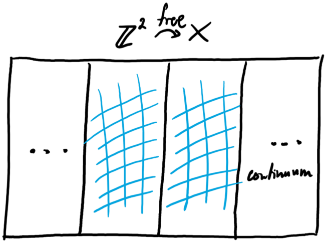

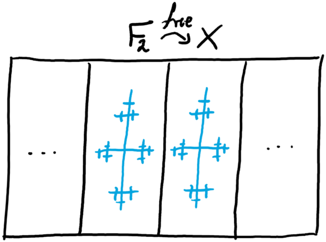

Smooth and Hyperfinite CBERs

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Theorem (Slaman-Steel, Weiss)

Let E be a CBER on a standard Borel space X. TFAE:

1. *E is hyperfinite.* *$E = \bigcup_n F_n$ where $F_0 \subseteq F_1 \subseteq \cdots$ are FBERs.*
2. *E is induced by a Borel $\mathbb{Z}$-action.* *$E = E_{\mathbb{Z}}^X$ for some $\mathbb{Z} \curvearrowright X$.*

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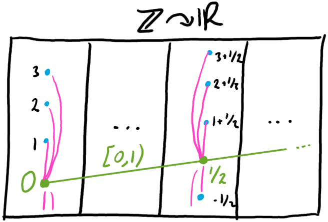

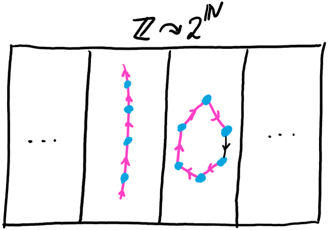

Graphing of a CBER

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A *graphing* of a CBER E on X is a Borel graph $G \subseteq X^2$ whose connected relation is E, i.e., $xEy \leftrightarrow xG \cdots Gy$ for all $x, y \in X$.

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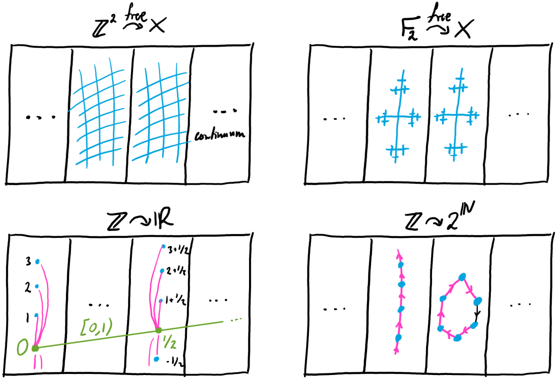
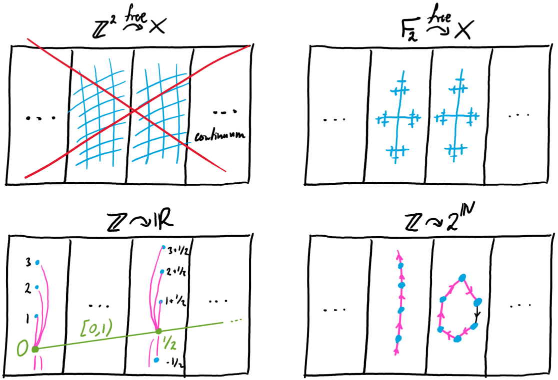


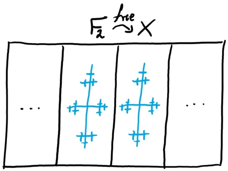
Definition

A *treeing* of a CBER E is an acyclic graphing, and a CBER E is said to be *treeable* if it admits a treeing.



Example (Free Actions)

Free actions of a free group $F_r \curvearrowright X$.



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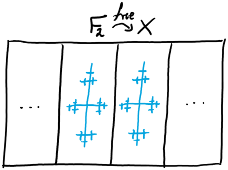

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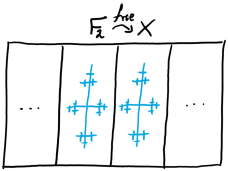

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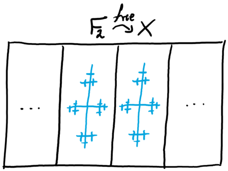

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Question (Robin Tucker-Drob; 2015)

Is the class of treeable CBERs robust under quasi-isometries?

Main Result

Theorem (Chen, Poulin, Tao, Tserunyan; 2023+)

If a CBER E admits a locally-finite graphing such that each component is a quasi-tree, then E is treeable.

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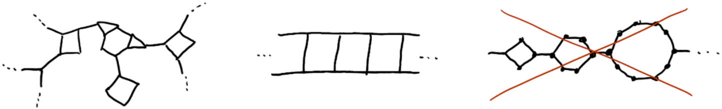
Two metric spaces X, Y are *quasi-isometric* if they are isometric up to a bounded multiplicative and additive error; X is a *quasi-tree* if it is quasi-isometric to a tree.



