huangchenzhou2029@u.northwestern.edu

OH: Lunt B14 Mon 2:30-4:00

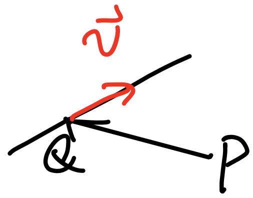
Notes will be posted on https://zhc515.github.io/teaching

Reminder: Midterm exam in the evening

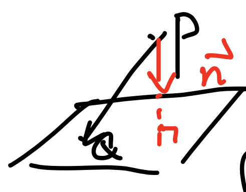
$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \mathrm{proj}_{\vec{w}}\vec{v} = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\cdot\vec{w}$$

1) Distance

- Point-Point: Pythagorean
- Point-Line: $d = \dfrac{|\vec{PQ}\times\vec{v}|}{|\vec{v}|}$



- Point-Plane: $d = |\mathrm{proj}_{\vec{n}}\vec{PQ}| = \dfrac{|\vec{PQ}\cdot\vec{n}|}{|\vec{n}|}$



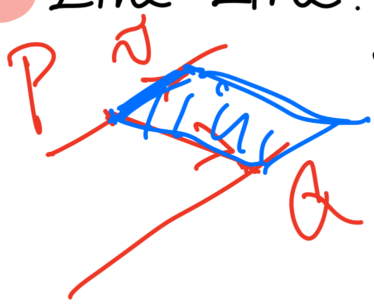
P

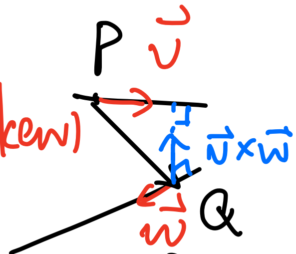
$ax+by+cz=d$

$\vec{n}=\langle a,b,c\rangle$

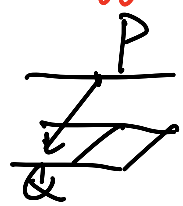
- Line-Line: $d = \dfrac{|\vec{PQ}\times\vec{v}|}{|\vec{v}|}$ (parallel)

  $d = |\mathrm{proj}_{\vec{v}\times\vec{w}}\vec{PQ}|$ (skew)



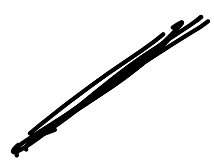


- Line-Plane: $d = |\mathrm{proj}_{\vec{n}}\vec{PQ}|$



- Plane-Plane: $d = \dfrac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$

$ax+by+cz=d_1$ & $ax+by+cz=d_2$



2) Past exam
(23)

2. Let v and w be two vectors in $\mathbb{R}^3$. You're not given the vectors themselves, but you are given that $v \cdot w = \sqrt{3}$ and $v \times w = \langle 1, 1, 1 \rangle$.

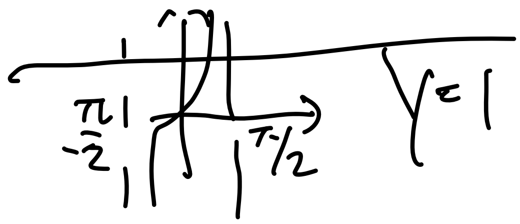
(a) (6 points) Find the angle θ between the vectors v and w. (You may reference the table of special angles on the front of the exam.)

$$\sqrt{3} = \vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cdot \cos\theta \quad ①$$

$$\sqrt{3} = |\vec{v} \times \vec{w}| = |\vec{v}| \cdot |\vec{w}| \cdot \sin\theta \quad ②$$

② ÷ ① $\quad 1 = \tan\theta \quad \theta \in [0, \pi)$

$$\theta = \pi/4.$$



(b) (4 points) Find a vector u such that $u \cdot v = u \cdot w = 0$, and such that the length of u is 3.

$$\vec{v} \times \vec{w} = \langle 1, 1, 1 \rangle$$

$$\vec{n} = \frac{\langle 1, 1, 1 \rangle}{\sqrt{1^2 + 1^2 + 1^2}}$$ normalized vector $|\vec{n}| = 1$

$3 \cdot \vec{n}$ is the answer $(= \langle \sqrt{3}, \sqrt{3}, \sqrt{3} \rangle)$.

