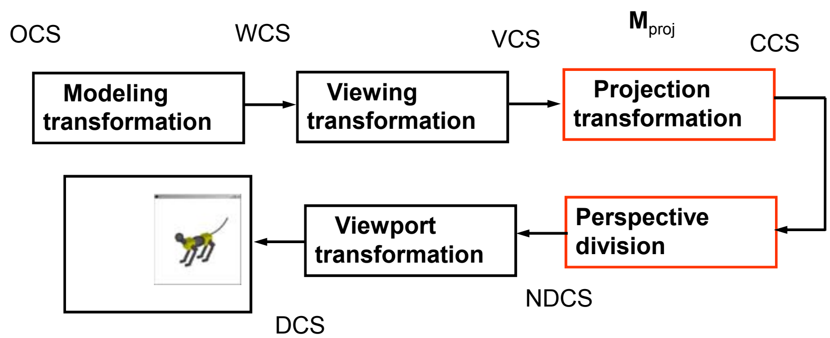
Graphics Pipeline

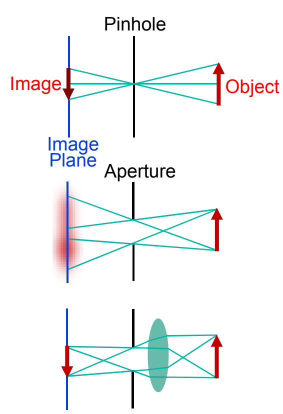


Reminder: Cameras (and the Eye)

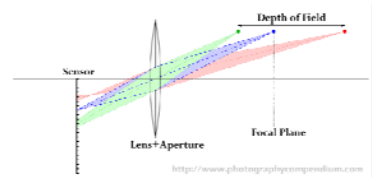
Ideal pinhole

Aperture

Aperture + lens



Depth of field



Projection Transformations

Mapping: $T : R^n \rightarrow R^m$

Projection: n > m

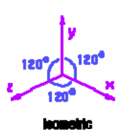
Planar Projection:

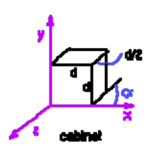
Projection onto a plane

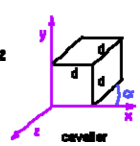
$R^3 \rightarrow R^2$ or
$R^4 \rightarrow R^3$ in homogenous coordinates

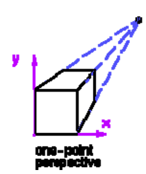
Taxonomy and Examples

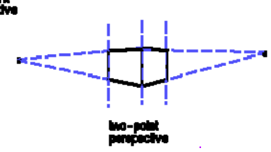
- Planar Projections
  - Parallel
    - Orthographic
      - Top
        Front
        Side
      - Axonometric:
        Isometric
        Dimetric
    - Oblique
      - Cabinet
      - Cavalier
  - Perspective
    - 1 Point
    - 2 Point
    - 3 Point





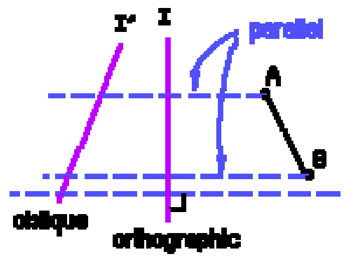




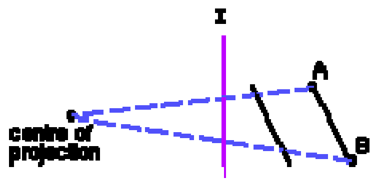


Basic Projections

Parallel



Perspective



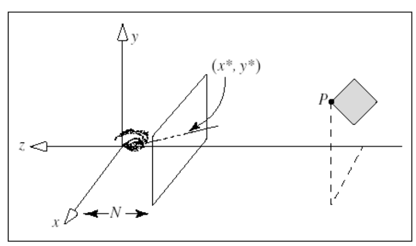
Camera Coordinate System

Camera at (0,0,0)

