

I/Q diagram of coupled system and logarithmic singularity

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Nov 18th 2019

Transmission of a single resonance

$$S_{21} = 1 - \frac{\kappa_1}{i(\omega - \omega_1) + \kappa + \gamma}$$

Realising the Denominator

$$\begin{aligned} S_{21} &= \frac{[i(\omega - \omega_1) + \gamma_1][i(\omega - \omega_1) + \kappa_1 + \gamma_1]}{[i(\omega - \omega_1) + \kappa_1 + \gamma_1][i(\omega - \omega_1) - \kappa_1 - \gamma_1]} \\ &= \frac{(\omega - \omega_1)^2 + \gamma_1(\gamma_1 + \kappa_1) + i\kappa_1(\omega - \omega_1)}{(\omega - \omega_1)^2 + (\kappa_1 + \gamma_1)^2} \end{aligned}$$

Define the variable

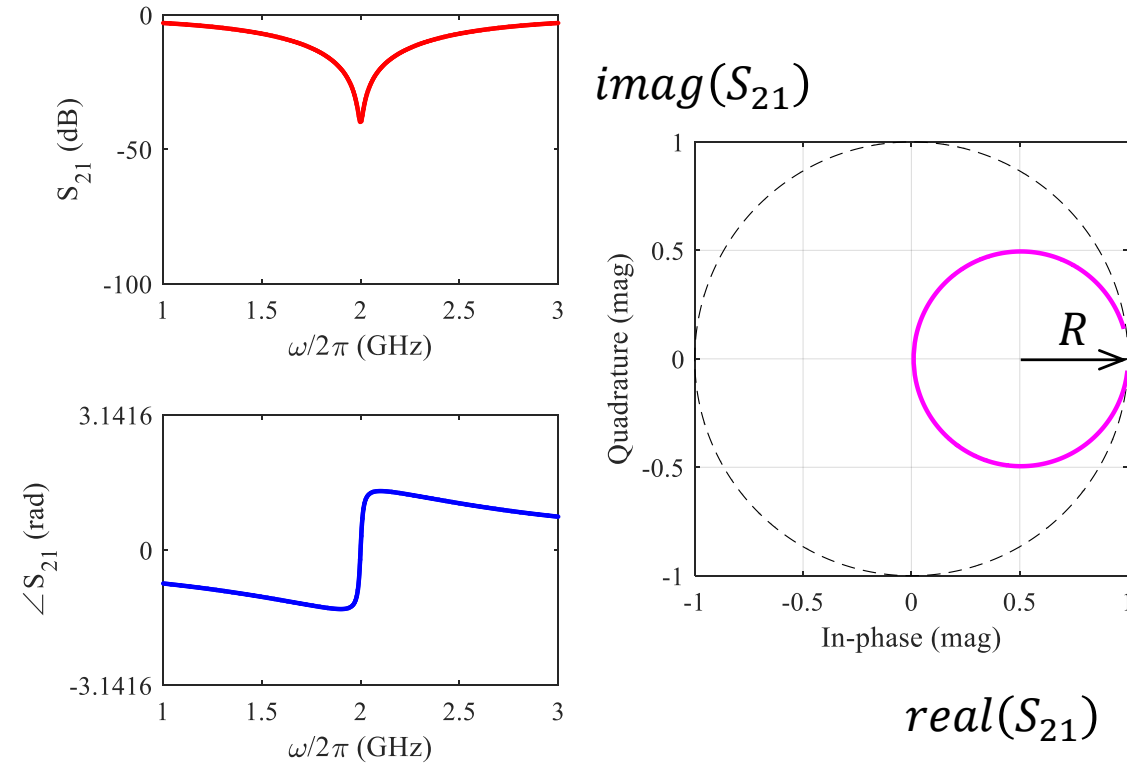
$$X = \text{real}(S_{21}) = \frac{(\omega - \omega_1)^2 + \gamma_1(\gamma_1 + \kappa_1)}{(\omega - \omega_1)^2 + (\kappa_1 + \gamma_1)^2}$$

$$Y = \text{imag}(S_{21}) = \frac{\kappa_1(\omega - \omega_1)}{(\omega - \omega_1)^2 + (\kappa_1 + \gamma_1)^2}$$

$$(X - X_C)^2 + Y^2 = R^2$$

$$X_C = 1 - \frac{\kappa_1}{2(\kappa_1 + \gamma_1)} \quad R = \frac{\kappa_1}{2(\kappa_1 + \gamma_1)}$$

Case 0: Empty Cavity



Case 1: Coherent coupling ($g \in \mathbb{R}$)

Transmission of a coupled system

$$S_{21} = 1 - \frac{\kappa_1}{i(\omega - \omega_1) + \kappa_1 + \gamma_1 + \frac{g^2}{i(\omega - \omega_2) + \kappa_2 + \gamma_2}}$$

Assume that $\omega \approx \omega_2$, near the resonance frequency of 2

$$S_{21} = 1 - \frac{\kappa_1}{i\Delta + \kappa_1 + \gamma_1 + \frac{g^2}{i(\omega - \omega_2) + \kappa_2 + \gamma_2}}$$

We set $\Delta = 0$ for simplification

$$S_{21} = \boxed{1 - \frac{\kappa_1}{\kappa_1 + \gamma_1}} + \boxed{\frac{1}{(\kappa_1 + \gamma_1)^2} \frac{g^2}{i(\omega - \omega_2) + \kappa_2 + \gamma_2}}$$

Constant
(fixed point on circle)

The trajectory of ω_2

Taylor expansion near ω_2

$$\frac{1}{c + x} = \frac{1}{c} - \frac{1}{c^2}x + \frac{1}{c^3}x^2 + \dots$$

Realising the Denominator

$$\frac{g^2[i(\omega - \omega_2) + \kappa_2 + \gamma_2]}{[i(\omega - \omega_2) + \kappa_2 + \gamma_2][i(\omega - \omega_2) - \kappa_2 - \gamma_2]}$$

