Bounded Arithmetic, Constant-depth Frege Proofs, and the Linear Time Hierarchy

COMP532 – Propositional Proof Complexity

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A weak subsystem of Peano Arithmetic

Let $L_{\mathrm{PA}} := \{0, +, \cdot, S, \leq\}$ be the first-order language of PA, whose axioms are the 'basic axioms' $\textsc{Basic}$ together with an axiom scheme $\textsc{Ind}$, consisting of the induction axioms

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Definition

The L_{PA}-theory $I\Delta_0$ is the subtheory of PA axiomatized by Basic and the restriction of Ind to Δ_0-formulas.

Polynomial Translation for $I\Delta_0$

Theorem (Paris-Wilkie, 1985; Krajíček, 1995)

Let R be a binary relation symbol and let $\varphi(x)$ be $\Delta_0(R)$-formula. If $I\Delta_0(R) \vdash \forall x \varphi(x)$, then there is $d \in \mathbb{N}$ such that $S_{\mathsf{F}_d}(\langle\varphi\rangle_n) = \mathsf{poly}(n)$.

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Theorem (Cook, 1975)

Let $\varphi(x)$ be a Σ^b_1-formula. If $S^1_2 \vdash \forall x \varphi(x)$, then $S_{\mathsf{EF}}(\|\varphi\|^n) = \mathsf{poly}(n)$.

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- $I\Delta_0 \leftrightarrow$ Functions in the Linear Time Hierarchy LTH.

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We call φ a *defining formula* for f, and say that f is *provably total in* T if it is Σ_1-definable function in T.

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Theorem (Buss's Witnessing Theorem, 1986)

For any function $f : \mathbb{N}^k \to \mathbb{N}$, the following are equivalent.

1. *f is Σ_1^b-definable in S_2^1, i.e., f is provably total in S_2^1.*
2. *There is a Σ_1^b-defining L-formula $\varphi(\overline{x}, y)$ for f and an L-term $t(\overline{x})$ such that $S_2^1 \vdash \forall \overline{x}\, \exists! y \leq t(\overline{x})\, \varphi(\overline{x}, y)$.*
3. *f is polynomial-time computable.*

The Engine of $I\Delta_0$: Parikh's Theorem

Theorem (Parikh, 1971)

Let $\varphi(\overline{x}, y)$ be a bounded formula. If $I\Delta_0 \vdash \forall\overline{x}\exists y\, \varphi(\overline{x}, y)$, then there is an L_{PA}-term $t(\overline{x})$ such that $I\Delta_0 \vdash \forall\overline{x}\exists y \leq t(\overline{x})\, \varphi(\overline{x}, y)$.

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so the theory $I\Delta_0 \cup \{\forall y \leq t(\overline{c})\, \neg\varphi(\overline{c}, y) : t \text{ term}\}$ is satisfiable by the Compactness Theorem, and thus has a model M. Define

$$N := \{a \in M : M \models a \leq t(\overline{c}), \text{ for some term } t\},$$

which models $I\Delta_0$.

The Engine of $I\Delta_0$: Parikh's Theorem

Theorem (Parikh, 1971)

Let $\varphi(\overline{x}, y)$ be a bounded formula. If $I\Delta_0 \vdash \forall\overline{x}\exists y\, \varphi(\overline{x}, y)$, then there is an L_{PA}-term $t(\overline{x})$ such that $I\Delta_0 \vdash \forall\overline{x}\exists y \leq t(\overline{x})\, \varphi(\overline{x}, y)$.

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which models $I\Delta_0$. But $N \models \exists\overline{x}\forall y \neg\varphi(\overline{x}, y)$, contradiction. ∎

Proof of the Witnessing Theorem for $I\Delta_0$, part 1

Theorem (Witnessing Theorem for $I\Delta_0$)

For any function $f : \mathbb{N}^k \to \mathbb{N}$, the following are equivalent.

1. *f is Σ_1-definable in $I\Delta_0$, i.e., f is provably total in $I\Delta_0$.*
2. *f is Δ_0-definable in $I\Delta_0$.*
3. *There is a Δ_0-defining formula $\varphi(\overline{x}, y)$ for f and a term $t(\overline{x})$ such that $I\Delta_0 \vdash \forall \overline{x}\, \exists! y \leq t(\overline{x})\, \varphi(\overline{x}, y)$.*
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The implications $3 \Rightarrow 2 \Rightarrow 1$ are clear.

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The Linear Time Hierarchy

Definition

$\mathsf{LTH} := \bigcup_{i=0}^{\infty} \Sigma_i^{\mathrm{lin}}$, where Σ_i^{lin} is the family of languages accepted in $O(n)$ time by an alternating Turing machine with i alternations.

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Proof of the Witnessing Theorem for $I\Delta_0$, part 2

Theorem ($3 \Leftrightarrow 4$)

For any $f : \mathbb{N}^k \to \mathbb{N}$, *there is a* Δ_0*-defining formula* $\varphi(\overline{x}, y)$ *for* f *and a term* $t(\overline{x})$ *such that* $I\Delta_0 \vdash \forall \overline{x}\, \exists! y \leq t(\overline{x})\, \varphi(\overline{x}, y)$ *iff* $f \in \mathsf{FLTH}$.

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References

Thank you!

- Buss S. R. (1986), *Bounded Arithmetic*, Naples, Bibliopolis. (Revision of the 1985 Princeton University Ph.D. thesis.)
- Buss S. R. (1999), *Bounded Arithmetic, Proof Complexity, and Two Papers of Parikh*, Annals of Pure and Applied Logic, **96**, pp. 43-55.
- Cook S. (1975), *Feasibly Constructive Proofs and the Propositional Calculus*, in: Proceedings of the 7th Annual ACM Symposium on Theory of Computing, pp. 83-97, ACM Press.
- Cook S. and Nguyen P. (2010), *Logical Foundations of Proof Complexity*, Perspectives in Logic, Cambridge University Press.
- Hájek P. and Pudlák P. (1993), *Metamathematics of First-order Arithmetic*, Perspectives in Mathematical Logic, Springer-Verlag.
- Krajíček J. (1995), *Bounded Arithmetic, Propositional Logic, and Complexity Theory*, Encyclopedia of Mathematics and its Applications, Cambridge University Press.
- Krajíček J. (1995), *On Frege and Extended Frege proof systems*, in: Feasible Mathematics II (Birkhäuser, Boston 1995), eds. P. Clote and J. Remmel, pp. 284-319.
- Parikh R. (1971), *Existence and Feasibility in Arithmetic*, J. Symbolic Logic, **36**, pp. 494-508.
- Paris J. and Wilkie A. (1985), *Counting Problems in Bounded Arithmetic*, in: Methods in Mathematical Logic (Caracas, Venezuela 1983), eds. A. Dold and B. Eckmann, LNM, vol. 1130, pp. 317-340, Springer-Verlag.
- Wrathall, C. (1978), *Rudimentary Predicates and Relative Computation*, SIAM J. Computing, **7**, pp. 194-209.