

# AN EXPOSITION OF THE RIBES-ZALESSKII PRODUCT THEOREM

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**ABSTRACT.** We give a detailed proof of the Ribes-Zalesskii product theorem for pro- $\mathcal{C}$  topologies, where  $\mathcal{C}$  is a pseudovariety of groups closed under extensions. The proof relies on geometric properties of the profinite Cayley graph of the pro- $\mathcal{C}$  completion of free groups, which we develop here following [Rib17].

**The pro- $\mathcal{C}$  topology.** A *pseudovariety of groups* is a non-empty class  $\mathcal{C}$  of finite groups that is closed under taking subgroups, finite direct products, and quotients.

A pseudovariety  $\mathcal{C}$  is said to be *closed under extensions (with abelian kernel)* if  $G \in \mathcal{C}$  whenever  $N, G/N \in \mathcal{C}$  for any normal (abelian) subgroup  $N \trianglelefteq G$ . Throughout, fix a pseudovariety  $\mathcal{C}$  of groups.

**Definition.** Let  $G$  be a group. The collection  $\mathcal{N}_{\mathcal{C}}(G)$  of all normal subgroups  $N \trianglelefteq G$  such that  $G/N \in \mathcal{C}$  forms a neighborhood base around the identity, which generates the *pro- $\mathcal{C}$  topology* on  $G$ . We say that  $G$  is *residually- $\mathcal{C}$*  if  $\bigcap \mathcal{N}_{\mathcal{C}}(G) = \{1\}$ , which occurs iff the pro- $\mathcal{C}$  topology on  $G$  is Hausdorff.

The goal of this note is to prove the following generalization of the *Ribes-Zalesskii Theorem* [RZ93].

**Theorem A** (Ribes-Zalesskii [RZ94, Theorem 5.1]). *Let  $\mathcal{C}$  be a pseudovariety of groups that is closed under extensions. If  $F$  is a free group of finite rank and  $H, K \leq F$  are finite generated subgroups which are closed in the pro- $\mathcal{C}$  topology of  $F$ , then the double coset  $HK$  is also closed in the pro- $\mathcal{C}$  topology of  $F$ .*

**Remark.** Ribes and Zalesskii actually proved that for any  $n \in \mathbb{N}$ , the  $n$ -product coset  $H_1 \cdots H_n$  is closed in the pro- $\mathcal{C}$  topology of  $F$  whenever  $H_1, \dots, H_n$  are finitely generated subgroups which are closed in the pro- $\mathcal{C}$  topology of  $F$ . We chose to present the proof only for the case  $n = 2$  since the general case requires some more careful bookkeeping, but all essential ideas of their proof are present here.

**Remark.** The condition that  $\mathcal{C}$  is closed under extensions is needed in two important parts of the proof.

1. It ensures that the pro- $\mathcal{C}$  topology on an open subgroup  $H \leq_o G$  coincides with the topology induced by the pro- $\mathcal{C}$  topology on  $G$ ; see Lemma 1.5.
2. A weaker condition that  $\mathcal{C}$  is closed under extensions *with abelian kernel* is needed to show that the profinite Cayley graph of free groups are  $\mathbb{Z}_{\mathcal{C}}$ -trees; see Theorem 2.12.

Auinger and Steinberg [AS05] discovered another proof of Theorem A, which, among other things, weakened the hypothesis by only requiring that  $\mathcal{C}$  is closed under extensions with abelian kernel.

**Remark.** When  $\mathcal{C}$  is the class of all finite groups (in which case we write *profinite* for ‘pro- $\mathcal{C}$ ’), we can drop the hypotheses that  $H$  and  $K$  are closed in the profinite topology of  $F$  since all finitely generated subgroups of  $F$  are closed; this is Hall’s Theorem (see, for instance, [Sta83, Section 6]).

This note is organized as follows. In Section 1, we gather some basic facts about the pro- $\mathcal{C}$  topology of a residually- $\mathcal{C}$  group  $G$  and the interaction between  $G$ , its subgroups, and its pro- $\mathcal{C}$  completion  $G_{\mathcal{C}}$ . Section 2 is a summary of results in profinite graph theory needed to develop the basic properties of profinite Cayley graphs of free groups, which we use to prove Theorem A in Section 3.

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1. PRO- $\mathcal{C}$  TOPOLOGIES AND COMPLETIONS

Throughout, fix a residually- $\mathcal{C}$  group  $G$  and equip it with its pro- $\mathcal{C}$  topology. The residually- $\mathcal{C}$  condition on  $G$  ensures that  $G$  embeds into its *pro- $\mathcal{C}$  completion* (see Section 1.2), which will be important later on. We will apply the results in this chapter to free groups, so we need free groups to be residually- $\mathcal{C}$ .

**Fact 1.1** ([RZ10, Proposition 3.3.15]). *If  $\mathcal{C}$  is closed under extensions, then every free group is residually- $\mathcal{C}$ .*

**1.1. Subgroups.** The following lemmas characterize when a subgroup  $H \leq G$  is open or closed in  $G$ . Let  $H_G := \bigcap_{g \in G} gHg^{-1}$  be the *normal core* of  $H$  in  $G$ . Note that if  $H$  has finite index in  $G$ , then so does  $H_G$ .

**Lemma 1.2.** *A subgroup  $H \leq G$  is open if and only if  $H_G \in \mathcal{N}_{\mathcal{C}}(G)$ .*

*Proof.* If  $H$  is open, then  $H$  has finite-index in  $G$ , so  $H_G$  is open as well. Thus  $N \leq H_G$  for some  $N \in \mathcal{N}_{\mathcal{C}}(G)$ , so  $G/N \twoheadrightarrow G/H_G$ , whence  $G/H_G \in \mathcal{C}$ . Conversely, if  $G/H_G \in \mathcal{C}$ , then  $H_G$  is open, and hence so is  $H$ . ■

**Lemma 1.3.** *A subgroup  $H \leq G$  is closed iff  $H$  is the intersection of all open subgroups of  $G$  containing  $H$ .*

*Proof.* First, note that the intersection of all open subgroups of  $G$  containing  $H$  coincides with  $\bigcap_{N \in \mathcal{N}_{\mathcal{C}}(G)} HN$  since if  $x \in \bigcap_{N \in \mathcal{N}_{\mathcal{C}}(G)} HN$  and  $H \subseteq K \leq_o G$ , then  $K_G \in \mathcal{N}_{\mathcal{C}}(G)$  by Lemma 1.2, so  $x \in HK_G \leq K$ .

If  $H$  is closed in the pro- $\mathcal{C}$ -topology on  $G$ , then for all  $x \in G \setminus H$ , take  $N \in \mathcal{N}_{\mathcal{C}}(G)$  such that  $xN \cap H = \emptyset$ , so  $x \notin HN$ , and hence  $H = \bigcap_{N \in \mathcal{N}_{\mathcal{C}}(G)} HN$ . Conversely, note that open subgroups are closed. ■

The pro- $\mathcal{C}$  topology on a subgroup  $H \leq G$  is, in general, finer than the topology induced from the pro- $\mathcal{C}$  topology on  $G$ : if  $N \in \mathcal{N}_{\mathcal{C}}(G)$ , then the natural map  $H/(H \cap N) \rightarrow G/N \in \mathcal{C}$  is injective, so  $H/(H \cap N) \in \mathcal{C}$ . The topologies need not coincide (see [RZ10, Example 3.1.3]). However, there are important cases where the two topologies on  $H$  do coincide, in which case we say that  $H$  is  *$\mathcal{C}$ -compatible* with  $G$ .