Define expanded variables

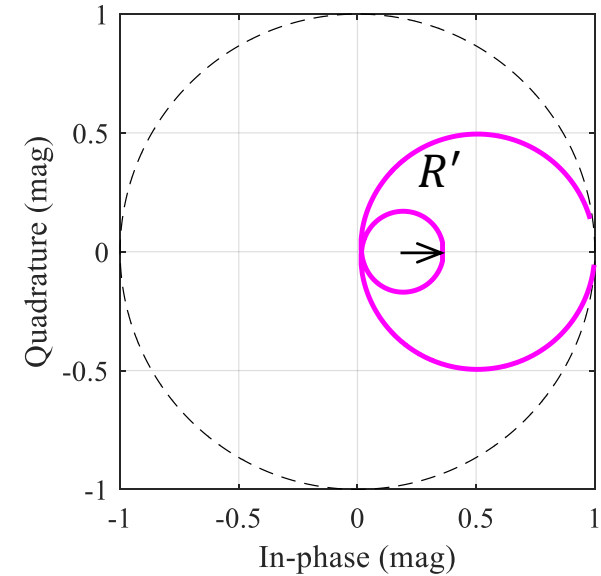
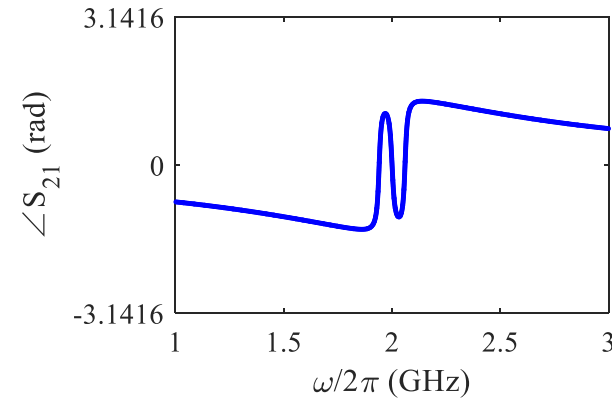
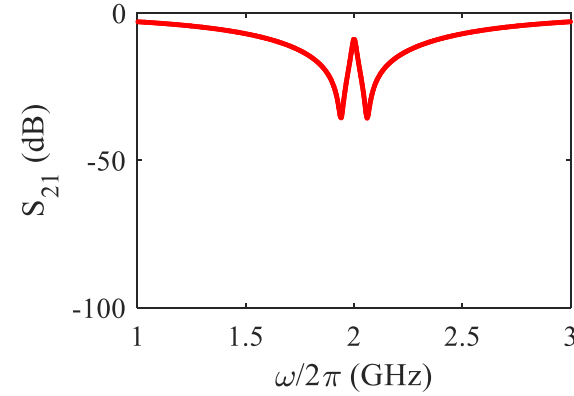
$$X' = \frac{g^2(\kappa_2 + \gamma_2)}{(\omega - \omega_2)^2 + (\kappa_2 + \gamma_2)^2}$$

$$Y' = \frac{g^2(\omega - \omega_2)}{(\omega - \omega_2)^2 + (\kappa_2 + \gamma_2)^2}$$

$$X'^2 + Y'^2 = R'^2$$

$$R' = g^2/(\kappa_1 + \gamma_1)^2$$

$$X_{c^2} = 1 - \frac{\kappa_1(\kappa_1 + \gamma_1) - g^2}{(\kappa_1 + \gamma_1)^2}$$



X_{c^2} inside circle 1 \rightarrow Incircle

Case 2: Dissipative coupling ($g \in \mathbb{I}$)

Expanded variables

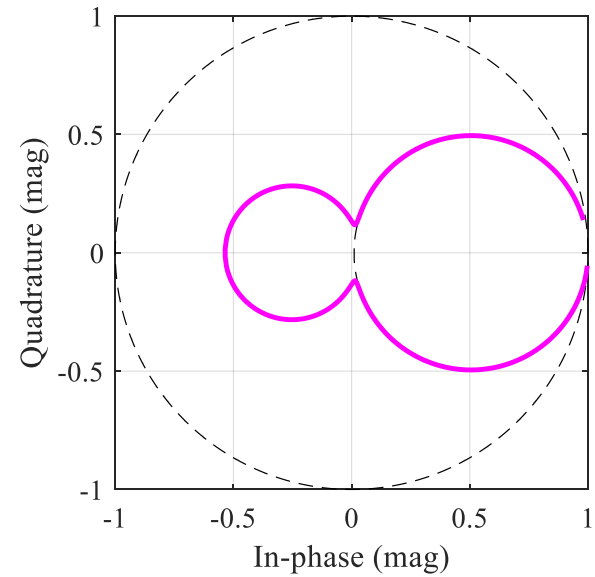
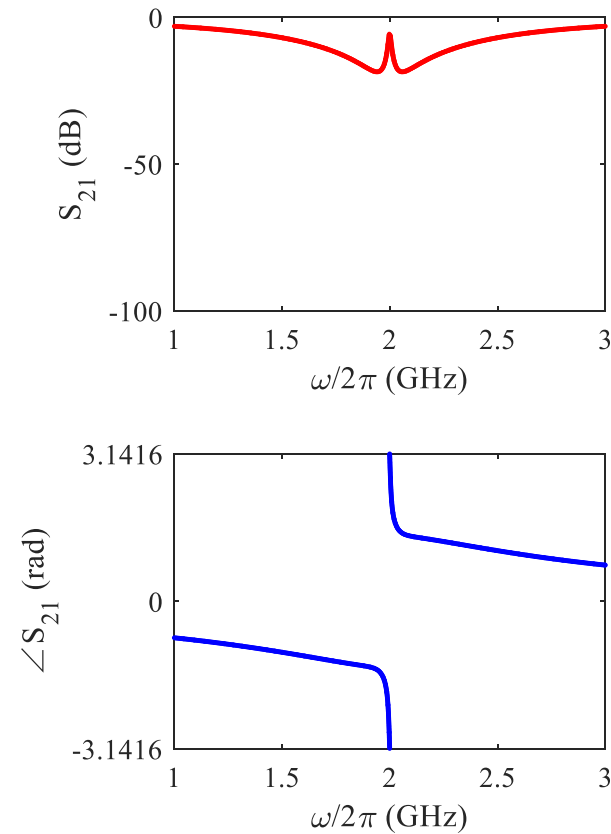
$$X' = -\frac{|g|^2(\kappa_2 + \gamma_2)}{(\omega - \omega_2)^2 + (\kappa_2 + \gamma_2)^2}$$

