1.5.4 Given the Neumann-problem

$$\begin{cases} \Delta u = f(x,y,z) & \text{in } D \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \mathrm{bd}D \end{cases} \quad (*)$$

(a) For any constant C, $\tilde{u} := u + C$ also satisfies $(*)$,

since $\Delta \tilde{u} = \Delta u + \cancel{\Delta C} = \Delta u = f(x,y,z)$ in D

$$\frac{\partial \tilde{u}}{\partial n} = (\nabla \tilde{u}) \cdot \vec{n} = \nabla(u+C) \cdot \vec{n} = \nabla u \cdot \vec{n} = 0 \text{ on } \mathrm{bd}D$$

(b)
$$\iiint_D f = \iiint_D \Delta u$$

$$= \iiint_D \nabla \cdot (\nabla u)$$

$$\overset{\text{divergence thm}}{=} \iint_{\mathrm{bd}D} \nabla u \cdot \vec{n} \, dS$$

$$= \iint_{\mathrm{bd}D} \frac{\partial u}{\partial n} dS$$

$$\overset{\text{by second eqn of } (*)}{=} \iint_{\mathrm{bd}D} 0 \, dS$$

(c) Any reasonable interpretation is OK.

For (a), temperature can be high

For (b), heat averages to zero, so that heat exchange is insulated.

1.5.6. Solve $u_x + 2xy^2 u_y = 0$

$$\begin{cases} \frac{dx}{dt} = 1 & (1) \\ \frac{dy}{dt} = 2xy^2 & (2) \\ \frac{du}{dt} = 0 & (3) \end{cases} \quad \text{s.t.} \quad \begin{cases} x(0) = 0 \\ y(0) = C \\ u(0) = f(C) \end{cases}$$

By (1), $x = t + c_1$. Since $x(0) = 0$, $c_1 = 0 \Rightarrow x = t$

By (2) and $x = t$, $\frac{dy}{dt} = 2ty^2$. $\frac{1}{y^2} dy = 2t\,dt \Rightarrow -\frac{1}{y} = t^2 + c_2$.

Since $y(0) = C$, $-\frac{1}{C} = c_2 \Rightarrow y = -\frac{1}{t^2 - \frac{1}{C}}$ (4)

By (3), $u = c_3$. Since $u(0) = f(C)$, $c_3 = f(C)$.

$$\Rightarrow u = f(C) \underset{\text{by } (4)}{=} f\left(\frac{1}{t^2 + \frac{1}{y}}\right) \overset{x=t}{=} f\left(\frac{1}{x^2 + \frac{1}{y}}\right)$$

$$= f\left(\frac{y}{x^2 y + 1}\right)$$

2.1.5 By d'Alembert formula, for the IVP

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < +\infty \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

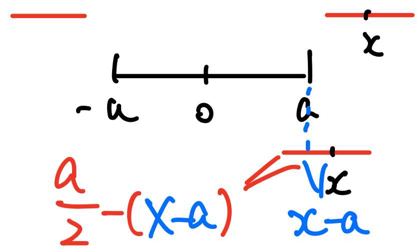
$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

In this problem, $\phi(x) \equiv 0$, $\psi(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| \geq a \end{cases}$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds = \frac{1}{2c}\,\text{length}\big((x-ct, x+ct) \cap (-a,a)\big)$$