**Lemma 1.4.** *If  $G = H * K$ , then (the image of)  $H$  is  $\mathcal{C}$ -compatible with  $G$ .*

*Proof.* Observe that  $G = H \rtimes K'$  where  $K'$  is the normal closure of  $K$  in  $G$ , so we instead prove the statement for when  $G = H \rtimes K$ . In this case, take  $N \in \mathcal{N}_{\mathcal{C}}(H)$  and note that  $G/KN \cong H/N \in \mathcal{C}$ , so  $KN$  is open in the pro- $\mathcal{C}$  topology of  $G$ . Observe that  $N = H \cap KN$ , so  $N$  is open in the induced topology. ■

**Lemma 1.5.** *If  $H \leq_o G$  is open and  $\mathcal{C}$  is closed under extensions, then  $H$  is  $\mathcal{C}$ -compatible with  $G$ .*

*Proof.* We show that  $\mathcal{N}_{\mathcal{C}}(H) \subseteq \mathcal{N}_{\mathcal{C}}(G)$ , so take  $N \in \mathcal{N}_{\mathcal{C}}(H)$ . It suffices to show that  $G/N_G \in \mathcal{C}$ , since then  $G/N \in \mathcal{C}$  as  $G/N_G \twoheadrightarrow G/N$ , and  $[G : N] = [G : H][H : N]$  is finite as  $H$  is open in  $G$ .

Since it is not necessarily the case that  $N \leq H_G$ , we work with  $M := H_G \cap N$  instead, which is harmless as  $N_G = M_G$ . Observe that  $G/H_G \cong (G/M_G)/(H_G/M_G)$ , which by Lemma 1.2 lies in  $\mathcal{C}$  since  $H$  is open in  $G$ , so it suffices to show that  $H_G/M_G \in \mathcal{C}$  as  $\mathcal{C}$  is closed under extensions. Since  $[G : M]$  is finite, there exist  $g_1, \dots, g_n \in G$  such that  $M_G = \bigcap_{i=1}^n g_i M g_i^{-1}$ , for some  $n \in \mathbb{N}$ . As  $H_G/M_G \hookrightarrow \prod_{i=1}^n H_G/g_i M g_i^{-1}$ , it suffices to show that each  $H_G/g_i M g_i^{-1} \in \mathcal{C}$ . To this end, note that

$$\frac{H_G}{M} = \frac{H_G}{H_G \cap N} \cong \frac{H_G N}{N} \leq \frac{H}{N} \in \mathcal{C},$$

where  $H/N \in \mathcal{C}$  by choice of  $N$ , so  $H_G/M \in \mathcal{C}$ , and hence  $H_G/g_i M g_i^{-1} \cong H_G/M \in \mathcal{C}$  via conjugation. ■

**1.2. Completions.** A pro- $\mathcal{C}$  group is an inverse limit of groups in  $\mathcal{C}$ . Given a residually- $\mathcal{C}$  group  $G$ , its *pro- $\mathcal{C}$  completion* is the group  $G_{\mathcal{C}} := \lim_{N \in \mathcal{N}_{\mathcal{C}}(G)} G/N$ , which is a pro- $\mathcal{C}$  group, and the canonical map  $G \rightarrow G_{\mathcal{C}}$  is injective. The first thing to check is that the pro- $\mathcal{C}$  topology on  $G$  is induced by the topology of  $G_{\mathcal{C}}$ . In the sequel, given a subset  $X \subseteq G$ , we let  $\bar{X}$  denote its closure in  $G_{\mathcal{C}}$ .

**Lemma 1.6.** *The topology of  $G_{\mathcal{C}}$  induces on  $G$  its pro- $\mathcal{C}$  topology.*

*Proof.* Recall that  $G \hookrightarrow G_{\mathcal{C}}$  via  $g \mapsto (gN)_{N \in \mathcal{N}_{\mathcal{C}}(G)}$ . We wish to show that  $H \subseteq G$  is open iff  $H = G \cap K$  for some open subset  $K \subseteq G_{\mathcal{C}}$ , for which it suffices to show this for open subgroups  $H \leq_o G$  and  $K \leq_o G_{\mathcal{C}}$ .

**Claim.** For any subgroup  $H \leq G$ , we have  $H = G \cap \overline{H}$  and  $[G : H] = [G_C : \overline{H}]$ .

*Proof.* Clearly we have  $H \subseteq G \cap \overline{H}$ , so take  $g \in G \cap \overline{H}$ . Since  $\overline{H} = \lim_N HN/N$ , we see that  $g \in HN$  for each  $N \in \mathcal{N}_C(G)$ . By Lemma 1.2, we have  $H_G \in \mathcal{N}_C(G)$ , so  $g \in HH_G = H$ . For the second claim, let  $\kappa := [G_C : \overline{H}]$ . Since  $G$  is dense in  $G_C$ , we have  $G\overline{H} = G_C$ , so take a left transversal  $\{g_\xi \in G : \xi < \kappa\}$  of  $\overline{H}$  in  $G_C$ . Note that  $g_\xi H = G \cap g_\xi \overline{H}$ , so  $G = G \cap \bigsqcup_{\xi < \kappa} g_\xi \overline{H} = \bigsqcup_{\xi < \kappa} g_\xi H$ , and thus  $\kappa = [G : H]$ .  $\square$

Thus if  $H \leq_o G$ , then  $[G_C : \overline{H}] = [G : H]$  is finite, and hence  $\overline{H} \leq_o G_C$  is an open subgroup too. Conversely, if  $K \leq_o G_C$  is an open subgroup of  $G_C$ , then  $G \cap K = G \cap \lim_N KN/N = \bigcap_N KN$  is closed by Lemma 1.3. Note that  $\overline{G \cap K} = K$  since  $G$  is dense in  $G_C$ . By the claim, we have  $[G : G \cap K] = [G_C : \overline{G \cap K}] = [G_C : K]$ , which is finite since  $K$  is open in  $G_C$ , and hence  $G \cap K$  is open in  $G$ .  $\blacksquare$

**Corollary 1.7.** A subset  $X \subseteq G$  is closed in the pro- $\mathcal{C}$  topology of  $G$  if and only if  $X = G \cap \overline{X}$ .