Game Plan

Quasi-treeing

Treeing

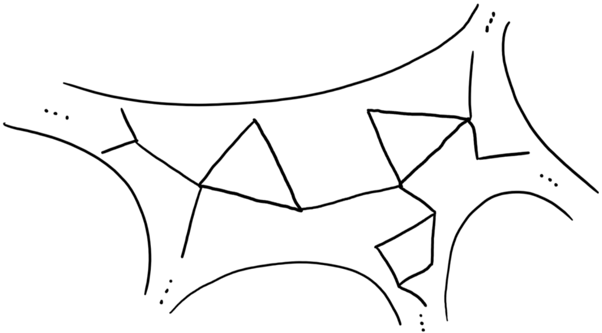
Game Plan

Quasi-treeing

$\downarrow$

Quasi-tree

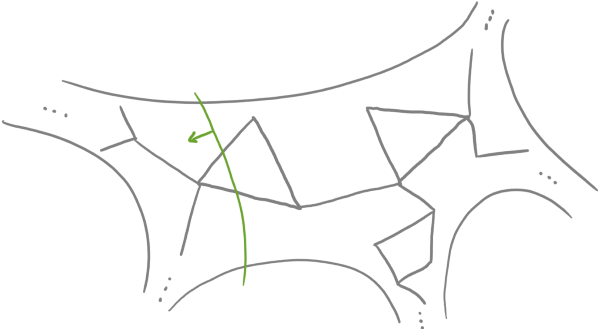
Treeing



Game Plan

Quasi-treeing → Quasi-tree → Finitely-sep. family of cuts

Treeing



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$$\begin{array}{ccc} \text{Quasi-treeing} & & \text{Treeing} \\ \downarrow & & \\ \text{Quasi-tree} & \longrightarrow & \text{Finitely-sep. family of cuts} \end{array}$$

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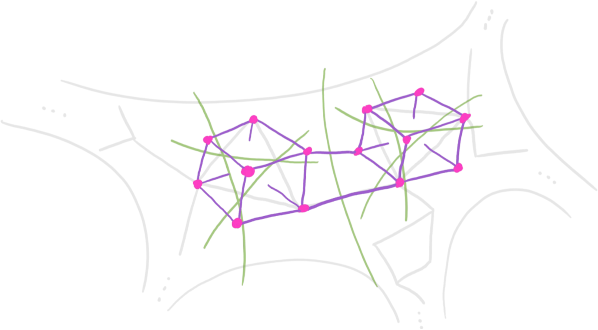
Quasi-treeing $\longrightarrow$ Treeing

Quasi-tree $\longrightarrow$ Finitely-sep. family of cuts $\longrightarrow$ Median graph w/ finite hyperplanes



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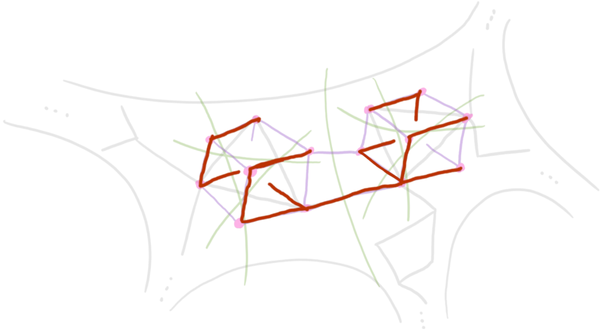
$$\begin{array}{ccccccc} \text{Quasi-treeing} & & & & & & \text{Treeing} \\ \downarrow & & & & & & \\ \text{Quasi-tree} & \longrightarrow & \begin{array}{c}\text{Finitely-sep.}\\ \text{family of cuts}\end{array} & \longrightarrow & \begin{array}{c}\text{Median graph w/}\\ \text{finite hyperplanes}\end{array} & \longrightarrow & \begin{array}{c}\text{`Canonical'}\\ \text{spanning tree}\end{array} \end{array}$$



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$$\begin{tikzcd}
\text{Quasi-treeing} \arrow[d] & \begin{array}{c}\text{Dense family}\\ \text{of cuts}\end{array} & & \text{Treeing} \\
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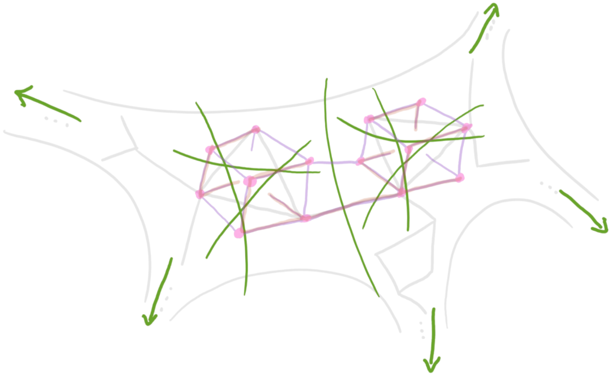

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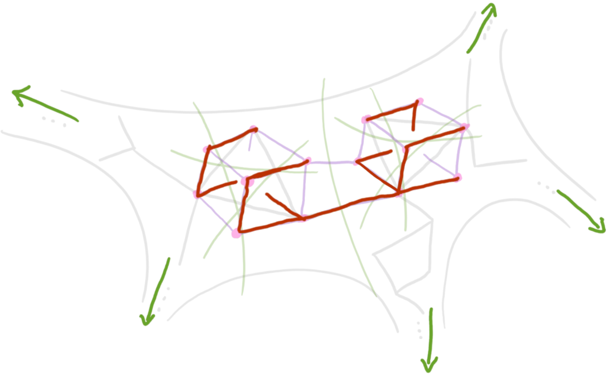

The End

Thank you!