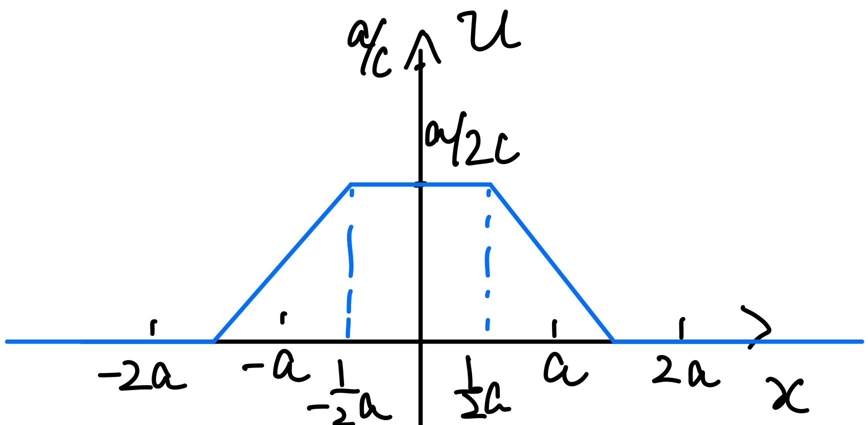
$t = \frac{a}{2c}$,

$$\text{length}\left(x-\frac{a}{2}, x+\frac{a}{2}\right) \cap (-a,a) = \begin{cases} 0 & |x| \geq \frac{3}{2}a \\ a & |x| \leq \frac{1}{2}a \\ \frac{a}{2}-(x-a) & \frac{1}{2}a \leq x \leq \frac{3}{2}a \\ \frac{a}{2}-(-a-x) & -\frac{3}{2}a \leq x \leq -\frac{1}{2}a \end{cases}$$

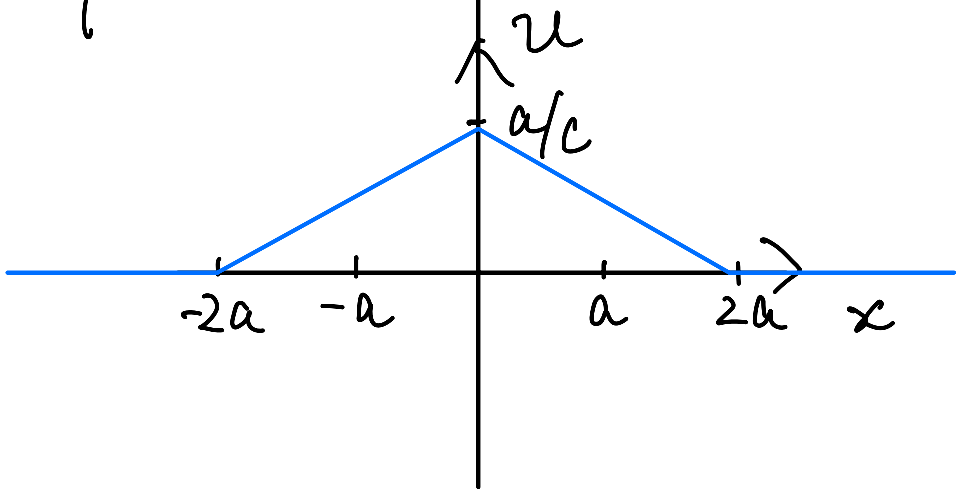


Thus
$$u\left(x, \frac{a}{2c}\right) = \begin{cases} 0 & |x| \geq \frac{3}{2}a \\ \frac{a}{2c} & |x| \leq \frac{1}{2}a \\ \frac{1}{2c}\left(\frac{3}{2}a - x\right) & \frac{1}{2}a \leq x \leq \frac{3}{2}a \\ \frac{1}{2c}\left(\frac{3}{2}a + x\right) & -\frac{3}{2}a \leq x \leq -\frac{1}{2}a \end{cases}$$