*Proof.* Let  $\overline{X}_G$  be the closure of  $X$  in  $G$ . Since  $X \subseteq G \cap \overline{X}$  is closed in  $G$ , we have  $\overline{X}_G \subseteq G \cap \overline{X}$ . Conversely, take  $x \in G \cap \overline{X}$  and note that every neighborhood of  $x$  in  $G$  contains a neighborhood of the form  $x(G \cap U)$  for some open subgroup  $U \leq_o G_C$ , and  $X \cap x(G \cap U) = X \cap (G \cap xU) = X \cap xU \neq \emptyset$  since  $x \in \overline{X}$ .  $\blacksquare$

## 2. PROFINITE GRAPHS AND TREES

In this section, we follow [Rib17] to give the necessary background on profinite graphs to study the geometric properties of the profinite Cayley graphs of free groups.

**2.1. Graphs.** A *graph* is a set  $\Gamma$  together with a subset  $V(\Gamma) \subseteq \Gamma$ , whose elements are called *vertices*, and maps  $d_0, d_1 : \Gamma \rightarrow V(\Gamma)$  such that  $d_i|_{V(\Gamma)} = \text{id}_{V(\Gamma)}$ ; the elements of  $E(\Gamma) := \Gamma \setminus V(\Gamma)$  are called *edges*. A *morphism* of graphs is a function  $\varphi : \Gamma \rightarrow \Gamma'$  such that  $\varphi d_i = d_i \varphi$  for  $i = 0, 1$  (so  $\varphi$  sends vertices to vertices, but does not necessarily send edges to edges). A *quotient* of  $\Gamma$  is a morphic image of  $\Gamma$ .

**Example 2.1.** If  $\Delta \subseteq \Gamma$  is a subgraph, then the map  $p : \Gamma \rightarrow \Gamma/\Delta$  obtained by *collapsing*  $\Delta$  to a point (as discrete spaces) gives rise to a graph  $\Gamma/\Delta$  with  $V(\Gamma/\Delta) := p(V(\Gamma))$  and  $d_i(p(m)) := p(d_i(m))$  for  $m \in \Gamma$  and  $i = 0, 1$ . Then  $p : \Gamma \rightarrow \Gamma/\Delta$  is a morphism sending each edge in  $\Delta$  to the distinguished vertex in  $V(\Gamma/\Delta)$ .

Every graph  $\Gamma$  has an *associated sequence*  $0 \rightarrow A[E(\Gamma)] \xrightarrow{\partial} A[V(\Gamma)] \xrightarrow{\varepsilon} A \rightarrow 0$  for any commutative ring  $A$ , where  $\partial$  and  $\varepsilon$  are defined by extending  $\partial(e) := d_1(e) - d_0(e)$  and  $\varepsilon(v) := 1$ . Clearly  $\text{im } \partial \leq \ker \varepsilon$ , so we define the homology  $A$ -modules as  $H_0(\Gamma, A) := \ker \varepsilon / \text{im } \partial$  and  $H_1(\Gamma, A) := \ker \partial$ .

**Fact 2.2** ([DD89, Section 6]). A finite graph  $\Gamma$  is connected iff  $H_0(\Gamma, A) = 0$  and is acyclic iff  $H_1(\Gamma, A) = 0$ , independently of  $A$ . In particular, a finite graph is a tree iff its associated sequence is exact.

We now pass to the profinite category by taking inverse limits in the above category of graphs. Explicitly, a *profinite graph*  $\Gamma$  is a profinite space with a closed subset  $V(\Gamma) \subseteq \Gamma$  and continuous maps  $d_0, d_1 : \Gamma \rightarrow V(\Gamma)$  such that  $d_i|_{V(\Gamma)} = \text{id}_{V(\Gamma)}$ ; it can be written as an inverse limit of its finite quotient graphs (see Lemma 2.6). A *morphism* of profinite graphs is a continuous map  $\varphi : \Gamma \rightarrow \Gamma'$  such that  $\varphi d_i = d_i \varphi$  for  $i = 0, 1$ .

**Example 2.3.** Let  $G = \lim_{N \leq_o G} G/N$  be a profinite group and let  $X \subseteq G$  be a finite subset of  $G$  such that  $1 \notin X$ . The *profinite Cayley graph* of  $G$  with respect to  $X$  is the profinite graph  $\Gamma(G, X) := G \times (X \cup \{1\})$  with  $V(\Gamma(G, X)) := G \times \{1\} \cong G$  and incidence maps  $d_0(g, x) := g$  and  $d_1(g, x) := gx$ .

Any continuous homomorphism  $\varphi : G \rightarrow H$  of profinite groups induces a morphism  $\Gamma(G, X) \rightarrow \Gamma(H, \varphi(X))$  of profinite Cayley graphs, and we have  $\Gamma(G, X) \cong \lim_{N \leq_o G} \Gamma(G/N, \pi_N(X))$ .

**Lemma 2.4.** Let  $p : \Gamma \rightarrow X$  be a surjection onto a profinite space  $X$ . There is at-most one graph structure on  $X$  such that  $p$  is a morphism, which exists iff  $p(d_i^\Gamma(m)) = p(d_i^\Gamma(m'))$  for all  $m, m' \in \Gamma$  such that  $p(m) = p(m')$ .

*Proof.* For  $p$  to be a morphism, it is necessary that we set  $V(X) := p(V(\Gamma))$  and  $d_i^X(x) := p(d_i^\Gamma(m))$  for any  $m \in p^{-1}(x)$ , which is well-defined iff the stated condition holds.  $\blacksquare$

**Remark 2.5.** In general, the edge set  $E(\Gamma) := \Gamma \setminus V(\Gamma)$  need not be closed in  $\Gamma$ , and so  $E(\Gamma)$  need not be a profinite graph. To remedy this, we use the (pointed) quotient graph  $E^*(\Gamma, *) := (\Gamma/V(\Gamma), *)$  where  $*$  is the image of  $V(\Gamma)$  under the projection, which is now a profinite graph with one vertex (see Figure 1).



Figure 1: A profinite graph  $\Gamma$  whose vertex set is the one-point compactification of  $\mathbb{N}$ . Its edge set is *not* closed since  $\lim_n e_n = \infty$ , so we collapse  $V(\Gamma)$  in  $\Gamma$  to obtain the rose with countably-many petals, which is now a pointed profinite graph encoding the edges of  $\Gamma$ .

**Lemma 2.6.** *Every profinite graph is the inverse limit of its finite quotient graphs.*

*Proof.* Let  $\Gamma$  be a profinite graph and let  $\mathcal{R}$  be the collection of all open equivalence relations on  $\Gamma$ , so  $\Gamma/R$  is finite for each  $R \in \mathcal{R}$ . Order  $\mathcal{R}$  by reverse inclusion, so  $(\mathcal{R}, \leq)$  is directed by taking common refinements. If  $R_1 \geq R_2$ , then there is a continuous map  $\varphi_{R_1, R_2} : \Gamma/R_1 \rightarrow \Gamma/R_2$  sending  $mR_1 \mapsto mR_2$ .

