## 4.计算下列积分

1

$$egin{aligned} \int_{-1}^1 x^m P_n(x) dx \ &\int_{-1}^1 x^m P_n(x) dx = \int_0^1 x^m P_n(x) dx + \int_{-1}^0 x^m P_n(x) dx \ &= (1 + (-1)^{m+n}) \int_0^1 x^m P_n(x) dx \end{aligned}$$

利用

$$\int_0^1 x^m P_n(x) dx = rac{m}{m+n+1} \int_0^1 x^{m-1} P_{n-1}(x) dx$$

m < n时

$$\int_0^1 x^m P_n(x) dx = rac{m!(n-m+1)!!}{(m+n+1)!!} \int_0^1 P_{n-m}(x) dx$$

利用

$$\int_0^1 P_n(x) dx = \left\{egin{array}{ll} 0, & n=2k \ rac{(-1)^k(2k-1)!!}{(2k+2)!!} & n=2k+1 \end{array}
ight.$$

若n-m为偶数,  $\int_0^1 P_n(x) dx = 0$ ; 若为奇数,  $(1+(-1)^{m+n}) = 0$ , 则得到

$$\int_{-1}^1 x^m P_n(x) dx = 0$$

 $m \geq n$ 时

$$\begin{split} \int_0^1 x^m P_n(x) dx &= \frac{m!}{(m-n)!} \frac{(m-n+1)!!}{(m+n+1)!!} \int_0^1 x^{m-n} dx \\ &= \frac{m!}{(m-n+1)!} \frac{(m-n+1)!!}{(m+n+1)!!} \\ &= \frac{m!}{(m-n+1)!!(m-n)!!} \frac{(m-n+1)!!}{(m+n+1)!!} \\ &= \frac{m!}{(m-n)!!(m+n+1)!!} \end{split}$$

得到结果

$$\int_{-1}^{1} x^{m} P_{n}(x) dx = \frac{m! [1 + (-1)^{m+n}]}{(m-n)!! (m+n+1)!!}$$

2.

$$\int_{-1}^{1}xP_{m}(x)P_{n}(x)dx$$
  $I=\int_{-1}^{1}xP_{m}(x)P_{n}(x)dx=\int_{-1}^{1}P_{n}(x)[rac{m+1}{2m+1}P_{m+1}(x)+rac{m}{2m+1}P_{m-1}(x)]dx$ 

$$I = rac{m+1}{2m+1} \int_{-1}^{1} P_{m+1}^2(x) dx = rac{m+1}{2m+1} rac{2}{2m+3}$$

$$I = rac{m}{2m+1} \int_{-1}^{1} P_{m-1}^2(x) dx = rac{m}{2m+1} rac{2}{2m-1}$$

若 $m-n \neq \pm 1$ 

$$I = 0$$

3.

$$\begin{split} \int_{-1}^{1} (1 - x^2) [P'_n(x)] dx \\ I &= \int_{-1}^{1} (1 - x^2) [P'_n(x)] dx \\ &= \int_{-1}^{1} (1 - x^2) P'_n(x) dP_n(x) \\ &= (1 - x^2) P'_n(x) P_n(x) |_{-1}^{1} - \int_{-1}^{1} P_n(x) [(1 - x^2) P_n(x)]' dx \end{split}$$

注意到 $P_n(x)$ 本身满足Legendre方程

$$[(1-x^2)P_n(x)]' = -n(n+1)P_n(x)$$

得到

$$I = n(n+1) \int_{-1}^{1} P_n^2(x) dx = rac{2n(n+1)}{2n+1}$$

## 5.把下列函数按Legendre函数系展开

1.

