HILBERT'S FIFTH PROBLEM

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Abstract. Following [Tao14], we provide a sketch of Hilbert's Fifth Problem, which asks for a topological characterization of Lie groups. To do so, we prove (part of) a deep structural result of Gleason and Yamabe on locally-compact groups, stating that they contain an inverse limit of Lie groups as an open subgroup.

Introduction

The purpose of this note is to sketch a roadmap for a positive solution to *Hilbert's Fifth Problem*, which asks for a purely topological characterization of Lie groups without direct reference to a smooth structure. As the notion of a topological group has not yet been established at the time, this question can be interpreted in a number of ways, but the most common interpretation, and the one answered affirmatively by Montgomery-Zippin [MZ52] and Gleason [Gle52], is the following

Theorem A (Hilbert's Fifth Problem)**.** *Every locally-Euclidean group is Lie.*

This theorem exemplifies a broad principle: if a group-like structure exhibits some weak regularity, then it often automatically exhibits a strong regularity as well. However, it is not directly useful in applications as it is often difficult to verify that a group is locally-Euclidean. In contrast, the property of being *locally-compact*, which is implied by being locally-Euclidean (see Lemma 1.2), is much easier to verify, so it would be nice if all locally-compact groups are Lie. This is of course not the case, but it is not extremely far from the truth[1], as shown by a theorem of Gleason [Gle51], later refined and strengthened by Yamabe [Yam53]:

Theorem B (Gleason-Yamabe)**.** *For every locally-compact group G and any neighborhood U of the identity, there is an open subgroup G' of G and a compact normal subgroup $K \trianglelefteq G'$ in U such that G'/K is Lie.*

Roughly speaking this theorem states that locally-compact groups are 'essentially' Lie groups after ignoring the very large scales (by passing to an open subgroup) and the very small scales (by quotienting by a small compact subgroup). Since open subgroups of G are also closed, the conclusion can be simplified in the case when G is connected, in which case there is a compact normal subgroup $K \trianglelefteq G$ in U such that G/K is Lie.

The 'dual' and more important special case is when G *has no small subgroups* (abbreviated G *has NSS*), that is to say that there is a small neighborhood U of the identity containing no non-trivial subgroups of G; in this case, K is trivial and we conclude that G' is Lie. In fact, since we can transport the chart at $e \in G'$ and cover G, any locally-compact group G having NSS is automatically Lie. Thus, after proving Theorem A (and Lemma 1.3), we will see that the following are equivalent for any locally-compact group:

$$\text{Locally-Euclidean} \quad \Leftrightarrow \quad \text{NSS} \quad \Leftrightarrow \quad \text{Lie}.$$

In addition to providing a positive solution to Hilbert's Fifth Problem, the Gleason-Yamabe Theorem can also be used to classify *approximate groups*, prove *Gromov's Theorem* on groups of polynomial growth, and also prove *Freiman's Theorem* for non-abelian groups; see [Hru12] and [BGT12] for statements and proofs.

The solution to Hilbert's Fifth Problem also opened up Lie theory to the wider world of locally-compact groups and *pro-Lie groups*, which are inverse limits of Lie groups (for example, connected locally-compact groups; see Theorem 1.11). Indeed, Lashof [Las57] first recognized that every locally-compact group has a suitable 'Lie algebra' and is supplied with a well-behaved exponential map. This then led to an attempt of establishing a 'Lie theory' for pro-Lie groups, for which we refer the interested reader to [HM07].

Date: December 15, 2024.

A project presented in partial fulfillment of the requirements of McGill University's Fall 2024 MATH599: Topics in Geometry and Topology (Lie Groups and their Representations), by Professor Joel Kamnitzer.

[1]See [Tao14, Section 1.1.1] for a brilliant introduction and for a series of examples that naturally lead to this theorem.

Organization. The goal of Section 1 is two-fold. First, we introduce the notion of a topological group and state some preliminary results, and state without proof a version of the *Baker-Campbell-Hausdorff* formula (Theorem 1.6) that will be used to tie up some loose ends in Theorem B.

Second, using techniques from the theory of totally-disconnected groups (namely, *van Danztig's Theorem*), we prove a slight strengthening of Theorem B, which makes the open subgroup G' uniform in U (see Theorem 1.11). With some fun topological juggling, this allows us to then deduce Theorem A in Section 1.4.

The rest of the note is then devoted to proving (a part of) Theorem B, which has three major steps.

1. The first step is to find a *subquotient* (a quotient of an open subgroup) of G that is locally-compact and has NSS. Somewhat surprisingly, we will need the compact case of Theorem B for this step, which we establish via the *Peter-Weyl Theorem*. Once this is known, it remains to show that locally-compact NSS groups are Lie, so we henceforth assume that G is locally-compact and has NSS.
2. The second step uses that G has NSS to construct a so-called *Gleason-metric* (see Definition 2.5) on G. This can be done by convoluting low-regularity metrics to higher-regularity ones.
3. The existence of a Gleason metric on G 'linearizes' its local behaviour, and allows one to define a vector space of one-parameter subgroups and its associated exponential map that act as proxies for their Lie-theoretic counterparts. Using exponential coordinates, we then apply a version of the BCH formula in Section 1.2 to conclude that G is Lie.

Step 3 is carried out in detail in Section 3, which is the main focus on this note. It turns out that Steps 1 and 2 are substantially harder, so we will only sketch the rough ideas of the proofs in Section 2.

1. Preliminaries and Consequences of the Gleason-Yamabe Theorem

1.1. **Basic notions.** A *topological group* is a group (G, e, m, ι) equipped with a topology making the operations $m : G \times G \to G$ and $\iota : G \to G$ continuous. It is *locally-compact* if the identity e admits a compact (not necessarily open) neighborhood, and *has no small subgroups (has NSS)* if there is a neighborhood U of the identity containing no non-trivial subgroups of G. Lastly, it is *locally-Euclidean* if around every point $g \in G$, there is a neighborhood U of g and a homeomorphism $\varphi : U \to V$ onto an open subset $V \subseteq \mathbb{R}^d$ for some $d \in \mathbb{N}$; the number d is unique by invariance-of-domain and is called the *dimension* of G, denoted $\dim G$.