$$Y' = -\frac{|g|^2(\omega - \omega_2)}{(\omega - \omega_2)^2 + (\kappa_2 + \gamma_2)^2}$$

$$X'^2 + Y'^2 = R'^2$$

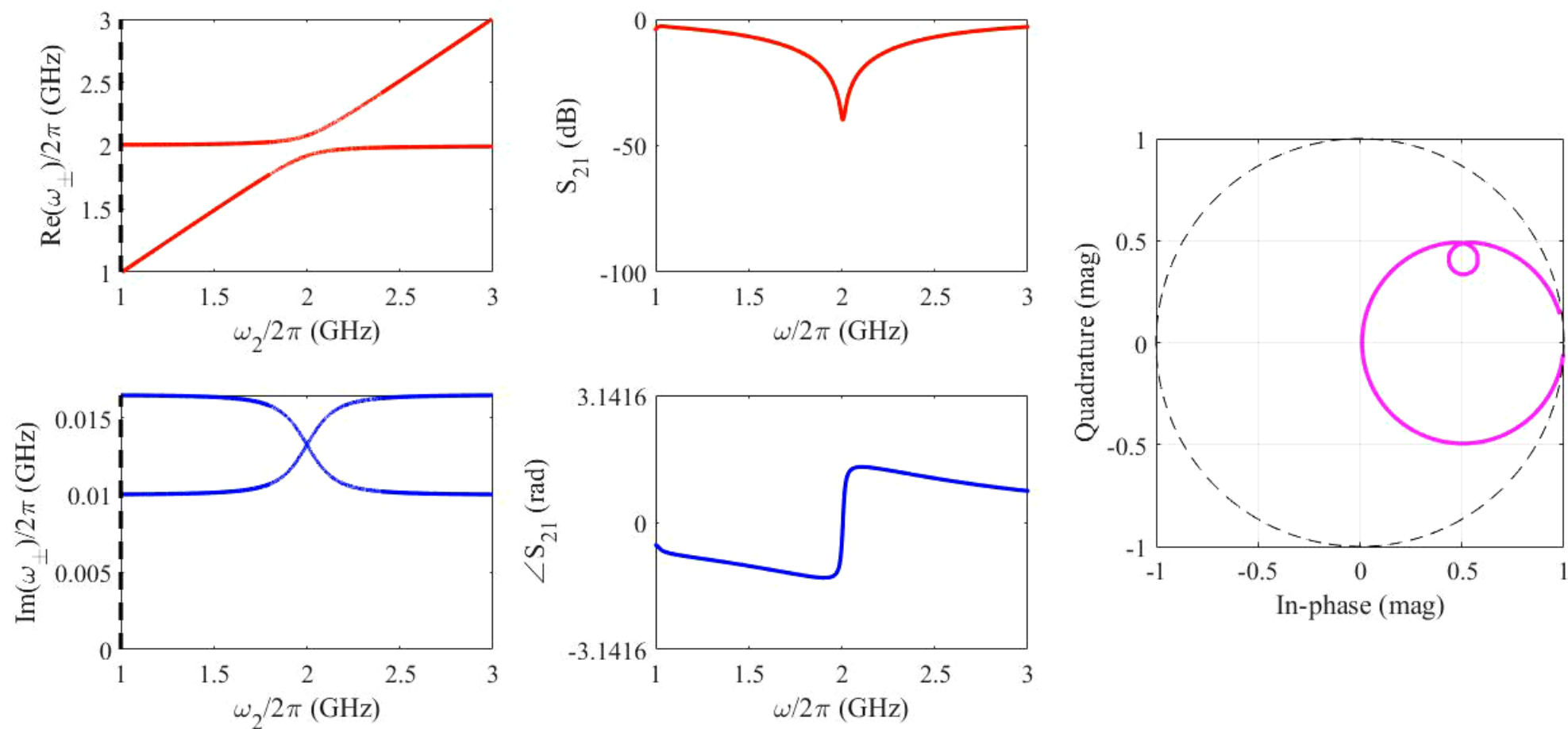
$$R' = -|g|^2/(\kappa_1 + \gamma_1)^2$$

$$X_{c^2} = 1 - \frac{g^2 + \kappa_1(\kappa_1 + \gamma_1)}{(\kappa_1 + \gamma_1)^2}$$

