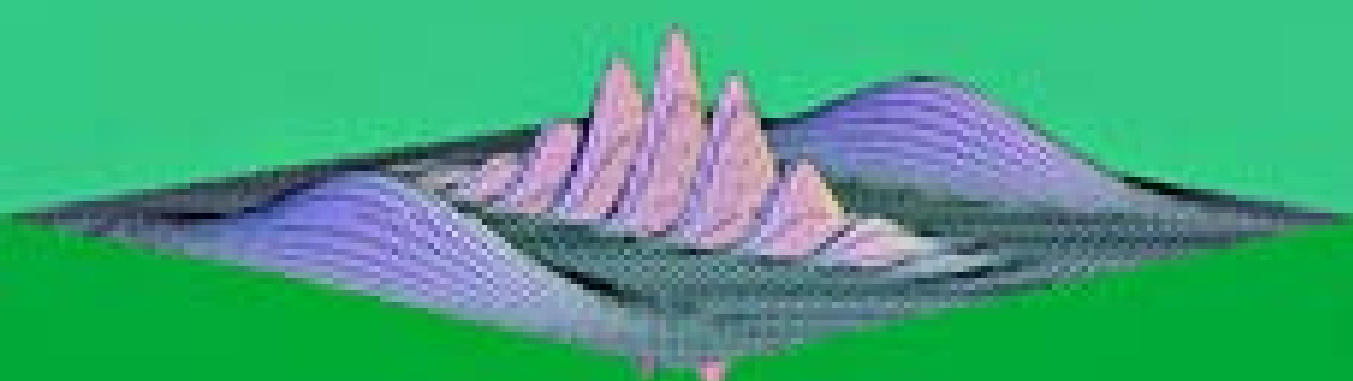
Christopher C. Gerry and Peter L. Knight

# Introductory Quantum Optics



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## Introductory Quantum Optics

This book provides an elementary introduction to the subject of quantum optics, the study of the quantum-mechanical nature of light and its interaction with matter.

The presentation is almost entirely concerned with the quantized electromagnetic field. Topics covered include single-mode field quantization in a cavity, quantization of multimode fields, quantum phase, coherent states, quasi-probability distribution in phase space, atom–field interactions, the Jaynes–Cummings model, quantum coherence theory, beam splitters and interferometers, nonclassical field states with squeezing etc., tests of local realism with entangled photons from down-conversion, experimental realizations of cavity quantum electrodynamics, trapped ions, decoherence, and some applications to quantum information processing, particularly quantum cryptography. The book contains many homework problems and a comprehensive bibliography.

This text is designed for upper-level undergraduates taking courses in quantum optics who have already taken a course in quantum mechanics, and for first- and second-year graduate students.

A solutions manual is available to instructors via solutions@cambridge.org.

Christopher Gerry is Professor of Physics at Lehman College, City University of New York. He was one of the first to exploit the use of group theoretical methods in quantum optics and is also a frequent contributor to *Physical Review A*. In 1992 he co-authored, with A. Inomata and H. Kuratsuji, *Path Integrals and Coherent States for Su (2) and SU (1, 1)*.

Peter Knight is a leading figure in quantum optics, and in addition to being President of the Optical Society of America in 2004, he is a Fellow of the Royal Society. In 1983 he co-authored *Concepts of Quantum Optics* with L. Allen. He is currently Head of the Physics Department of Imperial College and Chief Scientific Advisor at the UK National Physical Laboratory.

# Introductory Quantum Optics

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C. C. G. dedicates this book to his son, Eric.
P. L. K. dedicates this book to his wife Chris.

# Contents

# Acknowledgements

This book developed out of courses that we have given over the years at Imperial College London, and the Graduate Center of the City University of New York, and we are grateful to the many students who have sat through our lectures and acted as guinea pigs for the material we have presented here.

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As this book is intended as an introduction to quantum optics, we have not attempted to be comprehensive in our citations. We apologize to authors whose work is not cited.

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# Chapter 1
# Introduction

## 1.1 Scope and aims of this book

Quantum optics is one of the liveliest fields in physics at present. While it has been a dominant research field for at least two decades, with much graduate activity, in the past few years it has started to impact the undergraduate curriculum. This book developed from courses we have taught to final year undergraduates and beginning graduate students at Imperial College London and City University of New York. There are plenty of good research monographs in this field, but we felt that there was a genuine need for a straightforward account for senior undergraduates and beginning postgraduates, which stresses basic concepts. This is a field which attracts the brightest students at present, in part because of the extraordinary progress in the field (e.g. the implementation of teleportation, quantum cryptography, Schrödinger cat states, Bell violations of local realism and the like). We hope that this book provides an accessible introduction to this exciting subject.

