§1.1 - Calculus

Definition #1 (Limit)
Let $f : (a, b) \to \mathbf{R}$, $L \in \mathbf{R}$ and $c \in (a, b)$.

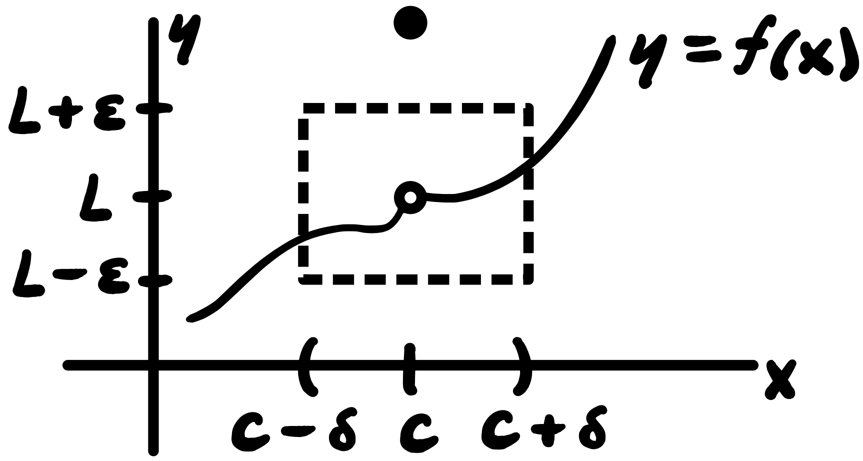
If for every $\epsilon > 0$ we have some $\delta > 0$ such that

$$0 < |x - c| < \delta \quad \text{implies} \quad |f(x) - L| < \epsilon$$

then the **limit** as x approaches c of $f(x)$ is L.

We denote this by

$$\lim_{x \to c} f(x) = L.$$



Avoid evaluating f at c with

$$0 < |x-c|$$

Locally,

$$f(x) \approx L \quad \text{for} \quad x \approx c \ \ \&\ \ x \neq c.$$

We have

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

from calculus rules or l'Hôpital.

Now

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

is continuous on $\mathbb{R}$:

$$\lim_{x \to c} f(x) = f(c)$$

for all $c \in \mathbb{R}$.

Definition #2 (Continuous)
Let $f : (a, b) \to \mathbf{R}$ and $c \in (a, b)$.

If
$$\lim_{x \to c} f(x) = f(c)$$
then f is **continuous** at c.

Theorem #1 (Intermediate Value Theorem (IVT))
Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$.

If L is some value between $f(a)$ and $f(b)$ then there exists some $c \in (a, b)$ such that

$$f(c) = L.$$

That is, f takes on every value between $f(a)$ and $f(b)$ at least once over the interval $[a, b]$.

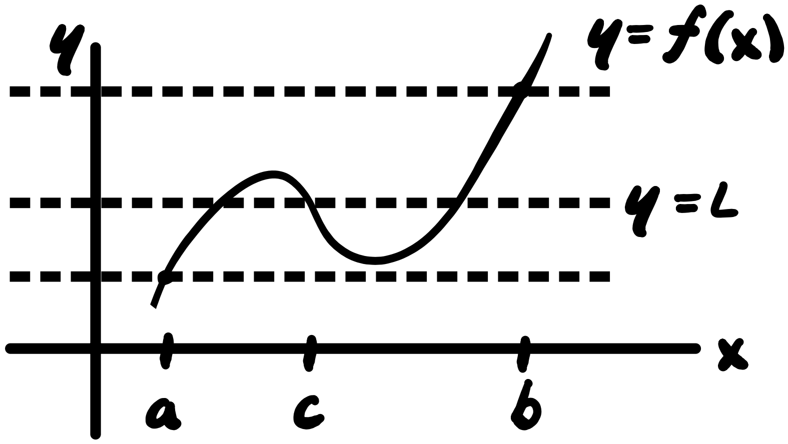
Definition #3 (Differentiable)
Let $f : (a, b) \to \mathbf{R}$ and $c \in (a, b)$.

