COMS 4771 Machine Learning (Spring 2018)
Problem Set #0

Yufei Zhao - `yz3170@columbia.edu`

January 21, 2018

Problem 1

(a)

X	1	2	3
P(X)	0.3	0.3	0.4

(b)

$$P(Y=1|X=2) = \frac{P(X=2, Y=1)}{P(X=2)} = \frac{2}{3}$$

(c)

Y	1	2
P(Y)	0.6	0.4

$$E[Y] = \sum y \cdot P(y) = 1.4$$

$$E[Y^2] = \sum y^2 \cdot P(y) = 2.2$$

$$D[Y] = E(Y - E(Y))^2 = E(Y^2) - E(Y)^2 = 0.24$$

(d)

$$P(X=1|Y=1) = \frac{0.1}{0.6} = \frac{1}{6}$$

$$P(X=2|Y=1) = \frac{0.2}{0.6} = \frac{1}{3}$$

$$P(X=3|Y=1) = \frac{0.3}{0.6} = \frac{1}{2}$$

$$E[f(X)|Y=1] = \sum x^2 \cdot P(X=x|Y=1) = 6$$

(e)

$$\begin{cases} P(X=1|Y=2)=0.5 \\ P(X=2|Y=2)=0.25 \\ P(X=3|Y=2)=0.25 \end{cases} \Rightarrow E[f(X)|Y=2]=3.75$$

$$\Rightarrow E[E[f(X)|Y]] = E[f(X)|Y=1]\cdot P(Y=1) + E[f(X)|Y=2]\cdot P(Y=2) = 5.096$$

Problem 2

(a)

$$\int_{-\infty}^{+\infty} \frac{1}{Z} f(x) dx = 1$$

$$\begin{aligned}
\Rightarrow Z &= \int_{0}^{+\infty} (e^{-\theta x} + e^{-2\theta x}) dx \\
&= (-\frac{1}{\theta} e^{-\theta x} - \frac{1}{2\theta} e^{-2\theta x})|_{0}^{+\infty} \\
&= \frac{1}{\theta} + \frac{1}{2\theta} \\
&= \frac{3}{2\theta}
\end{aligned}$$

(b)

$$\begin{aligned}
E[X] &= \int_{-\infty}^{+\infty} x \cdot f(x) dx \\
&= \frac{2\theta}{3} \int_{0}^{+\infty} x \cdot (e^{-\theta x} + e^{-2\theta x}) dx \\
&= \frac{2\theta}{3} (\frac{1}{\theta^2} + \frac{1}{4\theta^2}) \\
&= \frac{5}{6\theta}
\end{aligned}$$

(c)

$$\begin{aligned}
E[X^2] &= \frac{2\theta}{3} \int_{0}^{+\infty} x^2 \cdot (e^{-\theta x} + e^{-2\theta x}) dx \\
&= \frac{2\theta}{3} (\frac{2}{\theta^3} + \frac{1}{4\theta^3}) = \frac{3}{2\theta^2} \\
\Rightarrow D[X] &= E[X^2] - (E[X])^2 \\
&= \frac{3}{2\theta^2} - \frac{25}{36\theta^2} \\
&= \frac{29}{36\theta^2}
\end{aligned}$$

Problem 3

Let event
A: a person has the generic disease
B: test result of a person is "yes"
C: a person has red hair

$P(B) = 99\% \cdot P(A \cup C) + 1\% \cdot P(\overline{A} \cap \overline{C}) = 99\% \cdot (1\% + 1\% - 1\% \times 1\%) + 1\% \times 99\% \times 99\% = 2.97\%$
$P(A) = 1\%$
$P(B|A) = 99\%$

$$\Rightarrow P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} = 66.7\%$$

Problem 4

(a)

Let a, b to denote two matrix that produce A as in problem,
so that $A = a \cdot b \cdot a$

