CSC 503 Homework Assignment 6 Sample Solutions

Out: October 17, 2018
Due: October 26, 2018
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1. **[10 points]** Rewrite the formula

$$(P(x,y) \rightarrow (\neg Q(x,y) \wedge R(y,z))) \wedge (P(x,y) \wedge S(y,x))$$

in clausal form set notation.

Answer

To put the formula in clausal form, we first convert it to conjunctive normal form by rewriting the implication, and distributing the disjunction over the conjunction to obtain

$$(\neg P(x,y) \vee \neg Q(x,y)) \wedge (\neg P(x,y) \vee R(y,z)) \wedge P(x,y) \wedge S(y,x)$$

Rewriting this in set notation yields

$$\{\ \{\neg P(x,y), \neg Q(x,y)\},\ \{\neg P(x,y), R(x,y)\},\ \{P(x,y)\}\ \{S(y,x)\}\ \}$$

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2. **[10 points]** Rewrite the statement

$$\{\ \{R(x,y,z), S(y,z), S(x,y)\},\ \{P(x), S(x,y), \neg R(y,y,x)\},\ \{P(y), \neg S(y,x)\}\ \}$$

so that the variables in each clause are standardized apart.

Answer

The statement has three clauses. To standardize these apart, we rename the variables in each so that no two clauses share any variables. A simple way to do this is to rename each variable to a name with a subscript that indicates the clause number. Doing this produces

$$\{\ \{R(x_1,y_1,z_1), S(y_1,z_1), S(x_1,y_1)\},\ \{P(x_2), S(x_2,y_2), \neg R(y_2,y_2,x_2)\},\ \{P(y_3), \neg S(y_3,x_3)\}\ \}$$

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3. **[20 points]** Convert the formula

$$\forall x\, [P(x,y) \leftrightarrow (\forall y\, (R(x,y) \rightarrow Q(x,y)))]$$

to negation normal form, showing the steps of the conversion.

Answer

Note that y occurs both free and bound in this formula, so care must be taken to preserve the proper scope of the quantifier. We convert the formula starting with the principal connective of subformula following the initial quantifier, replacing the biimplication by two implications.

$$\forall x\, [\, [P(x,y) \rightarrow (\forall y\, (R(x,y) \rightarrow Q(x,y)))] \wedge [(\forall y\, (R(x,y) \rightarrow Q(x,y))) \rightarrow P(x,y)]\,].$$

We then rewrite the outermost implications as disjunctions. We reorder the second disjunction to clarify that the literal $P(x,y)$ lies outside the scope of $\forall y$.

$$\forall x\, [\, [\neg P(x,y) \vee (\forall y\, (R(x,y) \rightarrow Q(x,y)))] \wedge [P(x,y) \vee \neg(\forall y\, (R(x,y) \rightarrow Q(x,y)))]\,].$$

Rewrite the first implication to obtain

$$\forall x\, [\, [\neg P(x,y) \vee (\forall y\, (\neg R(x,y) \vee Q(x,y)))] \wedge [P(x,y) \vee \neg(\forall y\, (R(x,y) \rightarrow Q(x,y)))]\,].$$

Rewrite the negated quantified expression to obtain

$$\forall x\, [\, [\neg P(x,y) \vee (\forall y\, (\neg R(x,y) \vee Q(x,y)))] \wedge [P(x,y) \vee (\exists y\, \neg(R(x,y) \rightarrow Q(x,y)))]\,].$$

Convert the negated implication

$$\forall x\, [\, [\neg P(x,y) \vee (\forall y\, (\neg R(x,y) \vee Q(x,y)))] \wedge [P(x,y) \vee (\exists y\, \neg(\neg R(x,y) \vee Q(x,y)))]\,].$$

Apply de Morgan's laws to convert the negated disjunction

$$\forall x\, [\, [\neg P(x,y) \vee (\forall y\, (\neg R(x,y) \vee Q(x,y)))] \wedge [P(x,y) \vee (\exists y\, (R(x,y) \wedge \neg Q(x,y)))]\,].$$

The formula is now in negation normal form, containing only conjunctions, disjunctions, and negations applied to atomic predications.