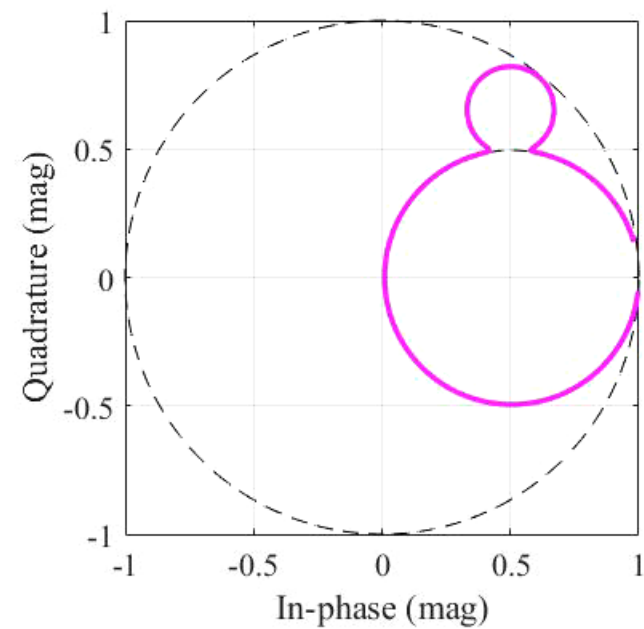
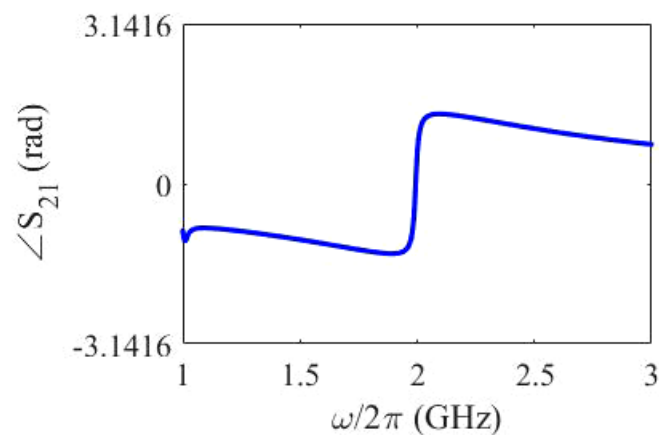
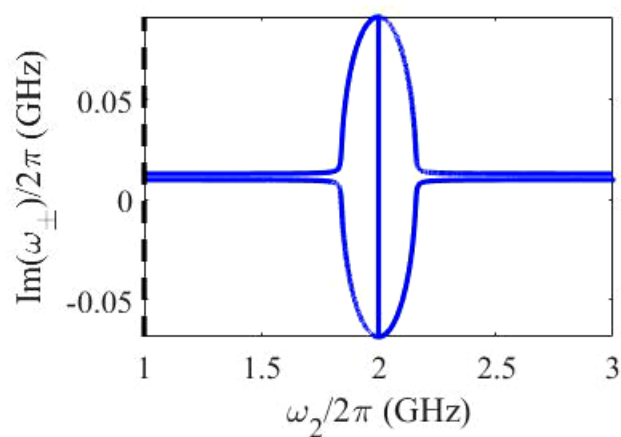
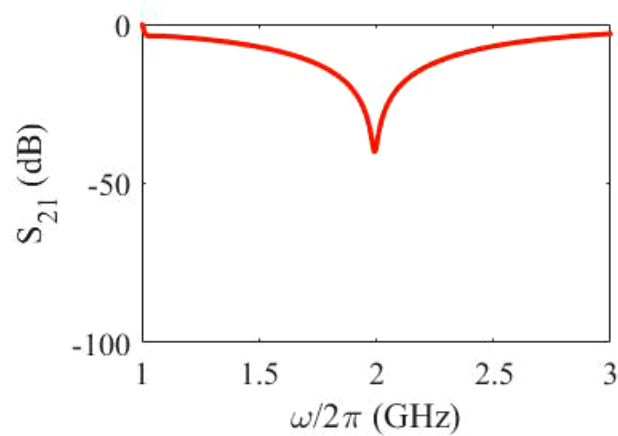
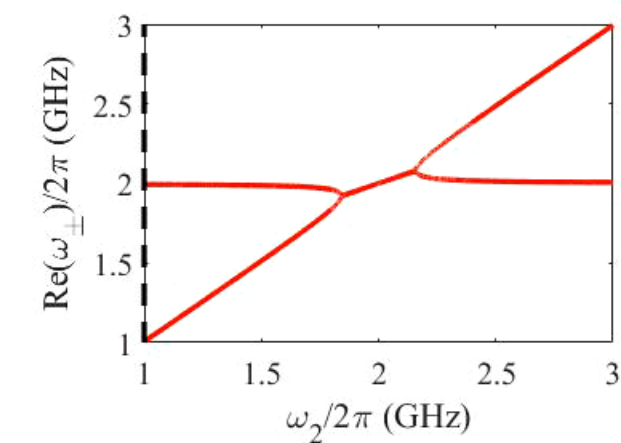


X_{c^2} outside circle 1 \rightarrow Excircle

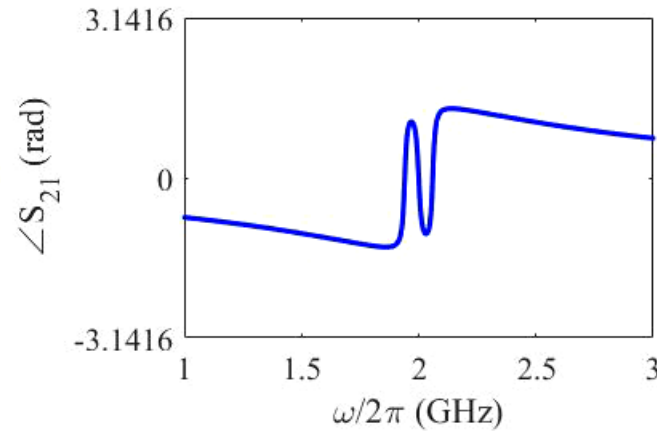
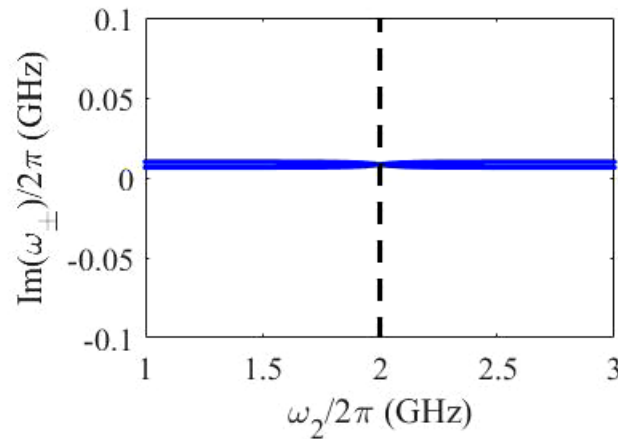
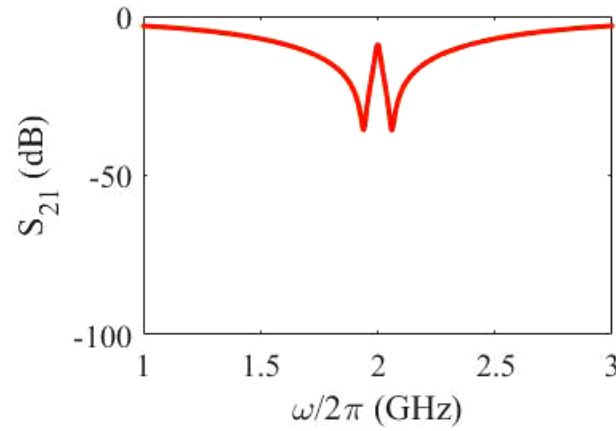
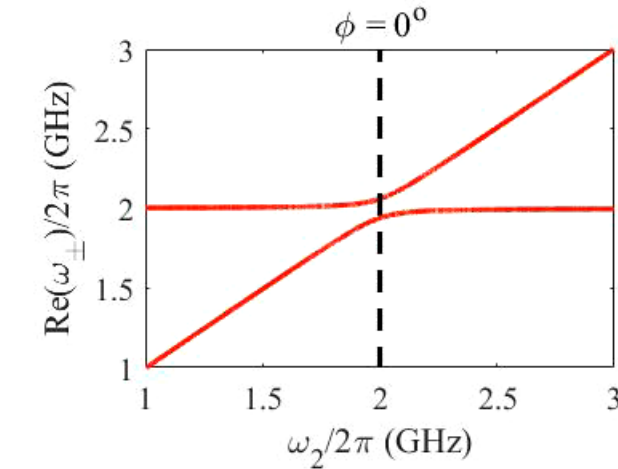
Case 1: Coherent coupling ($g \in \mathbb{R}$)