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & -1 < x < 0 \end{cases}$$

$$C_n=rac{2n+1}{2}\int_0^1xP_n(x)dx$$

n=0时

$$C_0 = \frac{1}{4}$$

n > 0时

$$C_n = rac{2n+1}{2(n+2)} \int_0^1 P_{n-1}(x) dx$$

利用

$$\int_0^1 P_n(x) dx = egin{cases} 1 & n=0 \ rac{1}{2} & n=1 \ 0 & n=2k \ rac{(-1)^k(2k-1)!!}{(2k+2)!!} & n=2k+1 \end{cases}$$

得到

$$C_n = egin{cases} rac{1}{2} & n=1 \ rac{5}{16} & n=2 \ 0 & n=2k+1 \ rac{4k+1}{4k+4}rac{(-1)^{k-1}(2k-3)!!}{(2k)!!} & n=2k,k=1,2,\ldots \end{cases}$$

得到结果

$$f(x) = rac{1}{4}P_0(x) + rac{1}{2}P_1(x) + rac{5}{16}P_2(x) + \sum_{n=1}^{\infty} rac{(-1)^{n-1}(2n-3)!!}{(2n+2)!!} rac{4n+1}{2}P_{2n}(x)$$

2.

$$f(x) = x^3, -1 < x < 1$$

$$C_n = rac{2n+1}{2} \int_{-1}^1 x^3 P_n(x) dx$$

利用第一题结果,得到

$$C_n=\left\{egin{array}{ll} 0 & n=0,2$$
 if  $n>3$   $rac{3}{5} & n=1 \ rac{2}{5} & n=3 \end{array}
ight.$ 

3.

$$f(x) = |x|, -1 < x < 1$$
  $C_n = \int_{-1}^1 |x| P_n(x) dx = rac{(2n+1)(1+(-1)^n)}{2} \int_0^1 x P_n(x) dx$ 

同1,得到结果

$$f(x) = rac{1}{2}P_0(x) + rac{5}{8}P_2(x) + \sum_{n=1}^{\infty} rac{(-1)^{n-1}(2n-3)!!}{(2n+2)!!}(4n+1)P_{2n}(x)$$

其中

$$\frac{(2n-3)!!}{(2n+2)!!}(4n+1) = \frac{(4n+1)(2n-2)!}{(2n+2)!!(2n-2)!!} = \frac{(4n+1)(2n-2)!}{2^{2n}(n-1)!(n+1)!}$$

得到答案结果.

## 6. 解下列定解问题

1.

$$\begin{cases} \Delta_3 u = 0 & r < a \\ u|_{r=s} = \cos^2 \theta \end{cases}$$

$$u(r, heta) = \sum_{n=0}^{\infty} C_n r^n P_n(\cos heta)$$

带入边界条件

$$\cos^2 heta = \sum_{n=0}^{\infty} C_n a^n P_n(\cos heta)$$

尝试将系数组合出来,已知

$$P_0(\cos \theta) = 1$$
  $P_2(\cos \theta) = \frac{3}{2}x^2 - \frac{1}{2}$ 

得到

$$\cos^2 heta = rac{1}{3} P_0(\cos heta) + rac{2}{3} P_2(\cos heta)$$

比对得到系数

$$C_0 = rac{1}{3}$$
  $C_2 = rac{2}{3} rac{1}{a^2}$ 

得到结果

$$u(r, heta)=rac{1}{3}+rac{2}{3}\Big(rac{r}{a}\Big)^2P_2(\cos heta)$$

2.

$$\left\{egin{array}{l} \Delta_3 u = 0, & r > 1 \ u|_{r=1} = \cos^2 heta, \ u|_{r=+\infty} = 0. \end{array}
ight.$$

球外通解为

$$u(r, heta) = C_0 + \sum_{n=0}^\infty D_n r^{-n-1} P_n(\cos heta)$$

由 $u|_{r=\infty}=0$ 可以得到

$$C_0 = 0$$

带入边界条件

$$\cos^2 heta = \sum_{n=0}^{\infty} D_n P_n (\cos heta)$$

同样对比系数得到

$$D_0 = \frac{1}{3}$$
  $D_2 = \frac{2}{3}$ 

得到解

$$u(r, heta)=rac{1}{3r}P_0(\cos heta)+rac{2}{3r^3}P_2(\cos heta)$$

3.

