§1.2 - Convergence

Definition #1 (Sequence)
Let $x_n \in \mathbf{R}$ for each $n \in \mathbf{N}$.

Then

$$(x_n) = (x_1, x_2, x_3, \ldots)$$

is a **sequence** in $\mathbf{R}$.

Definition #2 (Convergence)
Let (x_n) be a sequence in $\mathbf{R}$ and $x \in \mathbf{R}$.

Given any $\epsilon > 0$ we have some $r \in \mathbf{R}$ such that

$$n > r \quad \text{implies} \quad |x_n - x| < \epsilon$$

then (x_n) **converges** to x. We denote this by

$$\lim_{n \to \infty} x_n = x \quad \text{or} \quad x_n \to x.$$

#4

$$a_1 = 1/3$$
$$a_2 = 1/9$$
$$a_3 = 1/27$$
$$\vdots$$
$$a_n = \frac{1}{3^n}$$

Given $\varepsilon > 0$

$$\left|\frac{1}{3^n} - 0\right| = \frac{1}{3^n} < \frac{1}{n}$$

since $n < 3^n$ and so if

$$\frac{1}{n} < \varepsilon$$

we are done.

Now

$$\frac{1}{n} < \varepsilon \text{ if and only if } \frac{1}{\varepsilon} < n$$

and pick $r = \frac{1}{\varepsilon}$.

Hence

$$r < n \text{ implies } \left|\frac{1}{3^n} - 0\right| < \varepsilon$$

so

$$\lim_{n\to\infty} \frac{1}{3^n} = 0.$$

Remark
The book often uses x^* to denote the limit of (x_n) and $[x_n]$ to denote a sequence.

The phrase

"for all n-large"

means that we have some $r \in \mathbf{R}$ such that the given condition holds whenever $n > r$.

Example #1

Consider the sequences defined by

$$x_n = \frac{1}{2^n}, \quad y_n = \frac{1}{n!}, \quad \text{and} \quad z_n = \frac{1}{n^{2^n}}$$

for all $n \in \mathbf{N}$.

Each sequence converges to 0 but some do it faster: #7

$$\frac{x_{n+1}}{x_n} = \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \frac{1}{2}$$

$$\frac{y_{n+1}}{y_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{(n+1)}$$

$$\frac{z_{n+1}}{z_n} = \frac{\frac{1}{(n+1)^{(2^{n+1})}}}{\frac{1}{n^{2^n}}} = \frac{n^{2^n}}{(n+1)^{(2^{n+1})}} = \left(\frac{n}{n+1}\right)^{2n} \frac{1}{(n+1)}$$

The ratio

$$\frac{|x_{n+1} - x|}{|x_n - x|}$$

informs how fast $x_n \to x$. The smaller the ratio, the faster the convergence.

Definition #3 (Linear Convergence)
Let (x_n) be a sequence in $\mathbf{R}$ where $x_n \to x$.

If we have $c \in (0,1)$ such that

$$|x_{n+1} - x| \leq c|x_n - x|$$

for all n-large.

Then the convergence of (x_n) to x is at least **linear**.

Definition #4 (Superlinear Convergence)
Let (x_n) be a sequence in $\mathbf{R}$ where $x_n \to x$.

Suppose we have (ϵ_n) where $\epsilon_n \to 0$ such that

$$|x_{n+1} - x| \leq \epsilon_n |x_n - x|$$

for all n-large.

Then the convergence of (x_n) to x is at least **superlinear**.

Definition #5 (Quadratic Convergence)
Let (x_n) be a sequence in $\mathbf{R}$ where $x_n \to x$.

If we have $C > 0$ such that

$$|x_{n+1} - x| \leq C|x_n - x|^2$$

for all n-large.

Then the convergence of (x_n) to x is at least **quadratic**.

Definition #6 (Convergence of order α)
Let (x_n) be a sequence in $\mathbf{R}$ where $x_n \to x$ and $\alpha > 0$.

If we have $C > 0$ such that

$$|x_{n+1} - x| \leq C|x_n - x|^{\alpha}$$

for all n-large.

Then the convergence of (x_n) to x is at least of **order** α.

Now

$$\frac{x_{n+1}}{x_n} = \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \frac{1}{2} \quad \text{yields } |x_{n+1}-x| = \frac{1}{2}|x_n - x|$$

but

$$\varepsilon_n = \frac{1}{2}$$

does not converge to 0 : linear convergence

While

$$\frac{y_{n+1}}{y_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{1}{(n+1)} \quad \text{yields } |y_{n+1}-y| = \frac{1}{(n+1)}|y_n - y|$$

or at least super linear convergence.