**Claim.** *We have  $\Gamma \cong \lim_{R \in \mathcal{R}} \Gamma/R$  as topological spaces.*

*Proof.* Let  $\psi : \Gamma \rightarrow \lim_{R \in \mathcal{R}} \Gamma/R$  be the map induced by the quotients  $p_R : \Gamma \rightarrow \Gamma/R$ , so  $\psi$  is surjective. If  $m, m' \in \Gamma$ , then there is a clopen subset  $U \subseteq \Gamma$  such that  $m \in U \not\supseteq m'$ , so the equivalence relation on  $X$  with classes  $\{U, U^c\}$  separates  $m$  and  $m'$  in the quotient.  $\square$

Now, let  $\mathcal{R}_0 \subseteq \mathcal{R}$  be the subcollection of those equivalence relations  $R \in \mathcal{R}$  such that  $\Gamma/R$  admits a graph structure and  $p_R : \Gamma \rightarrow \Gamma/R$  is a morphism. We show that  $\Gamma \cong \lim_{R \in \mathcal{R}_0} \Gamma/R$ , for which it suffices to show that  $\mathcal{R}_0$  is cofinal in  $\mathcal{R}$ , so let  $R \in \mathcal{R}$ . The projection  $p_R$  induces a map  $\tilde{p}_R : \Gamma \rightarrow \Gamma/R \times \Gamma/R \times \Gamma/R$  sending  $m$  to  $(p_R(m), p_R(d_0 m), p_R(d_1 m))$ , whose image admits a unique graph structure making  $\tilde{p}_R$  a morphism by Lemma 2.4. Thus  $\tilde{p}_R(\Gamma) \cong \Gamma/R_0$  for some  $R_0 \in \mathcal{R}_0$ , whose equivalence classes are of the form  $\tilde{p}_R^{-1}(x)$  for  $x \in \tilde{p}_R(\Gamma)$ , and  $R_0 \geq R$  since if  $\tilde{p}_R(m) = \tilde{p}_R(m')$ , then in particular  $p_R(m) = p_R(m')$ .  $\blacksquare$

To define the associated sequence of a profinite graph  $\Gamma$ , we consider *profinite  $A$ -modules* over a profinite commutative ring  $A$  (see, for instance, [Ser97] or [RZ10]). The *free profinite  $A$ -module* over a profinite space  $X := \lim_i X_i$  is the profinite  $A$ -module  $A[[X]] := \lim_i A[X_i]$ , which satisfies the usual universal property: every continuous map  $\varphi : X \rightarrow M$  to a profinite  $A$ -module  $M$  extends uniquely to a homomorphism  $\bar{\varphi} : A[[X]] \rightarrow M$  of profinite  $A$ -modules. Letting  $\partial : A[[E^*(\Gamma, *)]] \rightarrow A[[V(\Gamma)]]$  and  $\varepsilon : A[[V(\Gamma)]] \rightarrow A$  extend  $e^* \mapsto d_1(e) - d_0(e)$  and  $v \mapsto 1$ , where  $e^* \in E^*(\Gamma, *)$  is the image of  $e \in E(\Gamma)$  in  $\Gamma/V(\Gamma)$ , the sequence

$$0 \longrightarrow A[[E^*(\Gamma, *)]] \xrightarrow{\partial} A[[V(\Gamma)]] \xrightarrow{\varepsilon} A \longrightarrow 0$$

is the *associated sequence* of a profinite graph  $\Gamma$ , with homology groups  $H_i(\Gamma, A)$  defined in the obvious way.

**Definition 2.7.** A profinite graph is  *$A$ -connected* if  $H_0(\Gamma, A) = 0$  and  *$A$ -acyclic* if  $H_1(\Gamma, A) = 0$ .

A morphism  $\varphi : \Gamma_1 \rightarrow \Gamma_2$  of profinite graphs induces homomorphisms  $\varphi_V : A[[V(\Gamma_1)]] \rightarrow A[[V(\Gamma_2)]]$  and  $\varphi_E : A[[E^*(\Gamma_1, *)]] \rightarrow A[[E^*(\Gamma_2, *)]]$  of profinite  $A$ -modules such that

$$\begin{array}{ccccc} A[[E^*(\Gamma_1, *)]] & \xrightarrow{\partial^{\Gamma_1}} & A[[V(\Gamma_1)]] & \xrightarrow{\varepsilon^{\Gamma_1}} & A \\ \downarrow \varphi_E & & \downarrow \varphi_V & & \parallel \\ A[[E^*(\Gamma_2, *)]] & \xrightarrow{\partial^{\Gamma_2}} & A[[V(\Gamma_2)]] & \xrightarrow{\varepsilon^{\Gamma_2}} & A \end{array}$$

commutes, so  $\varphi$  induce homomorphisms  $H_i(\Gamma_1, A) \rightarrow H_i(\Gamma_2, A)$  for  $i = 0, 1$ . Thus  $H_i(-, A)$  are functors, and a diagram chase show that both  $H_i(-, A)$  preserve limits,  $H_0(-, A)$  preserve epimorphisms, and  $H_1(-, A)$  preserve monomorphisms. This gives us the following results, which we will quote without further reference.

1. A profinite graph is  $A$ -connected iff all of its finite quotients are connected in the combinatorial sense. In particular,  $A$ -connectedness is independent of the ring  $A$ , so we have a notion of *profinite-connectedness*.
2. An inverse limit of  $A$ -acyclic finite graphs is  $A$ -acyclic. The converse fails precisely because the quotient of an  $A$ -acyclic graph need not be  $A$ -acyclic. It turns out that the dependence on  $A$  cannot be removed.
3. Every subgraph of an  $A$ -acyclic graph is a  $A$ -acyclic.

**Example 2.8.** Let  $G$  be a pro- $C$  group. If  $X \subseteq G$  is a topological generating set of  $G$  (i.e.,  $\overline{\langle X \rangle} = G$ ), then its profinite Cayley graph  $\Gamma(G, X)$  is profinitely-connected since  $\Gamma(G, X) \cong \lim_{N \trianglelefteq_o G} \Gamma(G/N, \pi_N(X))$ .

**2.2. Trees.** A profinite graph  $\Gamma$  is said to be an  $A$ -tree if  $\Gamma$  is profinitely-connected and  $A$ -acyclic. We give some lemmas justifying our usage of this term. Moreover, we show that under certain conditions on  $\mathcal{C}$ , the profinite Cayley graphs of the pro- $\mathcal{C}$  completions of free groups are  $\mathbb{Z}_{\mathcal{C}}$ -trees (see Theorem 2.12).