Our aim was to write an elementary book on the essentials of quantum optics directed to an audience of upper-level undergraduates, assumed to have suffered through a course in quantum mechanics, and for first- or second-year graduate students interested in eventually pursuing research in this area. The material we introduce is not simple, and will be a challenge for undergraduates and beginning graduate students, but we have tried to use the most straightforward approaches. Nevertheless, there are parts of the text that the reader will find more challenging than others. The problems at the end of each chapter similarly have a range of difficulty. The presentation is almost entirely concerned with the quantized electromagnetic field and its effects on atoms, and how nonclassical light behaves. One aim of this book is to connect quantum optics with the newly developing subject of quantum information processing.

Topics covered are: single-mode field quantization in a cavity, quantization of multimode fields, the issue of the quantum phase, coherent states, quasi-probability distributions in phase space, atom–field interactions, the Jaynes–Cummings model, quantum coherence theory, beam splitters and interferometers, nonclassical field states with squeezing, etc., test of local realism with entangled

photons from down-conversion, experimental realizations of cavity quantum electrodynamics, trapped ions, etc., issues regarding decoherence, and some applications to quantum information processing, particularly quantum cryptography. The book includes many homework problems for each chapter and bibliographies for further reading. Many of the problems involve computational work, some more extensively than others.

## 1.2 History

In this chapter we briefly survey the historical development of our ideas of optics and photons. A detailed account can be found in the "Historical Introduction" for example in the 6th edition of Born and Wolf. A most readable account of the development of quantum ideas can be found in a recent book by Whitaker [1]. A recent article by A. Muthukrishnan, M. O. Scully and M. S. Zubairy [2] ably surveys the historical development of our ideas on light and photons in a most readable manner.

The ancient world already was wrestling with the nature of light as rays. By the seventeenth century the two rival concepts of waves and corpuscles were well established. Maxwell, in the second half of the nineteenth century, laid the foundations of modern field theory, with a detailed account of light as electromagnetic waves and at that point classical physics seemed triumphant, with "minor" worries about the nature of black-body radiation and of the photoelectric effect. These of course were the seeds of the quantum revolution. Planck, an inherently conservative theorist, was led rather reluctantly, it seems, to propose that thermal radiation was emitted and absorbed in discrete quanta in order to explain the spectra of thermal bodies. It was Einstein who generalized this idea so that these new quanta represented the light itself rather than the processes of absorption and emission, and was able to describe how matter and radiation could come into equilibrium (introducing on the way the idea of stimulated emission), and how the photoelectric effect could be explained. By 1913, Bohr applied the basic idea of quantization to atomic dynamics and was able to predict the positions of atomic spectral lines.

Gilbert Lewis, a chemist, coined the word photon well after the light quanta idea itself was introduced. In 1926 Lewis said

> It would seem appropriate to speak of one of these hypothetical entities as a particle of light, a corpuscle of light, a light quantum, or light quant, if we are to assume that it spends only a minute fraction of its existence as a carrier of radiant energy, while the rest of the time it remains as an important structural element within the atom . . . I therefore take the liberty of proposing for this hypothetical new atom, which is not light but plays an important part in every process of radiation, the name photon [3].

Clearly Lewis's idea and ours are rather distantly connected!

De Broglie in a remarkable leap of imagination generalized what we knew about light quanta, exhibiting wave and particle properties to matter itself. Heisenberg, Schrödinger and Dirac laid the foundations of quantum mechanics in an amazingly short period from 1925 to 1926. They gave us the whole machinery we still use: representations, quantum-state evolution, unitary transformations, perturbation theory and more. The intrinsic probabilistic nature of quantum mechanics was uncovered by Max Born, who proposed the idea of probability amplitudes which allowed a fully quantum treatment of interference.

