STA 243 Assignment 4.5

5.

(a) Generate a random sample of size n = 100 for the ZIP model using parameters $p = 0.3$ and $\lambda = 2$.

According to the question, $X_i = R_i Y_i$ where $Y_i's$ have a Poisson(λ) distribution and the $R_i's$ have a Bernoulli(p) distribution, all independent of each other.

first generate 100 $R_i \sim Bernoulli(p)$ and 100 $Y_i \sim Poisson(\lambda)$ and let $X_i = R_i Y_i$

(b)

i. $(\lambda|p, r, x)$ treating p, r, x as fixed in $f(p, \lambda, r, x)$

$$\begin{aligned} f(\lambda|p, r, x) &\propto \lambda^{a-1} e^{-b\lambda} \prod_{i=1}^{n} e^{-\lambda r_i} \lambda^{x_i} \\ &\propto \lambda^{a+\sum_i x_i - 1} e^{-(b+\sum_i r_i)\lambda} \end{aligned}$$

which is the the pdf of Gamma($a + \sum_i x_i, b + \sum_i r_i$) with shape and rate parameters.

ii. $(p|\lambda, r, x)$ treating λ, r, x as fixed in $f(p, \lambda, r, x)$

$$\begin{aligned} f(p|\lambda, r, x) &\propto \prod_{i=1}^{n} p^{r_i} (1-p)^{1-r_i} \\ &\propto p^{(\sum_i r_i + 1) - 1} (1-p)^{(n - \sum_i r_i + 1) - 1} \end{aligned}$$

which is the the pdf of Beta($\sum_i r_i + 1, n - \sum_i r_i + 1$).

iii.

$(r_i|\lambda, p) \sim Bernoulli(p)$ and $(x_i|r, \lambda, p) \sim Poisson(\lambda r_i)$ and etc.

r_i's are independent.x_i's are independent.

$$\begin{aligned} f(r_i|\lambda, p, x) &= \frac{f(r_i, x|\lambda, p)}{f(x_i|\lambda, p)} \\ &= \frac{f(r_i, x, \lambda|p)}{f(x_i|\lambda, p) f(\lambda|p)} \\ &= \frac{f(r_i, x, \lambda, p)}{f(x_i|\lambda, p) f(\lambda|p) f(p)} \\ &= \frac{e^{-\lambda r_i} (\lambda r_i)^{x_i} p^{r_i} (1-p)^{1-r_i}}{x_i!(f(x|\lambda, p)} \\ &= \frac{e^{-\lambda r_i} (\lambda r_i)^{x_i} p^{r_i} (1-p)^{1-r_i}}{x_i!(f(x|r_i = 1, \lambda, p)p + f(x|r_i = 0, \lambda, p))(1-p)} \end{aligned}$$

so that

$$\begin{aligned} P(r_i = 1|\lambda, p, x) &= \frac{e^{-\lambda} \lambda^{x_i} p}{x_i!(\frac{(\lambda r_i)^{x_i} e^{-\lambda r_i}}{x_i!}|_{r_i=0}(1-p) + \frac{(\lambda r_i)^{x_i} e^{-\lambda r_i}}{x_i!}|_{r_i=1} p)} \\ &= \frac{e^{-\lambda} p}{(r_i)^{x_i} e^{-\lambda r_i}|_{r_i=0}(1-p) + (r_i)^{x_i} e^{-\lambda r_i}|_{r_i=1} p)} \\ &= \frac{e^{-\lambda} p}{I_{\{x_i=0\}}(1-p) + e^{-\lambda} p} \end{aligned}$$

so that $(r_i|\lambda, p, x) \sim Bernoulli(\frac{e^{-\lambda}p}{I_{\{x_i=0\}}(1-p)+e^{-\lambda}p})$

(c).