**Lemma 2.9.** *Let  $\{\Gamma_\alpha\}$  be a family of profinitely-connected subgraphs of an  $A$ -tree  $\Gamma$ . The intersection  $\bigcap_\alpha \Gamma_\alpha$  is an  $A$ -tree, and if  $\bigcap_\alpha \Gamma_\alpha \neq \emptyset$ , then their union  $\bigcup_\alpha \Gamma_\alpha$  is an  $A$ -tree too.*

*Proof.* Since  $\Gamma$  is an  $A$ -tree, it suffices to show that both  $\bigcap_\alpha \Gamma_\alpha$  and  $\bigcup_\alpha \Gamma_\alpha$  are profinitely-connected. Indeed, for the first, observe that  $A[V(\bigcap_\alpha \Gamma_\alpha)] = \bigcap_\alpha A[V(\Gamma_\alpha)]$  and  $A[E^*(\bigcap_\alpha \Gamma_\alpha, *)] = \bigcap_\alpha A[E^*(\Gamma_\alpha, *)]$ , so letting

$$0 \longrightarrow A[E^*(\Gamma, *)] \xrightarrow{\partial} A[V(\Gamma)] \xrightarrow{\varepsilon} A \longrightarrow 0$$

be the associated sequence of  $\Gamma$ , which is exact by hypothesis, we have

$$\ker(\varepsilon|_{\bigcap_\alpha \Gamma_\alpha}) = A[V(\bigcap_\alpha \Gamma_\alpha)] \cap \ker \varepsilon = \bigcap_\alpha A[V(\Gamma_\alpha)] \cap \ker \varepsilon = \bigcap_\alpha \ker(\varepsilon|_{\Gamma_\alpha}),$$

and similarly  $\text{im}(\partial|_{\bigcap_\alpha \Gamma_\alpha}) \subseteq \bigcap_\alpha \text{im}(\partial|_{\Gamma_\alpha})$ . Since  $\partial$  is injective, the preceding inclusion is an equality. Each  $\Gamma_\alpha$  is profinitely-connected, so  $\ker(\varepsilon|_{\Gamma_\alpha}) = \text{im}(\partial|_{\Gamma_\alpha})$  for each  $\alpha$ , and hence  $\ker(\varepsilon|_{\bigcap_\alpha \Gamma_\alpha}) = \text{im}(\partial|_{\bigcap_\alpha \Gamma_\alpha})$ .

For the second claim, let  $\alpha : \bigcup_\alpha \Gamma_\alpha \twoheadrightarrow \Delta$  be a morphism onto a finite graph  $\Delta$ , so for each  $\alpha$ , the image  $\alpha(\Gamma_\alpha) \subseteq \Delta$  is connected. Note that  $\bigcap_\alpha \alpha(\Gamma_\alpha) \supseteq \alpha(\bigcap_\alpha \Gamma_\alpha) \neq \emptyset$ , so  $\Delta = \bigcup_\alpha \alpha(\Gamma_\alpha)$  is connected too. ■

In view of the above lemma, for any two vertices  $v$  and  $w$  of an  $A$ -tree  $\Gamma$ , the intersection of all profinitely-connected subgraphs of  $\Gamma$  containing  $v$  and  $w$  is an  $A$ -tree; it is the *geodesic* between  $v$  and  $w$ , denoted  $[v, w]$ .

**Lemma 2.10.** *Let  $\Gamma$  be an  $A$ -tree. A subgraph  $\Delta \subseteq \Gamma$  is an  $A$ -tree iff  $[v, w] \subseteq \Delta$  for all  $v, w \in V(\Delta)$ .*

*Proof.* The forward direction is clear. Conversely, write  $\Gamma \cong \lim_i \varphi_i(\Gamma)$  as an inverse limit of its finite quotient graphs, so  $\Delta \cong \lim_i \varphi_i(\Delta)$ . We claim that each  $\varphi_i(\Delta)$  is connected, so  $\Delta$  is an  $A$ -tree as desired.

Indeed, take  $\bar{v}, \bar{w} \in V(\varphi_i(\Delta))$ , which are projections of some  $v, w \in V(\Delta)$ , so  $[v, w] \subseteq \Delta$  by hypothesis. Since  $[v, w]$  is profinitely-connected, its image  $\varphi_i([v, w])$  is a connected subgraph of  $\varphi_i(\Delta)$  containing  $\bar{v}$  and  $\bar{w}$ , so  $\varphi_i(\Delta)$  is connected. ■

**Lemma 2.11.** *Let  $\Gamma$  be a profinitely-connected profinite graph and let  $\Delta \subseteq \Gamma$  be an  $A$ -subtree of  $\Gamma$ . Then the map  $H_1(\Gamma, A) \rightarrow H_1(\Gamma/\Delta, A)$  induced from the collapsing map  $\Gamma \twoheadrightarrow \Gamma/\Delta$  is an isomorphism.*

*In particular, if  $\Gamma$  is an  $A$ -tree, then so is  $\Gamma/\Delta$ .*

*Proof.* Let  $i : \Delta \hookrightarrow \Gamma$  and consider the following commutative diagram of profinite  $A$ -modules, whose rows are exact since  $\Delta$  is an  $A$ -tree and  $\Gamma$  and  $\Gamma/\Delta$  are both connected. Since  $p_E : A[E^*(\Gamma, *)] \rightarrow A[E^*(\Gamma/\Delta, *)]$  has kernel  $i_E(A[E^*(\Delta, *)])$ , the first column is exact; similarly for the second column.

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A[E^*(\Delta, *)] & \xrightarrow{d^\Delta} & A[V(\Delta)] & \xrightarrow{\varepsilon^\Delta} & A \longrightarrow 0 \\ & & \downarrow i_E & & \downarrow i_V & & \parallel \\ & & A[E^*(\Gamma, *)] & \xrightarrow{d^\Gamma} & A[V(\Gamma)] & \xrightarrow{\varepsilon^\Gamma} & A \longrightarrow 0 \\ & & \downarrow p_E & & \downarrow p_V & & \parallel \\ & & A[E^*(\Gamma/\Delta, *)] & \xrightarrow{d^{\Gamma/\Delta}} & A[V(\Gamma/\Delta)] & \xrightarrow{\varepsilon^{\Gamma/\Delta}} & A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

It can be easily seen by diagram chasing (or, by applying a generalization of the Nine Lemma) that  $\ker d^\Gamma \rightarrow \ker d^{\Gamma/\Delta}$  is an isomorphism. ■

Recall that the Cayley graph of any free group of finite rank is a tree (or, in the homological language above, a  $\mathbb{Z}$ -tree). It is thus natural to ask whether the profinite Cayley graph of the pro- $\mathcal{C}$  completion of any free group of finite rank is a  $\mathbb{Z}_{\mathcal{C}}$ -tree. Let us call a pseudovariety  $\mathcal{C}$  *arborescent* [AW94] if this is the case.

**Theorem 2.12.** *A pseudovariety  $\mathcal{C}$  is arborescent iff  $\mathcal{C}$  is closed under extensions with abelian kernel.*

The converse direction, which is what we need for Theorem A, was first proved in [GR78, Theorem 1.2]; for the forward direction and the consequences of this equivalence, see [AW94, Theorem 2.1].



*Proof of Theorem 2.12 ( $\Leftarrow$ ).* Let  $F$  be a free group of finite rank and consider the profinite Cayley graph  $\Gamma := \Gamma(F_C)$  of the pro- $\mathcal{C}$  completion of  $F$ , with the standard generating set  $X \subseteq F$ . Observe that  $E(\Gamma) = F_C \times X$  is closed in  $\Gamma$ , so we can identify  $E(\Gamma)$  with  $E^*(\Gamma, *)$ . Since  $\Gamma \cong \lim_{N \trianglelefteq_o F_C} \Gamma(F_C/N, \pi_N(X))$  and each  $\Gamma(F_C/N, \pi_N(X))$  is connected in the combinatorial sense, we see that  $\Gamma$  is profinitely-connected.