If

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L$$

exists the f is **differentiable** at c and its derivative is denoted by

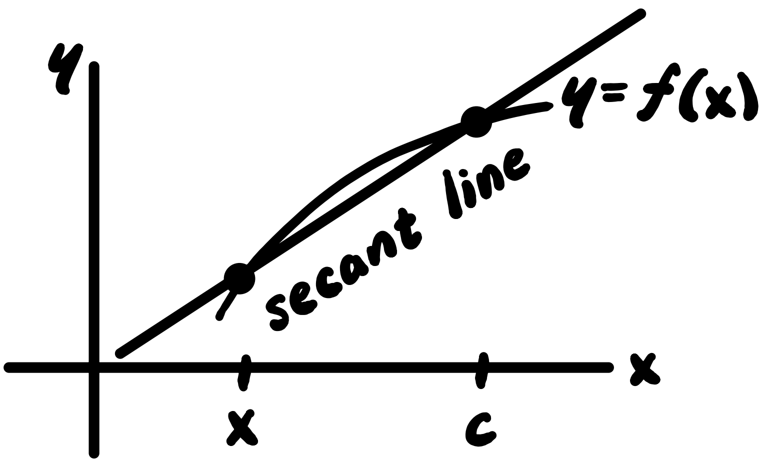
$$f'(c) = L.$$

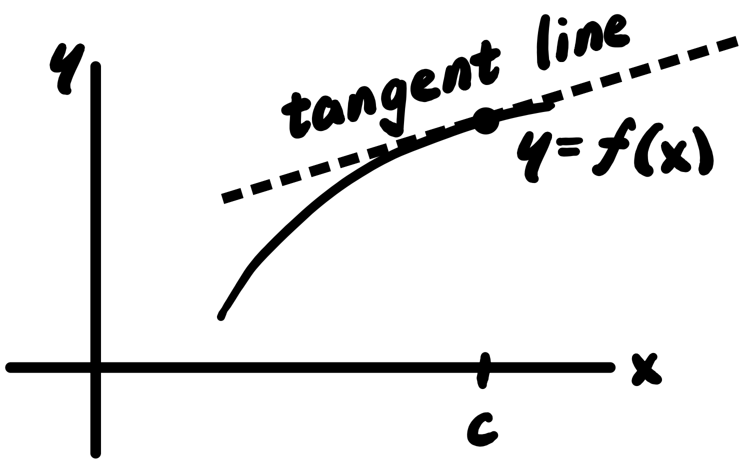


If $f(0) < 0$ and $f(1) > 0$ then

$$f(x) = 0$$

for some $x \in (0,1)$ provided f is continuous on $[0,1]$.





$$f(x) \approx \underbrace{f(c) + f'(c)(x-c)}_{\text{tangent line}} \quad \text{for } x \approx c$$

Lemma #1

Let $f : (a, b) \to \mathbf{R}$ be differentiable at c.

Then f is continuous at c.

Pf) Observe for $x \neq c$

$$f(x)-f(c) = \frac{f(x)-f(c)}{x-c} \cdot (x-c).$$

Since

$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c) \;\; \& \;\; \lim_{x \to c} (x-c) = 0$$

we have

$$\lim_{x \to c} \left(f(x)-f(c)\right) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \cdot \lim_{x \to c} (x-c) = 0$$

by limit laws. Hence f is continuous at c.

Let

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} = |x|$$

$f'(0)$ d.n.e but

$$\lim_{x \to 0} f(x) = f(0).$$

Warning

Continuity does **not** imply differentiability.

Definition #4 (Function Spaces)
Let $f : \mathbf{R} \to \mathbf{R}$.

If f is continuous on $\mathbf{R}$ then

$$f \in C(\mathbf{R}).$$

If f and f' are continuous on $\mathbf{R}$ then

$$f \in C^1(\mathbf{R})$$

or f is **continuously differentiable**.

Definition #5 (More Function Spaces)
Let $f : \mathbf{R} \to \mathbf{R}$ and $n \in \mathbf{N}$.

If $f, f', f'', \ldots, f^{(n)}$ are continuous on $\mathbf{R}$ then

$$f \in C^n(\mathbf{R}).$$

If f^n exists for every n then

$$f \in C^\infty(\mathbf{R})$$

or is f **infinitely differentiable.**

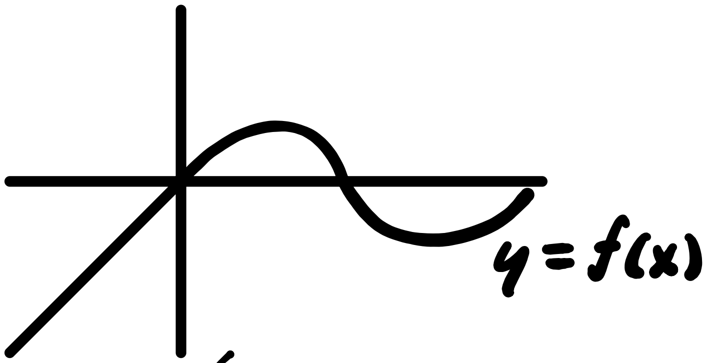
Lemma #2
We have

$$C^{\infty}(\mathbf{R}) \subset \ldots \subset C^{2}(\mathbf{R}) \subset C^{1}(\mathbf{R}) \subset C(\mathbf{R}).$$

Pf) Use prior lemma.