Lemma 1.1. *Every neighborhood U of the identity contains a symmetric neighborhood $V \subseteq U$ of the identity (i.e. $V = V^{-1}$) such that $V^2 \subseteq U$.*

Proof. First, $m^{-1}(U^\circ)$ is open and contains (e, e), so there are open subsets $W_1, W_2 \subseteq U$ such that $(e, e) \in W_1 \times W_2$. Set $W := W_1 \cap W_2$ so that $W^2 \subseteq U$. The neighborhood $V := W \cap W^{-1}$ is then as desired. ∎

Lemma 1.2. *Every locally-Euclidean group G is Hausdorff, locally-compact, and first-countable.*

Proof. Using a chart centered at e, we can pull back a compact neighborhood and a countable neighborhood base of the origin to ones in G, proving the latter two claims. For Hausdorffness, note that singletons are closed by the same argument, so given distinct $g, h \in G$, let U be a neighborhood containing e but not gh^{-1}. Letting V be as in Lemma 1.1, the translates $Vg \ni g$ and $Vh \ni h$ then separate g and h. ∎

We thus make the blanket assumption that all groups considered in this note are Hausdorff. Thus, if G is also first-countable, then it is metrizable by the Birkhoff-Kakutani Theorem.

Lemma 1.3. *Every Lie group G has NSS.*

Proof. Let $W \subseteq \mathfrak{g}$ and $U \subseteq G$ be open neighborhoods of the identity on which $\exp : W \to U$ is a homeomorphism, and shrink W so that $W \subseteq B_\varepsilon(0)$ for some $\varepsilon > 0$. Let $V := B_{\varepsilon/2}(0)$. We claim that $\exp(V)$ is as desired, so let $H \subseteq G$ be a subgroup contained in $\exp(V)$.

Fix $g \in H \subseteq \exp(V)$ so that $g = \exp(v)$ for some $v \in V$. But $g^2 \in H \subseteq \exp(V)$ too, so $g^2 = \exp(w)$ for some $w \in V$, and hence $\exp(w) = g^2 = \exp(v)^2 = \exp(2v)$. This shows that $2v = w \in V$, so by induction we have $2^n v \in V$ for all $n \in \mathbb{N}$. This is a contradiction since $\varepsilon > |2^n v| = 2^n |v| \to \infty$ as $n \to \infty$. ∎

1.2. **Local Groups and the BCH Formula.** We start from the very end of the journey and reduce the C^∞-regularity needed to build a Lie group to something strictly between a C^0 and a C^1-regularity.

To do so, it will be useful to consider a 'local' version of groups, since the regularity of charts centered at the identity can be translated around to cover the entire group. These *local groups* are important elsewhere in mathematics (particularly, in model theory and additive combinatorics; see [BGT12] and [Hru12]).

Definition 1.4. A *local group* $(G, \Omega, \Lambda, e, *, \iota)$ is topological space G equipped with an element $e \in G$ and continuous operations $* : \Omega \to G$ and $\iota : \Lambda \to G$ defined on some $\Omega \subseteq G^2$ and $\Lambda \subseteq G$, such that:

- (Closure). Ω is an open neighborhood of $G \times \{e\} \cup \{e\} \times G$ and Λ is an open neighborhood of e.
- (Associative). For each $g, h, k \in G$, whenever $(g*h)*k$ and $g*(h*k)$ are both defined, they coincide.
- (Identity). For each $g \in G$, we have $g * e = g = e * g$.
- (Inverse). If $g \in \Lambda$, then $g * \iota(g) = e = \iota(g) * g$.

A local group is a local *Lie* group if G is a smooth manifold and the operations are smooth on their domains.

Example 1.5. Restricting any (Lie) group G to an open neighborhood U of the identity yields a local (Lie) group $G|_U$, with $\Omega \coloneqq \{(g,h) \in G^2 : g, h, gh \in U\}$ and $\Lambda \coloneqq \{g \in G : g, g^{-1} \in U\}$.

In the Lie group case, fixing a chart $\varphi : U \to V$ centered at the identity induces a *pushforward* a local Lie group $\varphi_* G|_U$ whose underlying set is V, identity is $\varphi(e) = 0$, and whose group operations are defined by

$$x * y \coloneqq \varphi(\varphi^{-1}(x) \cdot \varphi^{-1}(y)) \quad \text{and} \quad \iota(x) \coloneqq \varphi(\varphi^{-1}(x)^{-1}).$$

for sufficiently small $x, y \in V$. Note that the operation $*$ is *different* from the usual additive group $V \subseteq \mathbb{R}^d$. For instance, if $G \coloneqq GL_n(\mathbb{R})$ and $U \coloneqq B_1(1)$, the chart $\varphi(g) \coloneqq g - 1$ centered at 1 induces a local group structure on $V \coloneqq \varphi(U)$ by the operation $x*y = x+y+xy$. In general, for any Lie group G, this pushforward local Lie group obeys the estimate $x * y = x + y + O(|x||y|)$ for sufficiently small x, y, by Taylor expansion.

Turning this example around, it is a remarkable fact that any local subgroup of $\mathbb{R}^d$ satisfying the estimate $x * y = x + y + O(|x||y|)$ for sufficiently small x, y can be restricted to a local Lie group:

Theorem 1.6 (Baker-Campbell-Hausdorff)**.** *The group operation $*$ of any local subgroup $V \subseteq \mathbb{R}^d$ satisfying the estimate $x*y = x+y+O(|x||y|)$ for sufficiently small x, y is real analytic near the identity. In particular, after restricting V to a sufficiently small neighborhood of the identity, one obtains a local Lie group.*

We refer to such local groups as $C^{1,1}$*-local groups*. Since the proof of this theorem is similar to its classical version, we omit the proof (see [Tao14, Section 1.2.5]). We summarize this section in the following

Corollary 1.7. *A topological group G is Lie iff there is a neighborhood U of the identity of G such that $G|_U$ is isomorphic to a $C^{1,1}$-local group. In particular, any topological group that is locally Lie is Lie.*

1.3. **Strengthening of the Gleason-Yamabe Theorem.**

One slight defect in the Gleason-Yamabe Theorem, as stated in Theorem B, is that the open subgroup G' of a locally-compact group can depend on the open neighborhood U. However, the following basic result from the theory of *totally-disconnected* locally-compact groups makes G' independent of U, which allows us to write G' as an *inverse limit* of Lie groups.