Thus it suffices to show that  $H_1(\Gamma, \mathbb{Z}_C) = 0$ , or, in other words, that the map  $\partial : \mathbb{Z}_C[F_C \times X] \rightarrow \mathbb{Z}_C[F_C]$  is a bijection onto the *augmentation ideal*  $I := \text{im } \partial = \ker \varepsilon$  of  $\varepsilon : \mathbb{Z}_C[F_C] \rightarrow \mathbb{Z}_C$ . To this end, let us make  $\mathbb{Z}_C[F_C]$  into a *profinite  $\mathbb{Z}_C$ -algebra* by taking the continuous linear extension of the multiplication on  $F_C$ , so it is in particular a profinite commutative ring. Observe then that  $\mathbb{Z}_C[F_C \times X]$  is a free profinite  $\mathbb{Z}_C[F_C]$ -module on  $\{1\} \times X$ , where  $\mathbb{Z}_C[F_C]$  acts on  $\mathbb{Z}_C[F_C \times X]$  by extending the left-multiplication action of  $F_C$  on  $F_C \times X$ . Since  $\partial(1, x) = x - 1$ , it suffices to show that  $I$  is freely-generated by  $\iota(X)$  as a profinite  $\mathbb{Z}_C[F_C]$ -module, where  $\iota : X \rightarrow M$  sends  $\iota(x) := x - 1$ .

**Claim.** *The augmentation ideal  $I$  is the profinite  $\mathbb{Z}_C[F_C]$ -module generated by  $\iota(X)$ .*

*Proof.* Since  $F_C = \lim_{N \in \mathcal{N}_C(F_C)} F_C/N$  and each  $F_C/N$  is a finite group generated by  $X$ , it suffices to prove that for a finite group  $G$ , the augmentation ideal  $\ker(\mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z})$  is generated by  $\{x - 1 : x \in X\}$ .

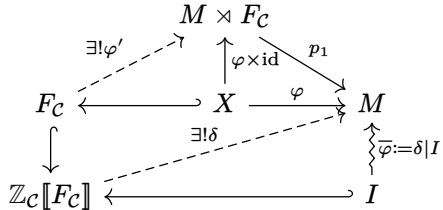
Indeed, if  $\sum_{g \in G} a_g = 0$  for  $a_g \in \mathbb{Z}$ , then for any fixed  $x \in X$ , we have  $a_x = -\sum_{g \neq x} a_g$ , and so

$$\sum_g a_g g = a_x x + \sum_{g \neq x} a_g g = \sum_{g \neq x} a_g (g - x) = \sum_{g \neq x} a_g x (x^{-1}g - 1) \in \langle h - 1 : h \in G \rangle.$$

Now write each  $h \in G$  as  $h = x_1 \cdots x_n$  for  $x_i \in X$ , so that  $h - 1 = \sum_{i \leq n} x_1 \cdots x_{i-1} (x_i - 1)$ .  $\square$

Thus it remains to show that  $I$  is freely-generated as a profinite  $\mathbb{Z}_C[F_C]$ -module by  $\iota(X)$ , so let  $\varphi : X \rightarrow M$  be a continuous map to a profinite  $\mathbb{Z}_C[F_C]$ -module  $M$ ; we need to show that there is a continuous  $\mathbb{Z}_C[F_C]$ -module homomorphism  $\bar{\varphi} : I \rightarrow M$  such that  $\bar{\varphi}\iota = \varphi$ , which is then unique by the claim.

Since  $M$  is a profinite  $\mathbb{Z}_C[F_C]$ -module, we have, in particular, a continuous action of  $F_C$  on  $M$ , and hence a semidirect product  $M \rtimes F_C$  such that the map  $\varphi \times \text{id} : X \rightarrow M \rtimes F_C$  is continuous. Now  $M$  is abelian as a pro- $\mathcal{C}$  group, so as  $\mathcal{C}$  is closed under extensions of abelian kernels, we see from an inverse limit argument that  $M \rtimes F_C$  is a pro- $\mathcal{C}$  group. Hence  $\varphi \times \text{id} : X \rightarrow M \rtimes F_C$  extends to a unique continuous group homomorphism  $\varphi' : F_C \rightarrow M \rtimes F_C$ , so the composite  $\delta_0 := p_1 \varphi' : F_C \rightarrow M$  is a continuous map such that  $\varphi'(x) = (\delta_0(x), x)$  for all  $x \in X$ . We have  $\delta_0(xy) = \delta_0(x) + x\delta_0(y)$  for all  $x, y \in X$ ; since  $F$  is topologically generated by  $X$  and  $\delta_0$  is continuous, we see that  $\delta_0$  is a *derivation* in the sense that  $\delta_0(gh) = \delta_0(g) + g\delta_0(h)$  for all  $g, h \in F_C$ .



All maps relevant to the construction of  $\bar{\varphi} : I \rightarrow M$ , fitted into a commutative diagram; the hooked arrows are subsets and the dashed arrows are induced by universal properties.

Let  $\delta : \mathbb{Z}_C[F_C] \rightarrow M$  be the unique continuous  $\mathbb{Z}_C$ -module homomorphism extending  $\delta_0$  and let  $\bar{\varphi} := \delta|_I$ , so we have  $\bar{\varphi}(x - 1) = \delta(x - 1) = \delta_0(x) - \delta_0(1) = \delta_0(x) = \varphi(x)$  for all  $x \in X$ . We claim that  $\bar{\varphi}$  is  $\mathbb{Z}_C[F_C]$ -linear, so it is the desired continuous  $\mathbb{Z}_C[F_C]$ -module homomorphism extending  $\varphi$ . Indeed, this follows since the set  $\mathbb{Z}_C[F_C]$  of all finite  $\mathbb{Z}_C$ -linear combinations of  $F_C$  is dense in  $\mathbb{Z}_C[F_C]$ , so by continuity, it suffices to observe that  $\delta(\alpha\beta) = \varepsilon(\beta)\delta(\alpha) + \alpha\delta(\beta)$  for all  $\alpha, \beta \in \mathbb{Z}_C[F_C]$ ; in particular,  $\beta := x - 1 \in I = \ker \varepsilon$  for all  $x \in X$ .  $\blacksquare$

**2.3. Actions.** Lastly, we need to study the behaviour of actions of a profinite group  $G$  on a profinite graph  $\Gamma$ , which are continuous actions  $G \curvearrowright \Gamma$  such that  $d_i(gm) = gd_i(m)$  for each  $m \in \Gamma$  and  $i = 0, 1$ .

