UNIVERSITY OF DERBY

5CM507 Graphics

Lecture A02b Transformation

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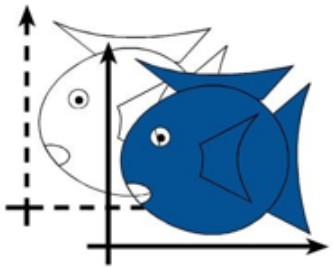
Contents

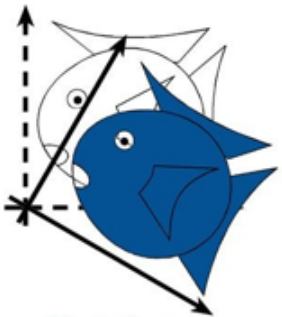
- 2D and 3D Scaling and Translation
- Homogeneous Coordinates
- 2D and 3D Rotation

Common Transformations

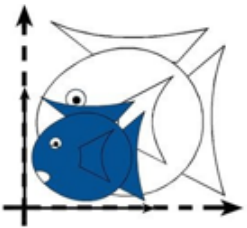
- We need to find transformations that follow certain rules



Translation



Rotation

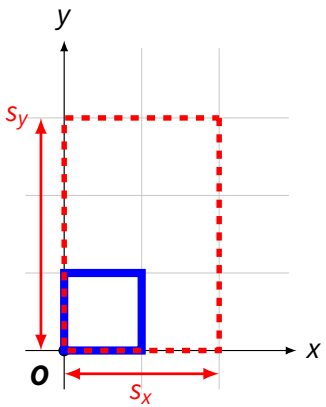


Isotropic (Uniform) Scaling

Transforms

2D and 3D Scaling

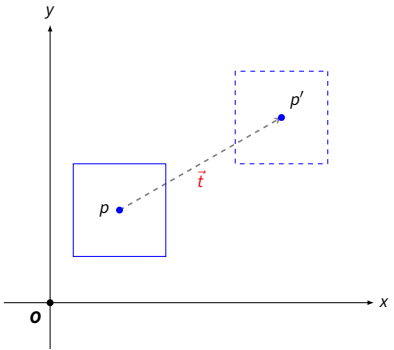
- When you add a model to a scene, you may need to resize it first by a scale factor of $S = S(s_x, s_y, s_z)$
- that is to transform every point $p(x, y, z)$ on the object to $p'(x', y', z') : x' = s_x x \quad y' = s_y y \quad y' = s_z z$
- in matrix form: $\mathbf{p}' = \mathbf{Sp}$
- 2D matrix : $\mathbf{S}\,(\mathrm{s}_x,\ \mathrm{s}_y)\,\mathbf{p} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
- 3D matrix : $\mathbf{S}\,(\mathrm{s}_x,\ \mathrm{s}_y,\ \mathrm{s}_z)\,\mathbf{p} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$



2D and 3D Translation

- Move or position (translate) a model to/at a location
  - $x' = x + t_x \quad y' = y + t_y \quad z' = z + t_z$
- Vector form : displacement determined by a vector

$$\vec{t}\,p' = p + \vec{t} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \end{bmatrix},$$



Homogenous Coordinates

Homogenous Coordinates

The problem

Translation cannot be represented as a matrix like

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ but in } \begin{cases} x' = x + t_x \\ y' = y + t_y \\ y' = y + t_z \end{cases} \quad t_x, t_y, t_z \text{ do not depend on } x, y, z$$

Idea: add an extra "scaling" dimension:

$\mathbf{P} = [x \quad y \quad z \quad 1]^T$

$\mathbf{P}' = \begin{bmatrix} x' & y' & z' & w' \end{bmatrix}^T \iff [\frac{x}{w} \quad \frac{y}{w} \quad \frac{z}{w} \quad 1]^T$

You will see $w \neq 1$ in perspective projection.

This is why you see this in the vertex shader: `gl_Position = vec4(pos, 1.0);`

3D Translation and Scaling Matrix

$$\begin{cases} x' = x + t_x \\ y' = y + t_y \\ z' = z + t_z \end{cases} \qquad \mathbf{T}(\mathrm{t_x}, \mathrm{t_y}, \mathrm{t_z}) = \begin{bmatrix} 1 & 0 & 0 & \mathrm{t_x} \\ 0 & 1 & 0 & \mathrm{t_y} \\ 0 & 0 & 1 & \mathrm{t_z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \mathrm{t_x} \\ 0 & 1 & 0 & \mathrm{t_y} \\ 0 & 0 & 1 & \mathrm{t_z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

```
glm::mat4 glm::translate(glm::vec3(tx, ty, tz))
```

$$\begin{cases} x' = s_x x \\ y' = s_y y \\ z' = s_z z \end{cases} \qquad \mathbf{S}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

```
glm::mat4 glm::scale(glm::vec3(sx, sy, sz))
```

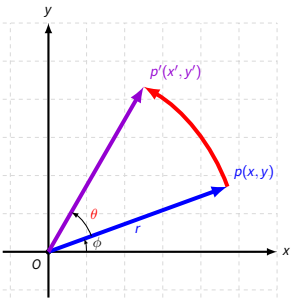
Now multiple transforms can be concatenated using matrix multiplication.

