

# Bounded Arithmetic, Constant-depth Frege Proofs, and the Linear Time Hierarchy

COMP532 – Propositional Proof Complexity

Nathan Acheampong and Zhaoshen Zhai

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# A weak subsystem of Peano Arithmetic

Let  $L_{\text{PA}} := \{0, +, \cdot, S, \leq\}$  be the first-order language of PA, whose axioms are the ‘basic axioms’ BASIC together with an axiom scheme IND, consisting of the induction axioms

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A formula  $\varphi$  is said to be *bounded* if every quantifier in  $\varphi$  is of the form  $\exists x \leq t$  or  $\forall x \leq t$ , for some  $L_{\text{PA}}$ -term  $t$ .

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The  $L_{\text{PA}}$ -theory  $I\Delta_0$  is the subtheory of PA axiomatized by BASIC and the restriction of IND to  $\Delta_0$ -formulas.

# Polynomial Translation for $I\Delta_0$

Theorem (Paris-Wilkie, 1985; Krajíček, 1995)

*Let  $R$  be a binary relation symbol and let  $\varphi(x)$  be  $\Delta_0(R)$ -formula. If  $I\Delta_0(R) \vdash \forall x \varphi(x)$ , then there is  $d \in \mathbb{N}$  such that  $S_{F_d}(\langle \varphi \rangle_n) = \text{poly}(n)$ .*

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## Theorem (Cook, 1975)

*Let  $\varphi(x)$  be a  $\Sigma_1^b$ -formula. If  $S_2^1 \vdash \forall x \varphi(x)$ , then  $S_{EF}(\|\varphi\|^n) = \text{poly}(n)$ .*

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- $I\Delta_0 \leftrightarrow$  Functions in the Linear Time Hierarchy LTH.

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We call  $\varphi$  a *defining formula* for  $f$ , and say that  $f$  is *provably total in*  $T$  if it is  $\Sigma_1$ -definable function in  $T$ .

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## Theorem (Buss's Witnessing Theorem, 1986)

*For any function  $f : \mathbb{N}^k \rightarrow \mathbb{N}$ , the following are equivalent.*

- 1.  $f$  is  $\Sigma_1^b$ -definable in  $S_2^1$ , i.e.,  $f$  is provably total in  $S_2^1$ .*
- 2. There is a  $\Sigma_1^b$ -defining  $L$ -formula  $\varphi(\bar{x}, y)$  for  $f$  and an  $L$ -term  $t(\bar{x})$  such that  $S_2^1 \vdash \forall \bar{x} \exists! y \leq t(\bar{x}) \varphi(\bar{x}, y)$ .*
- 3.  $f$  is polynomial-time computable.*

# The Engine of $I\Delta_0$ : Parikh's Theorem

## Theorem (Parikh, 1971)

*Let  $\varphi(\bar{x}, y)$  be a bounded formula. If  $I\Delta_0 \vdash \forall \bar{x} \exists y \varphi(\bar{x}, y)$ , then there is an  $L_{PA}$ -term  $t(\bar{x})$  such that  $I\Delta_0 \vdash \forall \bar{x} \exists y \leq t(\bar{x}) \varphi(\bar{x}, y)$ .*



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## Proof.

Suppose towards a contradiction that  $I\Delta_0 \vdash \forall \bar{x} \exists y \varphi(\bar{x}, y)$  but  $I\Delta_0 \not\vdash \forall \bar{x} \exists y \leq t(\bar{x}) \varphi(\bar{x}, y)$  for any term  $t(\bar{x})$ .

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*Let  $\varphi(\bar{x}, y)$  be a bounded formula. If  $I\Delta_0 \vdash \forall \bar{x} \exists y \varphi(\bar{x}, y)$ , then there is an  $L_{PA}$ -term  $t(\bar{x})$  such that  $I\Delta_0 \vdash \forall \bar{x} \exists y \leq t(\bar{x}) \varphi(\bar{x}, y)$ .*

## Proof.

Suppose towards a contradiction that  $I\Delta_0 \vdash \forall \bar{x} \exists y \varphi(\bar{x}, y)$  but  $I\Delta_0 \not\vdash \forall \bar{x} \exists y \leq t(\bar{x}) \varphi(\bar{x}, y)$  for any term  $t(\bar{x})$ . Then

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## Theorem (Witnessing Theorem for $I\Delta_0$ )

*For any function  $f : \mathbb{N}^k \rightarrow \mathbb{N}$ , the following are equivalent.*

- 1.  $f$  is  $\Sigma_1$ -definable in  $I\Delta_0$ , i.e.,  $f$  is provably total in  $I\Delta_0$ .*
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so  $I\Delta_0 \vdash \forall \bar{x} \exists y \varphi(\bar{x}, y)$  where  $\varphi(\bar{x}, y) := \exists z \leq y \exists \bar{w} \leq y \psi$ . ■

# The Linear Time Hierarchy

## Definition

$\text{LTH} := \bigcup_{i=0}^{\infty} \Sigma_i^{\text{lin}}$ , where  $\Sigma_i^{\text{lin}}$  is the family of languages accepted in  $O(n)$  time by an alternating Turing machine with  $i$  alternations.

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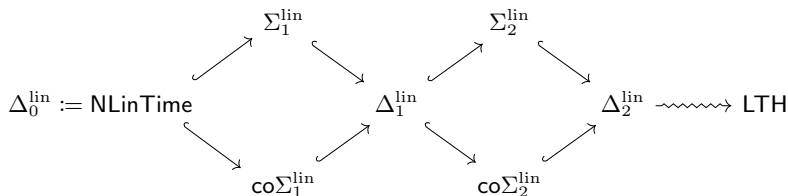
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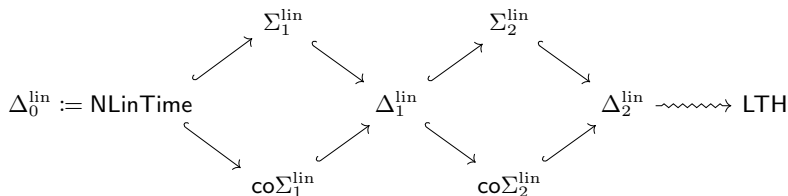


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## Theorem ( $3 \Leftrightarrow 4$ )

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Conversely, let  $G_f \in \text{LTH} = \Delta_0^{\mathbb{N}}$ . Observe that  $\mathbb{N} \models \forall \bar{x} \exists! y G_f(\bar{x}, y)$ , but we don't know if  $I\Delta_0$  proves this. Instead, let  $t(\bar{x})$  be a term such that  $f(\bar{x}) \leq t(\bar{x})$  and let  $\varphi_0(\bar{x}, y) := G_f(\bar{x}, y) \vee y = t(\bar{x}) + 1$ , so clearly  $I\Delta_0 \vdash \forall \bar{x} \exists y \varphi_0(\bar{x}, y)$ . Consider the formula

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## Theorem ( $3 \Leftrightarrow 4$ )

*For any  $f : \mathbb{N}^k \rightarrow \mathbb{N}$ , there is a  $\Delta_0$ -defining formula  $\varphi(\bar{x}, y)$  for  $f$  and a term  $t(\bar{x})$  such that  $I\Delta_0 \vdash \forall \bar{x} \exists! y \leq t(\bar{x}) \varphi(\bar{x}, y)$  iff  $f \in \text{FLTH}$ .*

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## Thank you!

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