**Lemma 2.13.** *If  $G \curvearrowright \Gamma$ , then  $\Gamma \cong \lim_i \Gamma_i$  where each  $\Gamma_i$  is a finite quotient  $G$ -graph of  $\Gamma$ .*

*Proof.* Proceed as in Lemma 2.6, taking  $\mathcal{R}_0$  to be the open  $G$ -invariant equivalence relations on  $\Gamma$ , so that  $p_R : \Gamma \rightarrow \Gamma/R$  is a  $G$ -map of  $G$ -spaces. Then  $\mathcal{R}_0$  is cofinal in the collection  $\mathcal{R}$  of all open equivalence relations on  $\Gamma$ , so that  $\Gamma \cong \lim_{R \in \mathcal{R}_0} \Gamma/R$  is as desired.  $\blacksquare$

**Corollary 2.14.** *Let  $G \curvearrowright \Gamma$  and suppose that  $G = \overline{\langle X \rangle}$  for some  $X \subseteq G$ . If  $\Delta \subseteq \Gamma$  is a subgraph such that  $\Delta \cap x\Delta \neq \emptyset$  for all  $x \in X$ , then  $G\Delta := \bigcup_{g \in G} g\Delta$  is profinitely-connected.*

*Proof.* Writing  $\Gamma \cong \lim_i \varphi_i(\Gamma)$  as an inverse limit of its finite quotient  $G$ -graphs, we have  $G\Delta = \lim_i G\varphi_i(\Delta)$  and  $\varphi_i(\Delta) \cap x\varphi_i(\Delta) \neq \emptyset$  for all  $x \in X$ , so we can assume without loss of generality that  $\Gamma$  is finite.

In this case, the kernel of  $G \rightarrow \text{Aut}(\Gamma)$  is an open normal subgroup of  $G$ , so after passing to the quotient, we may assume that  $G$  is finite. Now both  $G$  and  $\Gamma$  are finite, and the result is immediate. ■

Finally, we will need the existence of connected transversals of  $G \curvearrowright \Gamma$ . If  $\Gamma$  is finite, then such transversals always exist (see, for instance, [DD89]). This is not the case in general, but if  $G$  acts freely on  $\Gamma$  with finite quotient, then the existence of transversals follow from the same argument as in the finite case.

**Lemma 2.15.** *Let  $G$  be a profinite group acting on a profinite graph  $\Gamma$  and fix a vertex  $v_0 \in V(\Gamma)$ . If  $\Gamma/G$  is finite and  $G \curvearrowright \Gamma$  freely, then there is a (finite) connected transversal  $\Sigma \subseteq \Gamma$  of the action containing  $v_0$ .*

*Proof.* Let  $\mathcal{T}$  be the set of all finite subtrees  $T_0 \subseteq \Gamma$  containing  $v_0$  such that  $T_0 \hookrightarrow \Gamma \twoheadrightarrow \Gamma/G$  is an injection, and let  $T \in \mathcal{T}$  be maximal with respect to inclusion. Clearly  $T' := p(T)$  is a subtree of  $\Gamma/G$ .

We claim that  $T'$  is a spanning subtree of  $\Gamma/G$ . If not, then since  $\Gamma/G$  is finite and  $T'$  is connected, there is an edge  $e' \in \Gamma/G - T'$  such that  $d_i(e') \in T'$  but  $d_{1-i}(e') \notin T'$  for some  $i = 0, 1$ , say with  $i = 0$ . Choose  $v \in V(T)$  and  $e_0 \in E(T)$  such that  $p(v) = d_0(e')$  and  $p(e) = e'$ . Observe that  $p(v) = p(d_0(e))$ , so under the action of  $G$ , we can assume that  $v = d_0(e)$ . But  $T \sqcup \{e, d_1(e)\} \in \mathcal{T}$ , which contradicts the maximality of  $T$ .

Set  $\Sigma := d_0^{-1}(V(T'))$ , so clearly  $v_0 \in T \subseteq \Sigma$ . It remains to show that  $\Sigma$  is a transversal of  $G \curvearrowright \Gamma$ . Since  $p|_T : T \rightarrow T'$  is a bijection and  $V(T') = V(\Gamma/G)$ , it suffices to show that for each edge  $e' \in \Gamma/G - T'$ , there is a unique edge  $e \in \Sigma - T$  such that  $p(e) = e'$ . Indeed, let  $e \in E(\Gamma)$  project to  $e'$ , which under the action of  $G$  can be chosen so that  $d_0(e) \in V(T)$ , and hence  $e \in \Sigma - T$ . If  $e_1, e_2 \in \Sigma - T$  both project to  $e'$ , then  $e_2 = ge_1$  for some  $g \in G$ . But  $p(d_0(e_i)) = d_0(p(e_i)) = d_0(e') \in V(T')$  for both  $i = 1, 2$ , and since  $d_0(e_i) \in V(T)$ , we see from injectivity of  $p|_T$  that  $d_0(e_1) = d_0(e_2)$ , and hence  $d_0(e_1) = d_0(e_2) = d_0(ge_1) = gd_0(e_1)$ . This forces  $g = 1$  by freeness of the action, and hence  $e_1 = e_2$  as desired. ■

### 3. PROOF OF THE RIBES-ZALESSKII THEOREM

In this section, we prove the Ribes-Zalesskii Theorem for pro- $\mathcal{C}$  free groups, where  $\mathcal{C}$  is a pseudovariety of groups closed under extensions (Theorem A). Let  $F$  be a free group of finite rank. By Fact 1.1,  $F$  is residually- $\mathcal{C}$ , so  $F$  embeds into its pro- $\mathcal{C}$  completion  $F_{\mathcal{C}}$ . This embedding extends to an embedding  $\Gamma(F) \hookrightarrow \Gamma(F_{\mathcal{C}})$  of profinite graphs, where  $\Gamma(F_{\mathcal{C}})$  is a  $\mathbb{Z}_{\mathcal{C}}$ -tree by Theorem 2.12.

Let  $H, K \leq F$  be finitely generated subgroups which are closed in the pro- $\mathcal{C}$  topology of  $F$ . We will show that the double coset  $HK$  is also closed in the pro- $\mathcal{C}$  topology by studying actions on the profinite Cayley graph  $\Gamma(F_{\mathcal{C}})$ . To this end, we make some useful reductions, which require the following lemma.

**Lemma 3.1.** *If  $K \leq_c F$  is a finitely generated closed subgroup of  $F$ , then there is an open subgroup  $U \leq_o F$  containing  $K$  such that  $K$  is a free factor of  $U$ .*

*Proof.* Let  $\Gamma$  be the Cayley graph of  $F$  with respect to a fixed basis. By Lemma 1.3, we can write  $K = \bigcap_i U_i$  as the intersection of all open subgroups  $U_i \leq_o F$  containing  $K$ . For each  $i$ , let  $p_i : \Gamma/K \rightarrow \Gamma/U_i$  be the natural quotient map. Note that for any finite  $\Delta \subseteq \Gamma/K$ , there is some  $i$  such that  $p_i|_{\Delta}$  is injective.

