$Lagrangian$ of two pendulum oscillators coupled by a multi-modes reservoir

$$T = \frac{1}{2} m r_1^2 \dot{\phi_1}^2 + \frac{1}{2} m r_2^2 \dot{\phi_2}^2 + \frac{1}{2} \sum_k (mr^2)_k \, \dot{x}_k^2$$

O$_1$ O$_2$ Platform

$$U = mgr_1(1 - \cos\phi_1) + mgr_2(1 - \cos\phi_2) + \sum_k (mr)_k g(1 - \cos x_k) + \frac{1}{2}\sum_k \mathrm{K}_{1k}\,(r_1\phi_1 - x_k)^2 + \frac{1}{2}\sum_k \mathrm{K}_{2k}\,(r_2\phi_2 - x_k)^2$$

O$_1$ O$_2$ Platform Platform - O$_1$ Platform – O$_2$

$$L = T - U$$

Euler-Lagrangian equation on x coordinate

For small angle approximation
$\sin x_k = x_k$

$$\frac{\partial L}{d\dot{x_k}} = \sum_k (mr^2)_k \dot{x_k}$$

$$\frac{d}{dt}\left(\frac{\partial L}{d\dot{x_k}}\right) = \sum_k (mr^2)_k \ddot{x_k}$$

$$\frac{\partial L}{dx_k} = -\left[\sum_k (mr)_k g x_k - \sum_k \mathrm{K}_{1k}\,(r_1\phi_1 - x_k) - \sum_k \mathrm{K}_{2k}\,(r_2\phi_2 - x_k)\right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{d\dot{x_k}}\right) - \frac{\partial L}{dx_k} = 0$$

$$(mr^2)_k\ddot{x_k} + (mr)_k g x_k + \mathrm{K}_{1k}(r_1\phi_1 - x_k) + \mathrm{K}_{2k}(r_2\phi_2 - x_k) = 0$$

$$\ddot{x_k} + \left(\frac{g}{r}\right)_k x_k - \mathrm{K}_{1k}(r_1\phi_1 - x_k) - \mathrm{K}_{2k}(r_2\phi_2 - x_k) = 0$$

$$\omega_k = \sqrt{\left(\frac{g}{l}\right)_k}$$

$$\omega_k^2 \gg \mathrm{K}_{1k}, \mathrm{K}_{2k}$$

$$\boxed{\ddot{x_k} + \omega_k^2 x_k - g_{1k}\phi_1 - g_{2k}\phi_2 = 0}$$

$$g_{ik} = \frac{\mathrm{K}_{ik} r_i}{(mr^2)_k}$$

Euler-Lagrangian equation on ϕ_i coordinate

For small angle approximation

$$\sin\phi_i = \phi_i$$

$$\frac{\partial L}{d\dot{\phi}_i} = mr_i^2\dot{\phi}_i$$

$$\frac{d}{dt}\left(\frac{\partial L}{d\dot{x_k}}\right) - \frac{\partial L}{dx_k} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{d\dot{\phi}_i}\right) = mr_i^2\ddot{\phi}_i$$

$$\frac{\partial L}{d\phi_i} = -\left[mgr_i\sin\phi_i + \sum_k \mathrm{K}_{ik}\, r_i(r_i\phi_i - x_k)\right]$$

$$\omega_i = \sqrt{\frac{g}{r_i}}$$

$$mr_i^2\ddot{\phi}_i + mgr_i\phi_i + \sum_k \mathrm{K}_{ik} r_i\,(r_i\phi_i - x_k) = 0$$

$$\omega_i^2 \gg gik\,, \mathrm{K}_{2k}$$

$$\boxed{\ddot{\phi}_i + \omega_i^2\sin\phi_i - \sum_k g_{ik}\, x_k = 0}$$

Equations of Motion

$$\frac{d^2}{dt^2}\phi_1 + \omega_1^2\phi_1 + \sum_k g_{1k}x_k = 0$$

$$\frac{d^2}{dt^2}\phi_1 + \omega_1^2\phi_1 + \int_k g_{1k}x_k = 0$$

$$\frac{d^2}{dt^2}\phi_2 + \omega_2^2\phi_2 + \sum_k g_{2k}x_k = 0$$

$$\frac{d^2}{dt^2}\phi_2 + \omega_2^2\phi_2 + \int_k g_{2k}x_k = 0$$

$$\frac{d^2}{dt^2}x_k + \omega_k^2 x_k + (g_{1k}\phi_1 + g_{2k}\phi_2) = 0$$

$$\frac{d^2}{dt^2}x_k + \omega_k^2 x_k + \int_k (g_{1k}\phi_1 + g_{2k}\phi_2) = 0$$