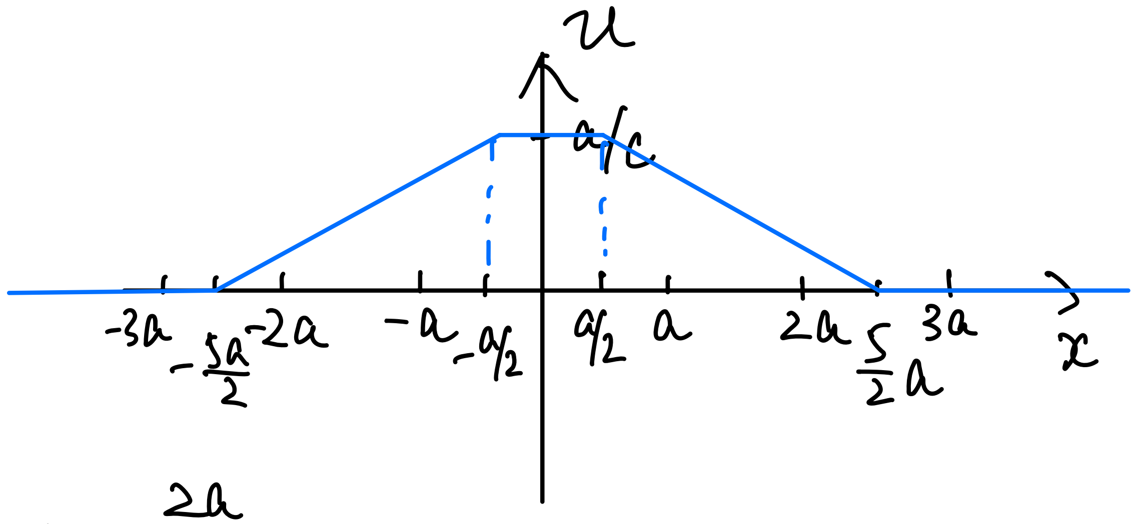
String profile at $t = \frac{a}{2c}$:



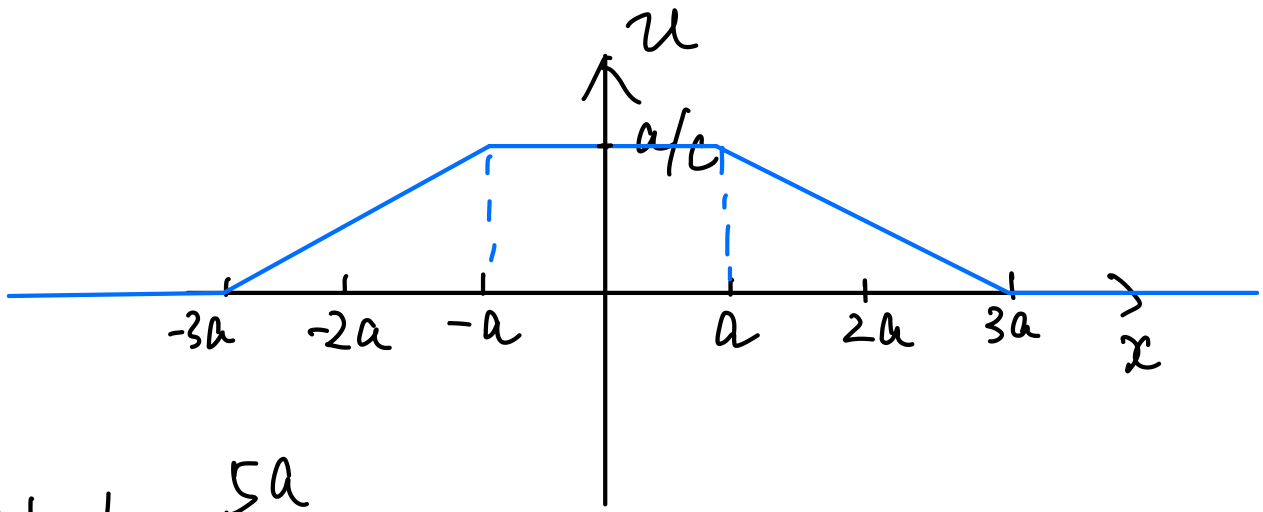
Similarly, at $t = \frac{a}{c}$,



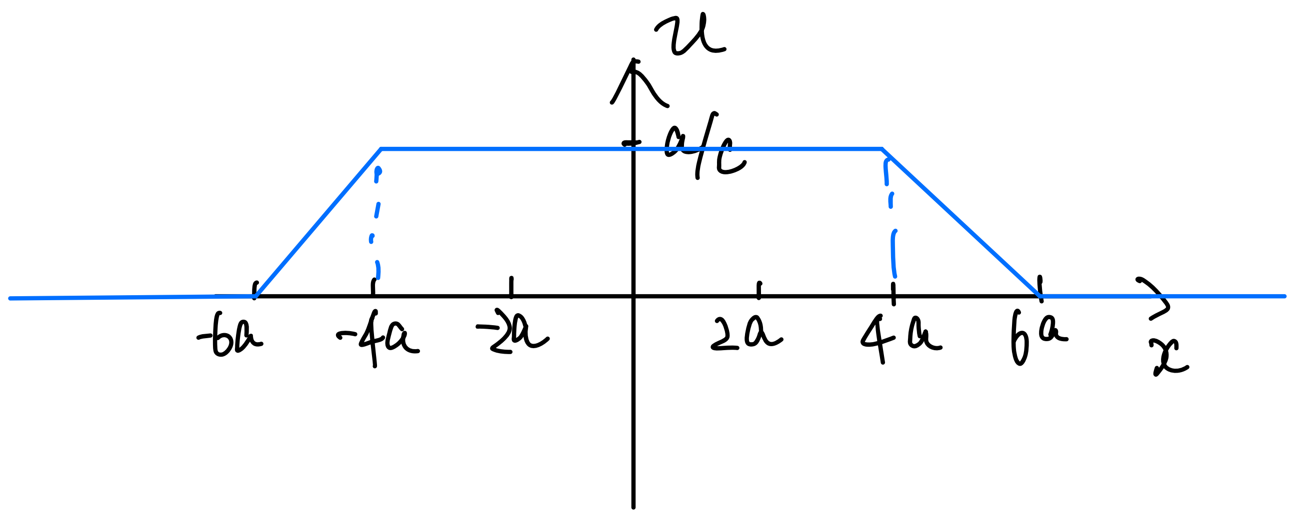
At $t = \frac{3a}{2c}$,



At $t = \frac{2a}{c}$,



At $t = \frac{5a}{c}$,



2.1.10 Solve $\begin{cases} u_{xx}+u_{xt}-20u_{tt}=0 & (1) \\ u(x,0)=\phi(x) & (2) \\ u_t(x,0)=\psi(x) & (3) \end{cases}$

By factorization (1) as $\left(\frac{\partial}{\partial x}-4\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x}+5\frac{\partial}{\partial t}\right)u=0$,
we have $u(x,t)=f(4x+t)+g(5x-t)$, (*)
with f, g to be determined

By (2), $\phi(x)=f(4x)+g(5x)$ (4)

By applying $\frac{\partial}{\partial t}$ to (*), $u_t(x,t)=f'(4x+t)-g'(5x-t)$.
By (3), the above equation becomes

$$\psi(x)=f'(4x)-g'(5x) \quad (5)$$

To solve for f and g, we apply $\frac{d}{dx}$ to (4), getting

$$\phi'(x)=4f'(4x)+5g'(5x) \quad (6)$$

Combining (5) and (6),

$$\begin{aligned} f'(4x)&=\tfrac{5}{9}\psi(x)+\tfrac{1}{9}\phi'(x) \\ g'(5x)&=\tfrac{1}{9}\phi'(x)-\tfrac{4}{9}\psi(x) \end{aligned} \xRightarrow[\text{scaling}]{\text{by}} \begin{aligned} f'(x)&=\tfrac{5}{9}\psi\left(\tfrac{x}{4}\right)+\tfrac{1}{9}\phi'\left(\tfrac{x}{4}\right) \\ g'(x)&=\tfrac{1}{9}\phi'\left(\tfrac{x}{5}\right)-\tfrac{4}{9}\psi\left(\tfrac{x}{5}\right) \end{aligned}$$

$$\xRightarrow[\text{w.r.t. } x]{\text{integrating}} \quad f(x)=\frac{5}{9}\int_0^x \psi\left(\frac{s}{4}\right)ds+\frac{1}{9}\int_0^x \phi'\left(\frac{s}{4}\right)ds+C_1$$

$$g(x)=\frac{1}{9}\int_0^x \phi'\left(\frac{s}{5}\right)ds-\frac{4}{9}\int_0^x \psi\left(\frac{s}{5}\right)ds+C_2$$