Case 2: Dissipative coupling ($g \in \mathbb{I}$)

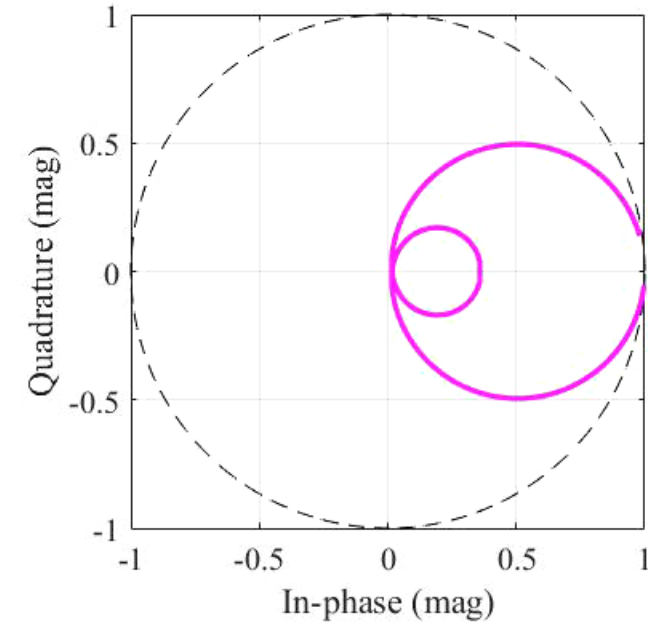


Case 3: From Coherent to Dissipative coupling ($g \in \mathbb{I} \rightarrow \mathbb{R}$ at $\Delta = 0$)



$$J = \text{real}(g \cdot e^{i\phi})$$

$$\Gamma = \text{imag}(g \cdot e^{i\phi})$$



The Fano resonance in plasmonic nanostructures and metamaterials

Boris Luk'yanchuk¹, Nikolay I. Zheludev², Stefan A. Maier³, Naomi J. Halas⁴, Peter Nordlander^{5*}, Harald Giessen⁶ and Chong Tow Chong^{1,7}

the classical formula for Rayleigh scattering:

Scattering efficiency $Q_{\text{sca}} \approx \frac{8}{3} \left| \frac{\varepsilon - 1}{\varepsilon + 2} \right|^2 q^4$

This formula holds for positive and negative ε except close to the surface plasmon resonance ($\varepsilon = -2$), where it exhibits a singularity in the absence of intrinsic damping. With increasing size, higher-

Grigoriev, V., et al. "Singular analysis of Fano resonances in plasmonic nanostructures." *Physical Review A* 88.6 (2013): 063805.

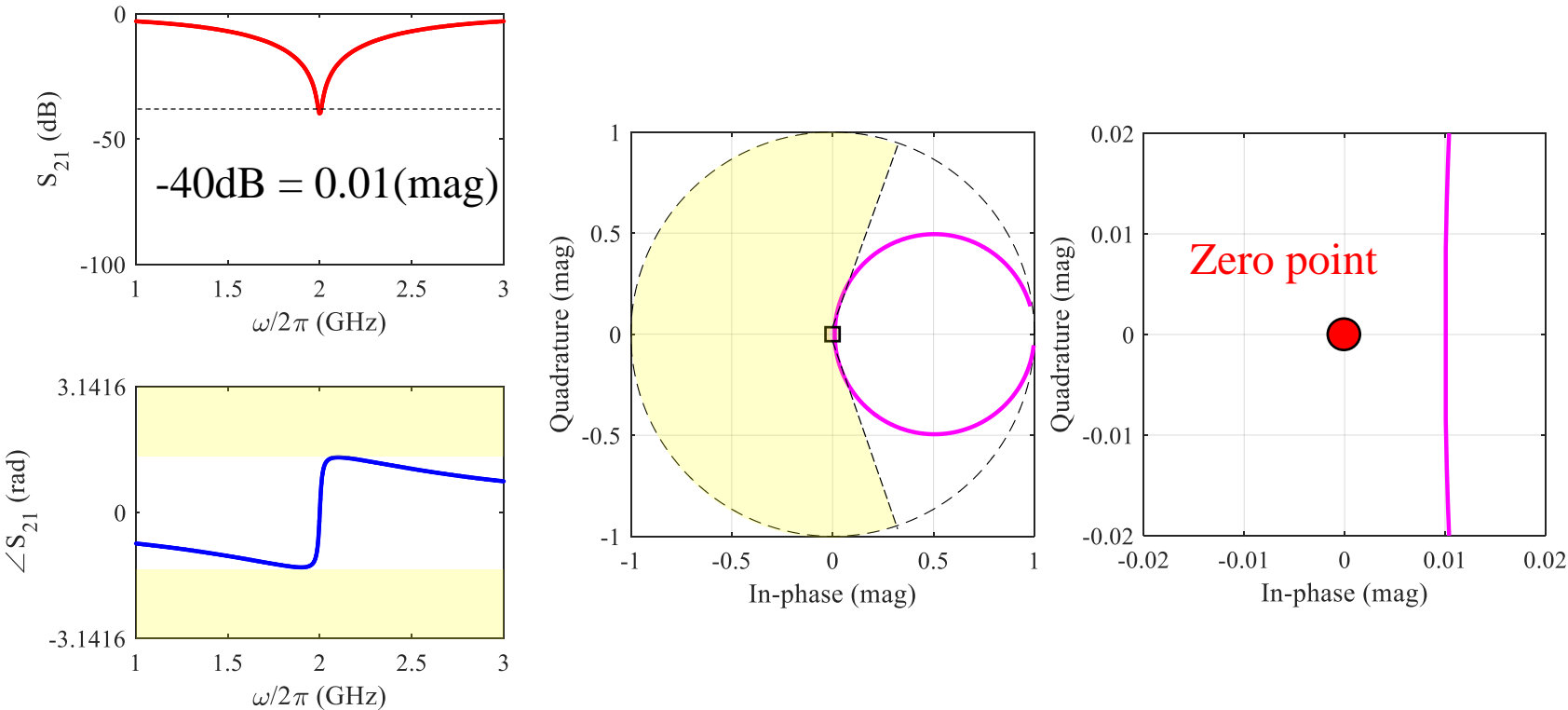
The complex function of S_{21} (transmission in log scale) is everywhere analytical except at the ZDCs.