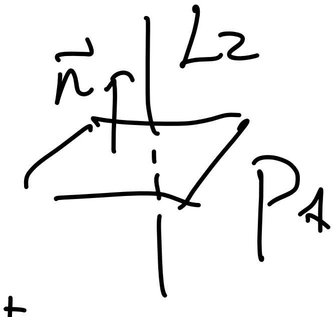
(23)

3. Let L_1 be the line parameterized by $\langle 1+2t, t, 1-3t\rangle$, and let L_2 be the line which goes through the origin and is perpendicular to the plane $\mathcal{P}_1$ given by $5x+2y-5z=4$.

(a) (2 points) Give a parametrization for line L_2.

direction vector (is the normal vector to P_1).

which is $\langle 5, 2, -5\rangle$. Answer: $\begin{cases} x=5t \\ y=2t \\ z=-5t \end{cases}$



(b) (4 points) Are lines L_1 and L_2 parallel, overlapping, ~~intersecting at a single point~~, or skew? Justify your answer.

Answer (only one solution for (s,t)).

$L_1: \begin{cases} x=1+2t \\ y=t \\ z=1-3t \end{cases}$ $\quad L_2: \begin{cases} x=5t \\ y=2t \\ z=-5t \end{cases}$

$\begin{cases} 1+2t=5s & (1) \\ t=2s & (2) \\ 1-3t=-5s & (3) \end{cases}$

By (1) & (2), Plug (2) inside (1)

$1+2\cdot(2s)=5s \Rightarrow s=1$

$\Rightarrow t=2\cdot 1=2$

(c) (4 points) Let $\mathcal{P}_2$ be the plane parallel to $\mathcal{P}_1$ and passing through $(0,1,0)$. Find the distance between $(1,2,1)$ and $\mathcal{P}_2$.

Plug in $s=1$, $t=2$ inside (3)

$LHS = 1-3\cdot 2 = -5$

$RHS = -5\cdot 1 = -5$ $\Rightarrow LHS=RHS$

(23)

5. Express the following polar equations in Cartesian coordinates. Then, identify which conic section the solution set is: a circle, an ellipse, a hyperbola, a parabola, two intersecting lines, or something else.

(a) (3 points) $\cos^2(\theta) - 4\sin^2(\theta) = \frac{1}{r^2}$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

Mult r^2, $\cos^2\theta \cdot r^2 - 4\sin^2\theta \cdot r^2 = 1$

$x^2 - 4\cdot y^2 = 1$

$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\left(\frac{y}{x}\right). \end{cases}$

$\frac{x^2}{1} - \frac{y^2}{1/4} = 1$

hyperbola

(b) (3 points) $\cos^2(\theta) - 4\sin^2(\theta) = 0$

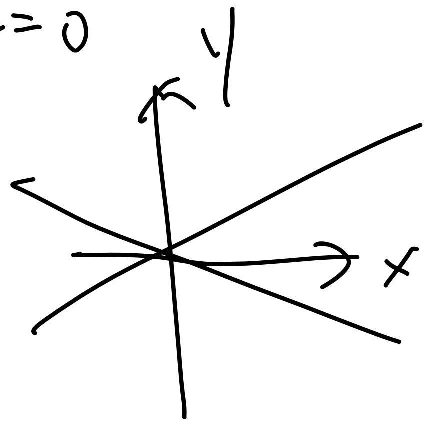
Mult r^2, $r^2\cos^2\theta - r^2 \cdot 4\sin^2\theta = 0$

