

1.2.6 Solve $\sqrt{1-x^2} u_x + u_y = 0$ $u(0,y) = y$

Proof:
$$\begin{cases} \frac{dx}{dt} = \sqrt{1-x^2} & (1) \\ \frac{dy}{dt} = 1 & (2) \\ \frac{du}{dt} = 0 & (3) \end{cases} \quad \text{s.t.} \quad \begin{cases} X(0) = 0 \\ Y(0) = C \\ u(0) = C \end{cases}$$

By (1), $\arcsin x = t + c_1$; (4)

By (2), $y = t + c_2$; (5)

By (3), $u = c_3$ (6)

Plug $X(0) = 0$ into (4), $c_1 = 0$, i.e. $\arcsin x = t$

Plug $Y(0) = C$ into (5), $c_2 = C$ i.e. $y = t + C$

Thus $u(0) = C$ into (6), $c_3 = C$ i.e. $u = C$

Thus (6) becomes $u(t) = C$
 describe t, C by x, y
 $\Rightarrow \underline{u(x,y) = y - \arcsin x}$

1.2.9. Solve $u_x + u_y = 1$.

Proof:
$$\begin{cases} \frac{dx}{dt} = 1 \\ \frac{dy}{dt} = 1 \\ \frac{du}{dt} = 1 \end{cases} \quad \text{s.t.} \quad \begin{cases} X(0) = 0 \\ Y(0) = C \\ u(0) = f(C) \end{cases}$$

Then $x = t + c_1$, $y = t + c_2$, $u = t + c_3$

By $X(0) = 0$, $c_1 = 0$; By $Y(0) = C$, $c_2 = C$; By $u(0) = f(C)$, $c_3 = f(C)$

Thus $x = t$, $y = t + C$, $u = t + f(C)$

Describe t, C by x, y . $\underline{u(x,y) = x + f(y-x)}$, where f is any function.

1.2.10 Solve $u_x + u_y + u = e^{x+2y}$ with $u(x,0) = 0$

Proof:
$$\begin{cases} \frac{dx}{dt} = 1 & (1) \\ \frac{dy}{dt} = 1 & (2) \\ \frac{du}{dt} = e^{x+2y} - u & (3) \end{cases} \quad \text{s.t.} \quad \begin{cases} x(0) = C \\ y(0) = 0 \\ u(0) = 0 \end{cases}$$

By (1), $x = t + C_1$. (4)

By (2), $y = t + C_2$. (5)

Plug (4), (5) into (3), $\frac{du}{dt} = e^{(t+C_1)+2(t+C_2)} - u = e^{3t+C_1+2C_2} - u$

i.e. $\frac{du}{dt} + u = e^{3t+C_1+2C_2}$ (*)

Integrating factor is e^t , so multiply e^t to (*):

$$\frac{d}{dt}(e^t \cdot u) = e^{4t+C_1+2C_2}$$

Integrate the above, $e^t \cdot u = \frac{1}{4} e^{4t+C_1+2C_2} + C_3$. (b)

By $x(0) = C$ and (4), $C_1 = C \Rightarrow x = t + C$ (+)

By $y(0) = 0$ and (5), $C_2 = 0 \Rightarrow y = t$

By $u(0) = 0$ and (b), and $C_1 = C, C_2 = 0$.

$$e^0 \cdot 0 = \frac{1}{4} \cdot e^{4 \cdot 0 + C + 2 \cdot 0} + C_3, \quad C_3 = -\frac{1}{4} e^C.$$

Thus (b) gives $u(t) = \frac{1}{4} e^{3t+C_1+2C_2} + e^{-t} \cdot C_3$

describe t, C by x, y using (+)

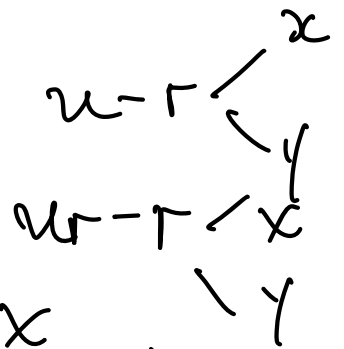
$$\begin{aligned} &= \frac{1}{4} e^{3y+(x-y)} + e^{-y} \cdot \left(-\frac{1}{4} e^C\right) \\ &= \frac{1}{4} e^{x+2y} - \frac{1}{4} e^{x-2y} \\ &= \frac{1}{4} e^x (e^{2y} - e^{-2y}) \end{aligned}$$

