Several improvements about BKZ algorithm

Ziyu Zhao

Tsinghua University

Abstract. todo

1 Introduction

Lattice based cryptography is nowadays a important part of post-quantum cryptography because it's fast and widely used in fully homomorphic encryption. The security of it mainly based on some lattice problems, such as learning with error problem (LWE) and ntru problem. These problems can be reduce to approximate shortest vector problem (ASVP), i.e. to find a relatively short vector given a lattice basis. And currently lattice reduction is the most efficient way to solve such problems, thus it is important to know the concrete hardness of ASVP.

Currently there are two types of algorithm to find short vector in the lattice given a lattice basis. One is SVP algorithms like enumeration and sieving, which can find almost the shortest vector in the lattice but the cost is exponential in the dimension of lattice. These SVP algorithms can only be applied on lattice with a small dimension. Another type of algorithm, for instance LLL algorithm and BKZ algorithm can work on high dimensional lattice in a realistic time. LLL algorithm is extremely fast and often used as preprocessing, BKZ algorithm gives a bridge from shortest vector in small dimension to short vector with the same root hermite factor in high dimension.

BKZ algorithm were first purposed by Schnorr in the 80's. It does enumeration on local blocks to find short vector then insert the new vector in the basis. Larger local blocksize gives shorter vector and cost more time. In 2011, Nguyen used some pruning technique in the enumeration step, it makes BKZ algorithm with a higer local blocksize practicable. In 2016, Yuntao Wang et al. purposed progressive BKZ, it start with a small blocksize, and increase it in a well organized manner, makes the algorithm significantly faster.

In this paper, we will give several further improvements of BKZ algorithm, and applied this techniques in lattice reduction with a SVP subroutine based on enumeration (these techniques can also be applied to sieving based SVP subroutine well). We implemented the new BKZ algorithm and tested it on several Ideal lattice challenges (the ideal lattice structure is never used), the running result shows we get a speed up with a factor $\approx 2^3$, which may be further improved since we did not used a well organized progressive BKZ (our blocksize are simply $80, 88, 96, \cdots$). Moreover our new BKZ algorithm is still easy to simulate (as BKZ 2.0) if we know the behavior of the SVP subroutine well.

Road map. In section 2, we present some basic facts about lattice and introduce the notations. Then in section 3, we will recall the developments of BKZ algorithm in the history. Our futher improvements of BKZ will be given in section 4. The infomation about the lattice challenge and the simulation of BKZ algorithm is in section 5.

2 Preliminaries

Lattice is discrete subgroup in $\mathbb{R}^m$. A lattice L admits a integral basis $\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n\}$ such that each vector $\mathbf{v}$ in L can be represented uniquely as linear combination with integral coefficients of $\mathbf{B}$. We say n is the dimension of the lattice. The determinant of the lattice is defined to be $\sqrt{\det(\mathbf{B}\mathbf{B}^T)}$ which is equal to the absolute value of $\det(\mathbf{B})$ if $m = n$.

Gaussian Heuristic Because of the discreteness, the shortest vector in L exists (not unique in general). It is extremely hard to compute the shortest vector (proved to be NP-hard problem), but the length of it is estimated to be

$$\frac{\Gamma(\frac{n}{2}+1)^{\frac{1}{n}}}{\sqrt{\pi}} \cdot \det(L)^{\frac{1}{n}} \approx \sqrt{\frac{n}{2\pi e}} \cdot \det(L)^{\frac{1}{n}}$$

when the lattice is *random* and n is not too small. In practice it works well if $n \geqslant 40$.