$$\begin{cases} \Delta_3 u = 0, & 1 < r < 2 \\ u|_{r=1} = \cos \theta, & \\ u|_{r=2} = 1 + \cos^2 \theta & \end{cases}$$

通解

$$u(r, heta) = \sum_{n=0}^{\infty} (C_n r^n + D_n r^{-n-1}) P_n(\cos heta)$$

首先尝试组合得到边界条件

$$\left\{egin{aligned} \cos heta &= P_1(\cos heta) \ 1+\cos^2 heta &= rac{4}{3}P_0(\cos heta) + rac{2}{3}P_2(\cos heta) \end{aligned}
ight.$$

带入边界条件

$$\begin{cases} \sum_{n=0}^{\infty} (C_n + D_n) P_n(\cos \theta) = \cos \theta = P_1(\cos \theta) \\ \sum_{n=0}^{\infty} (C_n 2^n + D_n 2^{-n-1}) P_n(\cos \theta) = 1 + \cos^2 \theta = \frac{4}{3} P_0(\cos \theta) + \frac{2}{3} P_2(\cos \theta) \end{cases}$$

得到系数

$$\begin{cases} C_0 + D_0 = 0 \\ C_0 + \frac{1}{2}D_0 = \frac{4}{3} \end{cases} \quad \begin{cases} C_1 + D_1 = 1 \\ 2C_1 + \frac{1}{4}D_1 = 0 \end{cases} \quad \begin{cases} C_2 + D_2 = 0 \\ 4C_2 + \frac{1}{8}D_2 = \frac{2}{3} \end{cases}$$

其余为0.解得

$$\begin{cases} C_0 = \frac{8}{3} \\ D_0 = -\frac{8}{3} \end{cases} \quad \begin{cases} C_1 = -\frac{1}{7} \\ D_1 = \frac{8}{7} \end{cases} \quad \begin{cases} C_2 = \frac{16}{93} \\ D_2 = -\frac{16}{93} \end{cases}$$

得到结果

$$u(r, heta) = rac{8}{3} - rac{8}{3r} - \left(rac{r}{7} - rac{8}{7r^2}
ight)P_1(\cos heta) + \left(rac{16}{93}r^2 - rac{16}{93r^3}
ight)P_2(\cos heta)$$

7.

半径为a的金属球壳, 用绝缘材料分成上下两个半球壳仍组成一个金属球壳, 经充电后, 上下半球壳的电位分别为 $u_1$ 和 $u_2$ , 计算球壳内部电位分布

列出方程

$$\begin{cases} \Delta_3 u = 0, & r < a \\ u(a, \theta) = \begin{cases} u_1 & 0 < \theta < \frac{\pi}{2} \\ u_2 & \frac{\pi}{2} < \theta < \pi \end{cases} \end{cases}$$

球内通解

$$u(r, heta) = \sum_{n=0}^{\infty} C_n r^n P_n(\cos heta)$$

带入边界条件

$$\sum_{n=0}^{\infty} C_n a^n P_n(\cos heta) = \left\{egin{array}{ll} u_1 & 0 < heta < rac{\pi}{2} \ u_2 & rac{\pi}{2} < heta < \pi \end{array}
ight.$$

求系数

$$egin{aligned} C_n &= rac{2n+1}{2a^n} igg( -\int_0^{rac{\pi}{2}} u_1 P_n(\cos heta) d\cos heta - \int_{rac{\pi}{2}}^{\pi} u_2 P_n(\cos heta) d\cos heta igg) \ &= rac{2n+1}{2a^n} igg( \int_0^1 u_1 P_n(x) dx + \int_{-1}^0 u_2 P_n(x) dx igg) \ &= rac{(2n+1)(u_1+(-1)^n u_2)}{2a^n} \int_0^1 P_n(x) dx \end{aligned}$$

利用

$$\int_0^1 P_n(x) dx = egin{cases} 1 & n=0 \ rac{1}{2} & n=1 \ 0 & n=2k \ rac{(-1)^k(2k-1)!!}{(2k+2)!!} & n=2k+1 \end{cases}$$