Fermi and Dirac, pioneers of quantum mechanics, were also among the first to address the question of how quantized light interacts with atomic sources and propagates. Fermi's *Reviews of Modern Physics* article in the 1930s, based on lectures he gave in Ann Arbor, summarize what was known at that time within the context of nonrelativistic quantum electrodynamics in the Coulomb gauge. His treatment of interference (especially Lipmann fringes) still repays reading today. It is useful to quote Willis Lamb in this context:

> Begin by deciding how much of the universe needs to be brought into the discussion. Decide what normal modes are needed for an adequate treatment. Decide how to model the light sources and work out how they drive the system [4].

This statement sums up the approach we will take throughout this book.

Weisskopf and Wigner applied the newly developed ideas of non-relativistic quantum mechanics to the dynamics of spontaneous emission and resonance fluorescence, predicting the exponential law for excited-state decay. This work already exhibited the self-energy problems, which were to plague quantum electrodynamics for the next 20 years until the development of the renormalization programme by Schwinger, Feynman, Tomonaga, and Dyson. The observation of the anomalous magnetic moment of the electron by Kusch, and of radiative level shifts of atoms by Lamb and Retherford, were the highlights of this era. The interested reader will find the history of this period very ably described by Schweber in his magisterial account of QED [5]. This period of research demonstrated the importance of considering the vacuum as a field which had observable consequences. In a remarkable development in the late 1940s, triggered by the observation that colloids were more stable than expected from considerations of van der Waals interactions, Casimir showed that long-range intermolecular forces were intrinsically quantum electrodynamic. He linked them to the idea of zero-point motion of the field and showed that metal plates in vacuum attract as a consequence of such zero-point motion.

Einstein had continued his study of the basic nature of quantum mechanics and in 1935 in a remarkable paper with Podolsky and Rosen was able to show how peculiar quantum correlations were. The ideas in this paper were to explode into one of the most active parts of modern physics with the development by Bohm and Bell of concrete predictions of the nature of these correlations; this laid the

foundations of what was to become the new subject of quantum information processing.

Optical coherence had been investigated for many years using amplitude interference: a first-order correlation. Hanbury Brown and Twiss in the 1950s worked on intensity correlations as a tool in stellar interferometry, and showed how thermal photon detection events were "bunched." This led to the development of the theory of photon statistics and photon counting and to the beginnings of quantum optics as a separate subject. At the same time as ideas of photon statistics were being developed, researchers had begun to investigate coherence in light–matter interactions. Radio-frequency spectroscopy had already been initiated with atomic beams with the work of Rabi, Ramsey and others. Sensitive optical pumping probes of light interaction with atoms were developed in the 1950s and 1960s by Kastler, Brossel, Series, Dodd and others.

By the early 1950s, Townes and his group, and Basov and Prokhorov, had developed molecular microwave sources of radiation: the new masers, based on precise initial state preparation, population inversion and stimulated emission. Ed Jaynes in the 1950s played a major role in studies of whether quantization played a role in maser operation (and this set the stage for much later work on fully quantized atom–field coupling in what became known as the Jaynes–Cummings model). Extending the maser idea to the optical regime and the development of lasers of course revolutionized modern physics and technology.

Glauber, Wolf, Sudarshan, Mandel, Klauder and many others developed a quantum theory of coherence based on coherent states and photodetection. Coherent states allowed us to describe the behaviour of light in phase space, using the quasi-probabilities developed much earlier by Wigner and others.

For several years after the development of the laser there were no tuneable sources: researchers interested in the details of atom–light or molecule–light interactions had to rely on molecular chance resonances. Nevertheless, this led to the beginning of the study of coherent interactions and coherent transients such as photon echoes, self-induced transparency, optical nutation and so on (well described in the standard monograph by Allen and Eberly). Tuneable lasers became available in the early 1970s, and the dye laser in particular transformed precision studies in quantum optics and laser spectroscopy. Resonant interactions, coherent transients and the like became much more straightforward to study and led to the beginnings of quantum optics proper as we now understand it: for the first time we were able to study the dynamics of single atoms interacting with light in a non-perturbative manner. Stroud and his group initiated studies of resonance fluorescence with the observation of the splitting of resonance fluorescence spectral lines into component parts by the coherent driving predicted earlier by Mollow. Mandel, Kimble and others demonstrated how the resonance fluorescence light was antibunched, a feature studied by a number of theorists including Walls, Carmichael, Cohen-Tannoudji, Mandel and Kimble. The observation of

antibunching, and the associated (but inequivalent) sub-Poissonian photon statistics laid the foundation of the study of "non-classical light". During the 1970s, several experiments explored the nature of photons: their indivisibility and the build up of interference at the single photon level. Laser cooling rapidly developed in the 1980s and 1990s and allowed the preparation of states of matter under precise control. Indeed, this has become a major subject in its own right and we have taken the decision here to exclude laser cooling from this text.