Looking at $-z$

Image plane = *near plane*

Image plane at $z = -N$

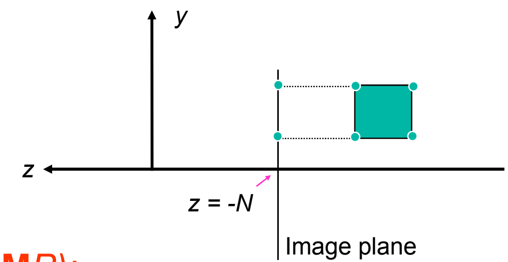


A Basic Orthographic Projection

$P'_x = P_x$

$P'_y = P_y$

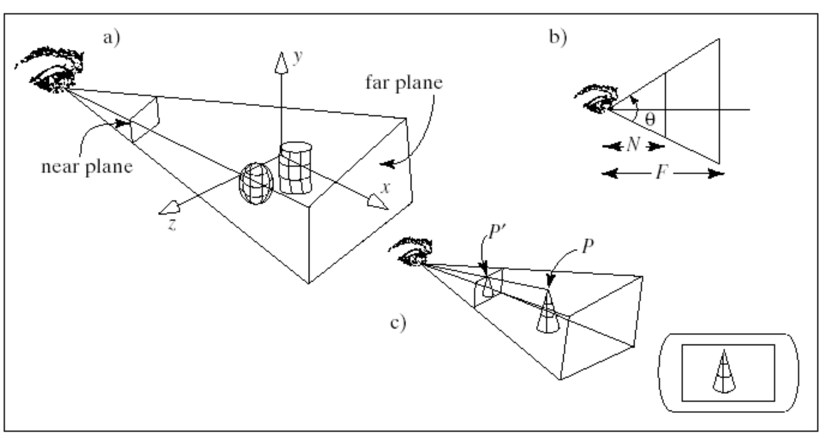
$P'_z = -N$



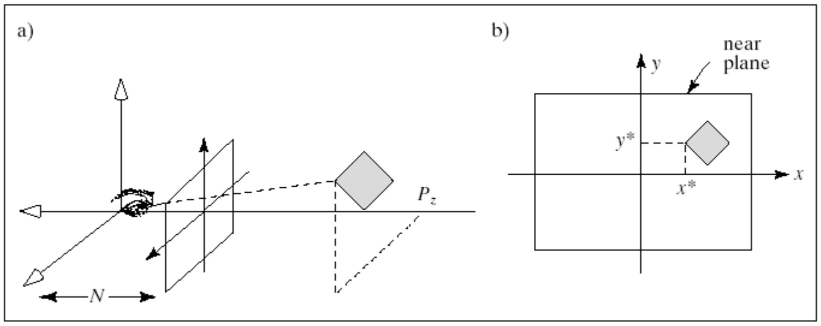
***Matrix Form** (P' = **M**P):*

$$\begin{bmatrix} P'_x \\ P'_y \\ P'_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -N \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix}$$

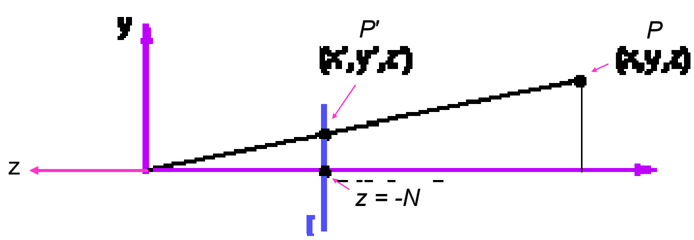
Perspective Projection



Perspective Projection of a Point



A Basic Perspective Projection



Similar triangles

$y'/N = y/-z \quad \Rightarrow \quad P_y' = P_y N / -P_z$

Similarly $\quad P_x' = P_x N / -P_z$

$P_z' = -N$

This is a non-linear transformation!