$$\boxed{\begin{aligned} x_k = c_1\cos(\omega_k t) + c_2\sin(\omega_k t) + \cos(\omega_k t)\int_{t_0}^{t}\frac{\sin(\omega_k t')\sum_k(g_{1k}\phi_1 + g_{2k}\phi_2)}{\omega_k}dt' \\ - \sin(\omega_k t)\int_{t_0}^{t}\frac{\cos(\omega_k t')\sum_k(g_{1k}\phi_1 + g_{2k}\phi_2)}{\omega_k}dt' \end{aligned}}$$

$$\phi_1 = A_1 \cos(\omega t + \theta_1)$$

$$\phi_2 = A_2 \cos(\omega t + \theta_2)$$

$$\boxed{\begin{aligned} x_k = c_1 \cos(\omega_k t) + c_2 \sin(\omega_k t) + \cos(\omega_k t) \int_{t_0}^{t} \frac{\sin(\omega_k t') \sum_k (g_{1k} A_1 \cos(\omega t' + \theta_1) + g_{2k} A_2 \cos(\omega t' + \theta_2))}{\omega_k} dt' \\ - \sin(\omega_k t) \int_{t_0}^{t} \frac{\cos(\omega_k t') \sum_k (g_{1k} A_1 \cos(\omega t' + \theta_1) + g_{2k} A_2 \cos(\omega t' + \theta_2))}{\omega_k} dt' \end{aligned}}$$

$$\int_0^t \sin(\omega_k t') \cos(\omega_1 t' + \theta_1)\ dt' = \frac{-2\omega_k \cos\theta_1 + (\omega + \omega_k) \cos\big((\omega_1 - \omega_k)t + \theta_1\big) + (\omega_k - \omega) \cos\big((\omega_1 + \omega_k)t + \theta_1\big)}{2(\omega_1 - \omega_k)(\omega_1 + \omega_k)}$$

$$\int_0^t \cos(\omega_k t') \cos(\omega_1 t' + \theta_1)\ dt' = \frac{-2\omega_k \sin\theta_1 + (\omega + \omega_k) \sin(\theta_1 + (\omega_1 - \omega_k)t) + (\omega_k - \omega) \sin\big((\omega_1 + \omega_k)t + \theta_1\big)}{2(\omega_1 - \omega_k)(\omega_1 + \omega_k)}$$

$$\omega \approx \omega_k \rightarrow (\omega - \omega_k) = \delta \text{ detuning } \& \ (\omega + \omega_k) \approx 2\omega_1$$

This does not converge !

$$\int_0^t \sin(\omega_k t') \cos(\omega t' + \theta_1)\ dt' = \frac{-2\omega_k \cos\theta_1 + 2\omega_1 \cos(\delta t + \theta_1) - \delta \cos(2\omega t + \theta_1)}{4\delta\omega}$$

$$= \frac{-\cos\theta_1}{2\delta} + \frac{\cos(\delta t + \theta_1)}{2\delta} - \frac{\cos(2\omega_1 t + \theta_1)}{4\omega_1}$$