Definition 1.8. A topological space is *totally-disconnected* if every non-empty connected subset is a singleton, and is *zero-dimensional* if it admits a basis of clopen sets.

Theorem 1.9 (van Danztig; [vDa36])**.** *Every totally-disconnected locally-compact group G admits a neighborhood base of the identity consisting of compact open subgroups.*

Proof. Let V be a compact neighborhood of the identity of G. A basic result in topology states that locally-compact totally-disconnected spaces are zero-dimensional, so there is a compact open neighborhood of the identity $U \subseteq V$, which we can take to be symmetric by Lemma 1.1.

For each $g \in U$, continuity of $h \mapsto gh$ furnishes an open neighborhood W_g of the identity with $gW_g \subseteq U$; applying Lemma 1.1, we obtain a symmetric open neighborhood L_g of the identity with $L_g^2 \subseteq W_g$. Covering $U \subseteq \bigcup_{i<n} g_i L_{g_i}$ by compactness for some $n \in \mathbb{N}$, and putting $L \coloneqq \bigcap_{i<n} L_{g_i}$, we have

$$UL \subseteq \bigcup_{i<n} g_i L_{g_i} L \subseteq \bigcup_{i<n} g_i L_{g_i}^2 \subseteq \bigcup_{i<n} g_i W_{g_i} \subseteq U.$$

Thus $L^n \subseteq U$ for all n by induction, so $K \coloneqq \bigcup_{n \geq 0} L^n$ is contained in U. Since L is symmetric, we see that K is an open subgroup of U, so it is clopen and is the desired compact open subgroup contained in V. ■

Corollary 1.10. *Every locally-compact group G contains an open subgroup $G' \subseteq G$ with $G'/(G')^0$ compact.*

Proof. Applying van Danztig's Theorem to the totally-disconnected locally-compact group G/G^0, we obtain a compact open subgroup of G/G^0, which pulls back to the desired open subgroup of G. ■

Theorem 1.11 (Strengthening of Gleason-Yamabe)**.** *Every locally-compact group G contains an open subgroup G' that is isomorphic to an inverse limit of Lie groups $(L_\alpha)_{\alpha\in I}$ for some directed set I.*

Furthermore, each L_α is isomorphic to G'/K_α for some compact normal subgroup $K_\alpha \trianglelefteq G'$, with $K_\beta \subseteq K_\alpha$ whenever $\alpha \le \beta$. Lastly, if G is also first-countable, we can take $I \coloneqq \mathbb{N}$ with the usual ordering.

Proof. Passing to an open subgroup as in Corollary 1.10, we may assume that G/G^0 is compact. We claim that every neighborhood U of the identity of G contains a compact normal subgroup $K \trianglelefteq G$ such that G/K is Lie. Letting I contain all finite collections of open neighborhoods of the identity, ordered by inclusion, the rest of the claims follow easily.

Indeed, let U be a neighborhood of the identity. By the Gleason-Yamabe Theorem (Theorem B), there is an open subgroup $H \subseteq G$ and a compact normal subgroup $L \trianglelefteq H$ contained in U such that H/L is Lie. In particular, H/L has NSS by Lemma 1.3, so let $V \subseteq G^0$ be a small enough neighborhood of the identity in G such that any subgroup of H in V lies in L. Since $H \subseteq G$ is open, we see that $G^0 \subseteq H$, and thus $[G : H] < \infty$ by compactness of G/G^0. Thus L has finitely-many conjugates in G, whose intersection $K \coloneqq \bigcap_{i=1}^n g_iLg_i^{-1}$ is then a compact normal subgroup in G. By our choice of V and normality of G^0, any subgroup G contained in the neighborhood $\bigcap_{i=1}^n g_iVg_i^{-1}$ of the identity is contained in $g_iLg_i^{-1}$ for each i, and hence in K, so G/K has NSS. A final application of the Gleason-Yamabe Theorem shows that G/K is Lie, as desired. ∎

1.4. **Proof of Theorem A.** Let G be a locally-Euclidean group, which by Lemma 1.2 is Hausdorff, locally-compact, and first-countable. By Theorem 1.11, G contains an open subgroup G' which is an inverse limit of Lie groups L_n, where each $L_n \cong G'/K_n$ for some compact normal subgroup $K_n \trianglelefteq G'$ with $K_{n+1} \subseteq K_n$. It suffices to show that G' is Lie, since then G is locally Lie, and hence Lie by Corollary 1.7; thus, we may assume without loss of generality that $G \cong \lim_n L_n$ and each $L_n \cong G/K_n$ in the first place.

The proof will consist of two steps. First, we show that $\dim L_n$ coincide for all but finitely-many n, which then forces K_n to be totally-disconnected for some large enough n. Then, we study the short exact sequence $0 \to K_n \to G \to L_n \to 0$ with K_n totally-disconnected, and use it to show that G is Lie.

