AN EXPOSITION OF THE RIBES-ZALESSKII PRODUCT THEOREM

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ABSTRACT. We give a detailed proof of the Ribes-Zalesskii product theorem for pro-$\mathcal{C}$ topologies, where $\mathcal{C}$ is a pseudovariety of groups closed under extensions. The proof relies on geometric properties of the profinite Cayley graph of the pro-$\mathcal{C}$ completion of free groups, which we develop here following [Rib17].

The pro-$\mathcal{C}$ topology. A *pseudovariety of groups* is a non-empty class $\mathcal{C}$ of finite groups that is closed under taking subgroups, finite direct products, and quotients.

A pseudovariety $\mathcal{C}$ is said to be *closed under extensions* (*with abelian kernel*) if $G \in \mathcal{C}$ whenever $N, G/N \in \mathcal{C}$ for any normal (abelian) subgroup $N \trianglelefteq G$. Throughout, fix a pseudovariety $\mathcal{C}$ of groups.

Definition. Let G be a group. The collection $\mathcal{N}_{\mathcal{C}}(G)$ of all normal subgroups $N \trianglelefteq G$ such that $G/N \in \mathcal{C}$ forms a neighborhood base around the identity, which generates the *pro-$\mathcal{C}$ topology* on G. We say that G is *residually-$\mathcal{C}$* if $\bigcap \mathcal{N}_{\mathcal{C}}(G) = \{1\}$, which occurs iff the pro-$\mathcal{C}$ topology on G is Hausdorff.

The goal of this note is to prove the following generalization of the *Ribes-Zalesskii Theorem* [RZ93].

Theorem A (Ribes-Zalesskii [RZ94, Theorem 5.1])**.** *Let $\mathcal{C}$ be a pseudovariety of groups that is closed under extensions. If F is a free group of finite rank and $H, K \leq F$ are finite generated subgroups which are closed in the pro-$\mathcal{C}$ topology of F, then the double coset HK is also closed in the pro-$\mathcal{C}$ topology of F.*

Remark. Ribes and Zalesskii actually proved that for any $n \in \mathbb{N}$, the coset $H_1 \cdots H_n$ is closed in the pro-$\mathcal{C}$ topology of F whenever $H_1, \ldots, H_n$ are finitely generated subgroups which are closed in the pro-$\mathcal{C}$ topology of F. We chose to present the proof only for the case $n = 2$ since the general case requires some more careful bookkeeping, but all essential ideas of their proof are present here.

Remark. The condition that $\mathcal{C}$ is closed under extensions is needed in two important parts of the proof.

1. It ensures that the pro-$\mathcal{C}$ topology on an open subgroup $H \leq_o G$ coincides with the topology induced by the pro-$\mathcal{C}$ topology on G; see Lemma 1.5.
2. A weaker condition that $\mathcal{C}$ is *closed under extensions with abelian kernel* is needed to show that the profinite Cayley graph of free groups are $\mathbb{Z}_{\mathcal{C}}$-trees; see Theorem 2.12.

Auinger and Steinberg [AS05] discovered another proof of Theorem A, which, among other things, weakened the hypothesis by only requiring that $\mathcal{C}$ is closed under extensions with abelian kernel.

Remark. When $\mathcal{C}$ is the class of all finite groups (in which case we write *profinite* for 'pro-$\mathcal{C}$'), we can drop the hypotheses that H and K are closed in the profinite topology of F since all finitely generated subgroups of F are closed; this is Hall's Theorem (see, for instance, [Sta83, Section 6]).

This note is organized as follows. In Section 1, we gather some basic facts about the pro-$\mathcal{C}$ topology of a residually-$\mathcal{C}$ group G and the interaction between G, its subgroups, and its pro-$\mathcal{C}$ completion $G_{\mathcal{C}}$. Section 2 is a summary of results in profinite graph theory needed to develop the basic properties of profinite Cayley graphs of free groups, which we use to prove Theorem A in Section 3.

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1. Pro-$\mathcal{C}$ topologies and Completions

Throughout, fix a residually-$\mathcal{C}$ group G and equip it with its pro-$\mathcal{C}$ topology. The residually-$\mathcal{C}$ condition on G ensures that G embeds into its *pro-$\mathcal{C}$ completion* (see Section 1.2), which will be important later on. We will apply the results in this chapter to free groups, so we need free groups to be residually-$\mathcal{C}$.

Fact 1.1 ([RZ10, Proposition 3.3.15]). *If $\mathcal{C}$ is closed under extensions, then every free group is residually-$\mathcal{C}$.*

1.1. Subgroups.

The following lemmas characterize when a subgroup $H \leq G$ is open or closed in G. Let $H_G := \bigcap_{g\in G} gHg^{-1}$ be the *normal core* of H in G. Note that if H has finite index in G, then so does H_G.

Lemma 1.2. *A subgroup $H \leq G$ is open if and only if $H_G \in \mathcal{N}_{\mathcal{C}}(G)$.*

Proof. If H is open, then H has finite-index in G, so H_G is open as well. Thus $N \leq H_G$ for some $N \in \mathcal{N}_{\mathcal{C}}(G)$, so $G/N \twoheadrightarrow G/H_G$, whence $G/H_G \in \mathcal{C}$. Conversely, if $G/H_G \in \mathcal{C}$, then H_G is open, and hence so is H. ■

Lemma 1.3. *A subgroup $H \leq G$ is closed iff H is the intersection of all open subgroups of G containing H.*

Proof. First, note that the intersection of all open subgroups of G containing H coincides with $\bigcap_{N\in\mathcal{N}_{\mathcal{C}}(G)} HN$ since if $x \in \bigcap_{N\in\mathcal{N}_{\mathcal{C}}(G)} HN$ and $H \subseteq K \leq_o G$, then $K_G \in \mathcal{N}_{\mathcal{C}}(G)$ by Lemma 1.2, so $x \in HK_G \leq K$.

If H is closed in the pro-$\mathcal{C}$-topology on G, then for all $x \in G\setminus H$, take $N \in \mathcal{N}_{\mathcal{C}}(G)$ such that $xN \cap H = \varnothing$, so $x \notin HN$, and hence $H = \bigcap_{N\in\mathcal{N}_{\mathcal{C}}(G)} HN$. Conversely, note that open subgroups are closed. ■

The pro-$\mathcal{C}$ topology on a subgroup $H \leq G$ is, in general, finer than the topology induced from the pro-$\mathcal{C}$ topology on G: if $N \in \mathcal{N}_{\mathcal{C}}(G)$, then the natural map $H/(H\cap N) \to G/N \in \mathcal{C}$ is injective, so $H/(H\cap N) \in \mathcal{C}$. The topologies need not coincide (see [RZ10, Example 3.1.3]). However, there are important cases where the two topologies on H do coincide, in which case we say that H is *$\mathcal{C}$-compatible* with G.