2D and 3D rotations

2D Rotation on XY plane => 3D rotation about Z

- Rotate a point $\mathbf{p}(x, y)$ about the origin for angle θ in counterclockwise direction to $\mathbf{p}'(x', y')$

$$\begin{cases} x' = \cos(\theta)x - \sin(\theta)y \\ y' = \sin(\theta)x + \cos(\theta)y \end{cases} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



- Add $z' = z$ => 3D rotation about Z axis

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

3D Rotation about global X and Y axes

Keep x unchanged => rotate about X axis

$$\mathbf{R_x}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Keep y unchanged => rotate about Y axis

$$\mathbf{R_y}(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Homogeneous coordinate is trivial:

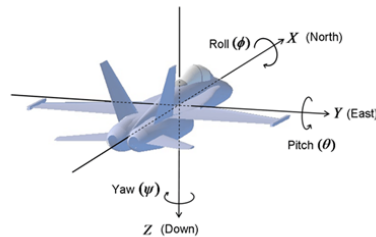
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} r_{00} & r_{01} & r_{00} & 0 \\ r_{10} & r_{11} & r_{12} & 0 \\ r_{20} & r_{21} & r_{22} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Idea: concatenating the rotations about global X, Y, Z, we can achieve arbitrary 3D rotation.

$\mathbf{R}(\mathbf{p}) = R(\theta_x)R(\theta_y)R(\theta_z)\mathbf{p}$ is a rotation of point $\mathbf{p}$ in ZYX order.

General 3D Rotation about 3 major axes - Euler angles

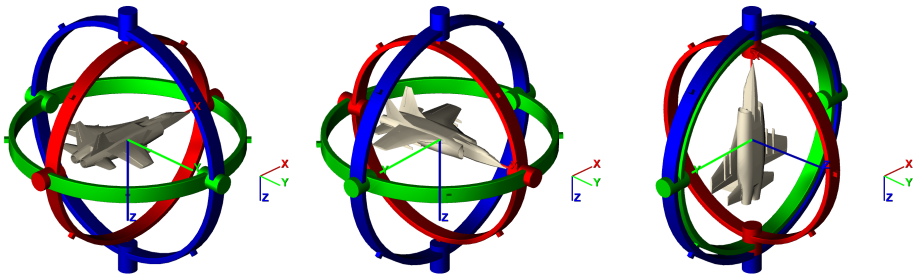
- Euler angles: 3 angles about 3 major axes
  - Local/body frame: intrinsic, e.g. Pitch, Yaw, Roll
  - Global/external frame: extrinsic
- $\mathbf{R} = R_a(\alpha)R_b(\beta)R_c(\gamma)$
- **Note**: for intrinsic rotation, matrix order is reversed: R_a 1st but multiplied last
  - We will revisit the topic in the next week.
- 12 orders of rotations, such as *XYZ*, *XZX*, Wiki



Euler Angles - Gimbal lock example

- A 90° second rotation => a third axis coincides with the first
- Lose a degree of freedom, a real gimbal gyroscope gets locked
- In CG: change multiple rotation angles to leave the state, but results in unintuitive motion

[Video of a real gimbal] [A Unity Gimbal lock demo]



$z = -x$ leading to a 2D rotation; for a real gimbal, Yaw is lost

More 3D Rotations - Rotate about arbitrary axis

- Rodrigues Formula

$$\mathbf{R} = \begin{bmatrix} \cos\theta + u_x^2(1-\cos\theta) & u_x u_y(1-\cos\theta) - u_z\sin\theta & u_x u_z(1-\cos\theta) + u_y\sin\theta \\ u_y u_x(1-\cos\theta) + u_z\sin\theta & \cos\theta + u_y^2(1-\cos\theta) & u_y u_z(1-\cos\theta) - u_x\sin\theta \\ u_z u_x(1-\cos\theta) - u_y\sin\theta & u_z u_y(1-\cos\theta) + u_x\sin\theta & \cos\theta + u_z^2(1-\cos\theta) \end{bmatrix}$$

- Quaternions : good for smooth rotation interpolation

$$\mathbf{R} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 + q_0 q_3) & 2(q_1 q_3 - q_0 q_2) \\ 2(q_1 q_2 - q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_0 q_1 + q_2 q_3) \\ 2(q_0 q_2 + q_1 q_3) & 2(q_2 q_3 - q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

