CSC 503 Homework Assignment 6

Out: October 17, 2018
Due: October 26, 2018
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1. **[10 points]** Rewrite the formula

$$(P(x,y) \rightarrow (\neg Q(x,y) \wedge R(y,z))) \wedge (P(x,y) \wedge S(y,x))$$

in clausal form set notation.

Answer

Remove implication:

$$(\neg P(x,y) \vee (\neg Q(x,y) \wedge R(y,z))) \wedge (P(x,y) \wedge S(y,x))$$

Convert to CNF: distribute disjunction over conjunctions:

$$(\neg P(x,y) \vee \neg Q(x,y)) \wedge (\neg P(x,y) \vee R(y,z)) \wedge P(x,y) \wedge S(y,x)$$

The clausal form set notation:

$$\{\ \{\neg P(x,y), \neg Q(x,y)\},\ \{\neg P(x,y), R(y,z)\},\ \{P(x,y)\},\ \{S(y,x)\}\ \}$$

2. **[10 points]** Rewrite the statement

$$\{\ \{R(x,y,z), S(y,z), S(x,y)\},\ \{P(x), S(x,y), \neg R(y,y,x)\},\ \{P(y), \neg S(y,x)\}\ \}$$

so that the variables in each clause are standardized apart.

Answer

Rename variables to different name in each clause:

$$\{\ \{R(x_1,y_1,z_1), S(y_1,z_1), S(x_1,y_1)\},\ \{P(x_2), S(x_2,y_2), \neg R(y_2,y_2,x_2)\},\ \{P(y_3), \neg S(y_3,x_3)\}\ \}$$

3. **[20 points]** Convert the formula

$$\forall x\ [P(x,y) \leftrightarrow (\forall y\ (R(x,y) \rightarrow Q(x,y)))]$$

to negation normal form, showing the steps of the conversion.

Answer

Insert parentheses to explicitly indicate y quantifier scope:

$$\forall x\ [P(x,y) \leftrightarrow\ [\forall y\ (R(x,y) \rightarrow Q(x,y))]\ ]$$

Rewrite $\leftrightarrow$ to $\rightarrow$:

$$\forall x\ [(P(x,y) \rightarrow [\forall y\ (R(x,y) \rightarrow Q(x,y))]) \wedge ([\forall y\ (R(x,y) \rightarrow Q(x,y))] \rightarrow P(x,y))]$$

Remove $\rightarrow$ with $\neg$ and $\vee$:

$$\forall x\ [(\neg P(x,y) \vee [\forall y\ (\neg R(x,y) \vee Q(x,y))]) \wedge (\neg[\forall y\ (\neg R(x,y) \vee Q(x,y))] \vee P(x,y))]$$

Then move negations in:

$$\forall x\ [(\neg P(x,y) \vee [\forall y\ (\neg R(x,y) \vee Q(x,y))]) \wedge ([\exists y \neg\neg(R(x,y) \wedge \neg Q(x,y))] \vee P(x,y))]$$

Remove doubles negations:

$$\forall x\ [(\neg P(x,y) \vee [\forall y\ (\neg R(x,y) \vee Q(x,y))]) \wedge ([\exists y(R(x,y) \wedge \neg Q(x,y))] \vee P(x,y))]$$

Now the formula is in NNF.

4. **[20 points]** Convert the sentence

$$\exists x\ \forall y\ \ ([P(x,y) \wedge (\forall z\ Q(y,z))] \vee [\exists u\ \exists v\ (P(x,u) \wedge R(y,v))])$$

to Skolem form, showing the steps of the conversion.

Answer

The given sentence is already in NNF:

$$\exists x\ \forall y\ \ ([P(x,y) \wedge (\forall z\ Q(y,z))] \vee [\exists u\ \exists v\ (P(x,u) \wedge R(y,v))])$$

Variable x has no preceding universally quantified variables, so replace x with constance c and drop $\exists x$:

$$\forall y\ \ ([P(c,y) \wedge (\forall z\ Q(y,z))] \vee [\exists u\ \exists v\ (P(c,u) \wedge R(y,v))])$$

Variable u has one preceding universally quantified variables y, so replace u with function $f_1(y)$ and drop $\exists u$:

$$\forall y\ \ ([P(c,y) \wedge (\forall z\ Q(y,z))] \vee [\exists v\ (P(c,f_1(y)) \wedge R(y,v))])$$

Variable v has one preceding universally quantified variables y, so replace v with function $f_2(y)$ and drop $\exists v$:

$$\forall y\ \ ([P(c,y) \wedge (\forall z\ Q(y,z))] \vee [(P(c,f_1(y)) \wedge R(y,f_2(y)))])$$

Now the sentence is in Skolem form.

5. **[20 points]** Convert the formula

$$\exists x\ [(\forall y\ P(x,y)) \rightarrow (\exists z\ Q(x,z))]$$

to prenex normal form, showing the steps of the conversion.

Answer

Convert to NNF first:

$$\exists x\ [\neg(\forall y\ P(x,y)) \vee (\exists z\ Q(x,z))]$$

$$\exists x\ [(\exists y \neg P(x,y)) \vee (\exists z\ Q(x,z))]$$

Move quantifier $\exists y$ out, rename y with u:

$$\exists x\ \exists u\ [\neg P(x,u) \vee (\exists z\ Q(x,z))]$$

Move quantifier $\exists z$ out, rename z with v:

$$\exists x\ \exists u\ \exists v\ [\neg P(x,u) \vee Q(x,v)]$$

Now the sentence is in prenex form.

6. **[20 points]** Compute the product substitution $\theta\sigma$ for

$$\begin{aligned} \theta &= \{a/x, b/y, f(y)/z, v/w, c/u\} \\ \sigma &= \{f(y)/x, g(z)/y, w/v\} \end{aligned}$$

Answer

$$\begin{aligned}
\theta\sigma &= \{a\sigma/x, b\sigma/y, f(y)\sigma/z, v\sigma/w, c\sigma/u, f(y)/x, g(z)/y, w/v\} \\
\theta\sigma &= \{a/x, b/y, f(g(z))/z, w/w, c/u, f(y)/x, g(z)/y, w/v\}
\end{aligned}$$

minus identical substitution in θ, and minus those in σ where have same target variable as in θ. So remove w/w, $f(y)/x$ and $g(z)/y$, now the product substitution is:

$$\theta\sigma = \{a/x, b/y, f(g(z))/z, c/u, w/v\}$$