Claim. *For each n, there is a continuous injection from a neighborhood of the identity of L_n into G.*

Proof. We have $L_n \cong G/K_n \cong (G/K_{n+1})/(K_n/K_{n+1}) \cong L_{n+1}/H_n$, where $H_n \coloneqq K_n/K_{n+1}$ is a compact normal subgroup of L_{n+1}, whence Lie by Cartan's Theorem. The natural projection $\mathfrak{l}_{n+1} \twoheadrightarrow \mathfrak{l}_{n+1}/\mathfrak{h}_n \cong \mathfrak{l}_n$ admits a continuous section $\mathfrak{l}_n \to \mathfrak{l}_{n+1}$, and passing to the inverse limit furnishes a continuous map $\eta_n : \mathfrak{l}_n \to L(G)$ to the set of *one-parameter subgroups* $L(G)$ of G, consisting of all continuous homomorphisms $\mathbb{R} \to G$ (see Definition 3.1). Since L_n is lie, the exponential map $\varphi \mapsto \varphi(1)$ from $L(L_n) \cong \mathfrak{l}_n$ is a local-homeomorphism, so $\varphi(1) \mapsto \varphi \mapsto \eta_n(\varphi)(1)$ composes to a map from a neighborhood of the identity of L_n to G, which is injective since $\eta_n(\varphi)(1) = \varphi(1) \bmod K_n$. □

In particular, since G is locally-Euclidean, and hence has a well-defined dimension of invariance-of-domain, we have $\dim L_n \le \dim G$ for each n as witnessed by the injections. But we have $\dim L_n \le \dim L_{n+1}$ for each n too, and being bounded by $\dim G$, this increasing chain of dimensions must stabilize to some $d \in \mathbb{N}$. Thus $\dim H_n = 0$ for some large n, and since H_n is compact, it is finite. Since K_n is the inverse limit of K_n/K_m as $m \to \infty$, we see that K_n a profinite group, and is in particular totally-disconnected.

Let $K \coloneqq K_n$ and $L \coloneqq L_n$. Since $L \cong G/K$, we have a short-exact sequence $0 \to K \to G \to L \to 0$, which by the previous claim, admits a partial splitting $\eta : U \hookrightarrow G$ for some neighborhood U of the identity of L.

Claim. *Normalizing $\eta(e) \coloneqq e$ by a translation, the map η restricts to a local group homomorphism from a neighborhood V of the identity of L into G, and $\pi^{-1}(V) \subseteq G$ is isomorphic to $V \times K$ as local groups.*

Proof. Since L is locally-connected, use Lemma 1.1 to shrink U to a *connected* symmetric neighborhood V of the identity with $V^2 \subseteq U$. The image of V under $\varphi \mapsto \eta(\varphi)\eta(\varphi^{-1})$ is then connected in K, which forces $\eta(\varphi)^{-1} = \eta(\varphi^{-1})$ for all $\varphi \in V$; similarly, considering the map $(\varphi, \psi) \mapsto \eta(\varphi)\eta(\psi)\eta(\varphi\psi)^{-1}$ shows that $\eta : V \hookrightarrow G$ is a homomorphism. Finally, for each $k \in K$, considering the map $\varphi \mapsto \eta(\varphi)k\eta(\varphi)^{-1}k^{-1}$ shows that $\eta(\varphi)$ commutes with K, so the preimage $\pi^{-1}(V)$ of the projection $\pi : G \twoheadrightarrow L$ is isomorphic as a local group to $V \times K$, after identifying $\eta(\varphi)k$ with (φ, k). □

Since G is locally-Euclidean, it is in particular locally-connected, and hence so is $V \times K$ by the claim. But then K is simultaneously totally-disconnected and locally-connected, forcing it to be discrete, and hence G is locally-isomorphic to V as a local group. Since V is a local Lie group, so is G, and hence G is Lie. ∎

2. Constructing Representations and Metrics

We start with a quick proof sketch of the first steps of the Gleason-Yamabe Theorem (Theorem B). For full proofs and discussion, see [Tao14] and [vDG15]; the latter takes an approach using non-standard analysis.

2.1. NSS from subgroup trapping.

The first step reduces the theorem to proving that all groups having NSS are Lie. Heuristically, we will use the compact neighborhood of the identity of G to 'trap' subgroups inside it, so that a certain quotient is NSS. This is done by first proving the compact case of Theorem B:

Theorem 2.1 (Gleason-Yamabe; compact case)**.** *For every compact group G and any neighborhood U of the identity, there is a compact normal subgroup $K \trianglelefteq G$ in U such that G/K is Lie.*

For this, we will need the *regular representation* $\tau : G \to U(L^2(G, \mu))$ of G, where $L^2(G, \mu)$ is the Hilbert space of square-integrable functions $f : G \to \mathbb{C}$ with respect to a fixed Haar measure μ, equipped with the inner product $\langle f, g\rangle \coloneqq \int_G f(x)\overline{g(x)}\,\mathrm{d}\mu$, and $\tau_g \in U(L^2(G))$ is defined by $\tau_g f(h) \coloneqq f(h^{-1}g)$.

This is an infinite-dimensional representation of G, whose invariant subspaces are described by the *Peter-Weyl Theorem*; roughly speaking, it states that each finite-dimensional irreducible representation $\rho_\lambda : G \to U(V_\lambda)$ sits inside $L^2(G)$ with multiplicity $\dim V_\lambda$, and gives rise to a decomposition $L^2(G) \cong \widehat{\bigoplus}_\lambda V_\lambda^{\dim V_\lambda}$ of $L^2(G)$ as a Hilbert direct sum. For us, we will only need a special case of this to prove Theorem 2.1:

Lemma 2.2 (Baby Peter-Weyl)**.** *If G is a compact group with Haar measure μ and $g \in G$ is not the identity, then there exists a finite-dimensional invariant subspace of $L^2(G)$ on which τ_g is not the identity.*

Proof of Theorem 2.1. Let $g \in G \setminus U$, so there is a finite-dimensional invariant subspace of $L^2(G)$ on which τ_g is not the identity, which we identify with $\mathbb{C}^n$. This gives us a continuous homomorphism $\rho : G \to GL_n(\mathbb{C})$ with $\rho_g \neq \mathrm{id}$, so $\rho_h \neq \mathrm{id}$ for some open neighborhood of g. By compactness of $G \setminus U$, there exist finitely-many continuous homomorphisms $\rho_i : G \to GL_{n_i}(\mathbb{C})$ such that for each $g \in G \setminus U$, we have $\rho_i(g) \neq \mathrm{id}$ for some i.

