CSC 503 Homework Assignment 4

Out: September 19, 2018
Due: September 26, 2018
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Let a and b be constant symbols, f a function symbol with three arguments, g a function symbol with one argument, h a function symbol with two arguments, P a predicate symbol with three arguments, and Q a predicate symbol with two arguments, and x, y, and z variable symbols.

Indicate, for each of the following strings, which strings are formulas in predicate logic, and state a reason for failure for strings which are not. No credit will be given for strings correctly identified as not being formulas without correct identification of a reason why the string is not a formula.

1. **[5 points]** $\exists y\ P(x, b, y) \wedge P(h(b, x), h(y, b), g(y)(a))$

   Answer

   Not a formula, $g(y)(a)$ is not a valid term.

2. **[5 points]** $\exists b\ \forall x\ Q(f(b, y, x), g(h(a, x)))$

   Answer

   Not a formula, b is not a variable where after the existential quatifier.

3. **[5 points]** $\forall z\ \exists x\ P(g(h(b, x), z), f(a, y))$

   Answer

   Not a formula, P needs 3 arguments.

4. **[5 points]** $\forall y\ g(Q(z, y)) \to P(h(f(y, z, z), z))$

   Answer

   Not a formula, implies only apply to valid formula, $g(Q(z, y))$ is not a valid formula.

5. **[5 points]** $P(a, g(g(f(a, \neg b, a))), a) \to P(a, f(a, \neg b, a), a)$

   Answer

   Yes, it is a valid formula.

Now let R be a predicate symbol with arity 2, f be a function of arity 2, and ϕ be the formula

$$\forall x\ [(R(x, z) \wedge \exists z\ \neg R(z, f(x, y))) \to \forall y\ R(y, z)]$$

6. **[5 points]** Indicate, for each occurrence of each variable in ϕ, whether that occurrence is free or bound.

   Answer

$$\forall x\ [(R(x_{bound}, z_{free}) \wedge \exists z\ \neg R(z_{bound}, f(x_{bound}, y_{free}))) \to \forall y\ R(y_{bound}, z_{free})]$$

7. **[5 points]** List all variables which occur both free and bound in ϕ.

   Answer

   (a) z
   (b) y

8. **[5 points]** Compute $\phi[t/x]$ for $t = h(f(g(y), x), a, x)$. Is t free for x in ϕ?

   Answer

   Yes, t is free for x in ϕ. Because all the occurrences of x are bounded, thus no substitution is performed, and therefor no free x leaf in ϕ occurs in the scope of $\forall y$ or $\exists y$ for any variable y occurring in t.

9. **[5 points]** Compute $\phi[t/y]$ for $t = h(f(g(y), x), a, x)$. Is t free for y in ϕ?

   Answer

   No, t is not free for y in ϕ. Because the free y is bound in the scope of $\forall x$, where variable x occurring in t .

10. **[5 points]** Compute $\phi[t/z]$ for $t = h(f(g(y), x), a, x)$. Is t free for z in ϕ?

    Answer

    No, t is not free for z in ϕ. Because the free z is bound in the scope of $\forall x$ and $\forall y$, where variable x and y occurring in t .

Now consider a language in which the only nonlogical symbols are a predicate symbol R of two arguments and a function symbol f of one argument. Let ϕ_1 and ϕ_2 be the sentences

$$\begin{aligned} \phi_1 &= \forall x\, \forall y\, \forall z\; R(x, y) \wedge R(y, z) \rightarrow R(x, z) \\ \phi_2 &= \forall x\, \forall y\, \forall z\; R(x, y) \rightarrow R(f(x, z), f(y, z)) \end{aligned}$$

Recall that an interpretation of the language identifies a domain of interpretation M (a set), and maps each of the predicate and function symbols to a set of tuples of elements of M of the appropriate size.

11. **[15 points]:** Give a formal interpretation I that makes ϕ_1 true and ϕ_2 false.

    Answer

    (a) Domain $M = \{a, b\}$
    (b) Function $f(a, a) = b$; $f(a, b) = b$; $f(b, a) = b$; $f(b, b) = b$;
    (c) Interpretation $R^I = \{(a, a)\}$

12. **[5 points]:** Briefly explain why I makes ϕ_1 true.

    Answer

    In order to make for all x, y, z, the ϕ_1 being true:

    (a) Case 1: if $x = y = z = a$, then $R(x, y), R(y, z), R(x, z)$ all are true.
    (b) Other than Case 1: any x, y, z combination will result a case either $R(x, y)$ or $R(y, z)$ being false, which makes the whole being true.

13. **[5 points]:** Briefly explain why I makes ϕ_2 false.

    Answer

    In order to make for all x, y, z, the ϕ_2 being false, just need to find there exist one false case:

    if $x = a, y = a$, then $R(x, y)$ is true, and $R(f(x, z), f(y, z))$ would be $R(b, b)$ which is false, therefore makes the ϕ_2 being false.

14. **[15 points]:** Give the formal definition of an interpretation J that makes ϕ_1 false and ϕ_2 true.

    Answer

    (a) Domain $M = \{a, b\}$
    (b) Function $f(a, a) = a$; $f(a, b) = a$; $f(b, a) = b$; $f(b, b) = b$;
    (c) Interpretation $R^I = \{(a, b), (b, a)\}$

15. **[5 points]:** Briefly explain why J makes ϕ_1 false.

    Answer

    In order to make for all x, y, z, the ϕ_1 being false, just need to find there exist one false case:

    if $x = a, y = b, z = a$, then $R(x, y)$ and $R(y, z)$ is true, and $R(x, z)$ is false, therefore makes the ϕ_1 being false.

16. **[5 points]:** Briefly explain why J makes ϕ_2 true.

Answer

In order to make for all x, y, z, the ϕ_2 being true:

(a) Case 1: if $x = y$, then $R(x, y)$ is either $R(a, a)$ or $R(b, b)$, which is false, makes the whole being true.

(b) Other 2: $x \neq y$, then $R(f(x, z), f(y, z))$ would be either $R(a, b)$ or $R(b, a)$ which is true in either case, which makes the whole being true.