In Homogeneous Matrix Form

Reminder:

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} \rightarrow \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} \xrightarrow[\times w]{} \begin{bmatrix} wP_x \\ wP_y \\ wP_z \\ w \end{bmatrix} \xrightarrow[\div w]{\text{homogenize}} \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

(a line in 4D space)

Perspective projection:

$$\begin{bmatrix} P'_x \\ P'_y \\ P'_z \\ 1 \end{bmatrix} = \begin{bmatrix} P_x N/(-P_z) \\ P_y N/(-P_z) \\ -N \\ 1 \end{bmatrix} \xrightarrow[\times\, -P_z/N]{} \begin{bmatrix} P_x \\ P_y \\ P_z \\ -P_z/N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/N & 0 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix}$$

Therefore: $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/N & 0 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} \xrightarrow[\div\, -P_z/N]{\text{and then: homogenize}} \begin{bmatrix} P'_x \\ P'_y \\ P'_z \\ 1 \end{bmatrix}$$

Matrix $\mathbf{M}$

$\div -P_z/N$

Homogenization step:
"Perspective Division"
(divide by $w = -P_z/N$)

Observations

- Projection undefined for $P_z = 0$
- If P is behind the eye,
  P_z changes sign
- Near plane just scales the picture
- Straight line → straight line
- Perspective foreshortening

$$P'_x = -N\frac{P_x}{P_z}$$
$$P'_y = -N\frac{P_y}{P_z}$$
$$P'_z = -N$$

Perspective Projection of a Line

$$L(t) = \mathbf{P} + \mathbf{v}t = \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} + \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix} t \xrightarrow{\text{Perspective Division \& drop fourth coordinate}}$$

Is it still a line?

Original: $L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} P_x + v_x t \\ P_y + v_y t \\ P_z + v_z t \end{bmatrix}$

Projected: $L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(P_x + v_x t)/(P_z + v_z t) \\ -N(P_y + v_y t)/(P_z + v_z t) \\ -N \end{bmatrix}$

$$x' = -N(P_x + v_x t)/(P_z + v_z t) \Rightarrow x'(P_z + v_z t) = -N(P_x + v_x t) \Rightarrow$$

$$x'P_z + x'v_z t = -NP_x - Nv_x t \Rightarrow \begin{cases} x'P_z + NP_x = -(x'v_z + Nv_x)t \\ \text{and similarly for y:} \\ y'P_z + NP_y = -(y'v_z + Nv_y)t \end{cases}$$

Is it still a line? (cont'd)

$$\left.\begin{aligned} x'P_z + NP_x &= -(x'v_z + Nv_x)t \\ y'P_z + NP_y &= -(y'v_z + Nv_y)t \end{aligned}\right| \Rightarrow \left.\begin{aligned} x'P_z + NP_x &= -(x'v_z + Nv_x)t \\ (y'v_z + Nv_y)t &= -(y'P_z + NP_y) \end{aligned}\right| \Rightarrow$$

$$(x'P_z + NP_x)(y'v_z + Nv_y) = (x'v_z + Nv_x)(y'P_z + NP_y) \Rightarrow$$

$$x'P_z y'v_z + x'P_z Nv_y + NP_x y'v_z + N^2 P_x v_y = x'v_z y'P_z + x'v_z NP_y + Nv_x y'P_z + N^2 P_y v_x \Rightarrow$$

$$(P_z Nc_y - v_z NP_y)x' + (NP_x v_z + Nv_x P_z)y' + N^2(P_x v_y + P_y v_x) = 0 \Rightarrow$$

$\Rightarrow$ $\boxed{ax' + by' + c = 0}$ which is the equation of a line in the x'-y' plane

But is There a Difference?

$$\text{Original:} \quad L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} P_x + v_x t \\ P_y + v_y t \\ P_z + v_z t \end{bmatrix}$$

$$\text{Projected:} \quad L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(P_x + v_x t)/(P_z + v_z t) \\ -N(P_y + v_y t)/(P_z + v_z t) \\ -N \end{bmatrix}$$

But is There a Difference?

The "speed along the lines" if v_z isn't 0

$$\text{Original:}\quad L(t)=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}P_x+v_xt\\P_y+v_yt\\P_z+v_zt\end{bmatrix}\Rightarrow \boxed{\frac{\partial L(t)}{\partial t}=\begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix}=\mathbf{v}}$$

$$\text{Projected:}\quad L'(t)=\begin{bmatrix}x'\\y'\\z'\end{bmatrix}=\begin{bmatrix}-Nx/z\\-Ny/z\\-N\end{bmatrix}=\begin{bmatrix}-N(P_x+v_xt)/(P_z+v_zt)\\-N(P_y+v_yt)/(P_z+v_zt)\\-N\end{bmatrix}\Rightarrow$$