得到

$$C_n = egin{cases} rac{u_1 + u_2}{2} & n = 0 \ rac{3(u_1 - u_2)}{4a} & n = 1 \ 0 & n = 2k \ rac{(u_1 - u_2)(4k + 3)}{2a^{2k + 1}} rac{(-1)^k(2k - 1)!!}{(2k + 2)!!} & n = 2k + 1 \end{cases}$$

得到结果

$$egin{aligned} u(r, heta) &= rac{u_1 + u_2}{2} + rac{3}{4}(u_1 - u_2)\left(rac{r}{a}
ight)P_1(\cos heta) \ &+ rac{u_1 - u_2}{2}\sum_{n=1}^{\infty}rac{(-1)^n(4n+3)(2n-1)!!}{(2n+2)!!}\Big(rac{r}{a}\Big)^{2n+1}P_{2n+1}(\cos heta) \end{aligned}$$

8.

半径为a, 表面黑色的均质球体, 在温度为0的空气中受阳光照射, 阳光的热流强度为q, 求此球体内的稳定温度分布

列出方程

$$\left\{egin{aligned} \Delta_3 u = 0, & r < a \ \left(rac{\partial u}{\partial r} + H u
ight)|_{r=a} = f( heta) = \left\{egin{aligned} q\cos heta & 0 \leq heta \leq rac{\pi}{2} \ 0 & rac{\pi}{2} < heta \leq \pi \end{aligned}
ight.$$

球内通解

$$u(r, heta) = \sum_{n=0}^{\infty} C_n r^n P_n(\cos heta)$$

带入边界条件

$$\sum_{n=0}^{\infty} C_n a^{n-1} (n+Ha) P_n(\cos heta) = f( heta)$$

将 $f(\theta)$ 展开,利用5(1)的结果

$$f(\theta) = \frac{1}{4}P_0(\cos\theta) + \frac{1}{2}P_1(\cos\theta) + \frac{5}{16}P_2(\cos\theta) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-3)!!}{(2n+2)!!} \frac{4n+1}{2}P_{2n}(\cos\theta)$$

比对系数,即可得到结果.不详写了.

一个半径为R,厚度为 $\frac{R}{2}$ 的半空心球,外球面和内球面上的温度始终保持为

$$f( heta) = A \sin^2 rac{ heta}{2}, \quad 0 \le heta \le rac{\pi}{2}$$

底面温度保持为 $\frac{A}{2}$ , 求半空心球内部个点的定常温度.

列出方程

$$\left\{egin{aligned} \Delta_3 u = 0, & rac{R}{2} \leq r \leq R, 0 \leq heta \leq rac{\pi}{2}, \ u|_{r=rac{R}{2}} = u|_{r=R} = A \sin^2rac{ heta}{2} \ u|_{ heta=rac{\pi}{2}} = rac{A}{2} \end{aligned}
ight.$$

通解为

$$u(r, heta) = \sum_{n=0}^{\infty} (C_n r^n + D_n r^{-n-1}) P_n(\cos heta)$$

将边界条件展开

$$A\sin^2rac{ heta}{2}=rac{A}{2}(1-\cos heta)=rac{A}{2}P_0(\cos heta)-rac{A}{2}P_1(\cos heta)$$

带入边界条件,得到

$$\begin{cases} C_0 + \frac{2}{R}D_0 = C_0 + \frac{1}{R}D_0 = \frac{A}{2} \\ \frac{R}{2}C_1 + \frac{4}{R^2}D_1 = RC_1 + \frac{1}{R^2}D_1 = -\frac{A}{2} \\ C_0 + D_0\frac{1}{r} = \frac{A}{2} \end{cases}$$

解得

$$\left\{ egin{array}{ll} C_0 = rac{A}{2} & D_0 = 0 \ C_1 = -rac{3A}{7R} & D_1 = -rac{AR^2}{14} \end{array} 
ight.$$

得到结果

$$u(r, heta) = rac{A}{2} - \left(rac{3r}{7R} + rac{R^2}{14r^2}
ight) A\cos heta$$