2.4.9 Solve $\begin{cases} u_t = k u_{xx} \quad (*) \\ u(x,0) = x^2 \end{cases}$ by the given method.

If u satisfies $(*)$, then

$$(u_{xxx})_t \underset{\substack{\text{exchange order}\\ \text{of differentiation}}}{=} (u_t)_{xxx} \underset{\text{by}(*)}{=} (k u_{xx})_{xxx} \underset{\substack{\text{exchange order}\\ \text{of differentiation}}}{=} k (u_{xxx})_{xx},$$

with $u_{xxx}(x,0) = \frac{d^3}{dx^3} u(x,0) = \frac{d^3}{dx^3} x^2 = 0$

i.e $v \overset{\text{def}}{=} u_{xxx}$ satisfies $\begin{cases} v_t = k v_{xx} \\ v(x,0) = 0 \end{cases}$

By uniqueness of heat equation with zero initial condition, we have $v \equiv 0$, i.e $u_{xxx} = 0$.

Integrating $v \equiv 0$ thrice, $u(x,t) = A(t)x^2 + B(t)x + C(t)$.

Since $u(x,0) = x^2$, $A(0) = 1$, $B(0) = C(0) = 0$.

Also by $u_t = k u_{xx}$, we have

$A'(t)x^2 + B'(t)x + C'(t) = 2kA(t) \Rightarrow \begin{cases} A'(t) = B'(t) = 0 \\ C'(t) = 2kA(t) \end{cases}$

Thus $A(t) \equiv 1$, $B(t) \equiv 0$, and $C'(t) = 2k$, $C(0) = 0$ give $C(t) = 2kt$

Thus $\underline{u(x,t) = x^2 + 2kt}$.

2.4.10. (a) By the formula given by the heat kernel.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot y^2 \, dy$$

Fix x. Let $p = \frac{x-y}{\sqrt{4kt}}$, then $dp = -\frac{1}{\sqrt{4kt}} dy$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{\infty}^{-\infty} e^{-p^2} \cdot (x - \sqrt{4kt} \cdot p)^2 \cdot (-\sqrt{4kt}) \, dp$$

$$= \frac{1}{\cancel{\sqrt{4\pi kt}}} \int_{-\infty}^{\infty} e^{-p^2} \cdot (x - \sqrt{4kt} \cdot p)^2 \cdot \cancel{\sqrt{4kt}} \, dp$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} (x^2 - \underbrace{\cancel{4xp\sqrt{kt}}}_{\text{by symmetry}} + 4kt \cdot p^2) \, dp$$

$$= \frac{1}{\sqrt{\pi}} \Big[\big(\underbrace{\int_{-\infty}^{\infty} e^{-p^2} dp}_{=\sqrt{\pi} \text{ (exercise)}} \big) \cdot x^2 + 4kt \int_{-\infty}^{\infty} e^{-p^2} \cdot p^2 \, dp \Big]$$

$$= x^2 + \frac{4kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \cdot p^2 \, dp$$

(b) By uniqueness (which is not so rigorous but OKAY),

$$u(x,t) = x^2 + \frac{4kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \cdot p^2 \, dp \underset{\text{by 2.4.9}}{=} x^2 + 2kt$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-p^2} \cdot p^2 \, dp = \frac{\sqrt{\pi}}{2}$$

2.4.14 $\phi(x)$ is continuous s.t. $|\phi(x)| \leq Ce^{ax^2}$ (*), then

$$\left| \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot \phi(y)\, dy \right|$$

$$\leq \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \left| e^{-\frac{(x-y)^2}{4kt}} \cdot \phi(y) \right| dy$$

by (*)

$$\leq \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot Ce^{ay^2}\, dy$$

$$= \frac{C}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt} + ay^2}\, dy \quad (**)$$

If $0 < t < \frac{1}{4ak}$, then $-\infty < -\frac{1}{4kt} < -a$ (Note $a>0$, and $k>0$)

