CSC 503 Homework Assignment 4 Sample Solutions

(Corrected September 24, 2018)

Out: September 19, 2018
Due: September 26, 2018
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Let a and b be constant symbols, f a function symbol with three arguments, g a function symbol with one argument, h a function symbol with two arguments, P a predicate symbol with three arguments, and Q a predicate symbol with two arguments, and x, y, and z variable symbols.

Indicate, for each of the following strings, which strings are formulas in predicate logic, and state a reason for failure for strings which are not. No credit will be given for strings correctly identified as not being formulas without correct identification of a reason why the string is not a formula.

1. **[5 points]** $\exists y\, P(x, b, y) \wedge P(h(b, x), h(y, b), g(y)(a))$

Answer
This is not a formula because $g(y)$ is a term, which cannot act as a function or predicate taking the argument a.

2. **[5 points]** $\exists b\, \forall x\, Q(f(b, y, x), g(h(a, x)))$

Answer
This is not a formula because b is a constant, and so cannot be used as a variable by the $\exists$ quantifier.

3. **[5 points]** $\forall z\, \exists x\, P(g(h(b, x), z), f(a, y))$

Answer
Any of the following reasons suffices. This is not a formula because g takes only one argument and is given two here. It is also not a formula because f takes three arguments and is given only two here. It is also not a formula because P takes three arguments and is given only two.

4. **[5 points]** $\forall y\, g(Q(z, y)) \rightarrow P(h(f(y, z, z), z))$

Answer
This is not a formula, because g is a function and so cannot take a formula as an argument.

5. **[5 points]** $P(a, g(g(f(a, \neg b, a))), a) \rightarrow P(a, f(a, \neg b, a), a)$

Answer
This is not a formula as the negations here precede terms rather than formulas, and appear in places in which terms are needed rather than formulas.

Now let R be a predicate symbol with arity 2, f be a function of arity 2, and ϕ be the formula

$$\forall x\ [(R(x,z) \wedge \exists z\ \neg R(z, f(x,y))) \rightarrow \forall y\ R(y,z)]$$

6. **[5 points]** Indicate, for each occurrence of each variable in ϕ, whether that occurrence is free or bound.

Answer
We number the quantifiers and mark each quantified variable with the number of the quantifier that binds it, leaving the unquantified variables unchanged. This marked formula is then

$$\forall x_1\ [(R(x_1,z) \wedge \exists z_2\ \neg R(z_2, f(x_1,y))) \rightarrow \forall y_3\ R(y_3,z)].$$

We see that in the antecedent, x and the second occurrence of z are bound, but y and the first occurrence of z are free. In the consequent, y is bound but z is free. x is also bound in the consequent, but does not appear in that subformula.

7. **[5 points]** List all variables which occur both free and bound in ϕ.

Answer
y and z

8. **[5 points]** Compute $\phi[t/x]$ for $t = h(f(g(y),x),a,x)$. Is t free for x in ϕ?

Answer
There is no free variable x in ϕ, so t is free for x in ϕ, and substituting t for x leaves the formula unchanged.

$$\phi[t/x] = \forall x\ [(R(x,z) \wedge \exists z\ \neg R(z, f(x,y))) \rightarrow \forall y\ R(y,z)]$$

9. **[5 points]** Compute $\phi[t/y]$ for $t = h(f(g(y),x),a,x)$. Is t free for y in ϕ?

Answer
Because t contains the variable x and the free occurrence of y appears within the scope of $\forall x$, t is not free for y in ϕ. Substituting t for y yields

$$\phi[t/y] = \forall x\ [(R(x,z) \wedge \exists z\ \neg R(z, f(x, h(f(g(y),x),a,x)))) \rightarrow \forall y\ R(y,z)]$$

which displays the capture of the x's in t by the first $\forall x$ quantifier.

10. **[5 points]** Compute $\phi[t/z]$ for $t = h(f(g(y),x),a,x)$. Is t free for z in ϕ?

Answer
t is not free for z in ϕ because t contains the variable x and the free occurrences of z appear within the scope of $\forall x$, and because t contains the variable y and the second free occurrence of z appears within the scope of $\forall y$. Substituting t for z yields

$$\phi[t/z] = \forall x\ [(R(x, h(f(g(y),x),a,x)) \wedge \exists z\ \neg R(z, f(x,y))) \rightarrow \forall y\ R(y, h(f(g(y),x),a,x))]$$

which displays the capture of x and y by two of the quantifiers.

Now consider a language in which the only nonlogical symbols are a predicate symbol R of two arguments and a function symbol f of two arguments. Let ϕ_1 and ϕ_2 be the sentences

$$\begin{aligned}\phi_1 &= \forall x\, \forall y\, \forall z\ R(x,y) \wedge R(y,z) \to R(x,z) \\ \phi_2 &= \forall x\, \forall y\, \forall z\ R(x,y) \to R(f(x,z), f(y,z))\end{aligned}$$

Recall that an interpretation of the language identifies a domain of interpretation M (a set), and maps each of the predicate and function symbols to a set of tuples of elements of M of the appropriate size.

11. **[15 points]:** Give a formal interpretation I that makes ϕ_1 true and ϕ_2 false.

Answer

Let I be the interpretation with domain $M = \{a, b\}$ such that
$R^I = \{(a, a)\}$ and $f^I = \{(a, a, b), (a, b, b), (b, a, b), (b, b, b)\}$.

12. **[5 points]:** Briefly explain why I makes ϕ_1 true.

Answer

In interpretation I, the only case in which the antecedent of ϕ_1 is true is when $x = y = z = a$, in which case the consequent requires $R(a, a)$, which is true, thus making the implication true. In all other cases, at least one of the conjuncts in the antecedent must be false, so the implication is true for all values of x, y, and z. Thus ϕ_1 is true.

13. **[5 points]:** Briefly explain why I makes ϕ_2 false.

Answer

In interpretation I, f has the value b on all arguments, and $R(b, b)$ is false, so the consequent is always false. The antecedent is true when x and y are both a, so the implication is false in this case, making the universally quantified statement ϕ_2 false.

14. **[15 points]:** Give the formal definition of an interpretation J that makes ϕ_1 false and ϕ_2 true.

Answer

Let J be the interpretation with domain $M = \{a, b\}$ such that
$R^J = \{(a, b), (b, a)\}$ and $f^J = \{(a, a, a), (a, b, a), (b, a, b), (b, b, b)\}$.

15. **[5 points]:** Briefly explain why J makes ϕ_1 false.

Answer

In interpretation J, R is intransitive, as $R(a, b)$ and $R(b, a)$ are true but $R(a, a)$ is false. This makes ϕ_1 false.

16. **[5 points]:** Briefly explain why J makes ϕ_2 true.

Answer

In J, the value of f is always its first argument. This means that whenever the antecedent of ϕ_2 is true, so is the consequent, because f makes the consequent into the same statement as the antecedent. Thus ϕ_2 is true.