No need to worry, you only need to call glm APIs to generate the rotation matrix.

```
glm::mat4 glm::rotate(glm::radians(angle_in_degree), glm::vec3(axis_x, axis_y,
                                                               axis_z));
```

Composite Transforms

Composite Transforms

- Composite transform matrix can be calculated by multiplying simple transformation matrices.
  - $\mathbf{M} = \mathbf{M_n} \ldots \mathbf{M_i} \ldots \mathbf{M_2} \mathbf{M_1}$
- Order of Matrix Multiplications (transforms)
  - OpenGL & GLSL (column-major matrices): right to left
    - $p' = \mathbf{T_3} \mathbf{T_2} \mathbf{T_1} p = \mathbf{T_3}(\mathbf{T_2}(\mathbf{T_1} p))$
  - Direct3D & HLSL (row-major matrices): left to right
    - $p'^T = p^T \mathbf{T_1}^T \mathbf{T_2}^T \mathbf{T_3}^T = ((p^T \mathbf{T_1}^T)\mathbf{T_2}^T)\mathbf{T_3}^T$

We will use the OpenGL convention throughout this module.

Composition of Translation and Rotation **TR**

A common transform composite is the Translation and Rotation couple.

Object view: rotates followed by a translation in body frame.

$$\mathbf{M} = \mathbf{R}T(t_x, t_y, t_z)$$

$$= \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_{00} & r_{01} & r_{02} & 0 \\ r_{10} & r_{11} & r_{12} & 0 \\ r_{00} & r_{01} & r_{02} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{00} & r_{01} & r_{02} & t_x \\ r_{10} & r_{11} & r_{12} & t_y \\ r_{00} & r_{01} & r_{02} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{bmatrix} \mathbf{R}_{3\times 3} & \vec{\mathbf{t}} \\ 0 & 1 \end{bmatrix}$$

A combined translation and rotation can be easily written into one matrix.

Chained in hierarchical transforms (the next week) $\mathbf{M_n} = \mathbf{T_1 R_1} \ldots \mathbf{T_i R_i} \ldots \mathbf{T_{n-1} R_{n-1} T_n R_n}$

Composition of Rotation and Translation **RT**

Object view: translates relative to the world frame then rotates in body frame, e.g. View Transform

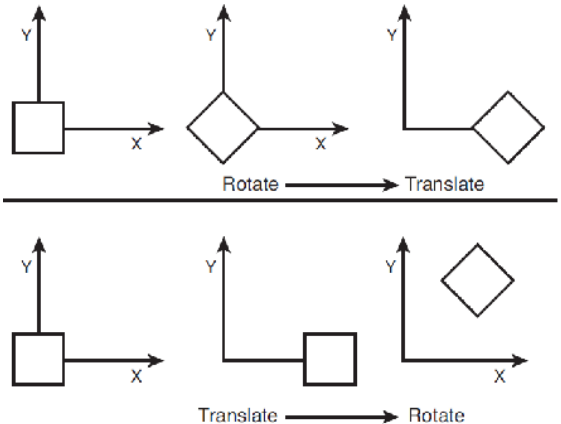
$$\mathbf{M} = \mathbf{RT}(t_x, t_y, t_z)$$

$$= \begin{pmatrix} r_{00} & r_{01} & r_{02} & 0 \\ r_{10} & r_{11} & r_{12} & 0 \\ r_{00} & r_{01} & r_{02} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{00} & r_{01} & r_{02} & r_{00}t_x + r_{01}t_y + r_{02}t_z \\ r_{10} & r_{11} & r_{12} & r_{10}t_x + r_{11}t_y + r_{12}t_z \\ r_{00} & r_{01} & r_{02} & r_{20}t_x + r_{21}t_y + r_{22}t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} r_{00} & r_{01} & r_{02} & \vec{r_0} \cdot \vec{\mathbf{t}} \\ r_{10} & r_{11} & r_{12} & \vec{r_1} \cdot \vec{\mathbf{t}} \\ r_{00} & r_{01} & r_{02} & \vec{r_2} \cdot \vec{\mathbf{t}} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{bmatrix} \mathbf{R}_{3\times3} & \mathbf{R}_{3\times3}\vec{\mathbf{t}} \\ 0 & 1 \end{bmatrix}$$

The Matrix Order Matters

- Be careful with transform/matrix orders.
- **TR** $\neq$ **RT**, both can happen, depending on the reference frame.
- Matrix multiplication is NOT commutative!
- The **T** and **R** matrix order depends on the reference frame of translation.
- We will revisit the topic the next week.



Summary

- Representing transforms as matrices
  - Homogeneous coordinates
- Divide and conquer
  - Decompose complex transformations into simple ones
  - 3D rotations (HARD math, easy APIs)
  - Composite translation and rotation
- Matrix order matters

Questions?

You are welcome to take the poll !