长为l的均匀弦,两端点固定,在距离一段(x=0)b处拨离平衡位置h后放开,求此弦的微小横振动。

列出方程

$$\left\{egin{aligned} u_{tt} &= a^2 u_{xx} & t > 0, 0 < x < l \ u(t,0) &= u(t,l) &= 0 \ u(0,x) &= \left\{egin{aligned} rac{h}{b}x & x < b \ rac{h}{b-l}(x-l) & x > b \end{aligned}
ight. & u_t(0,x) &= 0 \end{aligned}
ight.$$

令u = X(x)T(t), 分离变量

$$\frac{X''}{X} = \frac{1}{a^2} \frac{T''}{T} = -\lambda$$

得到方程

$$\begin{cases} X'' + \lambda X = 0 \\ T'' + a^2 \lambda T = 0 \end{cases}$$

由边界条件得到固有值问题

$$X(0) = X(l) = 0$$

由S-L定理,  $\lambda$ 非负. 设 $\lambda = \omega^2$ , 解出通解为

$$X = A\sin\omega x + B\cos\omega x$$

带入固有值问题

$$\begin{cases} X(0) = 0 \Rightarrow B = 0 \\ X(l) = 0 \Rightarrow A\sin\omega l + B\cos\omega l = 0 \end{cases}$$

得到

$$egin{aligned} \sin \omega l &= 0 \ \omega l &= n\pi \ \omega &= rac{n\pi}{l} \end{aligned} \qquad egin{aligned} n &= 0, 1, 2, \dots \ n &= 0, 1, 2, \dots \end{aligned}$$

解出

$$X_n = \sinrac{n\pi}{l}x \quad \lambda = rac{n^2\pi^2}{l^2} \quad n=1,2,\ldots$$

注意n=0时得到零解, 舍去. 将 $\lambda$ 带入解出T,

$$T_n = C_n \sin rac{n\pi a}{l} t + D_n \cos rac{n\pi a}{l} t$$

得到通解为

$$u(t,x) = \sum_{n=1}^{\infty} (C_n \sin rac{n\pi a}{l} t + D_n \cos rac{n\pi a}{l} t) \sin rac{n\pi}{l} x$$

带入初值条件得到

$$u_t(0,x) = \sum_{n=1}^{\infty} rac{n\pi a}{l} C_n \sinrac{n\pi}{l} x = 0 \Rightarrow C_n = 0$$
  $u(0,x) = \sum_{n=1}^{\infty} D_n \sinrac{n\pi}{l} x = egin{cases} rac{h}{b} x & 0 < x \leq b \ rac{h}{b-l} (x-l) & b < x \leq l \end{cases}$ 

则 $D_n$ 为

$$D_n \int_0^l \sin^2 rac{n\pi}{l} x dx = \int_0^b rac{h}{b} x \sin rac{n\pi}{l} x dx + \int_b^l rac{h}{b-l} (x-l) \sin rac{n\pi}{l} x dx$$
  $D_n = rac{2hl^2 \sin rac{n\pi}{l} b}{b(l-b)\pi^2 n^2}$ 

得到解为

$$u(t,x)=rac{2hl^2}{b(l-b)\pi^2}\sum_{n=1}^{+\infty}rac{1}{n^2}\sinrac{n\pi}{l}b\cosrac{n\pi a}{l}t\sinrac{n\pi}{l}x$$

6.

边长为11和12的矩形薄膜,长为11的两边固定,另两边自由,求此薄膜的固有振动.

列出方程

$$\left\{ egin{aligned} u_{tt} = a^2 
abla^2 u & t > 0, 0 < x < l_1, 0 < y < l_2 \ u|_{y=0} = u|_{y=l_2} = 0 \ u_x|_{x=0} = u_x|_{x=l_1} = 0 \end{aligned} 
ight.$$

分离变量u = X(x)Y(y)T(t), 得到方程组

$$\begin{cases} \frac{1}{a^2} \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} \\ Y(0) = Y(l_2) = 0 \\ X'(0) = X'(l_1) = 0 \end{cases}$$