Following the development of high-intensity pulses of light from lasers, a whole set of nonlinear optical phenomena were investigated, starting with the pioneering work in Ann Arbor by Franken and co-workers. Harmonic generation, parametric down-conversion and other phenomena were demonstrated. For the most part, none of this early work on nonlinear optics required field quantization and quantum optics proper for its description. But there were early signs that some could well do so: quantum nonlinear optics was really initiated by the study by Burnham and Weinberg (see Chapter 9) of unusual nonclassical correlations in down-conversion. In the hands of Mandel and many others, these correlations in down-conversion became the fundamental tool used to uncover fundamental insights into quantum optics.

Until the 1980s, essentially all light fields investigated had phase-independent noise; this changed with the production of squeezed light sources with phase-sensitive noise. These squeezed light sources enabled us to investigate Heisenberg uncertainty relations for light fields. Again, parametric down-conversion proved to be the most effective tool to generate such unusual light fields.

Quantum opticians realized quite early that were atoms to be confined in resonators, then atomic radiative transition dynamics could be dramatically changed. Purcell, in a remarkable paper in 1946 within the context of magnetic resonance, had already predicted that spontaneous emission rates, previously thought of as pretty immutable were in fact modified by enclosing the source atom within a cavity whose mode structure and densities are significantly different from those of free space. Putting atoms within resonators or close to mirrors became possible at the end of the 1960s. By the 1980s the theorists' dream of studying single atoms interacting with single modes of the electromagnetic field became possible. At this point the transition dynamics becomes wholly reversible, as the atom coherently exchanges excitation with the field, until coherence is eventually lost through a dissipative "decoherence" process. This dream is called the Jaynes–Cummings model after its proposers and forms a basic building block of quantum optics (and is discussed in detail in this book).

New fundamental concepts in information processing, leading to quantum cryptography and quantum computation, have been developed in recent years by Feynman, Benioff, Deutsch, Jozsa, Bennett, Ekert and others. Instead of using classical bits that can represent either the values 0 or 1, the basic unit of a

quantum computer is a quantum mechanical two-level system (qubit) that can exist in coherent superpositions of the logical values 0 and 1. A set of $n$ qubits can then be in a superposition of up to $2^n$ different states, each representing a binary number. Were we able to control and manipulate say 1500 qubits, we could access more states than there are particles in the visible universe. Computations are implemented by unitary transformations, which act on all states of a superposition simultaneously. Quantum gates form the basic units from which these unitary transformations are built up. In related developments, absolutely secure encryption can be guaranteed by using quantum sources of light.

The use of the quantum mechanical superpositions and entanglement results in a high degree of parallelism, which can increase the speed of computation exponentially. A number of problems which cannot feasibly be tackled on a classical computer can be solved efficiently on a quantum computer. In 1994 a quantum algorithm was discovered by Peter Shor that allows the solution of a practically important problem, namely factorization, with such an exponential increase of speed. Subsequently, possible experimental realizations of a quantum computer have been proposed, for example in linear ion traps and nuclear magnetic resonance schemes. Presently we are at a stage where quantum gates have been demonstrated in these two implementations. Quantum computation is closely related to quantum cryptography and quantum communication. Basic experiments demonstrating the in-principle possibility of these ideas have been carried out in various laboratories.

The linear ion trap is one of the most promising systems for quantum computation and is one we study in this book in detail. The quantum state preparation (laser cooling and optical pumping) in this system is a well-established technique, as is the state measurement by electron shelving and fluorescence. Singly charged ions of an atom such as calcium or beryllium are trapped and laser cooled to micro-Kelvin temperatures, where they form a string lying along the axis of a linear radio-frequency (r.f.) Paul trap. The internal state of any one ion can be exchanged with the quantum state of motion of the whole string. This can be achieved by illuminating the ion with a pulse of laser radiation at a frequency tuned below the ion's internal resonance by the vibrational frequency of one of the normal modes of oscillation of the string. This couples single phonons into and out of the vibrational mode. The motional state can then be coupled to the internal state of another ion by directing the laser onto the second ion and applying a similar laser pulse. In this way general transformations of the quantum state of all the ions can be generated. The ion trap has several features to recommend it. It can achieve processing on quantum bits without the need for any new technological breakthroughs, such as micro-fabrication techniques or new cooling methods. The state of any ion can be measured and re-prepared many times without problem, which is an important feature for implementing quantum error correction protocols.