$x^2 - 4y^2 = 0$

$x^2 = 4y^2$

$\pm\sqrt{\ }$ $\hookrightarrow$ $x = \pm 2y$

two intersecting lines



(c) (4 points) $r + 3r\cos^2\theta - 2\sin(\theta) = 0$

Mult r. $r^2 + 3r^2\cos^2\theta - 2r\sin\theta = 0$

$x^2 + y^2 + 3\cdot x^2 - 2y = 0$

$4x^2 + y^2 - 2y = 0$

$4x^2 + (y-1)^2 = 1$

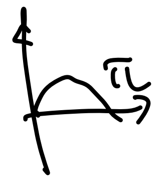
$\frac{x^2}{1/4} + \frac{(y-1)^2}{1} = 1$

ellipse

(23)

6. Let $\vec{r}(t) = \langle 3\sin(2t), 4\cos(2t)\rangle$ be the position of a particle in $\mathbb{R}^2$ at time t, where $0 \le t \le \pi$.

(a) (3 points) Sketch the trajectory of this particle. What is the name of this curve?

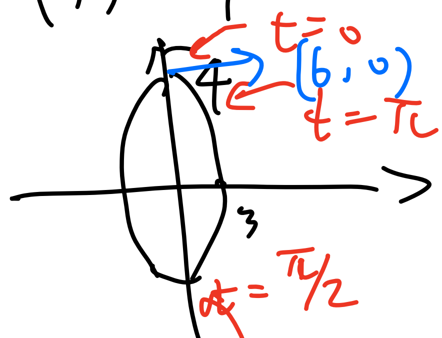


$x = 3\sin(2t)$
$y = 4\cdot\cos(2t)$ $\Rightarrow \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$

ellipse

$t=0$, then $x = 3\sin(2\cdot 0) = 0$
$y = 4\cdot\cos(2\cdot 0) = 4$

$t = \pi/2$, then $x = 3\cdot\sin(2\cdot\frac{\pi}{2}) = 0$



(b) (3 points) Compute the velocity of this particle at $t = 0$. In your sketch above, indicate where this particle is at $t = 0$, and draw the velocity vector at that point.

$y = 4\cdot\cos(2\cdot\pi/2) = -4$

$t = \pi$, then $x = 3\cdot\sin(2\cdot\pi) = 0$
$y = 4\cdot\cos(2\cdot\pi) = 4$

At $t=0$.
$\vec{r}\,'(0) = \langle 6, 0\rangle$

$\vec{r}\,'(t) = \langle (3\sin(2t))', (4\cos(2t))' \rangle$
$= \langle 6\cos(2t), -8\sin(2t) \rangle$

(c) (4 points) Find a point on the trajectory at which the tangent line has slope $-4/3$. You should report the value of t as well as where the particle is at that value of t.

$$\text{slope} = \frac{-8\sin(2t)}{6\cos(2t)} \overset{\text{set}}{=} -\frac{4}{3}$$

$\Rightarrow \tan(2t) = 1 \Rightarrow 2t = \pi/4$

$\Rightarrow t = \pi/8$

(24)

3. (a) (8 points) Transform the polar equation $r = \dfrac{\cos\theta}{(\sin\theta)^2}$ into Cartesian coordinates. Identify and sketch the resulting graph in rectangular coordinates, labeling your coordinate axes x and y.

(b) (12 points) Let a and b be constants with $\cos(b) \neq 0$, and consider the plane given by the equation

$$\cos(a)\sin(b)x + \sin(a)\sin(b)y + \cos(b)z = 1.$$

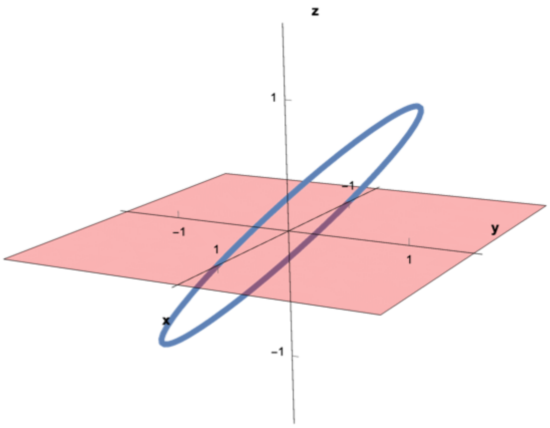
Find the normal vector and find where the plane intersects the z axis.
Finally, show that the distance from this plane to the origin is equal to 1, no matter what a and b are.