Lemma 1.4. *If $G = H * K$, then H is $\mathcal{C}$-compatible with G.*

Proof. Observe that $G = H \ltimes K'$ where K' is the normal closure of K in G, so we instead prove the statement for when $G = H \ltimes K$. In this case, take $N \in \mathcal{N}_{\mathcal{C}}(H)$ and note that $G/KN \cong H/N \in \mathcal{C}$, so KN is open in the pro-$\mathcal{C}$ topology of G. Observe that $N = H \cap KN$, so N is open in the induced topology. ■

Lemma 1.5. *If $H \leq_o G$ is open and $\mathcal{C}$ is closed under extensions, then H is $\mathcal{C}$-compatible with G.*

Proof. We show that $\mathcal{N}_{\mathcal{C}}(H) \subseteq \mathcal{N}_{\mathcal{C}}(G)$, so take $N \in \mathcal{N}_{\mathcal{C}}(H)$. It suffices to show that $G/N_G \in \mathcal{C}$, since then $G/N \in \mathcal{C}$ as $G/N_G \twoheadrightarrow G/N$, and $[G:N] = [G:H][H:N]$ is finite as H is open in G.

Since it is not necessarily the case that $N \leq H_G$, we work with $M := H_G \cap N$ instead, which is harmless as $N_G = M_G$. Observe that $G/H_G \cong (G/M_G)/(H_G/M_G)$, which by Lemma 1.2 lies in $\mathcal{C}$ since H is open in G, so it suffices to show that $H_G/M_G \in \mathcal{C}$ as $\mathcal{C}$ is closed under extensions. Since $[G:M]$ is finite, there exist $g_1,\dots,g_n \in G$ such that $M_G = \bigcap_{i=1}^n g_iMg_i^{-1}$, for some $n \in \mathbb{N}$. As $H_G/M_G \hookrightarrow \prod_{i=1}^n H_G/g_iMg_i^{-1}$, it suffices to show that each $H_G/g_iMg_i^{-1} \in \mathcal{C}$. To this end, note that

$$\frac{H_G}{M} = \frac{H_G}{H_G\cap N} \cong \frac{H_GN}{N} \leq \frac{H}{N} \in \mathcal{C},$$

where $H/N \in \mathcal{C}$ by choice of N, so $H_G/M \in \mathcal{C}$, and hence $H_G/g_iMg_i^{-1} \cong H_G/M \in \mathcal{C}$ via conjugation. ■

1.2. Completions.

A *pro-$\mathcal{C}$* group is an inverse limit of groups in $\mathcal{C}$. Given a residually-$\mathcal{C}$ group G, its *pro-$\mathcal{C}$ completion* is the group $G_{\mathcal{C}} := \lim_{N\in\mathcal{N}_{\mathcal{C}}(G)} G/N$, which is a pro-$\mathcal{C}$ group, and the canonical map $G \to G_{\mathcal{C}}$ is injective. The first thing to check is that the pro-$\mathcal{C}$ topology on G is induced by the topology of $G_{\mathcal{C}}$. In the sequel, given a subset $X \subseteq G$, we let $\overline{X}$ denote its closure in $G_{\mathcal{C}}$.

Lemma 1.6. *The topology of $G_{\mathcal{C}}$ induces on G its pro-$\mathcal{C}$ topology.*

Proof. Recall that $G \hookrightarrow G_{\mathcal{C}}$ via $g \mapsto (gN)_{N\in\mathcal{N}_{\mathcal{C}}(G)}$. We wish to show that $H \subseteq G$ is open iff $H = G \cap K$ for some open subset $K \subseteq G_{\mathcal{C}}$, for which it suffices to show this for open subgroups $H \leq_o G$ and $K \leq_o G_{\mathcal{C}}$.

Claim. *For any subgroup $H \leq G$, we have $H = G \cap \overline{H}$ and $[G : H] = [G_{\mathcal{C}} : \overline{H}]$.*

Proof. Clearly we have $H \subseteq G \cap \overline{H}$, so take $g \in G \cap \overline{H}$. Since $\overline{H} = \lim_N HN/N$, we see that $g \in HN$ for each $N \in \mathcal{N}_{\mathcal{C}}(G)$. By Lemma 1.2, we have $H_G \in \mathcal{N}_{\mathcal{C}}(G)$, so $g \in HH_G = H$. For the second claim, let $\kappa := [G_{\mathcal{C}} : \overline{H}]$. Since G is dense in $G_{\mathcal{C}}$, we have $G\overline{H} = G_{\mathcal{C}}$, so take a left transversal $\{g_\xi \in G : \xi < \kappa\}$ of $\overline{H}$ in $G_{\mathcal{C}}$. Note that $g_\xi H = G \cap g_\xi \overline{H}$, so $G = G \cap \bigsqcup_{\xi<\kappa} g_\xi \overline{H} = \bigsqcup_{\xi<\kappa} g_\xi H$, and thus $\kappa = [G : H]$. □

Thus if $H \leq_o G$, then $[G_{\mathcal{C}} : \overline{H}] = [G : H]$ is finite, and hence $\overline{H} \leq_o G_{\mathcal{C}}$ is an open subgroup too. Conversely, if $K \leq_o G_{\mathcal{C}}$ is an open subgroup of $G_{\mathcal{C}}$, then $G \cap K = G \cap \lim_N KN/N = \bigcap_N KN$ is closed by Lemma 1.3. Note that $\overline{G \cap K} = K$ since G is dense in $G_{\mathcal{C}}$. By the claim, we have $[G : G \cap K] = [G_{\mathcal{C}} : \overline{G \cap K}] = [G_{\mathcal{C}} : K]$, which is finite since K is open in $G_{\mathcal{C}}$, and hence $G \cap K$ is open in G. ■

Corollary 1.7. *A subset $X \subseteq G$ is closed in the pro-$\mathcal{C}$ topology of G if and only if $X = G \cap \overline{X}$.*

Proof. Let $\overline{X}_G$ be the closure of X in G. Since $X \subseteq G \cap \overline{X}$ is closed in G, we have $\overline{X}_G \subseteq G \cap \overline{X}$. Conversely, take $x \in G \cap \overline{X}$ and note that every neighborhood of x in G contains a neighborhood of the form $x(G \cap U)$ for some open subgroup $U \leq_o G_{\mathcal{C}}$, and $X \cap x(G \cap U) = X \cap (G \cap xU) = X \cap xU \neq \varnothing$ since $x \in \overline{X}$. ■

2. Profinite Graphs and Trees

In this section, we follow [Rib17] to give the necessary background on profinite graphs to study the geometric properties of the profinite Cayley graphs of free groups.