得到两个固有值问题

$$\begin{cases} X'' + \omega^2 X = 0 \\ X'(0) = X'(l_1) = 0 \end{cases} \quad \begin{cases} Y'' + \gamma^2 Y = 0 \\ Y(0) = Y(l_2) = 0 \end{cases}$$

分别解得

$$X_n = \cos rac{m\pi}{l_1} x \qquad \omega = rac{m\pi}{l_1} \quad m = 0, 1, 2, \dots$$
 $Y_n = \sin rac{n\pi}{l_2} y \qquad \gamma = rac{n\pi}{l_2} \quad n = 1, 2, \dots$ 

得到T满足方程

$$T'' + \left[ (rac{m}{l_1})^2 + (rac{n}{l_2})^2 
ight] a^2 \pi^2 T = 0$$

解得

$$T_{mn} = C_{mn} \cos \sqrt{(rac{m}{l_1})^2 + (rac{n}{l_2})^2} a \pi t + D_{mn} \sin \sqrt{(rac{m}{l_1})^2 + (rac{n}{l_2})^2} a \pi t$$

得到解

$$u_{mn}(t,x,y) = \left[ C_{mn} \cos \sqrt{(\frac{m}{l_1})^2 + (\frac{n}{l_2})^2} a\pi t + D_{mn} \sin \sqrt{(\frac{m}{l_1})^2 + (\frac{n}{l_2})^2} a\pi t \right] \cos \frac{m\pi}{l_1} x \sin \frac{n\pi}{l_2} y \quad n \neq 0$$

半径为a的无限长 $(-\infty < z < +\infty)$ 的圆柱体内无自由电荷分布,已知圆柱侧面电位如下,求圆柱内部点位分布

$$(1)u|_{r=a} = A\cos\theta; \quad (2)u|_{r=a} = A + B\sin\theta; \quad (3)u|_{r=a} = \sin 2\theta\cos\theta$$

列出方程,注意方程与z无关

$$\left\{egin{aligned} rac{1}{r}\partial_r(r\partial_r u) + rac{1}{r^2}\partial_ heta^2 u = 0 & 0 < r < a \ u|_{ heta = 0} = u|_{ heta = 2\pi} \end{aligned}
ight.$$

分离变量,得到

$$\left\{ egin{array}{l} rac{rR'+r^2R''}{R}+rac{1}{r^2}rac{\Theta''}{\Theta}=0 \ \Theta(0)=\Theta(2\pi) \end{array} 
ight.$$

得到固有值问题

$$\left\{egin{aligned} \Theta'' + \omega^2\Theta &= 0 \ \Theta(0) &= \Theta(2\pi) \end{aligned}
ight.$$

解得

$$\Theta(\theta) = A_n \cos n\theta + B_n \sin n\theta \quad \omega = n$$

得到 R满足方程

$$r^2 R'' + rR' - n^2 R = 0$$

这是个欧拉方程,用 $t = \ln r$ 代换,得到

$$\frac{d^2}{dt^2}R - n^2R = 0$$

注意讨论n=0, 得到解

$$R = egin{cases} C_n e^{nt} + D_n e^{-nt} = C_n r^n + rac{D_n}{r^n} & n 
eq 0 \ C_0 t + D_0 = C_0 \ln r + D_0 \end{cases}$$

已知r o 0时电势不发散, 舍去发散部分, 得到通解

$$u(r, heta) = \sum_{n=0}^{\infty} r^n (A_n \cos n heta + B_n \sin n heta)$$

分别带入边界条件, 得到解

(1)

$$u(a, heta) = \sum_{n=0}^{\infty} a^n (A_n \cos n heta + B_n \sin n heta) = A\cos heta$$

比对系数,得到

$$A_1=rac{A}{a}$$
 其余为 $0$ 

得到解

$$u(r,\theta) = A \frac{r}{a} \cos \theta$$

$$u(a, heta) = \sum_{n=0}^{\infty} a^n (A_n \cos n heta + B_n \sin n heta) = A + B \sin heta$$