Consider the map $\rho \coloneqq \bigoplus_i \rho_i : G \to GL_n(\mathbb{C})$, where $n \coloneqq \sum_i n_i$, which is faithful on $G \setminus U$ by construction. Thus $K \coloneqq \ker \rho$ is a compact normal subgroup of G contained in U, and by the Isomorphisms theorems for compact-Hausdorff spaces, we have $\rho(G) \cong G/K$. Finally, $\rho(G) \subseteq GL_n(\mathbb{C})$ is Lie by Cartan's Theorem. ■

With this tool in hand, one proves that every locally-compact group G has the *subgroup-trapping property*, which is mimics the NSS property after passing to a quotient:

Definition 2.3. A group G has the *subgroup-trapping property* if for every neighborhood U of the identity, there is an open sub-neighborhood $V \subseteq U$ of the identity such that the subgroup generated by all subgroups contained in V is contained ('trapped') in U.

Indeed, after passing to an open subgroup and a quotient, *subgroup-trapping* $\Leftrightarrow$ *NSS*:

Theorem 2.4. *For every locally-compact group G and any neighborhood U of the identity, there is an open subgroup $G' \subseteq G$ and a compact normal subgroup $K \trianglelefteq G'$ in U such that G'/K has NSS.*

2.2. Constructing Gleason metrics.

To prove Theorem B, it remains to show that every locally-compact NSS group is Lie. To do so, we will first convert the NSS property to a metric on G, called a *Gleason metric*, which is equivalent in strength but will be easier to use in concrete bounds and estimates. Its properties will be leveraged vigorously in Section 3.

Definition 2.5. A *Gleason metric*[2] on a locally-compact group G is a left-invariant metric d on G generating the topology on G, such that there exists a constant $C > 0$ satisfying the following *escape* property:

$$\text{If } g \in G \text{ and } n \geq 1 \text{ is such that } n\|g\| \leq 1/C, \text{ then } \|g^n\| \geq n\|g\|/C.$$

The main result proven by Yamabe, as an improvement to Gleason's earlier work, is the following

Theorem 2.6 ([Yam53])**.** *Every locally-compact NSS group admits a Gleason metric.*

Lastly, one can also obtain a commutator estimate using the Gleason metric, which will be very important for us in Section 3. The proof uses similar techniques as the proof of Theorem 2.6, but is considerably easier and, if not for lack of time, we would include it.

Proposition 2.7. *If d is a Gleason metric on a locally-compact group G, then there is a constant C' (possibly distinct from C) such that for all g, h with $\|g\|, \|h\| \leq 1/C'$, we have $\|[g, h]\| \leq C'\|g\|\|h\|$.*

[2] [Tao14] refers to this kind of metric as a *weak Gleason metric*, reserving the term *Gleason metric* for one that additionally satisfies the commutator estimate in Proposition 2.7.

3. One-parameter Subgroups and the Exponential Map

We have previously seen that every locally-compact group admits a subquotient that is locally-compact and has NSS (Theorem 2.4), and those subquotients admit Gleason metrics (Theorem 2.6).

To complete the proof of Theorem B, it remains to show that every locally-compact group equipped with a Gleason metric is Lie. The rough strategy of this step, which we find really satisfying, is to use the theory of Lie groups to identify the 'Lie-theoretic' structures that can be translated to our setting. As we shall see, the objects that will play a crucial are the *one-parameter subgroups*:

Definition 3.1. A *one-parameter subgroup* of G is a continuous homomorphism $\varphi : \mathbb{R} \to G$. We write $L(G)$ for the set of all one-parameter subgroups of G, and let $\exp : L(G) \to G : \varphi \mapsto \varphi(1)$ be the *exponential map*.

This is hardly surprising, since the Lie algebra of a Lie group G can be canonically identified with $L(G)$, and good control over $L(G)$ is sufficient for understanding the local behaviour of G; for an arbitrary locally-compact group, this control will thus be sufficient to build a global Lie structure by Corollary 1.7.

3.1. Cartan's Theorem and topological vector spaces.

To elucidate this idea, let us recall

Theorem 3.2 (Cartan)**.** *Every closed subgroup H of a Lie group G is Lie.*

Our proof will mimic Cartan's Theorem in many ways, so let us sketch it here for didactic reasons.

Proof sketch. Using the exponential map $\exp : L(G) \to G$, where $\mathfrak{g} \cong L(G)$ as mentioned, we will construct a candidate for the Lie algebra $\mathfrak{h}$ for H using the one-parameter subgroups

$$L(H) \coloneqq \{\varphi \in L(G) : \varphi(t) \in H \text{ for all } t \in \mathbb{R}\}. \tag{1}$$

First, we need to show that $L(H)$ is a vector space, which is easy since $t \mapsto e$ is the identity, $(\lambda\varphi)(t) \coloneqq \varphi(\lambda t)$ defines scalar multiplication, and addition can be defined since we have $\varphi(t) = \exp(tX)$ and $\psi(t) = \exp(tY)$ for unique $X, Y \in \mathfrak{g}$, and using this correspondence backwards we can define $\varphi + \psi \in L(H)$ by

$$(\varphi + \psi)(t) \coloneqq \exp(t(X + Y)) = \lim_{n\to\infty} (\exp(tX/n)\exp(tY/n))^n = \lim_{n\to\infty} (\varphi(t/n)\psi(t/n))^n. \tag{2}$$

Note that $(\psi + \psi)(t) \in H$ for all t since H is closed and each term in the limit is in H. The following (trivial) observation is what allows us to conclude that $L(G)$ is then a finite-dimensional vector space.

Observation. If W is a subspace of a finite-dimensional vector space V, then so is W.

The next step is to show that $L(H)$ is 'large' in the sense that $\exp : L(G) \to G$ restricts to a local homeomorphism between $L(H)$ and H. First, we show that $L(H)$ is non-trivial if H is not discrete.