2.1. Graphs.

A *graph* is a set Γ together with a subset $V(\Gamma) \subseteq \Gamma$, whose elements are called *vertices*, and maps $d_0, d_1 : \Gamma \twoheadrightarrow V(\Gamma)$ such that $d_i|V(\Gamma) = \mathrm{id}_{V(\Gamma)}$; the elements of $E(\Gamma) := \Gamma \setminus V(\Gamma)$ are called *edges*. A *morphism* of graphs is a function $\varphi : \Gamma \to \Gamma'$ such that $\varphi d_i = d_i \varphi$ for $i = 0, 1$ (so φ sends vertices to vertices, but does *not necessarily* send edges to edges). A *quotient* of Γ is a morphic image of Γ.

Example 2.1. If $\Delta \subseteq \Gamma$ is a subgraph, then the map $p : \Gamma \twoheadrightarrow \Gamma/\Delta$ obtained by *collapsing* Δ *to a point* (as discrete spaces) gives rise to a graph Γ/Δ with $V(\Gamma/\Delta) := p(V(\Gamma))$ and $d_i(p(m)) := p(d_i(m))$ for $m \in \Gamma$ and $i = 0, 1$. Then $p : \Gamma \twoheadrightarrow \Gamma/\Delta$ is a morphism sending each edge in Δ to the distinguished vertex in $V(\Gamma/\Delta)$.

Every graph Γ has an *associated sequence*

$$0 \longrightarrow A[E(\Gamma)] \xrightarrow{\partial} A[V(\Gamma)] \xrightarrow{\varepsilon} A \longrightarrow 0$$

for any commutative ring A, where ∂ and ε are defined by extending $\partial(e) := d_1(e) - d_0(e)$ and $\varepsilon(v) := 1$. Clearly $\operatorname{im}\partial \leq \ker\varepsilon$, so we define the homology A-modules as $H_0(\Gamma, A) := \ker\varepsilon / \operatorname{im}\partial$ and $H_1(\Gamma, A) := \ker\partial$.

Fact 2.2 ([DD89, Section 6])**.** *A finite graph Γ is connected iff $H_0(\Gamma, A) = 0$ and is acyclic iff $H_1(\Gamma, A) = 0$, independently of A. In particular, a finite graph is a tree iff its associated sequence is exact.*

We now pass to the profinite category by taking inverse limits in the above category of graphs. Explicitly, a *profinite graph* Γ is a profinite space with a closed subset $V(\Gamma) \subseteq \Gamma$ and continuous maps $d_0, d_1 : \Gamma \twoheadrightarrow V(\Gamma)$ such that $d_i|V(\Gamma) = \mathrm{id}_{V(\Gamma)}$; it can be written as an inverse limit of its finite quotient graphs (see Lemma 2.6). A *morphism* of profinite graphs is a continuous map $\varphi : \Gamma \to \Gamma'$ such that $\varphi d_i = d_i \varphi$ for $i = 0, 1$.

Example 2.3. Let $G = \lim_{N \in \mathcal{N}_{\mathcal{C}}(G)} G/N$ be a pro-$\mathcal{C}$ group and let $X \subseteq G$ be a finite subset of G such that $1 \notin X$. The *profinite Cayley graph* of G with respect to X is the profinite graph $\Gamma(G, X) := G \times (X \cup \{1\})$ with $V(\Gamma(G, X)) := G \times \{1\} \cong G$ and incidence maps $d_0(g, x) := g$ and $d_1(g, x) := gx$.

Any continuous homomorphism $\varphi : G \to H$ of profinite groups induces a morphism $\Gamma(G, X) \to \Gamma(H, \varphi(X))$ of profinite Cayley graphs, and we have $\Gamma(G, X) \cong \lim_{N \in \mathcal{N}_{\mathcal{C}}(G)} \Gamma(G/N, \pi_N(X))$.

Remark 2.4. In general, the edge set $E(\Gamma) := \Gamma \setminus V(\Gamma)$ need not be closed in Γ, and so $E(\Gamma)$ need not be a profinite graph. To remedy this, we use the (pointed) quotient graph $E^*(\Gamma, *) := (\Gamma/V(\Gamma), *)$ where $*$ is the image of $V(\Gamma)$ under the projection, which is now a profinite graph with one vertex (see Figure 1).

Lemma 2.5. *Let $p : \Gamma \twoheadrightarrow X$ be a surjection onto a profinite space X. There is at-most one graph structure on X such that p is a morphism, which exists iff $p(d_i^\Gamma(m) = p(d_i^\Gamma(m'))$ for all $m, m' \in \Gamma$ such that $p(m) = p(m')$.*

Figure 1: A profinite graph Γ whose vertex set is the one-point compactification of $\mathbb{N}$. Its edge set is *not* closed since $\lim_n e_n = \infty$, so we collapse $V(\Gamma)$ in Γ to obtain the rose with countably-many petals, which is now a pointed profinite graph encoding the edges of Γ.

Proof. For p to be a morphism, it is necessary that we set $V(X) := p(V(\Gamma))$ and $d_i^X(x) := p(d_i^\Gamma(m))$ for any $m \in p^{-1}(x)$, which is well-defined iff the stated condition holds. ∎

Lemma 2.6. *Every profinite graph is the inverse limit of its finite quotient graphs.*

Proof. Let Γ be a profinite graph and let $\mathcal{R}$ be the collection of all open equivalence relations on Γ, so Γ/R is finite for each $R \in \mathcal{R}$. Order $\mathcal{R}$ by reverse inclusion, so $(\mathcal{R}, \leq)$ is directed by taking common refinements. If $R_1 \geq R_2$, then there is a continuous map $\varphi_{R_1,R_2} : \Gamma/R_1 \to \Gamma/R_2$ sending $mR_1 \mapsto mR_2$.