$rank(b) = 5 \Rightarrow rank(A) \leq 5$
$rank(a) = 6 \Rightarrow rank(ab) = rank(ba) = rank(b) = 5$
$\Rightarrow rank(A) = rank(b) = 5$

So dimension of R is 5

(b)

$a^2 = I_6 \Rightarrow A^3 = a \cdot b^3 \cdot a$

$$\Rightarrow A^3 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & 0 \\ 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

if λ is eigenvalue of A, then $det(\lambda I_6 - A) = 0$

$$\Rightarrow \begin{vmatrix} \lambda+1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda-\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda-\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda-\frac{1}{8} & \frac{1}{8} & 0 \\ 0 & 0 & 0 & \frac{1}{8} & \lambda-\frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda-\frac{1}{4} \end{vmatrix} = 0$$

$$\Rightarrow \lambda(\lambda+1)(\lambda-\tfrac{1}{8})^2(\lambda-\tfrac{1}{64})^2 = 0$$

So, the largest eigenvalue of A^3 is $\frac{1}{8}$

(c)

with eigenvalue of $\frac{1}{2}$

$$(\tfrac{1}{2} \cdot I_6 - A) \cdot \vec{x} = \vec{0} \Rightarrow \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{8} & -\frac{1}{8} & 0 \\ 0 & 0 & 0 & -\frac{1}{8} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_6 = 0 \\ x_4 + x_5 = 0 \end{cases}$$

So, corresponding eigenvectors are $[0, 1, 0, 0, 0, 0]^T$ and $[0, 0, 1, 0, 0, 0]^T$

and $V = c_1 \cdot [0, 1, 0, 0, 0, 0]^T + c_2 \cdot [0, 0, 1, 0, 0, 0]^T$

Problem 5

(a)

$$\begin{aligned}
J(x) &= ||Ax-b||_2^2 + \frac{1}{4}||x||_2^2 \\
&= (-x_1-1)^2 + (\frac{1}{2}x_2-1)^2 + (\frac{1}{2}x_3-1)^2 + (\frac{1}{8}x_4+\frac{1}{8}x_5-1)^2 + (\frac{1}{8}x_4+\frac{1}{8}x_5-1)^2 + \\
&(\frac{1}{4}x_6-1)^2 + \frac{1}{4}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2) \\
&= \frac{5}{4}x_1^2 + 2x_1 + \frac{1}{2}x_2^2 - x_2 + \frac{1}{2}x_3^2 - x_3 + \frac{9}{32}x_4^2 + \frac{9}{32}x_5^2 + \frac{1}{16}x_4x_5 - \frac{1}{2}x_4 - \frac{1}{2}x_5 + \\
&\frac{5}{16}x_6^2 - \frac{1}{2}x_6 + 6
\end{aligned}$$

$$\Rightarrow \nabla J(x) = \begin{bmatrix} \frac{5}{2}x_1+2 \\ x_2-1 \\ x_3-1 \\ \frac{9}{16}x_4+\frac{1}{16}x_5-\frac{1}{2} \\ \frac{9}{16}x_5+\frac{1}{16}x_4-\frac{1}{2} \\ \frac{5}{8}x_6-\frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \nabla J(x)|_{x=(0,0,0,0,0,0)} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ -0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$$

(b)

$$\text{Let } \nabla J(x) = \vec{0} \Rightarrow \begin{cases} \frac{5}{2}x_1+2=0 \\ x_2-1=0 \\ x_3-1=0 \\ \frac{9}{16}x_4+\frac{1}{16}x_5-\frac{1}{2}=0 \\ \frac{9}{16}x_5+\frac{1}{16}x_4-\frac{1}{2}=0 \\ \frac{5}{8}x_6-\frac{1}{2}=0 \end{cases} \quad \Rightarrow \vec{x} = [-\frac{4}{5}, 1, 1, \frac{4}{5}, \frac{4}{5}, \frac{4}{5}]$$