1.2.6 Solve $\sqrt{1-x^2}\,u_x + u_y = 0 \quad u(0,y) = y$

Proof: $\begin{cases} \dfrac{dx}{dt} = \sqrt{1-x^2} & (1) \\ \dfrac{dy}{dt} = 1 & (2) \\ \dfrac{du}{dt} = 0 & (3) \end{cases}$ s.t. $\begin{array}{l} x(0) = 0 \\ y(0) = C \\ u(0) = C \end{array}$

By (1), $\arcsin x = t + c_1$; (4)

By (2), $y = t + c_2$; (5)

By (3), $u = c_3$ (6)

Plug $x(0) = 0$ into (4), $c_1 = 0$, i.e $\arcsin x = t$

Plug $y(0) = C$ into (5), $c_2 = C$ i.e $y = t + C$

Thus $u(0) = C$ into (6), $c_3 = C$ i.e $u = C$

Thus (6) becomes $u(t) = C$

describe t, C by x, y $\Longrightarrow$ $\underline{u(x,y) = y - \arcsin x}$

1.2.9. Solve $u_x + u_y = 1$.

Proof: $\begin{cases} \dfrac{dx}{dt} = 1 \\ \dfrac{dy}{dt} = 1 \\ \dfrac{du}{dt} = 1 \end{cases}$ s.t. $\begin{array}{l} x(0) = 0 \\ y(0) = C \\ u(0) = f(C) \end{array}$

Then $x = t + c_1$, $y = t + c_2$. $u = t + c_3$

By $x(0) = 0$, $c_1 = 0$; By $y(0) = C$, $c_2 = C$; By $u(0) = f(C)$, $c_3 = f(C)$

Thus $x = t$, $y = t + C$, $u = t + f(C)$

Describe t, C by x, y. $\underline{u(x,y) = x + f(y-x)}$, where f is any function.

1.2.10 Solve $u_x+u_y+u=e^{x+2y}$ with $u(x,0)=0$

Proof: $\begin{cases} \frac{dx}{dt}=1 & (1) \\ \frac{dy}{dt}=1 & (2) \\ \frac{du}{dt}=e^{x+2y}-u & (3) \end{cases}$ s.t. $\begin{cases} x(0)=C \\ y(0)=0 \\ u(0)=0 \end{cases}$

By (1), $x=t+c_1$. (4)

By (2), $y=t+c_2$. (5)

Plug (4), (5) into (3), $\frac{du}{dt}=e^{(t+c_1)+2\cdot(t+c_2)}-u=e^{3t+c_1+2c_2}-u$

i.e. $\frac{du}{dt}+u=e^{3t+c_1+2c_2}$ (*)

Integrating factor is e^t, so multiply e^t to (*):

$$\frac{d}{dt}(e^t\cdot u)=e^{4t+c_1+2c_2}$$

Integrate the above, $e^t\cdot u=\frac{1}{4}e^{4t+c_1+2c_2}+c_3$. (6)

By $x(0)=C$ and (4), $c_1=C \Rightarrow x=t+C$ (+)

By $y(0)=0$ and (5), $c_2=0 \Rightarrow y=t$

By $u(0)=0$ and (6), and $c_1=C$, $c_2=0$.

$$e^0\cdot 0=\frac{1}{4}\cdot e^{4\cdot 0+C+2\cdot 0}+c_3,\quad c_3=-\frac{1}{4}e^C.$$

Thus (6) gives $u(t)=\frac{1}{4}e^{3t+c_1+2c_2}+e^{-t}\cdot c_3$

$=\frac{1}{4}e^{3t+C}+e^{-t}\cdot(-\frac{1}{4}e^C)$

describe t,C by x,y using (+) $=\frac{1}{4}e^{3y+(x-y)}+e^{-y}\cdot(-\frac{1}{4}e^{x-y})$