Claim. *We have $\Gamma \cong \lim_{R\in\mathcal{R}} \Gamma/R$ as topological spaces.*

Proof. Let $\psi : \Gamma \to \lim_{R\in\mathcal{R}} \Gamma/R$ be the map induced by the quotients $p_R : \Gamma \twoheadrightarrow \Gamma/R$, so ψ is surjective. If $m, m' \in \Gamma$, then there is a clopen subset $U \subseteq \Gamma$ such that $m \in U \not\ni m'$, so the equivalence relation on X with classes $\{U, U^c\}$ separates m and m' in the quotient. □

Now, let $\mathcal{R}_0 \subseteq \mathcal{R}$ be the subcollection of those equivalence relations $R \in \mathcal{R}$ such that Γ/R admits a graph structure and $p_R : \Gamma \twoheadrightarrow \Gamma/R$ is a morphism. We show that $\Gamma \cong \lim_{R\in\mathcal{R}_0} \Gamma/R$, for which it suffices to show that $\mathcal{R}_0$ is cofinal in $\mathcal{R}$, so let $R \in \mathcal{R}$. The projection p_R induces a map $\widetilde{p}_R : \Gamma \to \Gamma/R \times \Gamma/R \times \Gamma/R$ sending m to $(p_R(m), p_R(d_0m), p_R(d_1m))$, whose image admits a unique graph structure making $\widetilde{p}_R$ a morphism by Lemma 2.5. Thus $\widetilde{p}_R(\Gamma) \cong \Gamma/R_0$ for some $R_0 \in \mathcal{R}_0$, whose equivalences classes are of the form $\widetilde{p}_R^{-1}(x)$ for $x \in \widetilde{p}_R(\Gamma)$, and $R_0 \geq R$ since if $\widetilde{p}_R(m) = \widetilde{p}_R(m')$, then in particular $p_R(m) = p_R(m')$. ∎

To define the associated sequence of a profinite graph, we consider *profinite A-modules* over a profinite commutative ring A. The *free profinite A-module* over a profinite space $X := \lim_i X_i$ is the profinite A-module $A[\![X]\!] := \lim_i A[X_i]$, which satisfies the usual universal property: for any continuous map $\varphi : X \to M$ to a profinite A-module M, there is a unique continuous homomorphism $\overline{\varphi} : A[\![X]\!] \to M$ extending φ. Then

$$0 \longrightarrow A[\![E^*(\Gamma,*)]\!] \xrightarrow{\partial} A[\![V(\Gamma)]\!] \xrightarrow{\varepsilon} A \longrightarrow 0$$

is the *associated sequence* of a profinite graph Γ, with homology groups $H_i(\Gamma, A)$ defined the same way.

Definition 2.7. A profinite graph is *A-connected* if $H_0(\Gamma, A) = 0$ and *A-acyclic* if $H_1(\Gamma, A) = 0$.

A morphism $\varphi : \Gamma_1 \to \Gamma_2$ of profinite graphs induces homomorphisms $\varphi_V : A[\![V(\Gamma_1)]\!] \to A[\![V(\Gamma_2)]\!]$ and $\varphi_E : A[\![E^*(\Gamma_1,*)]\!] \to A[\![E^*(\Gamma_2,*)]\!]$ of profinite A-modules such that

$$\begin{tikzcd}
A[\![E^*(\Gamma_1,*)]\!] \arrow[r, "\partial^{\Gamma_1}"] \arrow[d, "\varphi_E"] & A[\![V(\Gamma_1)]\!] \arrow[r, "\varepsilon^{\Gamma_1}"] \arrow[d, "\varphi_V"] & A \arrow[d, equal] \\
A[\![E^*(\Gamma_2,*)]\!] \arrow[r, "\partial^{\Gamma_2}"] & A[\![V(\Gamma_2)]\!] \arrow[r, "\varepsilon^{\Gamma_2}"] & A
\end{tikzcd}$$

commutes, so φ induce homomorphisms $H_i(\Gamma_1, A) \to H_i(\Gamma_2, A)$ for $i = 0, 1$. Thus $H_i(-, A)$ are functors, and a diagram chase show that both $H_i(-, A)$ preserve limits, $H_0(-, A)$ preserve epimorphisms, and $H_1(-, A)$ preserve monomorphisms. This gives us the following results, which we will quote without further reference.

1. A profinite graph is A-connected iff all of its finite quotients are connected in the combinatorial sense. In particular, A-connectedness is independent of the ring A, so we have a notion of *profinite-connectedness*.
2. An inverse limit of A-acyclic finite graphs is A-acyclic. The converse fails precisely because the quotient of an A-acyclic graph need not be A-acyclic. It turns out that the dependence on A cannot be removed.
3. Every subgraph of an A-acyclic graph is a A-acyclic.

Example 2.8. Let G be a pro-$\mathcal{C}$ group. If $X \subseteq G$ is a topological generating set of G (i.e., $\overline{\langle X \rangle} = G$), then its profinite Cayley graph $\Gamma(G, X)$ is profinitely-connected since $\Gamma(G, X) \cong \lim_{N\in\mathcal{N}_\mathcal{C}(G)} \Gamma(G/N, \pi_N(X))$.

2.2. **Trees.** A profinite graph Γ is said to be an *A-tree* if Γ is profinitely-connected and A-acyclic. We give some lemmas justifying our usage of this term. Moreover, we show that under certain conditions on $\mathcal{C}$, the profinite Cayley graphs of the pro-$\mathcal{C}$ completions of free groups are $\mathbb{Z}_{\mathcal{C}}$-trees (see Theorem 2.12).