For simplicity I denote $\lambda \overset{\text{def}}{=} -\frac{1}{4kt}$, so the exponent is

$$\lambda(x-y)^2 + ay^2 = (\lambda+a)y^2 - 2\lambda x \cdot y + \lambda x^2$$

$$= (\lambda+a)\left(y^2 - \frac{2\lambda x}{\lambda+a}y + \left(\frac{\lambda x}{\lambda+a}\right)^2\right) - \frac{(\lambda x)^2}{\lambda+a} + \lambda x^2$$

$$= (\lambda+a)\left(y - \frac{\lambda x}{\lambda+a}\right)^2 + \frac{\lambda a}{\lambda+a}$$

Fix x, make the change of variables $z = y - \frac{\lambda x}{\lambda+a}$.

then (**) becomes

$$\frac{C}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{(\lambda+a)z^2 + \frac{\lambda a}{\lambda+a}}\, dz$$

$$= \frac{C}{\sqrt{4\pi kt}} \cdot e^{\frac{\lambda a}{\lambda+a}} \cdot \int_{-\infty}^{\infty} e^{(\lambda+a)z^2}\, dz < \infty$$

if (and only if) $\underline{\lambda + a < 0}$, i.e. $0 < t < \frac{1}{4ak}$.

2.4.16 $u_t - ku_{xx} + bu = 0 \quad -\infty < x < \infty$

$u(x,0) = \phi(x)$.

where $b > 0$ is a constant.

Proof. By the hint, let $u(x,t) = e^{-bt} v(x,t)$,

then $u_t = -be^{-bt} \cdot v + e^{-bt} \cdot v_t$

$u_x = e^{-bt} \cdot v_x$ and $u_{xx} = e^{-bt} \cdot v_{xx}$

Thus $u_t - ku_{xx} + bu = 0$ reads

$\cancel{-be^{-bt} \cdot v} + e^{-bt} \cdot v_t - ke^{-bt} \cdot v_{xx} + \cancel{b \cdot e^{-bt} \cdot v} = 0$

i.e., $v_t - kv_{xx} = 0$ (Note $e^{-bt} > 0$)

Also by $u(x,0) = \phi(x)$, we have

$v(x,0) = \phi(x) \cdot e^{b \cdot 0} = \phi(x)$

Thus v satisfies $\begin{cases} v_t - kv_{xx} = 0 \\ v(x,0) = \phi(x). \end{cases}$

By the formula,

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \cdot \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot \phi(y)\,dy$$

$$\text{i.e. } u(x,t) = \frac{e^{-bt}}{\sqrt{4\pi kt}} \cdot \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot \phi(y)\,dy$$

2.5.4 Let $u(x,t)$ satisfy $u_{tt} - c^2 u_{xx} = 0$
$-\infty < x < \infty$.
with $|u''| \le C$ for some $C > 0$.
Let
$$v(x,t) = \frac{c}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-s^2c^2/4kt} \cdot u(x,s)ds$$
(a) Show that $v(x,t)$ solves $v_t - \frac{k}{c^2} v_{xx} = 0$.
Proof. By the hint, we may rewrite v
as $v(x,t) = \int_{-\infty}^{\infty} H(s,t)\, u(x,s)\, ds$, where
$$H(s,t) \stackrel{\text{def}}{=} \frac{c}{\sqrt{4\pi k t}} e^{-s^2c^2/4kt}.$$
Then $\frac{\partial}{\partial t} H(s,t)$
$$= \frac{c}{\sqrt{4\pi k}} \left(-\frac{1}{2} t^{-3/2} \cdot e^{-s^2c^2/4kt} + t^{-1/2} \cdot e^{-s^2c^2/4kt} \cdot \frac{(-s^2c^2)}{4k} \left(-\frac{1}{t^2}\right)\right)$$
$$= \frac{c}{\sqrt{4\pi k}} \cdot e^{-s^2c^2/4kt} \left(\frac{s^2c^2}{4k} \cdot \frac{1}{t^{5/2}} - \frac{1}{2} \cdot \frac{1}{t^{3/2}} \right)$$
$$\frac{\partial}{\partial s} H(s,t) = \frac{c}{\sqrt{4\pi k t}} \cdot e^{-s^2c^2/4kt} \cdot \left(-\frac{2sc^2}{4kt} \right)$$
$$\frac{\partial^2}{\partial s^2} H(s,t) = \frac{c}{\sqrt{4\pi k t}} \left[e^{-s^2c^2/4kt} \cdot \left(-\frac{2sc^2}{4kt} \right)^2 + e^{-s^2c^2/4kt} \cdot \left(-\frac{2c^2}{4kt} \right) \right]$$