→ logarithmic singularity when $S_{21} = 0$

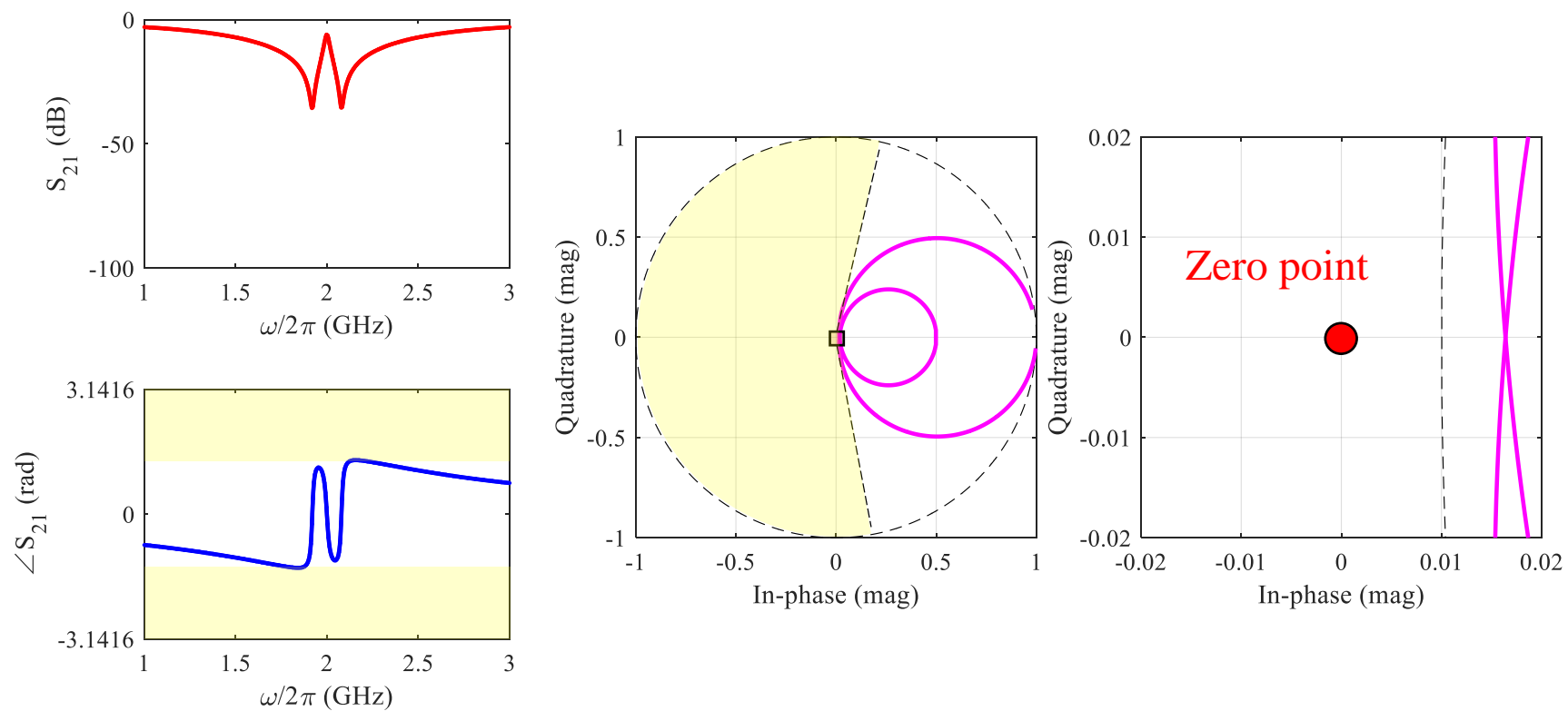
$$\lim_{\omega \rightarrow \omega_{ZDC}} \log_{10}(|S_{21}|) = -\infty$$