Lemma 2.9. *Let $\{\Gamma_\alpha\}$ be a family of profinitely-connected subgraphs of an A-tree Γ. The intersection $\bigcap_\alpha \Gamma_\alpha$ is an A-tree, and if $\bigcap_\alpha \Gamma_\alpha \neq \varnothing$, then their union $\bigcup_\alpha \Gamma_\alpha$ is an A-tree too.*

Proof. Since Γ is an A-tree, it suffices to show that both $\bigcap_\alpha \Gamma_\alpha$ and $\bigcup_\alpha \Gamma_\alpha$ are profinitely-connected. Indeed, for the first, observe that $A[\![V(\bigcap_\alpha \Gamma_\alpha)]\!] = \bigcap_\alpha A[\![V(\Gamma_\alpha)]\!]$ and $A[\![E^*(\bigcap_\alpha \Gamma_\alpha, *)]\!] = \bigcap_\alpha A[\![E^*(\Gamma_\alpha, *)]\!]$, so letting

$$0 \longrightarrow A[\![E^*(\Gamma,*)]\!] \xrightarrow{\partial} A[\![V(\Gamma)]\!] \xrightarrow{\varepsilon} A \longrightarrow 0$$

be the associated sequence of Γ, which is exact by hypothesis, we have

$$\ker\left(\varepsilon|\bigcap\nolimits_\alpha \Gamma_\alpha\right) = A\left[\!\left[V\left(\bigcap\nolimits_\alpha \Gamma_\alpha\right)\right]\!\right] \cap \ker\varepsilon = \bigcap\nolimits_\alpha A[\![V(\Gamma_\alpha)]\!] \cap \ker\varepsilon = \bigcap\nolimits_\alpha \ker(\varepsilon|\Gamma_\alpha),$$

and similarly $\operatorname{im}(\partial|\bigcap_\alpha \Gamma_\alpha) \subseteq \bigcap_\alpha \operatorname{im}(\partial|\Gamma_\alpha)$. Since ∂ is injective, the preceding inclusion is an equality. Each Γ_α is profinitely-connected, so $\ker(\varepsilon|\Gamma_\alpha) = \operatorname{im}(\partial|\Gamma_\alpha)$ for each α, and hence $\ker(\varepsilon|\bigcap_\alpha \Gamma_\alpha) = \operatorname{im}(\partial|\bigcap_\alpha \Gamma_\alpha)$.

For the second claim, let $\alpha : \bigcup_\alpha \Gamma_\alpha \twoheadrightarrow \Delta$ be a morphism onto a finite graph Δ, so for each α, the image $\alpha(\Gamma_\alpha) \subseteq \Delta$ is connected. Note that $\bigcap_\alpha \alpha(\Gamma_\alpha) \supseteq \alpha(\bigcap_\alpha \Gamma_\alpha) \neq \varnothing$, so $\Delta = \bigcup_\alpha \alpha(\Gamma_\alpha)$ is connected too. ∎

In view of the above lemma, for any two vertices v and w of an A-tree Γ, the intersection of all profinitely-connected subgraphs of Γ containing v and w is an A-tree; it is the *geodesic* between v and w, denoted $[v,w]$.

Lemma 2.10. *Let Γ be an A-tree. A subgraph $\Delta \subseteq \Gamma$ is an A-tree iff $[v,w] \subseteq \Delta$ for all $v, w \in V(\Delta)$.*

Proof. The forward direction is clear. Conversely, write $\Gamma \cong \lim_i \varphi_i(\Gamma)$ as an inverse limit of its finite quotient graphs, so $\Delta \cong \lim_i \varphi_i(\Delta)$. We claim that each $\varphi_i(\Delta)$ is connected, so Δ is an A-tree as desired.

Indeed, take $\overline{v}, \overline{w} \in V(\varphi_i(\Delta))$, which are projections of some $v, w \in V(\Delta)$, so $[v,w] \subseteq \Delta$ by hypothesis. Since $[v,w]$ is profinitely-connected, its image $\varphi_i([v,w])$ is a connected subgraph of $\varphi_i(\Delta)$ containing $\overline{v}$ and $\overline{w}$, so $\varphi_i(\Delta)$ is connected. ∎

Lemma 2.11. *Let Γ be a profinitely-connected profinite graph and let $\Delta \subseteq \Gamma$ be an A-subtree of Γ. Then the map $H_1(\Gamma, A) \to H_1(\Gamma/\Delta, A)$ induced from the collapsing map $\Gamma \twoheadrightarrow \Gamma/\Delta$ is an isomorphism.*

In particular, if Γ is an A-tree, then so is Γ/Δ.

Proof. Let $i : \Delta \hookrightarrow \Gamma$ and consider the following commutative diagram of profinite A-modules, whose rows are exact since Δ is an A-tree and Γ and Γ/Δ are both connected.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A[\![E^*(\Delta,*)]\!] \arrow[r, "d^\Delta"] \arrow[d, "i_E"] & A[\![V(\Delta)]\!] \arrow[r, "\varepsilon^\Delta"] \arrow[d, "i_V"] & A \arrow[r] \arrow[d, equal] & 0 \\
 & A[\![E^*(\Gamma,*)]\!] \arrow[r, "d^\Gamma"] \arrow[d, "p_E"] & A[\![V(\Gamma)]\!] \arrow[r, "\varepsilon^\Gamma"] \arrow[d, "p_V"] & A \arrow[r] \arrow[d, equal] & 0 \\
 & A[\![E^*(\Gamma/\Delta,*)]\!] \arrow[r, "d^{\Gamma/\Delta}"] \arrow[d] & A[\![V(\Gamma/\Delta)]\!] \arrow[r, "\varepsilon^{\Gamma/\Delta}"] \arrow[d] & A \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $p_E : A[\![E^*(\Gamma,*)]\!] \to A[\![E^*(\Gamma/\Delta,*)]\!]$ has kernel $i_E(A[\![E^*(\Delta,*)]\!])$, we see that the first column is exact, and similarly for the second column. It is now easily seen by diagram chasing (or, by applying a generalization of the Nine Lemma) that $\ker d^\Gamma \to \ker d^{\Gamma/\Delta}$ is an isomorphism. ∎

Recall that the Cayley graph of any free group of finite rank is a tree (or, in the homological language above, a $\mathbb{Z}$-tree). It is thus natural to ask whether the profinite Cayley graph of the pro-$\mathcal{C}$ completion of any free group of finite rank is a $\mathbb{Z}_{\mathcal{C}}$-tree. Let us call a pseudovariety $\mathcal{C}$ *arborescent* [AW94] if this is the case.

Theorem 2.12. *A pseudovariety $\mathcal{C}$ is arborescent iff $\mathcal{C}$ is closed under extensions with abelian kernel.*

The converse direction, which is what we need for Theorem A, was first proved in [GR78, Theorem 1.2]; for the forward direction and the consequences of this equivalence, see [AW94, Theorem 2.1] and [AW95].