$=\frac{1}{4}e^{x+2y}-\frac{1}{4}e^{x-2y}$

$=\frac{1}{4}e^x(e^{2y}-e^{-2y})$

1.3.6 From the 3-d heat eqn derive $u_t = k(u_{rr} + \frac{u_r}{r})$

By chain rule,

$$u_x = u_r \cdot \frac{\partial r}{\partial x} = u_r \cdot \frac{\partial \sqrt{x^2+y^2}}{\partial x}$$

$$= u_r \cdot \frac{x}{\sqrt{x^2+y^2}}$$

$$\begin{tikzcd}[row sep=small] & x \\ u \arrow[r, dash] & r \arrow[u, dash] \arrow[d, dash] \\ & y \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & x \\ u_r \arrow[r, dash] & r \arrow[u, dash] \arrow[d, dash] \\ & y \end{tikzcd}$$

$$u_{xx} = \frac{\partial}{\partial x}(u_r) \cdot \frac{x}{\sqrt{x^2+y^2}} + u_r \cdot \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$$

$$= \left(u_{rr} \cdot \frac{\partial r}{\partial x}\right) \cdot \frac{x}{\sqrt{x^2+y^2}} + u_r \cdot \frac{x' \cdot \sqrt{x^2+y^2} - x \cdot \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2}\right)}{\left(\sqrt{x^2+y^2}\right)^2}$$

$$= u_{rr} \cdot \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{x}{\sqrt{x^2+y^2}} + u_r \cdot \frac{\sqrt{x^2+y^2} - x \cdot \frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= u_{rr} \cdot \frac{x^2}{r^2} + u_r \cdot \left(\frac{1}{r} - \frac{x^2}{r^3}\right)$$

Similarly,

$$u_{yy} = u_{rr} \cdot \frac{y^2}{r^2} + u_r \cdot \left(\frac{1}{r} - \frac{y^2}{r^3}\right)$$

$u_{zz} = 0$ by assumption.

Thus $\Delta u = u_{xx} + u_{yy} + u_{zz}$

$$= u_{rr} \cdot \frac{x^2+y^2}{r^2} + u_r\left(\frac{2}{r} - \frac{x^2+y^2}{r^3}\right) + 0$$

$$= u_{rr} \cdot \frac{r^2}{r^2} + u_r\left(\frac{2}{r} - \frac{r^2}{r^3}\right)$$

$$= u_{rr} + u_r \cdot \frac{1}{r}$$

Thus the heat eqn $u_t = k \Delta u$ becomes

$$u_t = k\left(u_{rr} + \frac{u_r}{r}\right)$$

1.3.10. If $\vec{f}$ is continuous and $|\vec{f}(x)| \leq \frac{1}{|x|^3+1}$, then show ~~that~~ $\iiint_{\text{all space}} \nabla\cdot\vec{f}\,dx = 0$

Proof: Let D_R be the ball of radius R in the space

i.e. $D_R = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2\leq R^2\}$

Then by divergence theorem,

$$\iiint_{D_R} \nabla\cdot\vec{f}\,dx = \iint_{\substack{\text{boundary}\\ \text{of } D_R}} \vec{f}\cdot\vec{n}\,dS$$

$$= \iint_{|\vec{x}|=R} \vec{f}\cdot\vec{n}\,dS$$

Take absolute value, using $|\int f| \leq \int |f|$, and $|\vec{n}|=1$,

we have $$\left|\iiint_{D_R} \nabla\cdot\vec{f}\,dx\right| \leq \iint_{|\vec{x}|=R} |\vec{f}|\,dS$$

$$\overset{\text{by assumption on } \vec{f}}{\leq} \iint_{|\vec{x}|=R} \frac{1}{|x|^3+1}\,dS$$

$$= \iint_{|\vec{x}|=R} \frac{1}{R^3+1}\,dS$$

$$= \frac{1}{R^3+1}\iint_{|\vec{x}|=R} dS$$

$$= \frac{1}{R^3+1}\cdot 4\pi R^2$$

$$\to 0 \text{ as } R\to\infty$$

Thus $\iiint_{\text{all space}} \nabla\cdot\vec{f}\,dx = 0$.