Geometry of Image Formation

Pin hole camera: $x = \frac{-fX}{Z}, y = \frac{-fY}{Z}$

Projection of lines $\mathbf{X} = \mathbf{A} + \lambda\mathbf{D}$: $\lim_{\lambda\to\infty} \mathbf{p} = f\frac{\mathbf{A}+\lambda\mathbf{D}}{A_Z+\lambda D_Z} = f\frac{\mathbf{D}}{D_Z}$

Projection of planes $\mathbf{X}\cdot\mathbf{N} = d$:
$\lim_{Z\to\infty} \mathbf{p}\cdot\mathbf{N} = 0 \Rightarrow xN_x + yN_y + fN_z = 0$

Terrestrial perspective:
Ground plane has normal $\mathbf{N} = (0,1,0)$ and projects to $y = 0$.
Camera height h_c; an object of height δY with bottom at $(X, -h_c, Z)$ and top at $(X, \delta Y - h_c, Z)$. The bottom projects to $(fX/Z, -fh_c/Z)$ and top projects to $(fX/Z, f(\delta Y - h_c)/Z)$.

Geometric Transformations

Pose: the position and orientation of the object with respect to the camera. 6 degrees of freedom.
Shape: the coordinates of the points of an object relative to a coordinate frame on the object; invariant when the object undergoes rotations and translations.

Euclidean Transformations: rotations and translation are isometries or regid body motions that preserves distances between any pair of points.

Orthogonal Transformations:
preserves inner products: $\mathbf{a}\cdot\mathbf{b} = \psi(\mathbf{a})\cdot\psi(\mathbf{b})$
preserves norms: $\|\mathbf{a}\| = \|\psi(\mathbf{a})\|$
are isometries: $(\psi(\mathbf{a}) - \psi(\mathbf{b}))\cdot(\psi(\mathbf{a}) - \psi(\mathbf{b})) = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$

Orthogonal matrices:
$\mathbf{A}' = \mathbf{A}^{-1}$, $\det(\mathbf{A})^2 = 1 \Rightarrow \det(\mathbf{A}) = +1$ or -1.
$$\underbrace{\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}}_{\text{rotation, det} = +1}\underbrace{\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}}_{\text{rotation, det} = -1}$$

Group structure of isometries:
$\psi(\mathbf{a}) = \mathbf{A}\mathbf{a} + \mathbf{t}$, $\mathbf{A}$ is orthogonal matrix.
$\psi_1(\mathbf{a}) = \mathbf{A}_1\mathbf{a} + \mathbf{t}_1, \psi_2(\mathbf{a}) = \mathbf{A}_2\mathbf{a} + \mathbf{t}_2,$
$\psi_1\circ\psi_2(\mathbf{a}) = (\mathbf{A}_1\mathbf{A}_2)\mathbf{a} + (\mathbf{A}_1\mathbf{t}_2 + \mathbf{t}_1),$

Skew-symmetric matrix: $\mathbf{S} = -\mathbf{S}'$
Cross product Matrix:
$$\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\wedge\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} = \begin{bmatrix}a_2b_3 - a_3b_2\\ a_3b_1 - a_1b_3\\ a_1b_2 - a_2b_1\end{bmatrix} \Rightarrow \hat{\mathbf{a}} := \begin{bmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{bmatrix}$$

Roderigues Formula:
$\mathbf{R} = e^{\phi\hat{s}} = \mathbf{I} + \sin\phi\hat{\mathbf{s}} + (1-\cos\phi)\hat{\mathbf{s}}^2$,
$\mathbf{s}$ is a unit vector along ω and $\phi = \|\omega\|t$ is the total amount of rotation.

Affine transformations is a nonsingular linear transformation followed by a translation. $\psi(\mathbf{a}) = \mathbf{A}\mathbf{a} + \mathbf{t}$ ($\det\mathbf{A} \neq 0$) and preserves parallelism and midpoints and does **not** preserve lengths, angles, areas.
Ex. rotation, anisotropic scaling, sheer.

Degrees of freedom:
2D: isometry: $1+2=3$, affine: $4+2=6$
3D: isometry: $3+3=6$, affine: $9+3=12$

Optical Flow

Motion at 3D world projects to motion in the image. At every point (x,y) in the image we get a 2D vector, corresponding to the motion of the feature located at that point.
egomotion is the optical flow field of a moving observer.

$\mathbf{t}$ is translational velocity of camera, $\boldsymbol{\omega}$ is angular velocity of camera.
$$\begin{bmatrix}u\\v\end{bmatrix} = \frac{1}{Z}\begin{bmatrix}-1 & 0 & x\\ 0 & -1 & y\end{bmatrix}\begin{bmatrix}t_x\\t_y\\t_z\end{bmatrix} + \begin{bmatrix}xy & -(1+x^2) & y\\ 1+y^2 & -xy & -x\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}$$

Radiometry of Image Formation

Irradiance:
$I(x,y)$ measures how much light is captured at pixel (x,y).
Radiant power per unit area with units $\mathrm{W/m^2}$, denoted by E.

$L = \frac{\text{Power}}{(dAcos\theta)(d\omega)}$, units are $Wm^{-2}sr^{-1}$.

Lambertian model: $L = \rho\lambda n\cdot s$