比对系数,得到

$$A_0 = A$$
  $B_1 = \frac{B}{a}$ 

得到解

$$u(r, heta) = A + Brac{r}{a}\sin heta$$

(3)

$$egin{aligned} u(a, heta) &= \sum_{n=0}^\infty a^n (A_n \cos n heta + B_n \sin n heta) \ &= \sin 2 heta \cos heta \ &= rac{1}{2} \sin heta + rac{1}{2} \sin 3 heta \end{aligned}$$

比对系数, 得到解

$$u(r, heta)=rac{1}{2}rac{r}{a} ext{sin}\, heta+rac{1}{2}rac{r^3}{a^3} ext{sin}\,3 heta$$

## 9. 解下列定解问题

(1)

$$\left\{egin{array}{l} u_t = a^2 u_{xx} & t>0, 0 < x < 2l \ u_x(t,0) = u_x(t,2l) = 0 \ u(0,x) = arphi(x) = \left\{egin{array}{l} rac{1}{2A} & |x-l| < A < l \ 0 & rac{1}{2A} x \end{array}
ight. 
ight.$$

求解u(t,x), 并讨论当 $t\to +\infty$ 时及 $A\to 0$ 时解的极限

分离变量得到方程

$$\begin{cases} \frac{1}{a^2} \frac{T'}{T} = \frac{X''}{X} \\ X'(0) = X'(2l) = 0 \end{cases}$$

得到固有值问题

$$\begin{cases} X'' + \omega^2 X = 0 \\ X'(0) = X'(2l) = 0 \end{cases}$$

解得

$$X_n = \cos \frac{m\pi}{2l} x$$
  $w = \frac{m\pi}{2l}$   $m = 0, 1, 2, ...$ 

得到T满足方程

$$T' + \left(\frac{m\pi a}{2l}\right)^2 T = 0$$

注意讨论m=0. 解得

$$T_n = \exp\left[-\left(\frac{m\pi a}{2l}\right)^2 t\right]$$

得到通解

$$u(t,x) = \sum_{m=0}^{\infty} C_m \cos rac{m\pi}{2l} x \expigl[-(rac{m\pi a}{2l})^2 tigr]$$

带入初值条件

$$u(0,x) = \sum_{m=0}^{\infty} C_m \cos rac{m\pi}{2l} x = egin{cases} rac{1}{2A} & l-A < x < l+A \ 0 & ext{ $\sharp$ $x$} \end{cases}$$

注意讨论m=0, 得到系数

$$egin{aligned} C_m \int_0^{2l} \cos^2rac{m\pi}{2l}x dx &= \int_{l-A}^{l+A}rac{1}{2A}\cosrac{m\pi}{2l}x dx \ C_m &= egin{cases} rac{2}{m\pi A}\cos(rac{m\pi}{2})\sin(rac{m\pi A}{2l}) & n 
eq 0 \ rac{1}{2l} & n = 0 \end{cases} \end{aligned}$$

得到通解

$$u(t,x) = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{2l}{m\pi A} \cos(\frac{m\pi}{2}) \sin(\frac{m\pi A}{2l}) \exp\left[-(\frac{m\pi a}{2l})^2 t\right] \cos\frac{m\pi}{2l} x$$

注意m为奇数时,  $\cos(\frac{m\pi}{2})$ 为0, 设n=2m, 得到解

$$u(t,x)=rac{1}{2l}+\sum_{n=1}^{\infty}rac{1}{n\pi A}(-1)^n\sin(rac{n\pi A}{l})\expigl[-(rac{n\pi a}{l})^2tigr]\cosrac{n\pi}{l}x$$

取极限,得到

$$\lim_{t o\infty}u(t,x)=rac{1}{2l}\quad\lim_{A o\infty}u(t,x)=rac{1}{2l}+rac{1}{l}\sum_{n=1}^{\infty}(-1)^n\expigl[-(rac{n\pi a}{l})^2tigr]\cosrac{n\pi}{l}x$$

(3)

$$\left\{egin{aligned} u_{tt} + 2hu_t &= a^2u_{xx} \quad t > 0, 0 < x < l, h$$
为常数, $0 < h < rac{\pi a}{l} \ u(t,0) &= u(t,l) &= 0 \ u(0,x) &= arphi(x) \quad u_t(0,x) &= \psi(x) \end{aligned}
ight.$ 