Since  $K$  is finitely generated, the fundamental group of  $\Gamma/K$  is supported on a finite subgraph  $\Delta \subseteq \Gamma/K$ . Choosing  $i$  so that  $\Delta \hookrightarrow \Gamma/U_i$ , we see that  $K \cong \pi_1(\Gamma/K) \cong \pi_1(\Delta)$  is a free factor of  $\pi_1(\Gamma/U_i) \cong U_i$ . ■

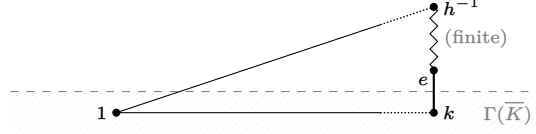
**Claim.** *We can assume that (1)  $K$  is a free factor of  $F$ , (2)  $K$  is  $\mathcal{C}$ -compatible with  $F$ , and (3)  $\overline{K} = K_{\mathcal{C}}$ .*

*Proof.* Applying Lemma 3.1 so that  $K$  is a free factor of an open subgroup  $U \leq_o F$ , we see that  $K$  is  $\mathcal{C}$ -compatible with  $U$  and  $U$  is  $\mathcal{C}$ -compatible with  $F$  by Lemmas 1.4 and 1.5, respectively. Let  $\{h_i\}$  be a finite transversal of  $H \cap U$  in  $H$ , so that  $HK = \bigsqcup_i h_i(H \cap U)K$  is closed in the pro- $\mathcal{C}$  topology of  $U$  since  $H \cap U$  is still finitely generated by Howson's Theorem (see [Sta83, Corollary 5.6]). □

In particular, we can assume that the following diagram on the left commutes, so that  $\Gamma(K) = \Gamma(\overline{K}) \cap \Gamma(F)$  by Corollary 1.7. Now, we start the actual proof. By Corollary 1.7 again, it suffices to show that  $\overline{HK} \cap F = HK$ . Since  $F_{\mathcal{C}}$  is compact, so are  $\overline{H}$  and  $\overline{K}$ , and hence  $\overline{H} \times \overline{K}$  is compact too by Tychonoff's Theorem. By continuity of multiplication, we see that  $\overline{HK}$  is compact too, hence closed, so  $\overline{HK} \subseteq \overline{H} \overline{K}$ . Thus, overall, it suffices to show that  $\overline{H} \overline{K} \cap F \subseteq HK$ . Take  $h \in \overline{H}$  and  $k \in \overline{K}$  such that  $hk \in F$ . We need to show that  $hk \in HK$ .

Since  $hk \in F$ , the geodesic  $[1, hk]$  in  $\Gamma(F_C)$  is a finite path, so  $[h^{-1}, k] = h^{-1}[1, hk]$  is a finite path as well. Let  $k_0 \in [h^{-1}, k]$  be closest to  $h^{-1}$  such that  $k_0 \in \Gamma(\overline{K})$ . If  $k_0 = h^{-1}$ , then  $h \in \overline{K}$ , so  $hk \in \overline{K} \cap F = K \subseteq HK$  and we are done. Otherwise, observe that  $k_0 k^{-1} \in K$  since  $[k_0, k] \subseteq \Gamma(\overline{K})$  is finite, so  $hk \in HK$  iff  $hk_0 \in HK$ . Hence, we can assume without loss of generality that  $k = k_0$ .

$$\begin{array}{ccc} \Gamma(F) & \hookrightarrow & \Gamma(\overline{F}) = \Gamma(F_C) \\ \uparrow & & \uparrow \\ \Gamma(K) & \hookrightarrow & \Gamma(\overline{K}) = \Gamma(K_C) \end{array}$$



(a) We can regard  $\Gamma(K)$  as a subgraph of  $\Gamma(F_C)$ , and  $\Gamma(K) \subseteq \Gamma(\overline{K}) \cap \Gamma(F) = \Gamma(\overline{K} \cap F)$  is an equality.

(b) Collapsing the two geodesics induces a cycle since  $e$  lies outside said geodesics and  $[h^{-1}, k]$  is finite.

**Claim.** *There is a free action of  $\overline{H} \cap \overline{K}$  on a profinite graph  $\Delta \subseteq \Gamma(\overline{K})$  containing  $k$  with finite quotient.*

*Proof.* Let  $\Lambda := \bigcup_{i=1}^n \overline{H}[1, h_i]$  where  $h_1, \dots, h_n$  are the free generators of  $H$ , and set  $\Delta := \Gamma(\overline{K}) \cap \Lambda$  so that  $\overline{H} \cap \overline{K}$  acts freely on  $\Delta$  by left-multiplication. Observe that  $\Lambda$  is  $\overline{H}$ -invariant and the quotient map  $p : \Lambda \rightarrow \Lambda/\overline{H}$  induces a map  $\tilde{p} : \Delta/(\overline{H} \cap \overline{K}) \rightarrow p(\Lambda)$ , which we claim is injective. Indeed, if  $t_2 = t_1 x$  for some  $x \in \overline{H}$  and  $t_1, t_2 \in \Delta$ , then the vertices of  $t_i$  lie in  $V(\Gamma(\overline{K}))$ , so  $t_2 = t_1 x'$  for some  $x' \in \overline{K}$ , which forces  $x = x' \in \overline{H} \cap \overline{K}$  by freeness. In particular, since  $p(\Lambda)$  is finite, so is  $\Delta/(\overline{H} \cap \overline{K})$ .

By Corollary 2.14,  $\Lambda$  is a  $\mathbb{Z}_C$ -tree since  $\Lambda = \bigcup_{h \in \overline{H}} h\Lambda_0$  and  $\Lambda_0 \cap h_i\Lambda_0 \neq \emptyset$ , where  $\Lambda_0 := \bigcup_{i=1}^n [1, h_i]$ . We claim that  $k \in [h^{-1}, 1]$ , so that  $k \in [h^{-1}, 1] \subseteq \Lambda$  by Lemma 2.10. Suppose otherwise, so there is an edge  $e \in [h^{-1}, k] \setminus [h^{-1}, 1]$  such that  $e \notin \Gamma(\overline{K})$  but  $k$  is a vertex of  $e$ . Then, collapsing the  $\mathbb{Z}_C$ -subtree  $[h^{-1}, 1] \cup [1, k]$  of the  $\mathbb{Z}_C$ -tree  $[h^{-1}, 1] \cup [1, k] \cup [h^{-1}, k]$  (see Lemma 2.9) induces a cycle at  $h \sim k$  since  $[1, k] \subseteq \Gamma(\overline{K})$  and  $e \notin [h^{-1}, 1] \cup [1, k]$ , which contradicts Lemma 2.11.  $\square$

By Lemma 2.15, there is a finite connected transversal  $\Sigma \subseteq \Delta$  containing 1. Since  $k \in \Delta$ , there is an element  $g \in \overline{H} \cap \overline{K}$  such that  $gk \in \Sigma$ . But  $\Sigma$  is finite and connected, so  $\Sigma \subseteq \Gamma(F)$ , and hence  $\Sigma \subseteq \Gamma(\overline{K}) \cap \Gamma(F) = \Gamma(K)$ . Thus we have  $gk \in K$ , so  $hg^{-1} \in \overline{H} \cap F = H$  by Corollary 1.7, and hence  $hk = (hg^{-1})(gk) \in HK$ .  $\blacksquare$

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