Gram-Schmidt Orthonormalization Process

If we have an orthonormal set of vectors $\vec{u_1}, \cdots, \vec{u_{k-1}}$,

then $\vec{w_k} = \vec{V_k} - \sum_{i=1}^{k-1} \vec{u_i} \cdot \vec{V_k} \times \vec{u_i}$

Example 1:

$$A = [\vec{V_1} \quad \vec{V_2} \quad \vec{V_3}] = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution:

Step 1: Normalize $\vec{V_1}$

$$\vec{u_1} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ 0 \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

Step 2: $\vec{W_2} = \vec{V_2} - \vec{u_1} \cdot \vec{V_2} \times \vec{u_1}$

$$= \begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{\sqrt{6}} \\ 0 \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{6}} \\ 0 \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ 2 \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

$\forall t \in 1, 2, \cdots, k-1, \quad \vec{W_k} \cdot \vec{u_t} = 0$

Proof: $(\vec{V_k} - \sum_{i=1}^{k-1} \vec{u_i} \cdot \vec{V_k} \times \vec{u_i}) \cdot \vec{u_t}$

$= \vec{V_k} \cdot \vec{u_t} - \underline{\sum_{i=1}^{k-1} (\vec{u_i} \cdot \vec{V_k}) \times (\vec{u_i} \cdot \vec{u_t})}$

$= 0$

if $t \neq i$, $\vec{u_i} \cdot \vec{u_t} = 0$

if $t = i$, we get:

$= (\vec{u_t} \cdot \vec{V_k}) \times (\vec{u_t} \cdot \vec{u_t})$

$= (\vec{u_t} \cdot \vec{V_k}) \times 1$

$= \vec{u_t} \cdot \vec{V_k}$

Step 3: Normalize $\vec{w}_2$ and get

$$\vec{u}_2 = \begin{bmatrix} \frac{\sqrt{2}}{6} \\ \frac{2\sqrt{2}}{3} \\ 0 \\ -\frac{\sqrt{2}}{6} \end{bmatrix}$$

Step 4: Compute u_3 in terms of u_1 and u_2

$$\vec{w}_3 = \vec{v}_3 - \vec{u}_1 \cdot \vec{v}_3 \times \vec{u}_1 - \vec{u}_2 \cdot \vec{v}_3 \times \vec{u}_2$$

$$= \begin{bmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ 0 \\ -\frac{4}{9} \end{bmatrix}$$

normalize $\vec{w}_3$:

$$\vec{u}_3 = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ -\frac{2}{3} \end{bmatrix}$$

Eigenvectors and Eigenvalues

A: a square matrix
$\vec{v}$: eigenvector
λ: eigenvalue

$$A\vec{v} = \lambda\vec{v}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} (2-\lambda)x_1 + x_2 &= 0 \\ x_1 + (2-\lambda)x_2 &= 0 \end{aligned}$$

To ensure that we have a nonzero vector

$$\Rightarrow \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (\lambda-3)(\lambda-1) = 0$$

when $\lambda = 3$:

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

when $\lambda = 1$:

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Full / Reduced SVD

① $U^T U = I$
② $V^T V = I$
③ $USS^T U^T = AA^T$
$\lambda = SS^T = \sum_{i=1}^{k} S_{ii}^2$ (a scalar)
$\Rightarrow \lambda u u^T = AA^T$
④ $VS^T S V = A^T A$

$$A_{mn} = U_{mm} S_{mn} V_{nn}^T$$

The columns of U:
orthonormal eigenvectors of AA^T
The columns of V:
orthonormal eigenvectors of $A^T A$
S:
a diagonal matrix contains the square roots of eigenvalues from U or V in descending order