Here

$$\frac{y_{n+1}}{y_n^2} = \frac{\frac{1}{(n+1)!}}{\left(\frac{1}{n!}\right)^2} = \frac{n!}{(n+1)} \quad \text{yields } |y_{n+1}-y| = \frac{n!}{(n+1)}|y_n - y|^2$$

so (y_n) lacks quadratic convergence: super linear convergence

Finally, #13

$$z_n^2 = \left(\frac{1}{n^{2^n}}\right)^2 = \frac{1}{n^{2^n} \cdot n^{2^n}} = \frac{1}{n^{2 \cdot 2^n}} = \frac{1}{n^{(2^{n+1})}}$$

$$z_{n+1} = \frac{1}{(n+1)^{(2^{n+1})}}$$

and

$$\frac{z_{n+1}}{z_n^2} = \frac{n^{(2^{n+1})}}{(n+1)^{(2^{n+1})}} = \left(\frac{n}{n+1}\right)^{(2^{n+1})} < 1.$$

Thus

$$|z_{n+1} - z| < |z_n - z|^2$$

so (z_n) has at least quadratic convergence.

Definition #7 (Big "Oh")
Let (x_n) and (α_n) be distinct sequences in $\mathbf{R}$.

Suppose we have $C > 0$ such that

$$|x_n| \leq C|\alpha_n|$$

for all n-large.

Then

$$x_n = \mathcal{O}(\alpha_n)$$

or (x_n) is "**big oh**" of (α_n).

Example #2

Show that if $x_n = \mathcal{O}(\alpha_n)$ and $y_n = \mathcal{O}(\alpha_n)$ then

$$x_n + y_n = \mathcal{O}(\alpha_n).$$

Let $\alpha_n = \frac{1}{n}$ and $x_n = \frac{n}{3n^2+1}$ then

$$0 < \frac{n}{3n^2+1} < \frac{n}{3n^2} = \frac{1}{3n} = \frac{1}{3}\alpha_n$$

for all n.

Thus

$$x_n = \mathcal{O}(\alpha_n).$$

Notice

$$\lim_{n\to\infty}\left|\frac{x_n}{\alpha_n}\right| \in (0,\infty).$$

If $x_n = \mathcal{O}(\alpha_n)$ and $y_n = \mathcal{O}(\alpha_n)$ then

$|x_n| \le C_1|\alpha_n|$ for $n > N_1$ & $|y_n| \le C_2|\alpha_n|$ for $n > N_2$

so

$$|x_n + y_n| \le |x_n| + |y_n|$$

and for $n > \max(N_1, N_2)$

$$|x_n + y_n| \le C_1|\alpha_n| + C_2|\alpha_n| = (C_1 + C_2)|\alpha_n|.$$

Thus

$$x_n + y_n = \mathcal{O}(\alpha_n).$$

Remark
If
$$\lim_{n\to\infty} \frac{x_n}{\alpha_n} = 0$$
then for n-large
$$x_n \ll \alpha_n$$
or x_n is “much smaller” than α_n.

However,
$$\frac{x_n}{\alpha_n}$$
is undefined if α_n is zero and we still want conditions to describe when x_n is “much smaller” than α_n.

Definition #8 (Little "Oh")
Let (x_n) and (α_n) be distinct sequences in $\mathbf{R}$.

Suppose (ϵ_n) is a nonnegative sequence where $\epsilon_n \to 0$ and

$$|x_n| \leq \epsilon_n |\alpha_n|$$

for n-large.

Then

$$x_n = o(\alpha_n)$$

or (x_n) is "**little oh**" of (α_n).

Lemma #1

Let (x_n) and (α_n) be distinct sequences in $\mathbf{R}$.

If (α_n) is nonzero and then

$$x_n = o(\alpha_n) \quad \text{if and only if} \quad \lim_{n\to\infty} \frac{x_n}{\alpha_n} = 0.$$

Pf) "=>" Let $x_n = o(\alpha_n)$ then we have some (#20)

(ε_n) with $\varepsilon_n \to 0$ and $N \in \mathbb{N}$ where

$$|x_n| \leq \varepsilon_n |\alpha_n|$$

for $n > N$.

Since $|\alpha_n| \neq 0$,

$$\frac{|x_n|}{|\alpha_n|} \leq \varepsilon_n$$

and by the Squeeze Theorem

$$\frac{|x_n|}{|\alpha_n|} \to 0$$

because $\varepsilon_n \to 0$.

"<=" If

$$\frac{|x_n|}{|\alpha_n|} \to 0$$

let

$$\varepsilon_n = \frac{|x_n|}{|\alpha_n|}$$

then

$$\varepsilon_n |\alpha_n| = |\alpha_n| \frac{|x_n|}{|\alpha_n|} = |x_n|$$

for all n. Naturally,

$$|x_n| \leq \varepsilon_n |\alpha_n|$$

and we are done.