Gram-Schmidt orthogonalization The Gram-Schmidt orthogonalization of $\mathbf{B}$ is given by $\mathbf{B}^* = (\mathbf{b}_1^*, \cdots, \mathbf{b}_n^*)$ where $\mathbf{b}_i^*$ is defined by

$$\mathbf{b}_i^* = \mathbf{b}_i - \sum_{j=1}^{i-1} \mu_{ij}\mathbf{b}_j^*,\ \mu_{ij} = \frac{\langle \mathbf{b}_i, \mathbf{b}_j^* \rangle}{\|\mathbf{b}_j^*\|^2}$$

we further denote by B_i the square of $\|\mathbf{B}_i^*\|$, we will call $[B_1, B_2, \cdots, B_n]$ the distance vector of the basis $\mathbf{B}$. The distance vector of a basis contains lots of infomation about this notation will be heavily used in the analysis of BKZ algorithm. we should also introduce the concept of local projected lattice. Let π_i be the orthogonal projection to $\mathrm{span}(\mathbf{b}_1, \cdots, \mathbf{b}_{i-1})^{\perp}$. Then we define the local projected lattice $L_{[i,j]}$ to be the lattice spanned by $B_{[i,j]} = (\pi_i(\mathbf{b}_i), \cdots, \pi_i(\mathbf{b}_j))$.

Root Hermite Factor For a vector $\mathbf{v}$ in a n dimensional lattice L, we define the root Hermite factor to be

$$\delta = \mathrm{rHF}(\mathbf{v}) = \frac{\|\mathbf{v}\|}{\det(L)}^{\frac{1}{n}}$$

the root Hermite factor measures the quality of the vector. The hardness to get a vector of certain length mainly depends on its root Hermite factor.

3 history of BKZ algorithm

3.1 the original algorithm

The first version of BKZ algorithm were purposed by Schnorr and Euchner as a generalization of the famous LLL algorithm. Briefy, LLL algorithm gives the basis an order, then always reduce the latter vector by the former ones, and after the reduction done, it trys to make the former one shorter (in the local projected lattice) by swaping contiguous vector pairs. The algorithm terminates when no more swap or reduction can be done. The first vector of the output basis is of length about 1.02^n times the Gaussian heuristic in practice, and the running time is polynomial in n, where n is the dimension of the lattice.

BKZ algorithm replace the *swap* in LLL algorithm by a full enumeration in the local projected lattice to get shorter vector. This vector will be inserted in the basis in a preselected place, and we use a LLL algorithm to remove the linear dependency. The size of the local projected lattice is fixed and the place to do enumeration is pre-specified. Same as LLL algorithm, BKZ algorithm terminates when no notrivial insertion can be done. The algorithm works as following:

Algorithm 1: BKZ algorithm

Input: a basis $B = (\mathbf{b}_1, \cdots, \mathbf{b}_n)$, blocksize d
Output: A BKZ-d reduced basis

LLL(B); **while** *last epoch did a nontrivial insertion* **do**
  for $i = 1, 2, \cdots, n-1$ **do**
    $h = \max(i + d - 1, n)$;
    $\mathbf{v} = \text{full\_enum}(L_{[i,h]})$; //find shortest vector by enumeration
    if $\|\mathbf{v}\| < \|\mathbf{b}_i\|$ **then**
      $\text{LLL}(\mathbf{b}_1, \cdots, \mathbf{b}_{i-1}, \mathbf{v}, \mathbf{b}_i, \cdots, \mathbf{b}_{\max(h+1,n)})$;
    else
      $\text{LLL}(\mathbf{b}_1, \cdots, \mathbf{b}_{\max(h+1,n)})$;

The running time of BKZ algorithm increase while the blocksize increase. It is not proved to be polynomial in n (the dimension) for fixed blocksize, even though for small blocksize d (for example $d < 20$), the algorithm always terminates in reasonable time and the output quality is significantly improved. For $d = 20$ and n sufficiently large, the shortest vector it founds has length around 1.0128^n times the Gaussian heuristic.

3.2 BKZ2.0

In 2011, Yuanmi and Nguyen gives several improvements of the original BKZ algorithm, which makes BKZ algorithm with a high blocksize ($d \sim 100$) practicable. The root hermite factor of the output vector is now improved to about 1.0095. They mainly did the following:

The first point is that, for a large blocksize d ($d \geqslant 40$), before no new insertion can be done, there is a long time that the quality of the basis improves poorly. So they used the *early abort* technique to stop the algorithm as soon as the quality of the basis not improve further. This provides an exponential speed up in practice without degenerating the output quality.

