# 5CM507 Graphics

## Lecture A02b 3D Transformations

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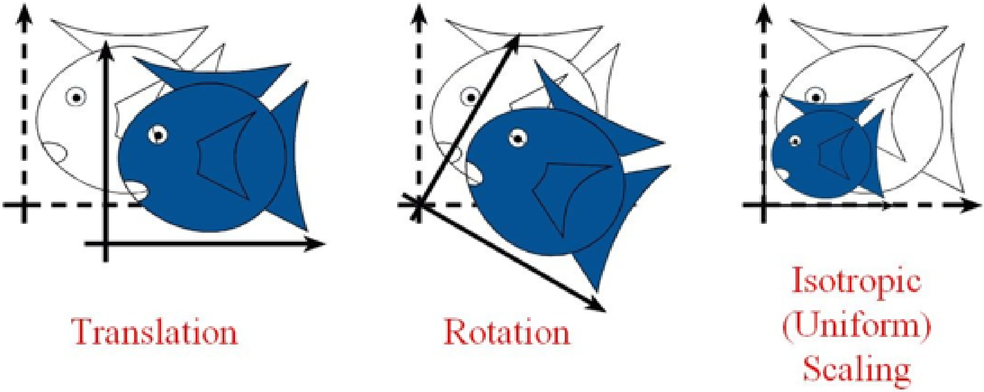
- 2D and 3D Scaling and Translation
- Homogeneous Coordinates
- 2D and 3D Rotation

# Common Transformations

- We need to find transformations that follow certain rules

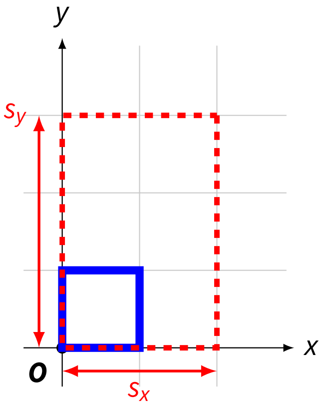


# Transformations

# 2D and 3D Scaling

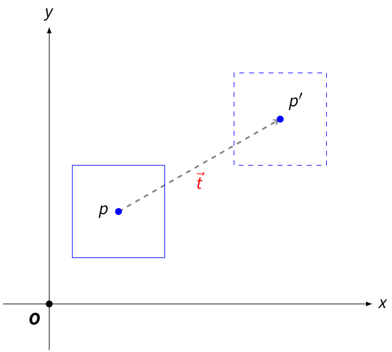
- If we want to scale a rigid object centred at the origin by a scale factor of $S = S(s_x, s_y)$
- that is to transform every point $p$ on the object to $p'$ with the following equations
  - $x' = s_x x \quad y' = s_y y$
- Matrix form: $\mathbf{p}' = \mathbf{S}\mathbf{p}$

$$\mathbf{S} = \mathbf{S}(\mathrm{s}_x, \ \mathrm{s}_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$



# 2D and 3D Translation

- Move (translate, displace) a point to a new location
  - $x' = x + t_x \quad y' = y + t_y$
- Vector form : displacement determined by a vector $\vec{T}$
  - $p' = p + \vec{t}$
  - $\mathbf{p} = \begin{bmatrix} x \\ y \end{bmatrix}, \mathbf{p}' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \mathbf{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$



# Homogenous Coordinates

# Homogenous Coordinates

## The problem

Translation cannot be represented as a matrix like

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{but in} \begin{cases} x' = x + t_x \\ y' = y + t_y \\ y' = y + t_z \end{cases} \quad t_x, t_y, t_z \text{ do not depend on } x, y, z$$

Idea: add an extra "scaling" dimension:

$$\mathbf{P} = [x \quad y \quad z \quad 1]^T$$

$$\mathbf{P}' = \begin{bmatrix} x' & y' & z' & w' \end{bmatrix}^T \iff [\tfrac{x}{w} \quad \tfrac{y}{w} \quad \tfrac{z}{w} 1]^T$$

This is why you see this in the vertex shader: `gl_Position = vec4(pos, 1.0);`

# 3D Translation and Scaling Matrix

$$\begin{cases} x' = x + t_x \\ y' = y + t_y \\ z' = z + t_z \end{cases} \qquad \mathbf{T}\left(\mathrm{t_x}, \mathrm{t_y}, \mathrm{t_z}\right) = \begin{bmatrix} 1 & 0 & 0 & \mathrm{t_x} \\ 0 & 1 & 0 & \mathrm{t_y} \\ 0 & 0 & 1 & \mathrm{t_z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \mathrm{t_x} \\ 0 & 1 & 0 & \mathrm{t_y} \\ 0 & 0 & 1 & \mathrm{t_z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

```
glm::mat4 glm::translate(glm::vec3(tx, ty, tz))
```

$$\begin{cases} x' = s_x x \\ y' = s_y y \\ z' = s_z z \end{cases} \qquad \mathbf{S}\left(s_x, s_y, s_z\right) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

```
glm::mat4 glm::scale(glm::vec3(sx, sy, sz))
```

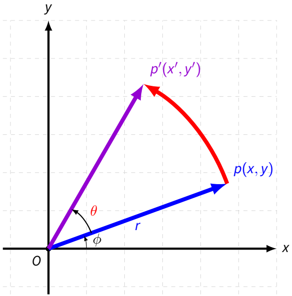
Now multiple transforms can be concatenated using matrix multiplication.