$$\frac{\partial x'}{\partial t}=-N\frac{\partial}{\partial t}((P_x+v_xt)/(P_z+v_zt))=-N\frac{v_x(P_z+v_zt)-(P_x+v_xt)v_z}{(P_z+v_zt)^2}=-N\frac{v_xP_z-P_xv_z}{(P_z+v_zt)^2}\Rightarrow$$

$$\boxed{\frac{\partial L'(t)}{\partial t}=\frac{-N}{(P_z+v_zt)^2}\begin{bmatrix}v_xP_z-P_xv_z\\v_yP_z-P_yv_z\\0\end{bmatrix}}$$

Effect of Perspective Projection on Lines

Line equations

$$\text{Original:}\quad L(t)=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}P_x+v_xt\\P_y+v_yt\\P_z+v_zt\end{bmatrix}$$

$$\text{Projected:}\quad L'(t)=\begin{bmatrix}x'\\y'\\z'\end{bmatrix}=\begin{bmatrix}-Nx/z\\-Ny/z\\-N\end{bmatrix}=\begin{bmatrix}-N(P_x+v_xt)/(P_z+v_zt)\\-N(P_y+v_yt)/(P_z+v_zt)\\-N\end{bmatrix}$$

What happens to lines parallel to the view plane?

Effect of Perspective Projection on Lines

Line equations

$$\text{Original:}\quad L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} P_x + v_x t \\ P_y + v_y t \\ P_z + v_z t \end{bmatrix}$$

$$\text{Projected:}\quad L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(P_x + v_x t)/(P_z + v_z t) \\ -N(P_y + v_y t)/(P_z + v_z t) \\ -N \end{bmatrix}$$

If lines are parallel to the view plane then:

$$v_z = 0 \rightarrow L'(t) = -\frac{N}{P_z}\begin{bmatrix} P_x + v_x t \\ P_y + v_y t \\ P_z \end{bmatrix}$$

slope of line: $\frac{v_y}{v_x}$ so, parallel lines parallel to the view plane remain parallel

Effect of Perspective Projection on Lines

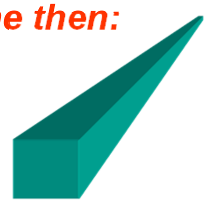
Line equations

$$\text{Original:}\quad L(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} P_x + v_x t \\ P_y + v_y t \\ P_z + v_z t \end{bmatrix}$$

$$\text{Projected:}\quad L'(t) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -Nx/z \\ -Ny/z \\ -N \end{bmatrix} = \begin{bmatrix} -N(P_x + v_x t)/(P_z + v_z t) \\ -N(P_y + v_y t)/(P_z + v_z t) \\ -N \end{bmatrix}$$

If lines are not parallel to the view plane then:

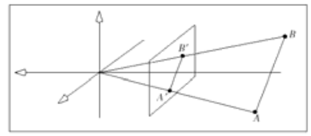
$$v_z \neq 0 \rightarrow \lim_{t\to\infty} L'(t) = \begin{bmatrix} -Nv_x/v_z \\ -Nv_y/v_z \\ -N \end{bmatrix}$$

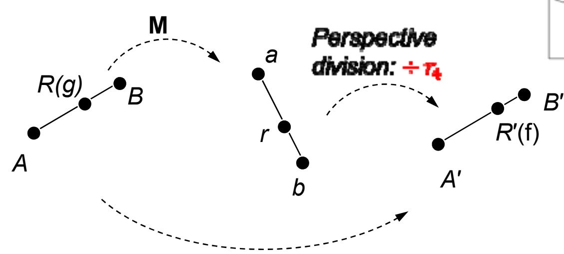


Lines converge to a **vanishing point**!

Foreshortening: In-Between Points on Perspective-Projected Lines

How do points on lines transform?