They also did some modification on the enumeration step. For a larger blocksize d, experiments shows the running time of BKZ is dominanted by the enumeration subroutine. They used pruned enumeration instead of the full enumeration, a proper pruning can gives an exponential speed up ($2^{d/4}$) while still output the shortest vector with a high probablity. They further gives a *extreme pruning* technique, which repeat a further pruned enumeration which output the shortest vector with a low probablity for several times. This leads to a speed up of $2^{d/2}$.

Another thing they did is to preprocessing the basis before the enumeration, since the nodes to enumerate will be fewer if the basis have a better quality. Taking some time to do a light reduction will largely reduce the enumeration time, in BKZ 2.0 they choose a BKZ algorithm with a small blocksize as preprocessing.

3.3 progressive BKZ

Progressive BKZ mainly means to progressively enlarge the blocksize while doing reduction. The key idea is if a enumeration with low dimension can further reduce the lattice, there is no need to use a much larger dimension since the cost for SVP is at least exponential in the dimension. This technique were mentioned in several studies including. These works mainly different in the way they increase the blocksize. In 2016, did a precise Cost Estimation of the progressive BKZ, and gives a optimized blocksize strategy. In their extimation, to do a BKZ-100 in a 800 dimensional lattice, their progressive BKZ is $2^2.7$ times faster than BKZ 2.0. And it's extimated 50 times faster than BKZ 2.0 to solve SVP challenge up do dimension 160.

4 Several improvements about BKZ algorithm

In these section we will give several techniques to further accelerate the BKZ algorithm. These techniques can be used to BKZ based on both sieving and enumeration type SVP subroutine. And one can use it almost for free (except for large final run for sieving, which requires more memory) to get a considerable acceleration in practice.

4.1 local basis processing instead of insertion

Currently We have two types of SVP algorithm, enumeration and sieving. Sieving is faster but requires large space which grows exponential in the sieving dimension. To do a large sieve or enumeration with the hope to find the shortest vector is generally not the best choice. The nodes to enumerate grows as the

basis get worse, so we always do some preprocessing as in BKZ2.0. So the whole enumeration process not only gives a short vector, but also a rather good basis. For sieving, we often use the left progressive sieve to accelerate, whose speed also relies on the quality of the basis. And we need the first several entries in the distance vector to be short to get a large dimension for free, which saves both time and memory. Thus it will not lead to much further cost to get a good basis.

Only insert one short vector like the original BKZ algorithm or BKZ2.0 will waste the almost *free* basis. It's generally better to compute the transform matrix of local processing (on the local projected lattice) then apply it on the vectors of the original basis (succeed by a size reduction). Then the next local basis to apply the SVP algorithms is already only little bit worse than an HKZ reduced basis. A obvious gain is we need no more preprocessing for it. And this is also the fundamental of the next technique.

4.2 jump by two or more

After we do a local basis processing with blocksize d, the fisrt vector will be short, and it's easy to see that the next few vector is not too long also. If we make our sieving context jump to right by two index or more, say we jump s steps after each SVP subroutine, we accelerate by factor s while not degenerating the quality much.

Here a crucial point is to use a blocksize slightly larger than we require. For instance, If we want to do a BKZ with blocksize d, we can choose a $d' = d + s$, then every time we jump s steps. The result will not be worse than do BKZ-d without jump (actually better), if the output basis has the similar quality as a *HKZ-reduced* basis. After the modification, the number of SVP subroutine is only $\frac{1}{s}$ as before, and for each subroutine, the cost is $2^{0.36s}$ (practically, when $d \sim 90$) as before if we use SVP algorithm based on 3-sieve. Take $s = 4$ we get a speed up of at least $2^0.56$, There is a detailed analysis of jump based on simulation of BKZ algorithm in the next section, It shows .

4.3 reduce only when we need

in practical use, we usually don't need the whole BKZ reduced basis. What we want is just a short vector (in lattice challenge) or make the tail of the distance vector large enough such that the key can be get by a size reduction (in real attack of lattce based cryptography). We only introduce the case of lattice challenge here since the another is totally similar.