> Let $e \neq h_n \to e$, so, since $\exp : \mathfrak{g} \to G$ is a local homomorphism, we obtain a sequence $X_n \neq 0$ converging to $0 \in \mathfrak{g}$ such that $\exp X_n = h_n$ for large enough n. Endow $\mathfrak{g}$ with an arbitrary norm so that $X_n/\|X_n\|$ lies on the (compact) unit sphere, and passing to a subsequence yields a limit X of unit norm.
>
> Now, $X_n \lfloor t\|X_n\|\rfloor \to tX$ for any $t \geq 0$, so $\exp(X_n)^{\lfloor t/\|X_n\|\rfloor} \to \exp(tX)$. Each term in the sequence lies in H, so $\exp(tX) \in H$ too. Setting $\exp(-tX) \coloneqq \exp(tX)^{-1} \in H$, we see that $(t \mapsto \exp(tX)) \in L(H)$ as desired.

Finally, that $\exp : L(H) \to H$ is a local homeomorphism follows along the same lines, which we omit. ■

This proof provides a precious guide for us which we will mimic very closely. However, our lack of ambient Lie structure (as opposed to in Cartan's Theorem, where H already sits inside a Lie group) forces us abandon the definition of $L(H)$ in (1), and instead build it up from scratch; this is done in Propositions 3.5 and 3.6, where addition in $L(H)$ is motivated by (2). Next, we will need an analogous observation that $L(H) \cong \mathbb{R}^n$ for some $n \geq 0$, which we provide now.

Lemma 3.3. *Every locally-compact Hausdorff TVS is isomorphic to $\mathbb{R}^n$ for some finite n.*

Proof. Let K be a compact neighborhood of the origin, which can be covered by dilates $K \subseteq S + K/2$ for some finite $S \subseteq V$. Then $K \subseteq \langle S\rangle + K/2$, which we can iterate so that $K \subseteq \langle S\rangle + 2^{-n}K$ for every $n \geq 1$. Thus $K \subseteq \langle S\rangle + U$ for every neighborhood U of the origin, so, since $\langle S\rangle$ is closed, we have $K \subseteq \langle S\rangle$. Hence $V = \langle S\rangle$, since for any $v \in V$, we have $2^{-n}v \in K$ for n large enough, so $v \in 2^nK \subseteq \langle S\rangle$. ■

Finally, and this is the hardest part, we show that $\exp : L(H) \to H$ is a local homeomorphism, and, again, it will be useful to show that $L(H)$ is non-trivial in the first place. We now embark on this journey.

3.2. **The vector space of one-parameter subgroups.** We come back to our original setting, where G is a locally-compact group equipped with a Gleason metric d. Writing $\|g\| \coloneqq d(g,e)$, we have by Proposition 2.7 that there is a constant $C > 0$ such that

1. (Escape): If $g \in G$ and $n \geq 1$ is such that $n\|g\| \leq 1/C$, then $\|g^n\| \geq n\|g\|/C$.
2. (Commutator): If $g, h \in G$ are such that $\|g\|, \|h\| \leq 1/C$, then $\|[g,h]\| \leq C\|g\|\|h\|$.

In what follows, it will not be necessary to keep track of the exact value of C, only that it stays uniform for all group elements. Thus, if X, Y are quantities depending on $g_1, \dots, g_n, m$, we adopt the notation $X \ll Y$ for $X \leq CY$ for some constant $C > 0$ uniform in $g_1, \dots, g_n, m$, and write $X \sim Y$ for $Y \ll X \ll Y$.

Lemma 3.4 (Basic bounds and estimates)**.** *Let $\varepsilon > 0$, chosen later depending on the constant C.*

1. *For all $g, g_1, \dots, g_n \in G$, we have $\|g\| = \|g^{-1}\|$ and $\|g_1 \cdots g_n\| \leq \sum_{i=1}^n \|g_i\|$.*
2. *If $g, h, k \in G$ are such that $\|g\|, \|h\|, \|k\| \leq \varepsilon$, then $d(gk, hk) \sim d(g,h)$ and $\|ghg^{-1}\| \sim \|h\|$.*
3. *If $n \geq 1$ and $\|g\|, \|h\| \leq \varepsilon/n$, then $d(g^n h^n, (gh)^n) \ll n^2\|g\|\|h\|$ and $d(g^n, h^n) \sim nd(g,h)$.*

Proof. The first claim follows from left-invariance and the triangle-inequality. The right-invariance estimate follows from the conjugation estimate as $d(gk, hk) = \|(hk)^{-1}(gk)\| = \|k^{-1}h^{-1}gk\| \sim \|h^{-1}g\| = d(g,h)$, and the conjugation estimate itself can be seen from the following computation and symmetry:

$$\|ghg^{-1}\| = \|[g,h]h\| \leq \|[g,h]\| + \|h\| \ll \|g\|\|h\| + \|h\| \leq (\varepsilon+1)\|h\| \sim \|h\|.$$

Let $n \geq 1$ and $\|g\|, \|h\| \leq \varepsilon/n$. Since $d(g^n h^n, (gh)^n) \leq \sum_{i<n} d((gh)^i g^{n-i} h^{n-i}, (gh)^{i+1} g^{n-i-1} h^{n-i-1})$ by the triangle-inequality, it suffices to show that each summand is bounded by $n\|g\|\|h\|$. Indeed, we have

$$\begin{aligned} d((gh)^i g^{n-i} h^{n-i}, (gh)^{i+1} g^{n-i-1} h^{n-i-1}) &= d(g^{n-i-1} h^{n-i}, hg^{n-i-1} h^{n-i-1}) \\ &\sim d(g^{n-i-1}h, hg^{n-i-1}) \\ &\ll \|g^{n-i-1}\|\|h\| \\ &\leq n\|g\|\|h\|. \end{aligned}$$