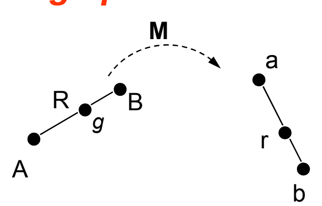


View coordinate system:	$R(g) = (1 - g)A + gB$
Projected homogeneous 4D:	$r = \mathbf{M}R$
Projected homogeneous 3D:	$R'(f) = (1 - f)A' + fB'$

***g* and *f* are not the same**
What is the relationship between *g* and *f* ?

First Step

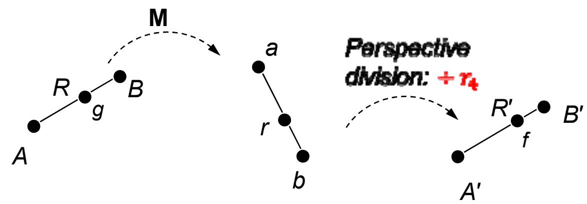
Viewing space to homogeneous space (4D)



$$R = (1 - g)A + gB$$
$$r = \mathrm{M}R = \mathrm{M}[(1 - g)A + gB] = (1 - g)\mathrm{M}A + g\mathrm{M}B \Rightarrow$$
$$r = (1 - g)a + gb$$
$$a = \mathrm{M}A = [a_1, a_2, a_3, a_4]^T$$
$$b = \mathrm{M}B = [b_1, b_2, b_3, b_4]^T$$

Second Step

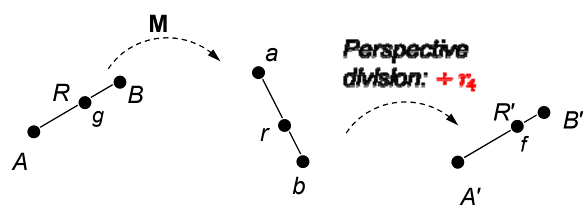
Perspective division



$$\left\{\begin{array}{l} r = (1-g)a + gb \\ a = [a_1, a_2, a_3, a_4]^T \\ b = [b_1, b_2, b_3, b_4]^T \end{array}\right\} \Rightarrow \boxed{R'_1 = \frac{r_1}{r_4} = \frac{(1-g)a_1 + gb_1}{(1-g)a_4 + gb_4}}$$

And similarly for R'_2 *and* R'_3

Putting it Together



$$R'_1 = \frac{(1-g)a_1 + gb_1}{(1-g)a_4 + gb_4} = \frac{\text{lerp}(a_1, b_1, g)}{\text{lerp}(a_4, b_4, g)}$$

lerp: linear Interpolation
(hardware acceleration)

Furthermore:

$$R' = (1-f)A' + fB' \Rightarrow R'_1 = (1-f)A'_1 + fB'_1$$

$$R'_1 = (1-f)\frac{a_1}{a_4} + f\frac{b_1}{b_4} = \text{lerp}\left(\frac{a_1}{a_4}, \frac{b_1}{b_4}, f\right)$$

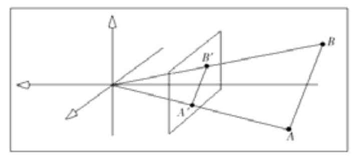
Relation Between the Fractions

$$\left.\begin{aligned} R'_1(f) &= \frac{\mathrm{lerp}(a_1, b_1, g)}{\mathrm{lerp}(a_4, b_4, g)} \\ R'_1(f) &= \mathrm{lerp}\left(\frac{a_1}{a_4}, \frac{b_1}{b_4}, f\right) \end{aligned}\right\} \Rightarrow g = \frac{f}{\mathrm{lerp}(\frac{b_4}{a_4}, 1, f)}$$

substituting this in $R(g) = (1-g)A + gB$ yields

$$R_1 = \frac{\mathrm{lerp}(\frac{A_1}{a_4}, \frac{B_1}{b_4}, f)}{\mathrm{lerp}(\frac{1}{a_4}, \frac{1}{b_4}, f)}$$

similarly for R_2 & R_3



THIS MEANS: For a given *f* in **image space** and *A, B* in **viewing space,** we can find the corresponding *R (or g)* in **viewing space** using the above formula

This works if "*A*", "*B*" are **positions**, **texture coordinates**, **color**, **normals**, etc.

So, it is generally VERY useful

Summary

Perspective projection is <u>non-linear</u>

Lines project to lines

Parallel lines either project to parallel lines or they intersect at the vanishing point

Foreshortening of projected lines and the "Inbetweeneness" relationship