For example, if we want to do BKZ-100 on a 700 dimensional lattice. For the last 6 tour of SVP subroutine, we don't need to visit all index. In fact we just need to visit the index in $[1, 600], [1, 500], [1, 400], [1, 300], [1, 200], [1, 100]$ reespectively since doing BKZ on $[1, m]$ only relates to the first $m + 99$ vectors. This way we can save half of the time for the last 6 epoch. Notice that if we use a progressively larger blocksize, even if we jump by two or more usually we stay on one dimension for only $10 \sim 20$ tours. It at least saves constant ratio of time since currently the best SVP algorithm takes exponential time.

4.4 a large final run

To solve SVP or find a short vector in low dimension, Nguyen's extreme pruning technique first reduce the basis (preprocessing) then do a heavy extremely pruned enumeration with the hope of finding vectors short enough. In BKZ for large dimensional lattice, we can choose a much larger dimension d in last SVP subroutine (working on $[1, d]$) to get a much shorter vector, to save the time for several tours of SVP subroutine with a normal blocksize. Since one tour costs $n/s \cdot T_{svp}$, which is much larger than a SVP subroutine, this method works well in practice (If we are search for the secret key hidden in the lattice, just do a large enumeration or sieving at the tail of the basis). A detailed analysis based on simulation of BKZ is presented in next section.

5 Lattice challenges

5.1 ideal lattice challenges

5.2 simulation and analysis

6 example

6.1 A Subsection Sample

Please note that the first paragraph of a section or subsection is not indented. The first paragraph that follows a table, figure, equation etc. does not need an indent, either.

Subsequent paragraphs, however, are indented.

Sample Heading (Third Level) Only two levels of headings should be numbered. Lower level headings remain unnumbered; they are formatted as run-in headings.

Sample Heading (Fourth Level) The contribution should contain no more than four levels of headings. Table 1 gives a summary of all heading levels.

Table 1. Table captions should be placed above the tables.

Heading level	Example	Font size and style
Title (centered)	**Lecture Notes**	14 point, bold
1st-level heading	**1 Introduction**	12 point, bold
2nd-level heading	**2.1 Printing Area**	10 point, bold
3rd-level heading	**Run-in Heading in Bold.** Text follows	10 point, bold
4th-level heading	*Lowest Level Heading.* Text follows	10 point, italic

Displayed equations are centered and set on a separate line.

$$x + y = z \tag{1}$$

Please try to avoid rasterized images for line-art diagrams and schemas. Whenever possible, use vector graphics instead (see Fig. 1).

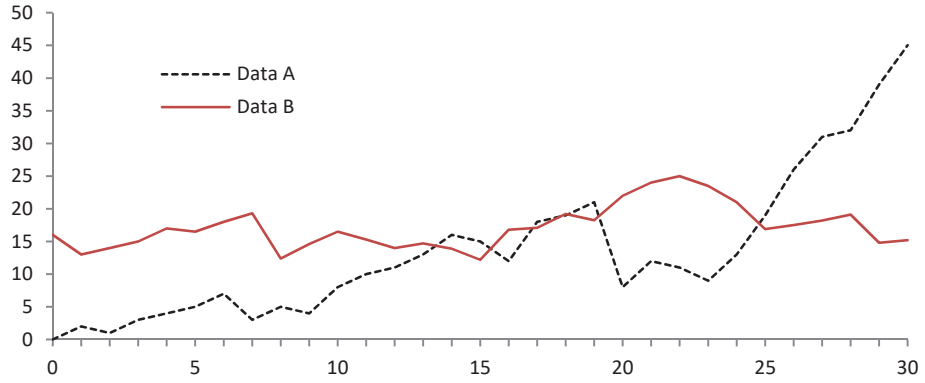

Fig. 1. A figure caption is always placed below the illustration. Please note that short captions are centered, while long ones are justified by the macro package automatically.

Theorem 1. *This is a sample theorem. The run-in heading is set in bold, while the following text appears in italics. Definitions, lemmas, propositions, and corollaries are styled the same way.*

Proof. Proofs, examples, and remarks have the initial word in italics, while the following text appears in normal font.

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