Finally, for the last estimate, first note that $d(g^n, h^n) = \|g^n\|\|h^n\| \geq (\varepsilon n)^2\|g\|\|h\| \gg nd(g,h)$ by the escape property. Conversely, set $k \coloneqq h^{-1}g$, so that $d(g^n, h^n) = d((hk)^n, h^n) \leq d((hk)^n, h^n k^n) + d(h^n k^n, h^n)$. Bound the first term by $n^2\|h\|\|k\| \leq \varepsilon n\|k\|$ and the second by $\|k^n\| \leq n\|k\|$, so overall, $d(g^n, h^n) \ll n\|k\|$. ■

With these bounds, which ultimately rely on the escape property of the Gleason metric, we begin to show that $L(G)$ can be made into a Euclidean space in such a way that $\exp : L(G) \to G$ is a local homeomorphism. To do so, equip $L(G)$ with the compact-open topology, which admits a basis of open sets

$$\left\{\varphi \in L(G) : \sup_{t \in I} d(\varphi(t), \varphi_0(t)) < r\right\}$$

ranging over all $\varphi_0 \in L(G)$, compact $I \subseteq \mathbb{R}$, and $r > 0$; this topology makes exp a continuous map. In fact, it suffices by the homomorphism property to fix $I \coloneqq [-1,1]$, so that $L(G)$ is metrizable with the supremum norm, and hence complete. Along with Lemma 3.3, the following propositions show that $L(G)$ is Euclidean.

Proposition 3.5. *$L(G)$ is locally-compact.*

Proof. Fix $\varphi_0 \in L(G)$ and let $\varepsilon > 0$ be as in Lemma 3.4. By continuity of φ_0, there is an interval $I \coloneqq [-T,T]$ such that $\|\varphi_0(t)\| \leq \varepsilon$ for all $t \in I$. We claim that $B_\varepsilon(\varphi_0)$ is equicontinuous, so it is totally-bounded by the Arzelà-Ascoli Theorem, and hence precompact by the completeness of $L(G)$.

To this end, let $\varphi \in B_\varepsilon(\varphi_0)$, so $\|\varphi(t)\| \leq 2\varepsilon$ for all $t \in I$. By the escape property, we have for ε small enough that $\|\varphi(t/n)\| \ll \varepsilon/n$ for all $t \in I$ and $n \geq 1$, and so $\|\varphi(t)\| \ll |t|/T$ for all $t \in I$. This gives us the Lipschitz property $d(\varphi(t), \varphi(t')) = \|\varphi(t-t')\| \ll |t-t'|/T$ for sufficiently small $t, t' \in I$, so in particular, $B_\varepsilon(\varphi_0)$ is equicontinuous as desired. ■

Proposition 3.6. *$L(G)$ is a topological vector space.*

Proof. The trivial map $0 : t \mapsto e$ is a one-parameter subgroup, and so are scalar multiples $(c\varphi)(t) \coloneqq \varphi(ct)$. For addition, recall the formula $\exp(t(X+Y)) = \lim_{n\to\infty}(\exp(tX/n)\exp(tY/n))^n$ in (2), valid for Lie groups, which we mimic by setting

$$(\varphi + \psi)(t) \coloneqq \lim_{n\to\infty} (\varphi(t/n)\psi(t/n))^n$$

for all $\varphi, \psi \in L(G)$. It suffices to show that $\varphi + \psi \in L(G)$; the continuity of $+$ and $\cdot$ are relatively easy, and associativity follows along similar lines of reasoning.

Indeed, we have for any $m, n \geq 1$ that $\|\varphi(t/nm)\|, \|\psi(t/nm)\| \ll \varepsilon/(nm)$ for sufficiently small t, and thus

$$d(\varphi(t/nm)^m\psi(t/nm)^m, (\varphi(t/nm)\psi(t/nm))^m) \ll m^2(\varepsilon/mn)^2 = \varepsilon^2/n^2$$

by Lemma 3.4. Thus $d(\varphi(t/n)\psi(t/n), (\varphi(t/nm)\psi(t/nm))^m) \ll \varepsilon^2/n^2$, and applying it again gives

$$d((\varphi(t/n)\psi(t/n))^n, (\varphi(t/nm)\psi(t/nm))^{nm}) \sim n(\varepsilon^2/n^2) = \varepsilon^2/n.$$

Fixing $m \geq 1$ and sending $n \to \infty$ shows that $(\varphi(t/n)\psi(t/n))^n$ is a Cauchy sequence, and hence is convergent, for all sufficiently small t; the general case follows since we can replace t by $t/2$ by continuity of multiplication. This argument shows that this sequence is uniformly Cauchy, so the pointwise limit $\varphi + \psi$ is continuous. It is a group homomorphism by the density of the rationals and the computation

$$(\varphi + \psi)(at) = \lim_{n\to\infty}(\varphi(at/n)\psi(at/n))^n = \lim_{n\to\infty}(\varphi(t/n)\psi(t/n))^{an}$$
$$(\varphi + \psi)(-t)^{-1} = \lim_{n\to\infty}(\varphi(-t/n)\psi(-t/n))^{-n} = \lim_{n\to\infty}(\psi(t/n)\varphi(t/n))^n$$

for all $a, b \in \mathbb{N}$, so $(\varphi+\psi)(at)(\varphi+\psi)(bt) = (\varphi+\psi)((a+b)t)$, where the second computation can be conjugated by $\psi(t/n)$, whose norm goes to 0, so that $(\varphi + \psi)(-t)^{-1} = (\varphi + \psi)(t)$ by Lemma 3.4. ■

Corollary 3.7. *$L(G)$ is isomorphic to $\mathbb{R}^n$ for some $n \geq 0$.*

3.3. **The exponential map.** This brings us to the last step of our endeavor to use $L(G)$ as a proxy of the Lie algebra of the to-be-Lie-group G. It remains to show that exp is a local homeomorphism.