# 2D and 3D rotations

# 2D Rotation on $XY$ to 3D rotation about $Z$

- Rotate a point $\mathbf{p}(x, y)$ about the origin for angle $\theta$ in counterclockwise direction to $\mathbf{p}'(x', y')$
  - $x' = \cos(\theta)x - \sin(\theta)y$
  - $y' = \sin(\theta)x + \cos(\theta)y$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$




- Add $z' = z$ => 3D rotation about $Z$ axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

# 3D Rotation about global $X$ and $Y$ axes

For rotation about $X$ axis, x is unchanged

$$\mathbf{R_x}(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For rotation about $Y$ axis, y is unchanged

$$\mathbf{R_y}(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

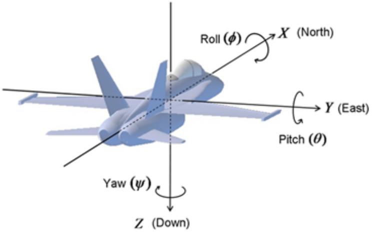
Note the sign of $\sin(\theta)$ changed

Concatenating the rotations about global $X$, $Y$, $Z$, we can achieve arbitrary 3D rotation.

$\mathbf{R} = R(\theta_x)R(\theta_y)R(\theta_z)$ is a rotation first about $Z$, then $Y$, then $X$

# General 3D Rotation about local axis - Euler angles

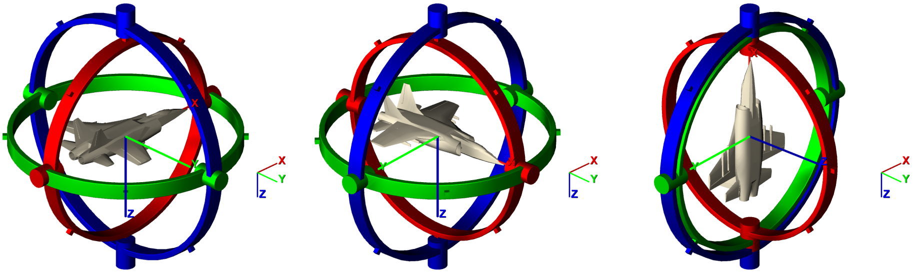
- Euler angles: 3 angles about local axes, such as Pitch, Yaw, Roll of an aircraft
- 3 rotation matices produce the global rotation
- $\mathbf{R} = R_a(\alpha)R_b(\beta)R_c(\gamma)$
- In reverse order for Local->Global transform : $R_a$ : $1^{st}$, $R_b$: $2^{nd}$, $R_c$ :$3^{rd}$
- **Note: transform matrices about the local/body system multiplied in reverse order**
- Why reversed: Imagine transforming the final local frame back to the original frame
- Can be X-Y-X or other order of rotations



# Euler Angles - Gimbal lock example

- A $90^{\circ}$ second rotation => a third axis coincides with the first
- Lose a degree of freedom, a real gimbal gyroscope get locked
- For graphics: change multiple rotation angles to leave the state, but result in unintuitive motion

Video of a real gimbal   A Unity Gimbal lock demo




$z = -x$ leading to a 2D rotation; for a real gimbal, Yaw is lost

# More 3D Rotations - Rotate about arbitrary axis

- Rodrigues Formula

$$\mathbf{R} = \begin{bmatrix} \cos\theta + u_x^2(1-\cos\theta) & u_x u_y(1-\cos\theta) - u_z\sin\theta & u_x u_z(1-\cos\theta) + u_y\sin\theta \\ u_y u_x(1-\cos\theta) + u_z\sin\theta & \cos\theta + u_y^2(1-\cos\theta) & u_y u_z(1-\cos\theta) - u_x\sin\theta \\ u_z u_x(1-\cos\theta) - u_y\sin\theta & u_z u_y(1-\cos\theta) + u_x\sin\theta & \cos\theta + u_z^2(1-\cos\theta) \end{bmatrix}$$

- Quaternions : good for smooth rotation interpolation

$$\mathbf{R} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2\,(q_1 q_2 + q_0 q_3) & 2\,(q_1 q_3 - q_0 q_2) \\ 2\,(q_1 q_2 - q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2\,(q_0 q_1 + q_2 q_3) \\ 2\,(q_0 q_2 + q_1 q_3) & 2\,(q_2 q_3 - q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

No need to worry, you only need to call glm APIs to generate the rotation matrix.

```
glm::mat4 glm::rotate(glm::radians(angle_in_degree), glm::vec3(axis_x, axis_y,
axis_z));
```

# Summary

- Representing affine transformations as matrices
  - Homogeneous coordinates
- Divide and conquer
  - Decompose complex transformations into simple ones
  - 3D rotations
  - Hierarchical modelling (the next week)
- 3D rotations are HARD, but using APIs is not !
- Matrix order is critical: transform about the local/body system multiplied in reverse order

Questions?