For $t \gg 0$ & $t = 2n\pi/\omega$, we take the average over time, we get for n cycles:

$$= -\frac{n\pi}{\omega} \sin\theta_1$$

$$\int_0^{2n\pi/\omega} \cos(\omega_k t') \cos(\omega_1 t' + \theta_1)\ dt' = \frac{n\pi}{\omega} \cos\theta_1$$

$$\boxed{x_k = c_k \cos(\omega_k t + \theta_k) - \cos(\omega_k t) \frac{\left(g_{1k} A_1 \frac{\pi}{\omega_k} \sin\theta_1 + g_{2k} A_2 \frac{\pi}{\omega_k} \sin\theta_2\right)}{\omega_k} - \sin(\omega_k t) \frac{\left(g_{1k} A_1 \frac{\pi}{\omega_k} \cos\theta_1 + g_{2k} A_2 \frac{\pi}{\omega_k} \cos\theta_2\right)}{\omega_k}}$$

$$x_k = c_k \cos(\omega_k t + \theta_k) - \frac{\pi}{\omega_k^2}[\cos(\omega_k t)\,(g_{1k} A_1 \sin\theta_1 + g_{2k} A_2 \sin\theta_2) + \ \sin(\omega_k t)\,(g_{1k} A_1 \cos\theta_1 + g_{2k} A_2 \cos\theta_2)]$$

$$\boxed{x_k = c_k \cos(\omega_k t + \theta_k) - \frac{\pi}{\omega_k^2}[(g_{1k} A_1 \sin(\omega_k t + \theta_1) + g_{2k} A_2 \sin(\omega_k t + \theta_2))]}$$

$$\frac{d^2}{dt^2}\phi_1 + \omega_1^2 \phi_1 + \sum_k g_{1k} x_k = 0$$

$$\frac{d^2}{dt^2}\phi_2 + \omega_2^2 \phi_2 + \sum_k g_{2k} x_k = 0$$

$$\frac{d^2}{dt^2}\phi_1 + \omega_1^2 \phi_1 + \sum_k g_{1k} c_k \cos(\omega_k t + \theta_k) - \frac{\pi}{\omega_k^2}[(g_{1k} A_1 \sin(\omega_k t + \theta_1) + g_{2k} A_2 \sin(\omega_k t + \theta_2))] = 0$$

$$\frac{d^2}{dt^2}\phi_2 + \omega_2^2 \phi_2 + \sum_k g_{2k} c_k \cos(\omega_k t + \theta_k) - \frac{\pi}{\omega_k^2}[(g_{1k} A_1 \sin(\omega_k t + \theta_1) + g_{2k} A_2 \sin(\omega_k t + \theta_2))] = 0$$

$$\frac{d^2}{dt^2}\phi_1 + \omega_1^2\phi_1 + \sum_k g_{1k}c_k\cos(\omega_k t + \theta_k) - \frac{\pi}{\omega_k^2}[(g_{1k}A_1\sin(\omega_k t + \theta_1) + g_{2k}A_2\sin(\omega_k t + \theta_2))] = 0$$

$$\frac{d^2}{dt^2}\phi_2 + \omega_2^2\phi_2 + \sum_k g_{2k}c_k\cos(\omega_k t + \theta_k) - \frac{\pi}{\omega_k^2}[(g_{1k}A_1\sin(\omega_k t + \theta_1) + g_{2k}A_2\sin(\omega_k t + \theta_2))] = 0$$

$$\frac{d^2}{dt^2}\phi_1 + \omega_1^2\phi_1 + \sum_k g_{1k}c_k\cos(\omega_k t + \theta_k) - \frac{\pi}{\omega_k^2}[(g_{1k}A_1'\sin(\omega_k t + \theta_1 + \theta_k'))] = 0$$

$$\frac{d^2}{dt^2}\phi_2 + \omega_2^2\phi_2 + \sum_k g_{2k}c_k\cos(\omega_k t + \theta_k) - \frac{\pi}{\omega_k^2}[(g_{1k}A_1'\sin(\omega_k t + \theta_1 + \theta_k'))] = 0$$

```
In[27]:= DSolve[{x''[t] + w^2 x[t] + Integrate[a * Cos[wk[k] + p[k]] - b / wk[k]^2 * Sin[wk[k] * t] + p2[k], k] == 0}, x[t], t]
         求解微分方程                    积分           余弦                        正弦
```