Example #3

Show

$$\frac{1}{n\ln(n)} = o\left(\frac{1}{n}\right).$$

Hence

$$\frac{1}{n\ln(n)} \to 0$$

faster than

$$\frac{1}{n} \to 0.$$

Observe

$$\lim_{n\to\infty} \frac{\frac{1}{n\ln(n)}}{\frac{1}{n}} = 0$$

since

$$\frac{\frac{1}{n\ln(n)}}{\frac{1}{n}} = \frac{1}{\ln(n)} \to 0.$$

Remark

If a property holds in some arbitrary open interval about c we will say the condition holds "**near**" c.

The function given by $f(x) = \sin(x)$ is increasing "near" 0 but is not increasing on $\mathbf{R}$.

Definition #9 (Big "Oh" for Functions)
Let $f, g : \mathbf{R} \to \mathbf{R}$ and $c \in \mathbf{R}$.

Suppose we have $M > 0$ such that

$$|f(x)| \leq M|g(x)|$$

for all x "near" c and $x \neq c$.

Then

$$f(x) = \mathcal{O}(g(x)) \quad \text{as} \quad x \to c.$$

Example #4

Determine the best integer value of n in the equation

$$\arctan(x) = x + \mathcal{O}(x^n)$$

as $x \to 0$.

Notice #26

$$\arctan(x) = \sum_{K=0}^{\infty} \frac{(-1)^K}{2K+1} x^{2K+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \quad *$$

so

$$\arctan(x) = x + E_3(x)$$
$$E_3(x) = \frac{1}{3} f^{(3)}(\xi)\, x^3$$

by Taylor's Remainder Theorem. With $|f^{(3)}|$ continuous/bounded near 0 we have $\arctan(x) - x = \mathcal{O}(x^3)$ or $\arctan(x) = x + \mathcal{O}(x^3)$ as $x \to 0$.

* Since

$$\frac{1}{1+x^2} = \sum_{K=0}^{\infty} (-1)^K x^{2K}$$

for $x \in (-1, 1]$ and so

$$\arctan(x) + C = \int \frac{1}{1+x^2}\, dx = \sum_{K=0}^{\infty} (-1)^K \int x^{2K}\, dx$$
$$= \sum_{K=0}^{\infty} \frac{(-1)^K}{2K+1} x^{2K+1}.$$

Now

$$\arctan(0) = 0 \Rightarrow C = \sum_{K=0}^{\infty} \frac{(-1)^K}{2K+1} 0^{2K+1} = 0$$

implies

$$\arctan(x) = \sum_{K=0}^{\infty} \frac{(-1)^K}{2K+1} x^{2K+1}.$$

Definition #10 (Little "Oh" for Functions)
Let $f, g : \mathbf{R} \to \mathbf{R}$ and $c \in \mathbf{R}$.

Suppose we have a function e, which is nonnegative and

$$\lim_{x \to c} e(x) = 0,$$

where

$$|f(x)| \leq e(x)|g(x)|$$

for all x "near" c and $x \neq c$.

Then

$$f(x) = o(g(x)) \quad \text{as} \quad x \to c.$$

Lemma #2

Let $f, g : \mathbf{R} \to \mathbf{R}$ and $c \in \mathbf{R}$.

Suppose $g(x)$ is nonzero for all x "near" c and $x \neq c$ then

$$f(x) = o(g(x)) \text{ as } x \to c \quad \text{if and only if} \quad \lim_{x \to c} \frac{f(x)}{g(x)} = 0.$$

Remark
We can also define

$$f(x) = o(g(x)) \quad \text{as} \quad x \to \infty$$

and

$$f(x) = \mathcal{O}(g(x)) \quad \text{as} \quad x \to \infty$$

by modifying our sequence definitions to "for x-large" instead of "for n-large."

Problem

We want to evaluate

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

at x_0 by reducing the amount of computation.

Example #5

Observe

$$\begin{aligned} f(x) &= -7 + 3x - 4x^2 + 5x^3 \\ &= -7 + 3x - 4x^2 + 5x \cdot x^2 \\ &= -7 + x(3 + x(-4 + 5x)) \end{aligned}$$

Definition #11 (Nested Form)
Let

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

then

$$p(x) = a_0 + x(a_1 + x(a_2 + \ldots + x(a_{n-1} + a_n x) \ldots)$$

is the **nested form** of p.

Lemma #3

To evaluate

$$p(x) = a_0 + x(a_1 + x(a_2 + \ldots + x(a_{n-1} + a_n x) \ldots)$$

requires n additions and n multiplications.