Proof of Theorem 2.12 ($\Leftarrow$). Let F be a free group of finite rank and consider the profinite Cayley graph $\Gamma \coloneqq \Gamma(F_\mathcal{C})$ of the pro-$\mathcal{C}$ completion of F, with the standard generating set $X \subseteq F$. Observe that $E(\Gamma) = F_\mathcal{C} \times X$ is closed in Γ, so we can identify $E(\Gamma)$ with $E^*(\Gamma, *)$. Since $\Gamma \cong \lim_{N \in \mathcal{N}_\mathcal{C}(F_\mathcal{C})} \Gamma(F_\mathcal{C}/N, \pi_N(X))$ and each $\Gamma(F_\mathcal{C}/N, \pi_N(X))$ is connected in the combinatorial sense, we see that Γ is profinitely-connected.

Thus it suffices to show that $H_1(\Gamma, \mathbb{Z}_\mathcal{C}) = 0$, or, in other words, that the map $\partial : \mathbb{Z}_\mathcal{C}[\![F_\mathcal{C} \times X]\!] \to \mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!]$ is a bijection onto the *augmentation ideal* $I \coloneqq \operatorname{im}\partial = \ker\varepsilon$ of $\varepsilon : \mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!] \to \mathbb{Z}_\mathcal{C}$. Observe that $\mathbb{Z}_\mathcal{C}[\![F_\mathcal{C} \times X]\!]$ is a free profinite $\mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!]$-module on $\{1\} \times X$, where $\mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!]$ acts on $\mathbb{Z}_\mathcal{C}[\![F_\mathcal{C} \times X]\!]$ by the continuous linear extension of the left-multiplication action of $F_\mathcal{C}$ on $F_\mathcal{C} \times X$. Since $\partial(1, x) = x - 1$, it suffices to show that I is freely-generated by $\iota(X)$ as a profinite $\mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!]$-module, where $\iota : X \to M$ sends $\iota(x) \coloneqq x - 1$.

Claim. *The augmentation ideal I is the profinite $\mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!]$-module generated by $\iota(X)$.*

Proof. Since $F_\mathcal{C} = \lim_{N \in \mathcal{N}_\mathcal{C}(F_\mathcal{C})} F_\mathcal{C}/N$ and each $F_\mathcal{C}/N$ is a finite group generated by X, it suffices to prove that for a finite group G, the augmentation ideal $\ker(\mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z})$ is generated by $\{x - 1 : x \in X\}$.

Indeed, if $\sum_{g \in G} a_g = 0$ for $a_g \in \mathbb{Z}$, then for any fixed $x \in X$, we have $a_x = -\sum_{g \neq x} a_g$, and so

$$\sum\nolimits_g a_g g = a_x x + \sum\nolimits_{g \neq x} a_g g = \sum\nolimits_{g \neq x} a_g(g - x) = \sum\nolimits_{g \neq x} a_g x(x^{-1}g - 1) \in \langle h - 1 : h \in G \rangle.$$

Now write each $h \in G$ as $h = x_1 \cdots x_n$ for $x_i \in X$, so that $h - 1 = \sum_{i \leq n} x_1 \cdots x_{i-1}(x_i - 1)$. □

Thus it remains to show that I is freely-generated as a profinite $\mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!]$-module by $\iota(X)$, so let $\varphi : X \to M$ be a continuous map to a profinite $\mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!]$-module M; we need to show that there is a continuous $\mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!]$-module homomorphism $\overline{\varphi} : I \to M$ such that $\overline{\varphi}\iota = \varphi$, which is then unique by the claim.

To this end, observe that since M is a profinite $\mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!]$-module, we have a semidirect product $M \rtimes \mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!]$ such that the map $\varphi \times \mathrm{id} : X \to M \rtimes \mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!]$ is continuous. Note that M is abelian as a pro-$\mathcal{C}$ group, so since $\mathcal{C}$ is closed under extensions of abelian kernels, we see after passing to an inverse limit that $M \rtimes \mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!]$ is a pro-$\mathcal{C}$ group, and hence there is a unique continuous homomorphism $\varphi' : F_\mathcal{C} \to M \rtimes \mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!]$ such that $\varphi'(x) = (\varphi(x), x)$ for all $x \in X$. Thus $\delta_0 \coloneqq p_1\varphi' : F_\mathcal{C} \to M$ is a continuous map such that $\delta_0(x) = \varphi(x)$ for all $x \in X$, and is a *derivation* in the sense that $\delta_0(gh) = \delta_0(g) + g\delta_0(h)$ for all $g, h \in F_\mathcal{C}$.

$$\begin{tikzcd}
 & M \rtimes \mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!] \arrow[dr, "p_1"] & \\
F_\mathcal{C} \arrow[ur, dashed, "\exists!\varphi'"] \arrow[d, hook] & X \arrow[l, hook'] \arrow[u, "\varphi\times\mathrm{id}"'] \arrow[r, "\varphi"] & M \\
\mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!] \arrow[urr, dashed, "\exists!\delta"] & & I \arrow[ll, hook'] \arrow[u, squiggly, "\overline{\varphi}:=\delta|I"']
\end{tikzcd}$$

All maps relevant to the construction of $\overline{\varphi} : I \to M$, fitted into a commutative diagram; the hooked arrows are subsets and the dashed arrows are induced by universal properties.

Now, let $\delta : \mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!] \to M$ be the unique $\mathbb{Z}_\mathcal{C}[\![F_\mathcal{C}]\!]$-module homomorphism extending δ_0, so its restriction $\overline{\varphi}$ to I is as desired since $\overline{\varphi}(x - 1) = \delta(x - 1) = \delta_0(x) - \delta_0(1) = \delta_0(x) = \varphi(x)$ for all $x \in X$. ■

2.3. **Actions.** Lastly, we need to study the behaviour of actions of a profinite group G on a profinite graph Γ, which are continuous actions $G \curvearrowright \Gamma$ such that $d_i(gm) = gd_i(m)$ for each $m \in \Gamma$ and $i = 0, 1$.