If $g(x) = x^{3/2}$ for all $x \in \mathbb{R}$:

$$g \in C^1[0,\infty), \quad g \notin C^2[0,\infty), \quad g \in C^2(0,1)$$



If

$$f(x) = \begin{cases} \sin(x), & x \geq 0 \\ x, & x < 0 \end{cases}$$

for all $x \in \mathbb{R}$ then

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$$

and

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{x}{x} = 1$$

implies

$$f'(0) = 1.$$

Thus

$$f'(x) = \begin{cases} \cos(x), & x \geq 0 \\ 1, & x < 0 \end{cases}$$

and

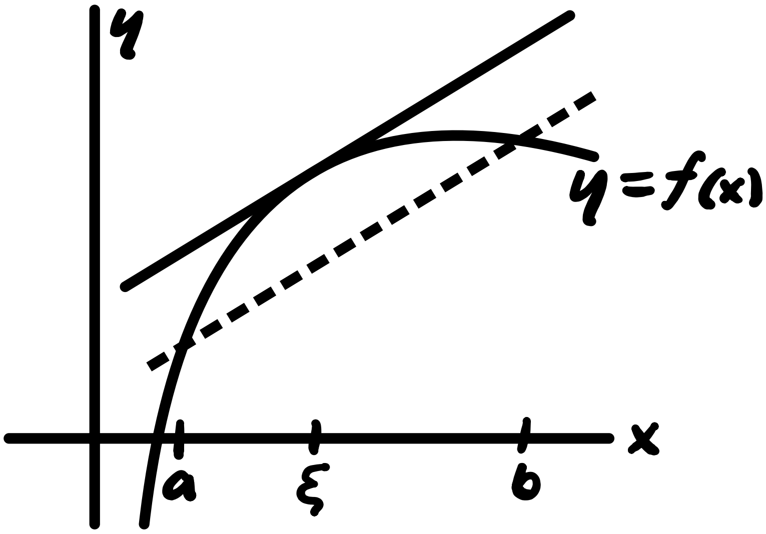
$$\lim_{x \to 0} f'(x) = f'(0)$$

tells us $f \in C^1(\mathbb{R})$. Is $f \in C^2(\mathbb{R})$? $f \in C^3(\mathbb{R})$?

Theorem #2 (Mean Value Theorem (MVT))
Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b).

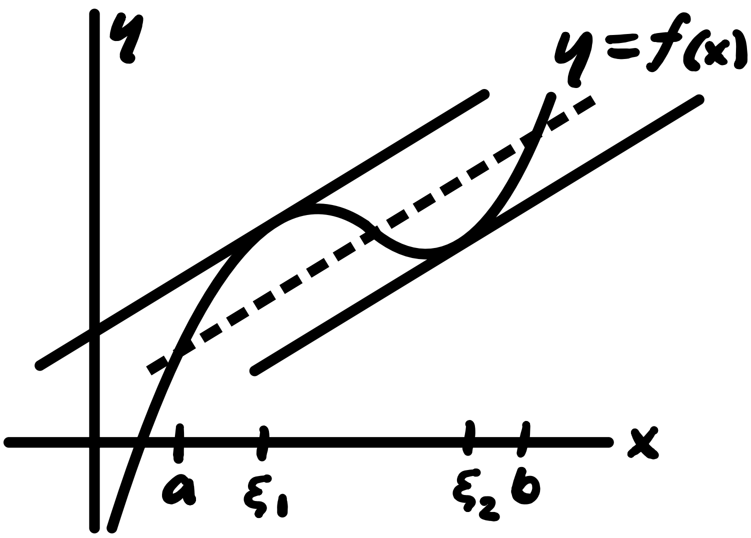
Then there exists $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$



We have a tangent line parallel to the secant line from $(a,f(a))$ to $(b,f(b))$.

But we may have several:



Remark

The Mean Value Theorem can also be expressed as

$$f(x) = f(a) + f'(\xi)(x - a)$$

where $x \in [a, b]$ and ξ is between a and x.

Corollary #1 (Rolle's Theorem)
Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b).

If $f(a) = f(b)$ then there exists $\xi \in (a, b)$ such that

$$f'(\xi) = 0.$$

Theorem #3 (Taylor's Remainder Theorem (Lagrange Form))
Let $f \in C^n(\mathbf{R})$ and assume $f^{(n+1)}$ exists on $\mathbf{R}$.