分离变量得到方程

$$\begin{cases} \frac{T''+2hT'}{T} = a^2 \frac{X''}{X} \\ X(0) = X(l) = 0 \end{cases}$$

得到固有值问题

$$\begin{cases} X'' + \gamma^2 X = 0 \\ X(0) = X(l) = 0 \end{cases}$$

解得

$$X_n = \sin \frac{n\pi}{l} x$$
  $\gamma = \frac{n\pi}{l}$   $n = 1, 2, \dots$ 

得到T满足方程

$$T''+2hT'+\left(rac{n\pi a}{l}
ight)^2T=0$$

今 $T = e^{\lambda t}$ , 得到

$$\lambda^2 + 2h\lambda + \left(rac{n\pi a}{l}
ight)^2 = 0$$

解得

$$\lambda = -h \pm i \sqrt{\left(rac{n\pi a}{l}
ight)^2 - h^2}$$

记
$$\omega_n = \sqrt{\left(rac{n\pi a}{l}
ight)^2 - h^2}$$
, 得到

$$T=e^{-ht}(a_n\cos\omega_n t+b_n\sin\omega_n t)$$

得到通解

$$u(t,x) = \sum_{n=1}^{+\infty} e^{-ht} (a_n \cos \omega_n t + b_n \sin \omega_n t) \sin rac{n\pi}{l} x$$

得到系数为

$$a_n = rac{2}{l} \int_0^l arphi(x) \sin rac{n\pi}{l} x dx \quad b_n = rac{a_n}{\omega_n} h + rac{2}{l \omega_n} \int_0^l \psi(x) \sin rac{n\pi}{l} x dx$$

## 10. 解下列边值问题

(1)

$$\begin{cases} \Delta_2 u = 1 & 1 < r < 2 \\ u|_{r=1} = \frac{5}{4} + \cos^2 \theta \\ u|_{r=2} = 1 + \sin^2 \theta \end{cases}$$

极坐标下方程为

$$rac{1}{r}\partial_r(r\partial_r u)+rac{1}{r^2}\partial_ heta^2 u=1$$

选取特解 $v=\frac{1}{4}r^2$ , 令u=w+v, 则w满足

$$\begin{cases} \Delta_2 w = 0 & 1 < r < 2 \\ w|_{r=1} = 1 + \cos^2 \theta \\ w|_{r=2} = \sin^2 \theta \end{cases}$$

和第七题相同, 通解为

$$w=C_0 \ln r + D_0 + \sum_{n=1}^{\infty} (C_n r^n + \frac{D_n}{r^n}) (A_n \cos n heta + B_n \sin n heta)$$

带入边界条件

$$egin{aligned} w|_{r=1} &= rac{3}{2} + rac{1}{2} \cos 2 heta = D_0 + \sum_{n=1}^{\infty} (C_n + D_n) (A_n \cos n heta + B_n \sin n heta) \ w|_{r=2} &= rac{1}{2} - rac{1}{2} \cos 2 heta = C_0 \ln 2 + D_0 + \sum_{n=1}^{\infty} (2^n C_n + rac{D_n}{2^n}) (A_n \cos n heta + B_n \sin n heta) \end{aligned}$$

比对系数, 得到

$$D_0 = rac{3}{2} \quad C_0 = -rac{1}{\ln 2}$$

$$\begin{cases} A_2(C_2 + D_2) = \frac{1}{2} \\ A_2(4C_2 + \frac{D_2}{4}) = -\frac{1}{2} \end{cases} \Rightarrow \begin{cases} A_2C_2 = -\frac{1}{6} \\ A_2D_2 = \frac{2}{3} \end{cases}$$

得到解

$$w=rac{3}{2}-rac{\ln r}{\ln 2}-rac{1}{6}igg(r^2-rac{4}{r^2}igg)\cos 2 heta$$

加上特解,得到

$$u(r, heta) = w = rac{3}{2} - rac{\ln r}{\ln 2} - rac{1}{6}igg(r^2 - rac{4}{r^2}igg)\cos 2 heta + rac{r^2}{4}$$