Lemma 2.13. *If $G \curvearrowright \Gamma$, then $\Gamma \cong \lim_i \Gamma_i$ where each Γ_i is a finite quotient G-graph.*

Proof. Proceed as in Lemma 2.6, taking $\mathcal{R}_0$ to be the open G-invariant equivalence relations on Γ, so that $p_R : \Gamma \twoheadrightarrow \Gamma/R$ is a G-map of G-spaces. Then $\mathcal{R}_0$ is cofinal in the collection $\mathcal{R}$ of all open equivalence relations on Γ, so that $\Gamma \cong \lim_{R \in \mathcal{R}_0} \Gamma/R$ is as desired. ■

Corollary 2.14. *Let $G \curvearrowright \Gamma$ and suppose that $G = \overline{\langle X \rangle}$ for some $X \subseteq G$. If $\Delta \subseteq \Gamma$ is a subgraph such that $\Delta \cap x\Delta \neq \varnothing$ for all $x \in X$, then $G\Delta \coloneqq \bigcup_{g \in G} g\Delta$ is profinitely-connected.*

Proof. Writing $\Gamma \cong \lim_i \varphi_i(\Gamma)$ as an inverse limit of its finite quotient G-graphs, we have $G\Delta = \lim_i G\varphi_i(\Delta)$ and $\varphi_i(\Delta) \cap x\varphi_i(\Delta) \neq \varnothing$ for all $x \in X$, so we can assume without loss of generality that Γ is finite.

In this case, the kernel of $G \to \mathrm{Aut}(\Gamma)$ is an open normal subgroup of G, so after passing to the quotient, we may assume that G is finite. Now both G and Γ are finite, and the result is immediate. ■

Finally, we will need the existence of connected transversals of $G \curvearrowright \Gamma$. If Γ is finite, then such transversals always exist (see, for instance, [DD89]). This is not the case in general, but if G acts freely on Γ with finite quotient, then the existence of transversals follow from the same argument as in the finite case.

Lemma 2.15. *Let G be a profinite group acting on a profinite graph Γ and fix a vertex $v_0 \in V(\Gamma)$. If Γ/G is finite and $G \curvearrowright \Gamma$ freely, then there is a (finite) connected transversal $\Sigma \subseteq \Gamma$ of the action containing v_0.*

Proof. Let $\mathcal{T}$ be the set of all finite subtrees $T_0 \subseteq \Gamma$ containing v_0 such that $T_0 \hookrightarrow \Gamma \twoheadrightarrow \Gamma/G$ is an injection, and let $T \in \mathcal{T}$ be maximal with respect to inclusion. Clearly $T' \coloneqq p(T)$ is a subtree of Γ/G.

We claim that T' is a spanning subtree of Γ/G. If not, then since Γ/G is finite and T' is connected, there is an edge $e' \in \Gamma/G - T'$ such that $d_i(e') \in T'$ but $d_{1-i}(e') \notin T'$ for some $i = 0, 1$, say with $i = 0$. Choose $v \in V(T)$ and $e_0 \in E(T)$ such that $p(v) = d_0(e')$ and $p(e) = e'$. Observe that $p(v) = p(d_0(e))$, so under the action of G, we can assume that $v = d_0(e)$. But $T \sqcup \{e, d_1(e)\} \in \mathcal{T}$, which contradicts the maximality of T.

Set $\Sigma \coloneqq d_0^{-1}(V(T))$, so clearly $v_0 \in T \subseteq \Sigma$. It remains to show that Σ is a transversal of $G \curvearrowright \Gamma$. Since $p|T : T \to T'$ is a bijection and $V(T') = V(\Gamma/G)$, it suffices to show that for each edge $e' \in \Gamma/G - T'$, there is a unique edge $e \in \Sigma - T$ such that $p(e) = e'$. Indeed, let $e \in E(\Gamma)$ project to e', which under the action of G can be chosen so that $d_0(e) \in V(T)$, and hence $e \in \Sigma - T$. If $e_1, e_2 \in \Sigma - T$ both project to e', then $e_2 = ge_1$ for some $g \in G$. But $p(d_0(e_i)) = d_0(p(e_i)) = d_0(e') \in V(T')$ for both $i = 1, 2$, and since $d_0(e_i) \in V(T)$, we see from injectivity of $p|T$ that $d_0(e_1) = d_0(e_2)$, and hence $d_0(e_1) = d_0(e_2) = d_0(ge_1) = gd_0(e_1)$. This forces $g = 1$ by freeness of the action, and hence $e_1 = e_2$ as desired. ■

3. Proof of the Ribes-Zalesskii Theorem

In this section, we prove the Ribes-Zalesskii Theorem for pro-$\mathcal{C}$ free groups, where $\mathcal{C}$ is a pseudovariety of groups closed under extensions. Let F be a free group of finite rank. By Fact 1.1, F is residually-$\mathcal{C}$, so F embeds into its pro-$\mathcal{C}$ completion $F_{\mathcal{C}}$. This embedding extends to an embedding $\Gamma(F) \hookrightarrow \Gamma(F_{\mathcal{C}})$ of profinite graphs, where $\Gamma(F_{\mathcal{C}})$ is a $\mathbb{Z}_{\mathcal{C}}$-tree by Theorem 2.12.

Let $H, K \leq F$ be finitely generated subgroups which are closed in the pro-$\mathcal{C}$ topology of F. We will show that the double coset HK is also closed in the pro-$\mathcal{C}$ topology by studying actions on the profinite Cayley graph $\Gamma(F_{\mathcal{C}})$. To this end, we make some useful reductions, which require the following lemma.

Lemma 3.1. *If $K \leq_c F$ is a finitely generated closed subgroup of F, then there is an open subgroup $U \leq_o F$ containing K such that K is a free factor of U.*

Proof. Let Γ be the Cayley graph of F with respect to a fixed basis. By Lemma 1.3, we can write $K = \bigcap_i U_i$ as the intersection of all open subgroups $U_i \leq_o F$ containing K. For each i, let $p_i : \Gamma/K \twoheadrightarrow \Gamma/U_i$ be the natural quotient map. Note that for any finite $\Delta \subseteq \Gamma/K$, there is some i such that $p_i|\Delta$ is injective.

