CSC 503 Homework Assignment 5 Sample Solutions

Out: September 25, 2018
Due: October 3, 2018
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If using the Fitch macros to typeset proofs in first order logic, one introduces a dummy variable x by means of the command `\open[x]`.

1. **[20 points]** Using only the basic natural deduction rules, prove the sequent

$$\forall x \forall y(\neg R(y) \to S(x)) \vdash (\exists y \neg R(y)) \to (\forall x S(x)).$$

Answer
The dummy variables can all be introduced on the same line rather than on separate lines as in the following.

1	$\forall x \forall y(\neg R(y) \to S(x))$	Premise
2	$\exists y \neg R(y)$	Assumption
3	x_0	
4	y_0 $\neg R(y_0)$	Assumption
5	$\forall y(\neg R(y) \to S(x_0))$	$\forall$e, 1
6	$\neg R(y_0) \to S(x_0)$	$\forall$e, 5
7	$S(x_0)$	$\to$e, 4, 6
8	$S(x_0)$	$\exists$e, 2, 4–7
9	$\forall x S(x)$	$\forall$i, 3–8
10	$(\exists y \neg R(y)) \to (\forall x S(x))$	$\to$i, 2–9

2. **[20 points]** Using only the basic natural deduction rules, prove the sequent

$$\forall x(P(x) \to \neg R(x)) \vdash \neg(\exists x(P(x) \land R(x))).$$

Answer

1	$\forall x(P(x) \to \neg R(x))$	Premise
2	$\exists x(P(x) \land R(x))$	Assumption
3	x_0 $P(x_0) \land R(x_0)$	Assumption
4	$P(x_0)$	$\land$e$_1$, 3
5	$R(x_0)$	$\land$e$_2$, 3
6	$P(x_0) \to \neg R(x_0)$	$\forall$e, 1
7	$\neg R(x_0)$	$\to$e, 4, 6
8	$\bot$	$\neg$e, 5, 7
9	$\bot$	$\exists$e, 2, 3–8
10	$\neg(\exists x(P(x) \land R(x)))$	$\neg$i, 2–9

3. **[10 points]** Using only the basic natural deduction rules, prove the sequent

$$\forall x\, P(a, f(a), x), \forall z \forall x \forall y\, (P(x, y, z) \to P(f(x), y, f(f(z)))) \vdash P(f(a), f(a), f(f(f(a)))).$$

Answer

1	$\forall x\, P(a, f(a), f(x))$	Premise
2	$\forall z \forall x \forall y\, (P(x, y, z) \to P(f(x), y, f(f(z))))$	Premise
3	$\forall x \forall y\, (P(x, y, f(a)) \to P(f(x), y, f(f(f(a)))))$	$\forall$e, 2
4	$\forall y\, (P(a, y, f(a)) \to P(f(a), y, f(f(f(a)))))$	$\forall$e, 3
5	$P(a, f(a), f(a)) \to P(f(a), f(a), f(f(f(a))))$	$\forall$e, 4
6	$P(a, f(a), f(a))$	$\forall$e, 1
7	$P(f(a), f(a), f(f(f(a))))$	$\to$e, 5, 6

4. **[20 points]** Using only the basic natural deduction rules, prove the sequent

$$\exists x \forall y \forall z P(x, y, z) \vdash \forall z \forall y \exists x P(x, y, z).$$

Answer

The dummy variables can all be introduced on the same line rather than on separate lines as in the following.

1	$\exists x \forall y \forall z P(x, y, z)$				Premise
2	z_0				
3		y_0			
4			x_0	$\forall y \forall z P(x_0, y, z)$	Assumption
5				$\forall z P(x_0, y_0, z)$	$\forall$e, 4
6				$P(x_0, y_0, z_0)$	$\forall$e, 5
7				$\exists x P(x, y_0, z_0)$	$\exists$i, 6
8			$\exists x P(x, y_0, z_0)$		$\exists$e, 1, 4–7
9		$\forall y \exists x P(x, y, z_0)$			$\forall$i, 3–8
10	$\forall z \forall y \exists x P(x, y, z)$				$\forall$i, 2–9

5. **[10 points]** Using only the basic natural deduction rules, prove the sequent

$$\forall x \forall y \, [S(x,y) \rightarrow \neg S(y,x)] \vdash \forall x \, \neg S(x,x)$$

Answer

1		$\forall x \forall y \, [S(x,y) \rightarrow \neg S(y,x)]$	Premise
2	x_0	$\forall y \, [S(x_0,y) \rightarrow \neg S(y,x_0)]$	$\forall$e, 1
3		$(S(x_0,x_0) \rightarrow \neg S(x_0,x_0))$	$\forall$e, 2
4		$S(x_0,x_0)$	Assumption
5		$\neg S(x_0,x_0)$	$\rightarrow$e, 4, 3
6		$\bot$	$\neg$e, 4, 5
7		$\neg S(x_0,x_0)$	$\neg$i, 4–6
8		$\forall x \, \neg S(x,x)$	$\forall$i, 7

6. **[20 points]** Using only the basic natural deduction rules, prove the sequent

$$\forall x \forall y \forall z[(S(x,y) \wedge S(y,z)) \rightarrow S(x,z)], \forall x \neg S(x,x) \vdash \forall x \forall y[S(x,y) \rightarrow \neg S(y,x)]$$

Answer

1		$\forall x \forall y \forall z[(S(x,y) \wedge S(y,z)) \rightarrow S(x,z)]$	Premise
2		$\forall x \neg S(x,x)$	Premise
3	x_0	$\forall y \forall z[(S(x_0,y) \wedge S(y,z)) \rightarrow S(x_0,z)]$	$\forall$e, 1
4	y_0	$\forall z[(S(x_0,y_0) \wedge S(y_0,z)) \rightarrow S(x_0,z)]$	$\forall$e, 3
5		$(S(x_0,y_0) \wedge S(y_0,x_0)) \rightarrow S(x_0,x_0)$	$\forall$e, 4
6		$S(x_0,y_0)$	Assumption
7		$S(y_0,x_0)$	Assumption
8		$S(x_0,y_0) \wedge S(y_0,x_0)$	$\wedge$i, 6, 7
9		$S(x_0,x_0)$	$\rightarrow$e, 8, 5
10		$\neg S(x_0,x_0)$	$\forall$e, 2
11		$\bot$	$\neg$e, 9, 10
12		$\neg S(y_0,x_0)$	$\neg$i, 7–11
13		$S(x_0,y_0) \rightarrow \neg S(y_0,x_0))$	$\rightarrow$i, 6–12
14		$\forall y \, S(x_0,y) \rightarrow \neg S(y,x_0))$	$\forall$i, 13
15		$\forall x \forall y \, S(x,y) \rightarrow \neg S(y,x))$	$\forall$i, 14