1.3.6 From the 3-d heat eqn derive $u_t = k(u_{rr} + \frac{u_r}{r})$

By chain rule,

$$u_x = u_r \cdot \frac{\partial r}{\partial x} = u_r \cdot \frac{\partial \sqrt{x^2 + y^2}}{\partial x}$$

$$= u_r \cdot \frac{x}{\sqrt{x^2 + y^2}}$$



$$u_{xx} = \frac{\partial}{\partial x}(u_r) \cdot \frac{x}{\sqrt{x^2 + y^2}} + u_r \cdot \frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2 + y^2}} \right)$$

$$= (u_{rr} \cdot \frac{\partial r}{\partial x}) \cdot \frac{x}{\sqrt{x^2 + y^2}} + u_r \cdot \frac{x' \sqrt{x^2 + y^2} - x \cdot \frac{\partial}{\partial x}(\sqrt{x^2 + y^2})}{(\sqrt{x^2 + y^2})^2}$$

$$= u_{rr} \cdot \frac{x}{\sqrt{x^2 + y^2}} \cdot \frac{x}{\sqrt{x^2 + y^2}} + u_r \cdot \frac{\sqrt{x^2 + y^2} - x \cdot \frac{x}{\sqrt{x^2 + y^2}}}{x^2 + y^2}$$

$$= u_{rr} \cdot \frac{x^2}{r^2} + u_r \cdot \left(\frac{1}{r} - \frac{x^2}{r^3} \right)$$

Similarly,

$$u_{yy} = u_{rr} \cdot \frac{y^2}{r^2} + u_r \cdot \left(\frac{1}{r} - \frac{y^2}{r^3} \right)$$

$$u_{zz} = 0 \text{ by assumption.}$$

$$\text{Thus } \Delta u = u_{xx} + u_{yy} + u_{zz}$$

$$= u_{rr} \cdot \frac{x^2 + y^2}{r^2} + u_r \left(\frac{2}{r} - \frac{x^2 + y^2}{r^3} \right) + 0$$

$$= u_{rr} \cdot \frac{r^2}{r^2} + u_r \left(\frac{2}{r} - \frac{r^2}{r^3} \right)$$

$$= u_{rr} + u_r \cdot \frac{1}{r}$$

Thus the heat eqn $u_t = k \Delta u$ becomes

$$u_t = k(u_{rr} + \frac{u_r}{r})$$

1.3. 10. If \vec{f} is continuous and $|\vec{f}(x)| \leq \frac{1}{(|x|^3+1)}$,
 then show that $\iiint_{\text{all space}} \nabla \cdot \vec{f} \, dx = 0$

Proof. Let D_R be the ball of radius R in the space
 i.e. $D_R = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq R^2\}$
 Then by divergence theorem,

$$\begin{aligned} \iiint_{D_R} \nabla \cdot \vec{f} \, dx &= \iint_{\text{boundary of } D_R} \vec{f} \cdot \vec{n} \, dS \\ &= \iint_{|\vec{x}|=R} \vec{f} \cdot \vec{n} \, dS \end{aligned}$$

Take absolute value, using $\int |f| \leq \int |f|$, and $|\vec{n}|=1$,
 we have $\left| \iiint_{D_R} \nabla \cdot \vec{f} \, dx \right| \leq \iint_{|\vec{x}|=R} |\vec{f}| \, dS$

$$\begin{aligned} \text{by assumption on } \vec{f} &\rightarrow \leq \iint_{|\vec{x}|=R} \frac{1}{|\vec{x}|^3+1} \, dS \\ &= \iint_{|\vec{x}|=R} \frac{1}{R^3+1} \, dS \\ &= \frac{1}{R^3+1} \iint_{|\vec{x}|=R} dS \\ &= \frac{1}{R^3+1} \cdot 4\pi R^2 \\ &\rightarrow 0 \text{ as } R \rightarrow \infty \end{aligned}$$

Thus $\iiint_{\text{all space}} \nabla \cdot \vec{f} \, dx = 0$.