Out[27]= $$\left\{\left\{x[t] \to C[1]\,\mathrm{Cos}[t\,w] + \mathrm{Cos}[t\,w]\int_1^t \frac{\left(\int\left(a\,\mathrm{Cos}[p[k] + wk[k]] + p2[k] - \frac{b\,\mathrm{Sin}[K[1]\,wk[k]]}{wk[k]^2}\right)dk\right)\mathrm{Sin}[w\,K[1]]}{w}\,dK[1] + C[2]\,\mathrm{Sin}[t\,w] + \left(\int_1^t -\frac{\mathrm{Cos}[w\,K[2]]\int\left(a\,\mathrm{Cos}[p[k] + wk[k]] + p2[k] - \frac{b\,\mathrm{Sin}[K[2]\,wk[k]]}{wk[k]^2}\right)dk}{w}\,dK[2]\right)\mathrm{Sin}[t\,w]\right\}\right\}$$

$$x_1 = c_1\cos(\omega_1 t + \phi_1) + \cos\omega_1 t\int_1^t \frac{\int\left(a\cos(\theta_k + \omega_k) + \theta_k' \;-\right)dk}{\omega_1}\,dt'$$

```
In[4]:= (*
  p1 = A1*Cos[w1*t+a1];
  p2 = A2*Cos[w2*t+a2];*)
  DSolve[{x''[t] + wk[k]^2 x[t] + Sum[g1k[k] * p1[t] + g2k[k] * p2[t], k] == 0}, x[t], t]
  求解微分方程              求和
```

Out[4]=

$$\left\{\left\{x[t]\to C[2]\,\mathrm{Cos}[t\,wk[k]]+\mathrm{Cos}[t\,wk[k]]\int_1^t\frac{\mathrm{Sin}[K[2]\,wk[k]]\sum_k\left(g1k[k]\,p1[K[2]]+g2k[k]\,p2[K[2]]\right)}{wk[k]}\,dK[2]+C[1]\,\mathrm{Sin}[t\,wk[k]]+\left(\int_1^t-\frac{\mathrm{Cos}[K[1]\,wk[k]]\sum_k\left(g1k[k]\,p1[K[1]]+g2k[k]\,p2[K[1]]\right)}{wk[k]}\,dK[1]\right)\mathrm{Sin}[t\,wk[k]]\right\}\right\}$$

```
In[5]:= (*
  p1 = A1*Cos[w1*t+a1];
  p2 = A2*Cos[w2*t+a2];*)
  DSolve[{x''[t] + wk[k]^2 x[t] + Integrate[g1k[k] * p1[t] + g2k[k] * p2[t], k] == 0}, x[t], t]
  求解微分方程              积分
```

Out[5]=

$$\left\{\left\{x[t]\to C[2]\,\mathrm{Cos}[t\,wk[k]]+\mathrm{Cos}[t\,wk[k]]\int_1^t\frac{\left(\int\left(g1k[k]\,p1[K[2]]+g2k[k]\,p2[K[2]]\right)dk\right)\mathrm{Sin}[K[2]\,wk[k]]}{wk[k]}\,dK[2]+C[1]\,\mathrm{Sin}[t\,wk[k]]+\left(\int_1^t-\frac{\mathrm{Cos}[K[1]\,wk[k]]\int\left(g1k[k]\,p1[K[1]]+g2k[k]\,p2[K[1]]\right)dk}{wk[k]}\,dK[1]\right)\mathrm{Sin}[t\,wk[k]]\right\}\right\}$$

```
In[6]:= Integrate[Sin[wk * t] * Cos[w * t + p], {t, 0, tf}]
        积分      正弦         余弦
```

Out[6]=

$$\frac{-2\,wk\,\mathrm{Cos}[p]+(w+wk)\,\mathrm{Cos}[p+tf\,w-tf\,wk]+(-w+wk)\,\mathrm{Cos}[p+tf\,(w+wk)]}{2\,(w-wk)\,(w+wk)}$$

```
In[7]:= Integrate[Cos[wk * t] * Cos[w * t + p], {t, 0, tf}]
        积分      余弦         余弦
```

Out[7]=

$$\frac{-2\,w\,\mathrm{Sin}[p]+(w+wk)\,\mathrm{Sin}[p+tf\,w-tf\,wk]+(w-wk)\,\mathrm{Sin}[p+tf\,(w+wk)]}{2\,(w-wk)\,(w+wk)}$$