$$\lim_{\omega \rightarrow \omega_{ZDC}} \frac{d\phi}{d\omega} = \infty$$

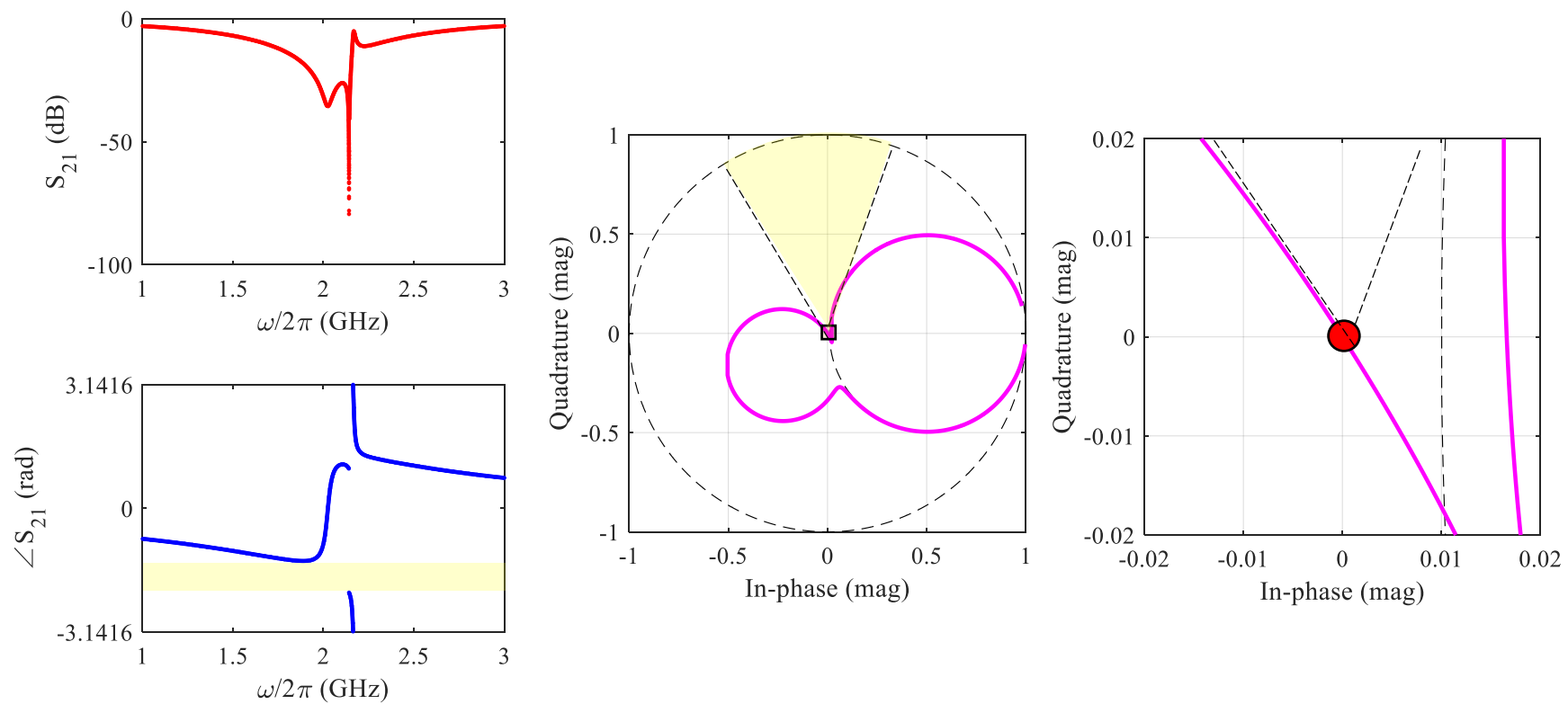
Case 0: Empty Cavity



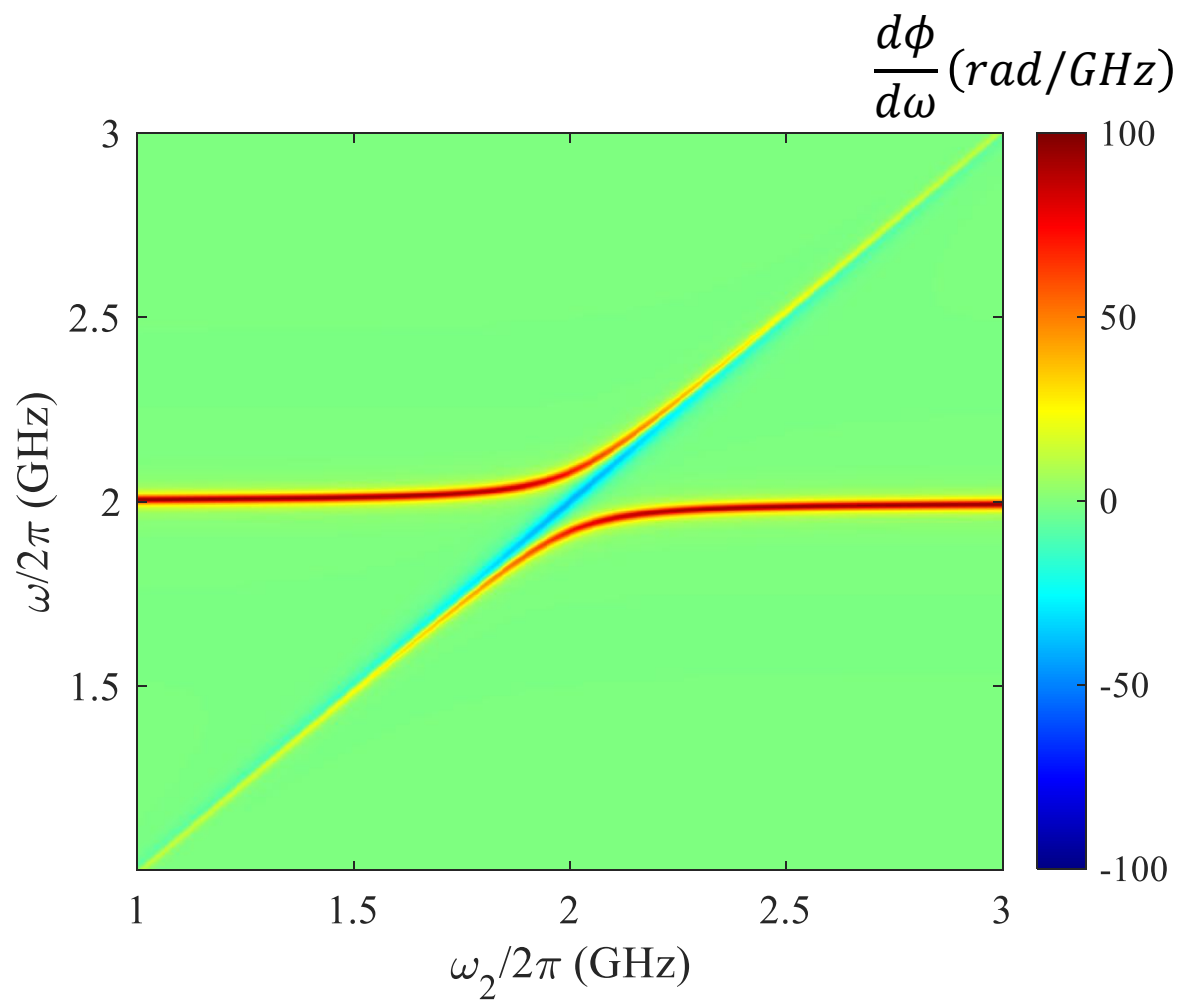
Case 1: Coherent coupling ($g \in \mathbb{R}$)



