W03 - Bayesian hypothesis comparison

Vaccine vs control treatment

f_V = prob of disease in V group

f_C = " " in C group

H_1 : V is better than control : $f_V < f_C$

H_0 : V = C : $f_V = f_C$

$$P(H_1|D) = \frac{P(D|H_1)\cdot P(H_1)}{P(D)}$$

$$P(H_2|D) = \frac{P(D|H_2)\cdot P(H_2)}{P(D)}$$

$$P(D) = \sum_{\text{all hypotheses}} P(D|H_i)\cdot P(H_i)$$

but

$$\frac{P(H_1|D)}{P(H_2|D)} = \frac{P(D|H_1)\cdot P(H_1)}{P(D|H_2)\cdot P(H_2)}$$

if $P(H_1) = P(H_2)$ prior

$$\boxed{\frac{P(H_1|D)}{P(H_2|D)} = \frac{P(D|H_1)}{P(D|H_2)}}$$

$H_1 \longleftrightarrow H_0$ → compare the evidences of D under the two hypotheses

$P(D|H)$ → integrate to all values of the parameters that define H.

$$P(D|H_1) = \int_{f_c} \int_{\substack{f_v \\ f_v < f_c}} P(D|f_v f_c) \cdot \underset{\substack{\shortparallel \\ 1}}{P(f_c f_v)} \, df_c \, df_v$$

$$P(D|H_0) = \int_f P(D|f_v = f_c) \; \underset{\substack{\shortparallel \\ 1}}{P(f f)} \, df$$

$$\frac{P(D|H_1)}{P(D|H_2)} = \frac{\int_0^1 df_c \int_0^{f_c} df_v \; f_v^{n_v} (1-f_v)^{N_v - n_v} \, f_c^{n_c} (1-f_c)^{N_c - n_c}}{\int_0^1 df \; f^{n_v} (1-f)^{N_v - n_v} \, f^{n_c} (1-f)^{N_c - n_c}}$$

$$\left.\begin{matrix} N_c = 10 \\ n_c = 3 \end{matrix}\right] \quad \left.\begin{matrix} N_v = 30 \\ n_v = 3 \end{matrix}\right] \quad \simeq \frac{0.003}{0.001} \simeq 3$$

$H_1 : H_0$

$1 : 1$ ← data

$\downarrow$

$3 : 1$

→ go to code

Occam's razor

If several models can explain the observations, always go with the simpler model.

why?
- simplicity
- aesthetics
- avoid over fitting
- bayes tells us so

Consider two nested hypotheses

H_1 # param H_1 > # param H_2.

H_2 $H_2 \subset H_1$

(x_i, y_i) H_1 : $y_i = w_0 + w_1 x_i$

H_2 : $y_i = w_0$

* If ML fit of parameters to data.
  H_1 will always be more favorable

* If bayesian comparison of H_1, H_2?

$$\frac{P(H_1|D)}{P(H_2|D)} = \frac{P(D|H_1)}{P(D|H_2)}$$

$$P(D|H_1) = \int_{P_1} P(D|P_1)\,P(P_1)\,dP_1$$

$$P(D|H_2) = \int_{P_2} P(D|P_2)\cdot P(P_2)\,dP_2$$

$$\log P(D|P_1) \simeq \log P(D|P_1^*) + \frac{1}{2}\left.\frac{\delta^2 \log P(D|P_1)}{\delta P_1^2}\right|_{P_1=P_1^*}(P_1-P_1^*)^2$$

$$= \log P(D|P_1^*) - \frac{(P_1-P_1^*)^2}{2\sigma_1^{*2}}$$

$$\sigma_1^{*2} =: -\frac{1}{\left.\frac{\delta^2 \log P(D|P_1)}{\delta P_1^2}\right|_{P_1=P_1^*}}$$

Then

$$P(D|P_1) \simeq P(D|P_1^*)\, e^{-\frac{(P_1-P_1^*)^2}{2\sigma_1^2}}$$

$$P(P_1) = \begin{cases} \frac{1}{P^+ - P^-} \equiv \frac{1}{\sigma_1} & P^- \le P_1 \le P^+ \\ 0 & \end{cases}$$

$$P(D|H_1) = P(D|P_1^*) \int_{P^-}^{P^+} e^{-\frac{(P_1-P_1^*)^2}{2(\sigma_1^*)^2}} \frac{1}{\sigma_1}\,dP_1$$

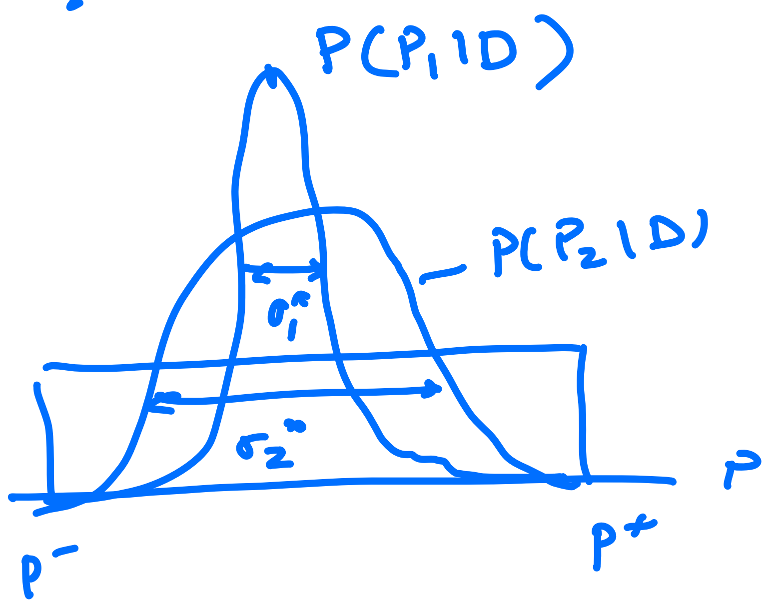
$$\simeq P(D|P_1^*)\sqrt{2\pi}\;\sigma_1^*/\sigma_1$$

$$\frac{P(H_1|D)}{P(H_2|D)} = \underbrace{\frac{P(D|P_1^*)}{P(D|P_2^*)}}_{\text{Maximum Likelihood term}} \; \underbrace{\frac{\sigma_1^*/\sigma_1}{\sigma_2^*/\sigma_2}}_{\text{Bayesian Occam's razor}}$$

$H_1 \supset H_2$ $P(D|P_1^*) > P(D|P_2^*)$

depends on σ_1^*/σ_2

if $\sigma_1^* < \sigma_2^*$ H_2 has chance to win



Example

One or two different bacterial colonies?