Lemma 3.8. *If $g, h \in G$ are sufficiently small, then $g^2 = h^2$ forces $g = h$.*

Proof. If $\|g\|, \|h\| \leq \varepsilon/2$, then $d(g, h) \sim 2d(g^2, h^2) = 0$ by Lemma 3.4. ■

Lemma 3.9. *If $g_n \to e$ and $N_n \coloneqq \lfloor \varepsilon/\|g_n\| \rfloor$, then $\varphi_n(t) \coloneqq g_n^{\lfloor tN_n \rfloor}$ converges to an element $\varphi \in L(G)$.*

Proof. Since $\|\varphi_n(t)\| \leq \varepsilon$ for all $n \in \mathbb{N}$ and $|t| \leq 1$, we can pass to a subsequence so that $\varphi_n(t)$ converges to some $\varphi(1) \in G$. In general, note that if $\varphi_n(t) \to h$ and $\varphi_n(r) \to k$ for $h, k \in G$, then since $\lfloor (t + r)N_n \rfloor$ and $\lfloor tN_n \rfloor + \lfloor rN_n \rfloor$ differ by at most one, and $g_n \to e$, we see that $\varphi_n(t + r) \to hk$.

With this observation, we claim that $\varphi_n(t)$ converges when t is a dyadic rational, so it converges for every $|t| \leq 1$ by density. Indeed, if $\varphi_n(1/2)$ have two subsequential limits h, k, then $h^2 = \varphi(1) = k^2$, and so $h = k$ by Lemma 3.8. Iterating and using the same observation, the claim follows.

Finally, that $\varphi : [-1, 1] \to G$ extends to a one-parameter subgroup $\varphi \in L(G)$ follows along the same lines as the proof of Proposition 3.6, which we omit. ■

Proposition 3.10 ([Hir90])**.** *For any neighborhood L of the origin in $L(G)$, its image $\exp(L)$ is a neighborhood of the identity in G.*

Proof. Shrinking L if necessary, we may assume that L is a compact star-shaped neighborhood of the identity in $L(G)$ and that $K \coloneqq \exp L$ is contained in a ball of radius ε around the origin. Since L is compact, so is K. Suppose for sake of contradiction that K is not a neighborhood of the origin of G, so there is a sequence $g_n \in G \setminus K$ such that $\|g_n\| \to 0$ as $n \to \infty$. By compactness of K, there exist some $h_n \in K$ that minimizes $d(g_n, h_n)$, and we let $\psi_n \in L$ be such that $\exp \psi_n = h_n$ for each n; note that $\psi_n \to 0$ as $n \to \infty$.

Writing $k_n \coloneqq h_n^{-1}g_n$, we have $\|k_n\| = d(g_n, h_n) \leq d(g_n, e) = \|g_n\|$ for all n, and hence $\|k_n\|, \|h_n\| \to 0$ as $n \to \infty$. Let $N_n \coloneqq \lfloor \varepsilon/\|k_n\| \rfloor$, so by Lemma 3.9, $k_n^{\lfloor tN_n \rfloor}$ converges to an element $\varphi \in L(G)$.

We claim that with $\gamma_n \coloneqq \psi_n + \varphi_n$, where $\varphi_n \coloneqq \frac{1}{N_n}\varphi$, the element $\exp \gamma_n$ is closer to g_n than h_n is, and since $\gamma_n \in L$ for large enough n, this gives the desired contradiction. Indeed, we have from Lemma 3.4 that

$$\begin{aligned} d((\psi_n(1/m)\varphi_n(1/m))^m, \psi_n(1)\varphi_n(1)) &\ll m^2 \|\psi_n(1/m)\| \|\varphi_n(1/m)\| \\ &\sim \|\psi_n(1)\| \|\varphi_n(1)\| \\ &\sim \|h_n\| \|\varphi(1)\| / N_n \\ &\sim \|h_n\| / N_n \end{aligned}$$

for all m, so $d(\exp\gamma_n, \exp\psi_n \exp\varphi_n) \ll \|h_n\|/N_n$ by definition of addition in $L(G)$. But $\exp\psi_n = h_n$ and $d(\exp\varphi_n, k_n) \sim d(\exp\varphi, k_n^{N_n})/N_n$ by Lemma 3.4 again, so

$$d(\exp\gamma_n, g_n) \ll \|h_n\|/N_n + d(h_n \exp\varphi_n, g_n) = \|h_n\|/N_n + d(\exp\varphi_n, k_n) \sim \left(\|h_n\| + d(\exp\varphi, k_n^{N_n})\right)/N_n$$

for all n. We now obtain the desired contradiction since $\|h_n\|, d(\exp\varphi, k_n^{N_n}) \to 0$ and $N_n \to \infty$ as $n \to \infty$, so $d(\exp\gamma_n, g_n)$ is arbitrarily small, which contradicts the choice of h_n for large n. ■

Theorem 3.11. *Every locally-compact group G admitting a Gleason metric has a neighborhood of the identity that is a $C^{1,1}$-local group. In particular, G is Lie by Corollary 1.7.*

Proof. By Lemma 3.8, let L be a compact neighborhood of the identity of $L(G)$ small enough so that $\exp|_L$ is injective. Thus $L \cong \exp(L) =: K$ is a homeomorphism, and since K is a neighborhood of the identity of G by Proposition 3.10, we can consider the local group $G|_K$ which pulls-back to a local group structure on L. Identifying $L(G) \cong \mathbb{R}^n$ by Corollary 3.7, we claim that L is a $C^{1,1}$-local group, and hence so is K.

Indeed, since $\exp : L \to K$ is bilipschitz, it suffices to show that $d(\varphi(1)\psi(1), (\varphi+\psi)(1)) \ll \|\varphi(1)\| \|\psi(1)\|$ for sufficiently small $\varphi, \psi \in L(G)$. By definition of $\varphi + \psi$, this will follow if we have

$$d(\varphi(1)\psi(1), (\varphi(1/n)\psi(1/n))^n) \ll \|\varphi(1)\| \|\psi(1)\|,$$

for all $n \geq 1$, which we get by applying Lemma 3.4 since $\|\varphi(1/n)\| \ll \|\varphi(1)\|/n$ and similarly for ψ. ■

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