Since $\phi(x) = f(4x) + g(5x)$ by (4), we have

$$f(4x) + g(5x) = \frac{5}{9}\int_0^{4x} \psi\left(\frac{s}{4}\right)ds + \frac{1}{9}\int_0^{4x} \phi'\left(\frac{s}{4}\right)ds + C_1$$

$$+ \frac{1}{9}\int_0^{5x} \phi'\left(\frac{s}{5}\right)ds - \frac{4}{9}\int_0^{5x} \psi\left(\frac{s}{5}\right)ds + C_2$$

by change of variables $t = s/4,\ w = s/5$

$$= \cancel{\frac{5}{9}\cdot 4\int_0^{x} \psi(s)ds} + \frac{1}{9}\cdot 4\int_0^{x} \phi'(s)ds + C_1$$

$$+ \frac{1}{9}\cdot 5\int_0^{x} \phi'(s)ds - \cancel{\frac{4}{9}\cdot 5\int_0^{x} \psi(s)ds} + C_2$$

by FTC

$$= \phi(x) + C_1 + C_2$$

$$\Rightarrow C_1 + C_2 = 0.$$

Hence by (*), $u(x,t) = f(4x+t) + g(5x-t)$

$$= \frac{5}{9}\int_0^{4x+t} \psi\left(\frac{s}{4}\right)ds + \frac{1}{9}\int_0^{4x+t} \phi'\left(\frac{s}{4}\right)ds + C_1$$

$$+ \frac{1}{9}\int_0^{5x-t} \phi'\left(\frac{s}{5}\right)ds - \frac{4}{9}\int_0^{5x-t} \psi\left(\frac{s}{5}\right)ds + C_2$$

$$= \frac{4}{9}\int_0^{\frac{4x+t}{4}} \phi'(s)ds + \frac{5}{9}\int_0^{\frac{5x-t}{5}} \phi'(s)ds$$

$$+ \frac{20}{9}\int_0^{\frac{4x+t}{4}} \psi(s)ds - \frac{20}{9}\int_0^{\frac{5x-t}{5}} \psi(s)ds$$

$$= \frac{4}{9}\phi\left(x+\frac{t}{4}\right) + \frac{5}{9}\phi\left(x-\frac{t}{5}\right) + \frac{20}{9}\int_{x-\frac{t}{5}}^{x+\frac{t}{4}} \psi(s)ds$$

2.2.1 Use the energy conservation of the wave equation to prove that the only solution to

$$\begin{cases} \rho u_{tt} = T u_{xx} & (1) \\ u(x,0) = 0 & (2) \\ u_t(x,0) = 0 & (3) \end{cases}$$

is $u=0$. ($\rho > 0$, $T>0$ are constants)

Proof: We already know that

$$E(t) \overset{def}{=} \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)\,dx$$

is a constant independent of the time t (cf. Chp2.2 of Strauss).

In particular, $E(t) = E(0)$ for any $t \geq 0$, i.e. for any $t \geq 0$

$$E(t) = E(0) = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2(x,0) + T u_x^2(x,0))\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \rho \cdot \underset{\text{by (3)}}{0^2} + T \cdot \underset{\text{by applying } \frac{\partial}{\partial x} \text{ to (2)}}{0^2}\,dx = 0$$

Thus $E(t) \equiv 0$. By vanishing theorem, for a fixed t,

$$\rho u_t^2(x,t) + T u_x^2(x,t) = 0 \text{ for any } -\infty < x < \infty.$$

$$\Rightarrow u_t(x,t) = u_x(x,t) = 0.$$

$$\Rightarrow u(x,t) \equiv 0$$