If $x, c \in \mathbf{R}$ then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_n(x)$$

where

$$E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-c)^{n+1}$$

for some ξ between x and c. We call E_n the **error term**.

Definition #6 (Taylor Polynomial)
Let $f \in C^m(\mathbf{R})$ and $c \in \mathbf{R}$.

If $n \in \mathbf{Z}$ with $0 \leq n \leq m$ then

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

is the degree (at most) n **Taylor polynomial** of f centered at c.

$$f(x) = e^x \quad c = 0$$ #21

$$f^{(k)}(x) = e^x \quad \text{for all } k \in \mathbb{N}$$

$$f^{(k)}(0) = 1$$

Hence

$$e^x = \sum_{k=0}^{10} \frac{x^k}{k!} + E_{11}(x)$$

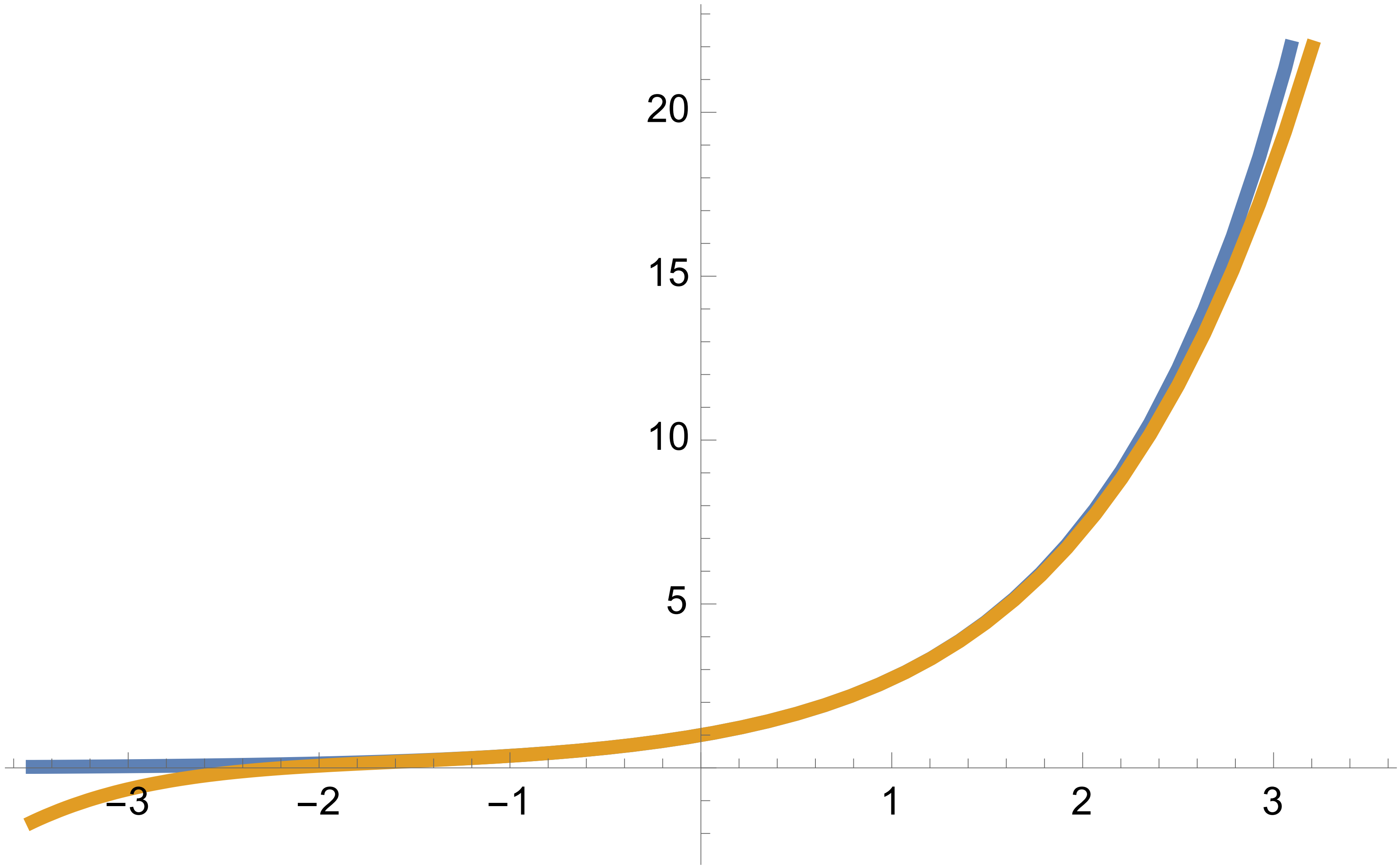
$$E_{11}(x) = \frac{f^{(11)}(\xi)}{11!} x^{11} = \frac{e^{\xi}}{11!} x^{11}$$

where ξ is between x and 0, $|\xi| < |x|$.