$$= \frac{c}{\sqrt{4\pi k t}} e^{-s^2c^2/4kt} \cdot \left[\frac{s^2c^4}{4k^2t^2} + \frac{c^2}{2kt} \right]$$

$$\Rightarrow \frac{\partial}{\partial t} H(s,t) - \frac{k}{c^2}\frac{\partial^2}{\partial s^2} H(s,t)$$

$$= \frac{c}{\sqrt{4\pi k t}} e^{-s^2c^2/4kt} \left(\frac{s^2c^2}{4kt^2} - \frac{1}{2t} \right)$$

$$- \frac{k}{c^2} \cdot \frac{c}{\sqrt{4\pi k t}} e^{-s^2c^2/4kt} \cdot \left[\frac{s^2c^4}{4k^2t^2} + \frac{c^2}{2kt} \right] = 0$$

i.e $H(s,t)$ satisfies $\underline{H_t - \frac{k}{c^2} H_{ss} = 0}$

Then $$v_t = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} H(s,t) \cdot u(x,s)\,ds$$

$$= \frac{k}{c^2} \int_{-\infty}^{\infty} \left(\frac{\partial^2}{\partial s^2} H(s,t) \right) \cdot u(x,s)\,ds$$

IBP
$$= \frac{k}{c^2} \cdot \left[\underbrace{\left(\frac{\partial}{\partial s} H(s,t) \right)}_{=0 \text{ as } t \to \pm\infty \text{ by L'Hopitel}} \cdot u(x,s) \Big|_{s=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial}{\partial s} H(s,t) \frac{\partial}{\partial s} u(x,s)\,ds \right]$$

$$= -\frac{k}{c^2} \left(\int_{-\infty}^{\infty} \frac{\partial}{\partial s} H(s,t) \cdot \frac{\partial}{\partial s} u(x,s)\,ds \right)$$

IBP
$$= -\frac{k}{c^2} \left[\underbrace{H(s,t)}_{=0 \text{ as } t \to \pm\infty \text{ by L'Hopitel}} \cdot \frac{\partial}{\partial s} u(x,s) \Big|_{s=-\infty}^{\infty} - \int_{-\infty}^{\infty} H(s,t) \frac{\partial^2}{\partial s^2} u(x,s)\,ds \right]$$

$$= \frac{k}{c^2}\int_{-\infty}^{\infty} H(s,t)\cdot\frac{\partial^2}{\partial s^2}u(x,s)\,ds.$$

While $v_{xx} = \int_{-\infty}^{\infty} H(s,t)\cdot\frac{\partial^2}{\partial x^2}u(x,s)\,ds$

Thus $v_t - k v_{xx}$

$$= \int_{-\infty}^{\infty} H(s,t)\cdot\left(\frac{k}{c^2}\frac{\partial^2}{\partial s^2}u(x,s) - k\frac{\partial^2}{\partial x^2}u(x,s)\right)ds$$

$$= \int_{-\infty}^{\infty} H(s,t)\cdot\frac{k}{c^2}\left(\underbrace{u_{ss} - c^2 u_{xx}}_{=0 \text{ by assumption of } u}\right)ds$$

$$= \int_{-\infty}^{\infty} H(s,t)\cdot\frac{k}{c^2}\cdot 0\,ds = 0.$$

$$\Rightarrow v_t - k v_{xx} = 0.$$

(b) Full marks will be given if you state (by the hint),

for any s, $\lim_{t\to 0^+} H(s,t) = \delta(s)$,

i.e. the dirac delta "function" at s.

Then performing some formal computation based on $\lim_{t\to 0^+} H(s,t) = \delta(s)$. Namely,

$$\lim_{t\to 0} v(x,t) = \lim_{t\to 0}\int_{-\infty}^{\infty} H(s,t)u(x,s)\,ds$$

$$= \int_{-\infty}^{\infty} \lim_{t\to 0} H(s,t)u(x,s)\,ds$$

$$= \int_{-\infty}^{\infty} \delta(s)\, u(x,s)\, ds$$

$$= u(x,0).$$

This is, essentially the definition of $\delta(s)$.
For a rigorous treetment, see Strauss Chp3.5, Theorem 1.