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4. **[20 points]** Convert the sentence

$$\exists x\, \forall y\ \left([P(x,y) \wedge (\forall z\, Q(y,z))] \vee [\exists u\, \exists v\, (P(x,u) \wedge R(y,v))]\right)$$

to Skolem form, showing the steps of the conversion.

Answer

The formula is in negation normal form, so we may replace existential quantifiers with functions directly. We first replace the outer $\exists x$. This quantifier does not appear within the scope of any universal quantifier, so the Skolem function will be a constant (function of no arguments) $f_1()$, yielding the formula

$$\forall y\ \left([P(f_1(),y) \wedge (\forall z\, Q(y,z))] \vee [\exists u\, \exists v\, (P(f_1(),u) \wedge R(y,v))]\right)$$

The other existentials are in the second disjunct, and appear only in the scope of the outer $\forall y$ quantifier, so we replace u in the second conjunct with a Skolem function $f_2(y)$, and replace v in the second conjunct with a Skolem function $f_3(y)$, yielding the formula

$$\forall y\ \left([P(f_1(),y) \wedge (\forall z\, Q(y,z))] \vee [(P(f_1(),f_2(y)) \wedge R(y,f_3(y)))]\right).$$

The formula is now in Skolem form.

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5. **[20 points]** Convert the formula

$$\exists x\, [(\forall y\, P(x,y)) \rightarrow (\exists z\, Q(x,z))]$$

to prenex normal form, showing the steps of the conversion.

Answer

The initial $\forall x$ quantifies the entire formula, so it is already in the desired place. There are two embedded quantifiers Neither of these lies within the scope of the other, so one can pull these out in either order. To move the $\forall y$ quantifier, note that it occurs within the antecedent of an implication, and so changes to an existential quantifier when pulled out of the implication.

$$\exists x\, \exists y\, [P(x,y) \rightarrow (\exists z\, Q(x,z))].$$

The conversion rule calls for renaming the quantified variable y, but in this case there are no other occurrences (free or bound) of a variable written y, so we can just keep the name the same. Alternatively, we could have employed a new variable w and obtained

$$\exists x\, \exists w\, [P(x,w) \rightarrow (\exists z\, Q(x,z))].$$

We next pull out the $\exists z$ quantifier, which, occurring within the consequent of an implication, does not change type. Again, there are no competing using of the variable name z, so we retain it and obtain

$$\exists x\, \exists y\, \exists z\, [P(x,y) \rightarrow Q(x,z)].$$

The formula is now in prenex normal form, as all the quantifers precede the quantifier-free matrix of the formula. If we had pulled out the second embedded quantifier first, we would obtain the formula

$$\exists x\, \exists z\, \exists y\, [P(x,y) \rightarrow Q(x,z)]$$

that differs only in the order of the y and z quantifiers.

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6. **[20 points]** Compute the product substitution $\theta\sigma$ for

$$\begin{aligned} \theta &= \{a/x, b/y, f(y)/z, v/w, c/u\} \\ \sigma &= \{f(y)/x, g(z)/y, w/v\} \end{aligned}$$

Answer

By definition, this product is

$$\{[a\sigma]/x, [b\sigma]/y, [f(y)\sigma]/z, [v\sigma]/w, [c\sigma/u]\} \cup \sigma =$$
$$\{[a\sigma]/x, [b\sigma]/y, [f(y)\sigma]/z, [v\sigma]/w, [c\sigma/u]\} \cup \{f(y)/x, g(z)/y, w/v\}$$

minus any elements of

$$\{[a\sigma]/x, [b\sigma]/y, [f(y)\sigma]/z, [v\sigma]/w, [c\sigma]/u\}$$

that represent an identity substitution for one of the substitution targets in θ (that is, any elements x/x, y/y, z/z, v/v, w/w, or u/u), and minus any elements of σ that have the same target variable as one of the elements of θ (that is, substitute for x, y, z, w, or u).

Simplifying the σ-substituted terms in θ, we have

$$\theta\sigma = \{a/x, b/y, f(g(z))/z, w/w, c/u\} \cup \{f(y)/x, g(z)/y, w/v\}$$

In accordance with the conditions of the definition, we remove w/w from the set on the left and remove $f(y)/x$ and $g(z)/y$ from the set on the right. We thus obtain the product substitution

$$\theta\sigma = \{a/x, b/y, f(g(z))/z, c/u, w/v\}.$$