For x "small"

$$\frac{e^{\xi}}{11!} x^{11} \approx 0$$

$$e^x \quad \text{and} \quad T_5(x)$$



Remark

If f is "well-behaved"

$$f(x) \approx \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

or $E_n(x)$ is relatively small.

Theorem #4 (Taylor's Remainder Theorem (Restated))
Let $f \in C^{n+1}(\mathbf{R})$ and $h \in \mathbf{R}$.

For $x \in \mathbf{R}$ fixed we have

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + E_n(h)$$

where

$$E_n(h) = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

for some ξ between x and $x+h$.

Let $x = 0$ and $f \in C^3$ then #25

$$f(h) = f(0) + hf'(0) + h^2 \frac{f''(0)}{2} + h^3 \frac{f'''(\xi)}{6}$$

where ξ depends on h.

So

$$f(h) - f(0) = hf'(0) + h^2 \frac{f''(0)}{2} + h^3 \frac{f'''(\xi)}{6}$$

and for $h \neq 0$

$$\frac{f(h) - f(0)}{h} = f'(0) + h \frac{f''(0)}{2} + h^2 \frac{f'''(\xi)}{6}.$$

Hence

$$\frac{f(h) - f(0)}{h} - f'(0) = h \frac{f''(0)}{2} + h^2 \frac{f'''(\xi)}{6}.$$

and

$$\frac{f(h) - f(0)}{h}$$

is an approximation of $f'(0)$ with error

$$h \frac{f''(0)}{2} + h^2 \frac{f'''(\xi)}{6}.$$

Remark

Just a normal function of x:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_n(x).$$

With x-fixed,

$$g(h) = f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}h^k + E_n(h)$$

is acting as a function of h. So the function value depends on how far away from x we are.

Lemma #3 (Kincaid Cheney - §1.1 Problem #21)
Let $f \in C^n(\mathbf{R})$.

If

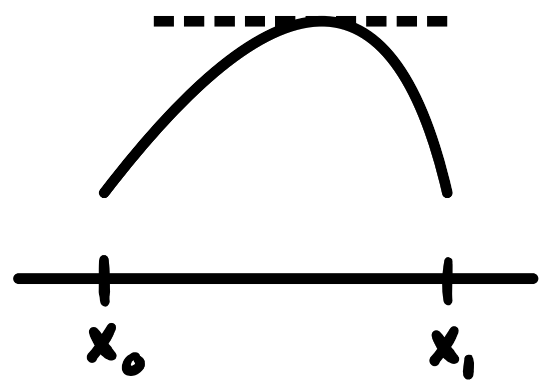
$$f(x_0) = f(x_1) = \ldots = f(x_n)$$

where $x_0 < x_1 < \ldots < x_n$ then

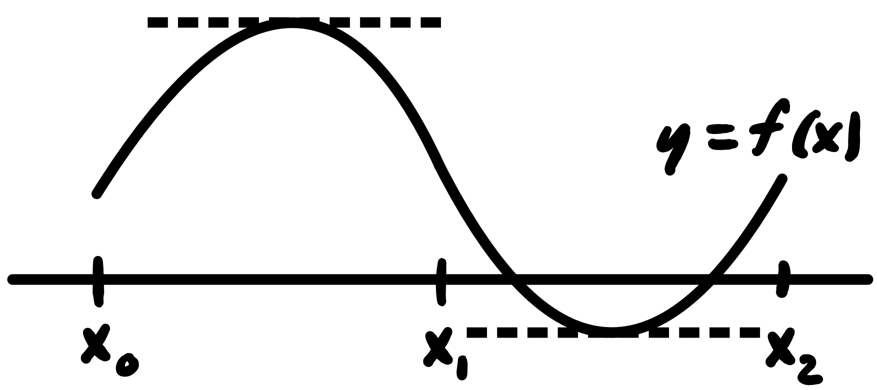
$$f^{(n)}(\xi) = 0$$

for some $\xi \in (x_0, x_n)$.

Pf) If $n=1$ we have Rolle's Theorem #28



If $n=2$



We have c_1 and c_2 with

$$f'(c_1)=f'(c_2)=0$$

and by Rolle's Theorem

$$f''(\xi)=0.$$