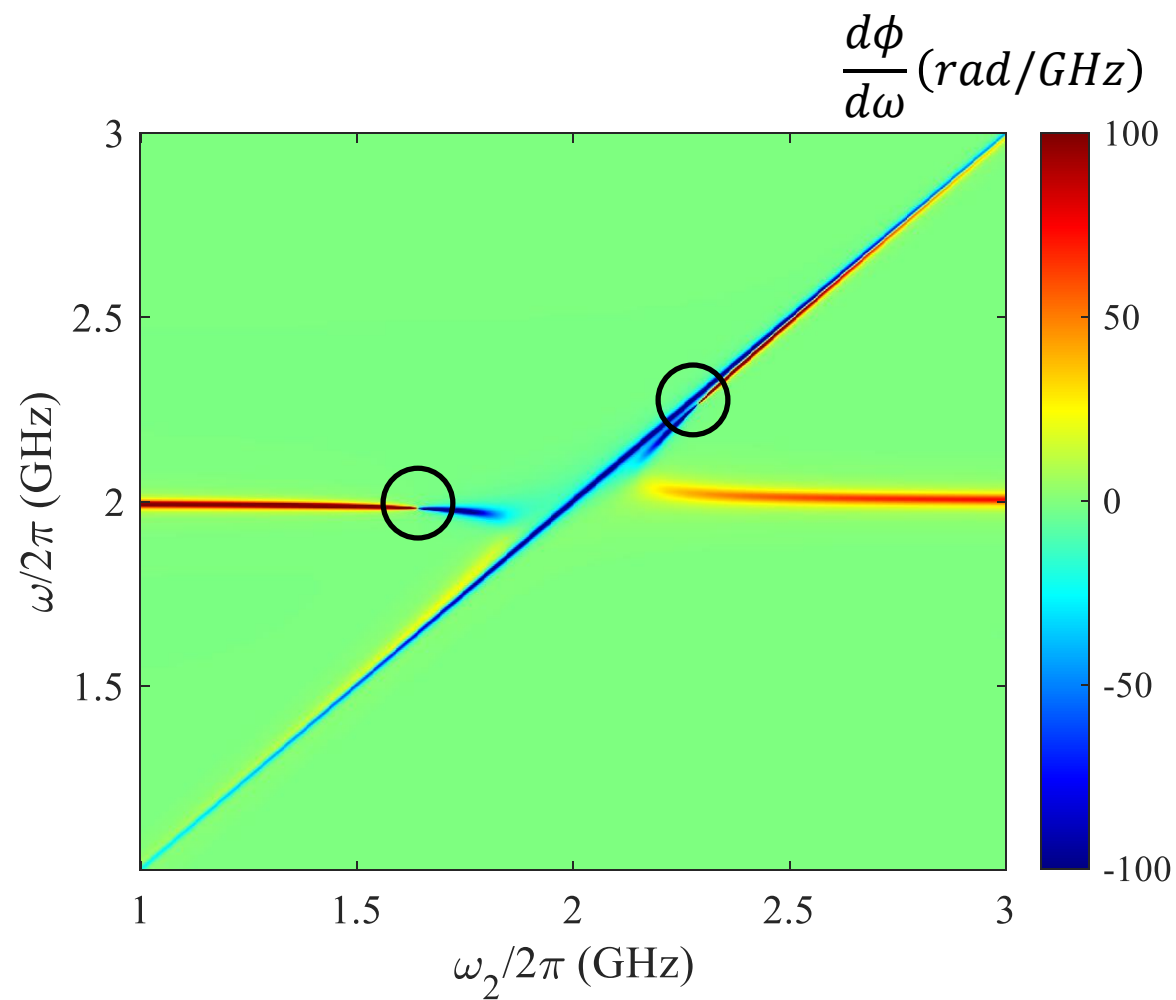
Case 2: Dissipative coupling ($g \in \mathbb{I}$)



Case 1: Coherent coupling ($g \in \mathbb{R}$)



Case 2: Dissipative coupling ($g \in \mathbb{I}$)



Conclusion

- Coherent coupling is limited by the original circle of resonator 1.
- Dissipative coupling breaks the limitation of the coherent coupling and can reach logarithmic singular point that produce ZDC.
- The two π phase jump come from the I/Q curve includes the origin.

Equivalent Q factor of hybridized modes

$$S_{21} = 1 - \frac{\kappa}{i(\omega_c - \omega) + \kappa + \gamma}$$

$$\frac{d|S_{21}|}{d\omega}(\gamma) = \frac{d|S_{21}|}{d|S_{21}|^2} \frac{d|S_{21}|^2}{d\omega}$$

Numerical result:

$$Q_{Loaded} \sim \kappa$$

$$Q_{unLoaded} \sim \gamma$$

For $S = 200$ dB/MHz, the equivalent $Q_u = 46,000$