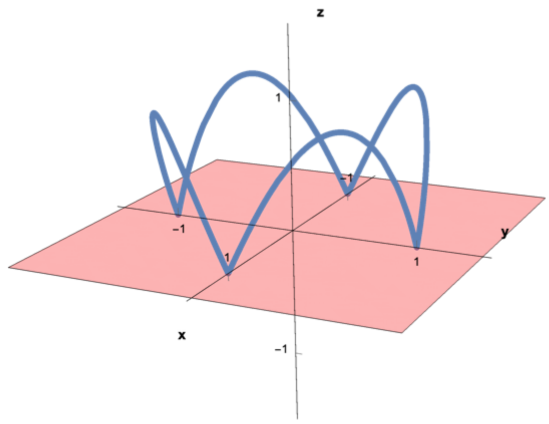
(Prac1)

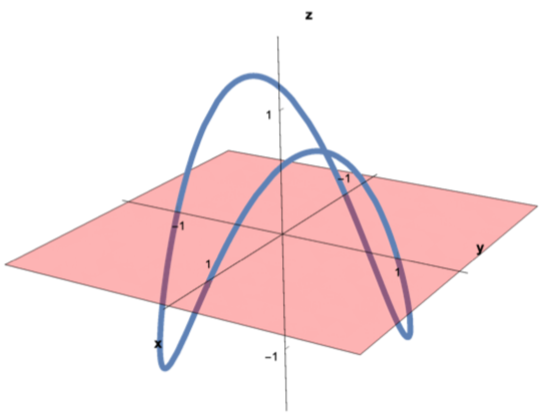
4. (15 points) Let $\mathcal{C}$ be the curve with parametrization $\mathbf{r}(t) = \langle \cos t, \sin t, \sin(2t) \rangle$.

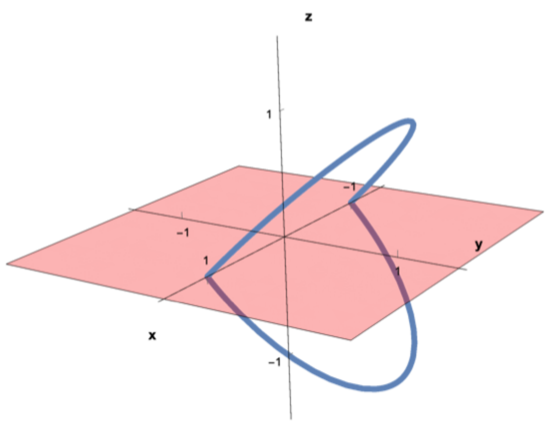
   (a) Exactly one of the figures below is a graph of $\mathbf{r}(t)$ for $0 \leq t \leq 2\pi$. Identify which is correct via a process of elimination: that is, indicate each incorrect graph with an 'X' and briefly explain why it cannot be a graph of $\mathbf{r}(t)$; then indicate the correct graph with a checkmark.

   Note: I've included a shaded portion of the xy-plane in each figure to help you visualize the curve.









(Prac1)

6. (10 points) Let $\mathcal{S}$ be the surface with equation $x^2 + y^2 + 4z^2 - 2x + 4y + 1 = 0$.

   (a) Identify $\mathcal{S}$ as one of our familiar named surfaces. You should first do some algebra to bring the equation into a more standard form.

   Justify your answer. You may reference your work in (b) if you like.

$$(x-1)^2+(y+4)^2+4z^2 = 4$$

   (b) Find equations for the $(x = 1)$-, $(y = -2)-$ and $(z = 0)$-cross sections, and sketch these in the coordinate system below. Each cross section sketch must include at least 4 plotted points.