Since K is finitely generated, the fundamental group of Γ/K is supported on a finite subgraph $\Delta \subseteq \Gamma/K$. Choosing i so that $\Delta \hookrightarrow \Gamma/U_i$, we see that $K \cong \pi_1(\Gamma/K) \cong \pi_1(\Delta)$ is a free factor of $\pi_1(\Gamma/U_i) \cong U_i$. ■

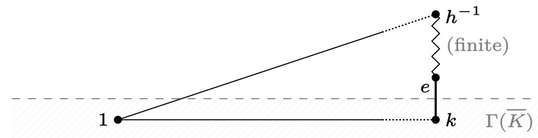
Claim. *We can assume that (1) K is a free factor of F, (2) K is $\mathcal{C}$-compatible with F, and (3) $\overline{K} = K_{\mathcal{C}}$.*

Proof. Applying Lemma 3.1 so that K is a free factor of an open subgroup $U \leq_o F$, we see that K is $\mathcal{C}$-compatible with U and U is $\mathcal{C}$-compatible with F by Lemmas 1.4 and 1.5, respectively. Let $\{h_i\}$ be a finite transversal of $H \cap U$ in H, so that $HK = \bigsqcup_i h_i(H \cap U)K$ is closed in the pro-$\mathcal{C}$ topology of U since $H \cap U$ is still finitely generated by Howson's Theorem (see [Sta83, Corollary 5.6]). □

In particular, we can assume that the following diagram on the left commutes, so that $\Gamma(K) = \Gamma(\overline{K}) \cap \Gamma(F)$ by Corollary 1.7. Now, we start the actual proof. By Corollary 1.7 again, it suffices to show that $\overline{HK} \cap F = HK$. Since $F_{\mathcal{C}}$ is compact, so are $\overline{H}$ and $\overline{K}$, and hence $\overline{H} \times \overline{K}$ is compact too by Tychonoff's Theorem. By continuity of multiplication, we see that $\overline{H}\,\overline{K}$ is compact too, hence closed, so $\overline{HK} \subseteq \overline{H}\,\overline{K}$. Thus, overall, it suffices to show that $\overline{H}\,\overline{K} \cap F \subseteq HK$. Take $h \in \overline{H}$ and $k \in \overline{K}$ such that $hk \in F$. We need to show that $hk \in HK$.

Since $hk \in F$, the geodesic $[1, hk]$ in $\Gamma(F_{\mathcal{C}})$ is a finite path, so $[h^{-1}, k] = h^{-1}[1, hk]$ is a finite path as well. Let $k_0 \in [h^{-1}, k]$ be closest to h^{-1} such that $k_0 \in \Gamma(\overline{K})$. If $k_0 = h^{-1}$, then $h \in \overline{K}$, so $hk \in \overline{K} \cap F = K \subseteq HK$ and we are done. Otherwise, observe that $k_0 k^{-1} \in K$ since $[k_0, k] \subseteq \Gamma(\overline{K})$ is finite, so $hk \in HK$ iff $hk_0 \in HK$. Hence, we can assume without loss of generality that $k = k_0$.

$$\begin{tikzcd}
\Gamma(F) \arrow[r, hook] & \Gamma(\overline{F}) = \Gamma(F_{\mathcal{C}}) \\
\Gamma(K) \arrow[r, hook] \arrow[u, hook] & \Gamma(\overline{K}) = \Gamma(K_{\mathcal{C}}) \arrow[u, hook]
\end{tikzcd}$$



(a) We can regard $\Gamma(K)$ as a subgraph of $\Gamma(F_{\mathcal{C}})$, and $\Gamma(K) \subseteq \Gamma(\overline{K}) \cap \Gamma(F) = \Gamma(\overline{K} \cap F)$ is an equality.

(b) Collapsing the two geodesics induces a cycle since e lies outside said geodesics and $[h^{-1}, k]$ is finite.

Claim. *There is a free action of $\overline{H} \cap \overline{K}$ on a profinite graph $\Delta \subseteq \Gamma(\overline{K})$ containing k with finite quotient.*

Proof. Let $\Lambda := \bigcup_{i=1}^{n} \overline{H}[1, h_i]$ where $h_1, \dots, h_n$ are the free generators of H, and set $\Delta := \Gamma(\overline{K}) \cap \Lambda$ so that $\overline{H} \cap \overline{K}$ acts freely on Δ by left-multiplication. Observe that Λ is $\overline{H}$-invariant and the quotient map $p : \Lambda \to \Lambda / \overline{H}$ induces a map $\widetilde{p} : \Delta / (\overline{H} \cap \overline{K}) \to p(\Lambda)$, which we claim is injective. Indeed, if $t_2 = t_1 x$ for some $x \in \overline{H}$ and $t_1, t_2 \in \Delta$, then the vertices of t_i lie in $V(\Gamma(\overline{K}))$, so $t_2 = t_1 x'$ for some $x' \in \overline{K}$, which forces $x = x' \in \overline{H} \cap \overline{K}$ by freeness. In particular, since $p(\Lambda)$ is finite, so is $\Delta / (\overline{H} \cap \overline{K})$.

By Corollary 2.14, Λ is a $\mathbb{Z}_{\mathcal{C}}$-tree since $\Lambda = \bigcup_{h \in \overline{H}} h\Lambda_0$, where $\Lambda_0 := \bigcup_{i=1}^{n} [1, h_i]$ and $\Lambda_0 \cap h_i \Lambda_0 \neq \varnothing$ for each $i \leq n$. We claim that $k \in [h^{-1}, 1]$, so $k \in [h^{-1}, 1] \subseteq \Lambda$ by Lemma 2.10. Suppose not, so there is an edge $e \in [h^{-1}, k] \setminus [h^{-1}, 1]$ such that $e \notin \Gamma(\overline{K})$ but k is a vertex of e. Then, collapsing the $\mathbb{Z}_{\mathcal{C}}$-subtree $[h^{-1}, 1] \cup [1, k]$ of the $\mathbb{Z}_{\mathcal{C}}$-tree $[h^{-1}, 1] \cup [1, k] \cup [h^{-1}, k]$ (see Lemma 2.9) induces a cycle at $h \sim k$ since $[1, k] \subseteq \Gamma(\overline{K})$ and $e \notin [h^{-1}, 1] \cup [1, k]$, which contradicts Lemma 2.11. $\square$

By Lemma 2.15, there is a finite connected transversal $\Sigma \subseteq \Delta$ containing 1. Since $k \in \Delta$, there is an element $g \in \overline{H} \cap \overline{K}$ such that $gk \in \Sigma$. But Σ is finite and connected, so $\Sigma \subseteq \Gamma(F)$, and hence $\Sigma \subseteq \Gamma(\overline{K}) \cap \Gamma(F) = \Gamma(K)$. Thus we have $gk \in K$, so $hg^{-1} \in \overline{H} \cap F = H$ by Corollary 1.7, and hence $hk = (hg^{-1})(gk) \in HK$. ■

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