Trapped atoms or ions can be strongly coupled to an electromagnetic field mode in a cavity, which permits the powerful combination of quantum processing and long-distance quantum communication. This suggests ways in which we may construct quantum memories. These systems can in principle realize a quantum processor larger than any which could be thoroughly simulated by classical computing but the decoherence generated by dephasing and spontaneous emission is a formidable obstacle.

Entangled states are the key ingredient for certain forms of quantum cryptography and for quantum teleportation. Entanglement is also responsible for the power of quantum computing, which, under ideal conditions, can accomplish certain tasks exponentially faster than any classical computer. A deeper understanding of the role of quantum entanglement in quantum information theory will allow us to improve existing applications and to develop new methods of quantum information manipulation. These are all described in later chapters.

What then is the future of quantum optics? It underpins a great deal of laser science and novel atomic physics. It may even be the vehicle by which we can realize a whole new technology whereby quantum mechanics permits the processing and transmission of information in wholly novel ways. But of course, whatever we may predict now to emerge will be confounded by the unexpected: the field remains an adventure repeatedly throwing up the unexpected.

## 1.3 The contents of this book

The layout of this book is as follows. In Chapter 2, we show how the electromagnetic field can be quantized in terms of harmonic oscillators representing modes of the electromagnetic field, with states describing how many excitations (photons) are present in each normal mode. In Chapter 3 we introduce the coherent states, superposition states carrying phase information. In Chapter 4 we describe how light and matter interact. Chapter 5 quantifies our notions of coherence in terms of optical field correlation functions. Chapter 6 introduces simple optical elements such as beam splitters and interferometers, which manipulate the states of light. Chapter 7 describes those nonclassical states whose basic properties are dictated by their fundamental quantum nature. Spontaneous emission and decay in an open environment are discussed in Chapter 8. Chapter 9 describes how quantum optical sources of radiation can be used to provide tests of fundamental quantum mechanics, including tests of nonlocality and Bell inequalities. Chapter 10 discusses how atoms confined in cavities and trapped laser-cooled ions can be used to study basic interaction phenomena. Chapter 11 applies what we have learnt to the newly emerging problems of quantum information processing. Appendices set out some mathematical ideas needed within the main body of the text. Throughout we have tried to illustrate the ideas we have been developing through homework problems.

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## Suggestions for further reading

*Many books on quantum optics exist, most taking the story much further than we do, in more specialized monographs*.

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Y. Yamamoto and A. İmamoğlu, *Mesoscopic Quantum Optics* (New York: Wiley-Interscience, 1999).

*A useful reprint collection of papers on coherent states, including the early work by Glauber, Klauder, and others, is the following*.

J. R. Klauder and B.-S. Skagerstam (editors), *Coherent States* (Singapore: World Scientific, 1985).

*The history of quantum optics and of fundamental tests of quantum theory can be found in a number of places. We have found the following invaluable*.

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# Chapter 2
# Field quantization

In this chapter we present a discussion of the quantization of the electromagnetic field and discuss some of its properties with particular regard to the interpretation of the photon as an elementary excitation of a normal mode of the field. We start with the case of a single-mode field confined by conducting walls in a one-dimensional cavity and later generalize to multimode fields in free space. The photon number states are introduced and we discuss the fluctuations of the field observables with respect to these states. Finally, we discuss the problem of the quantum description of the phase of the quantized electromagnetic field.

## 2.1 Quantization of a single-mode field

We begin with the rather simple but very important case of a radiation field confined to a one-dimensional cavity along the $z$-axis with perfectly conducting walls at $z = 0$ and $z = L$ as shown in Fig. 2.1.

The electric field must vanish on the boundaries and will take the form of a standing wave. We assume there are no sources of radiation, i.e. no currents or charges nor any dielectric media in the cavity. The field is assumed to be polarized along the $x$-direction, $\mathbf{E}(\mathbf{r}, t) = \mathbf{e}_x E_x(z, t)$, where $\mathbf{e}_x$ is a unit polarization vector. Maxwell's equations without sources are, in SI units,

$$\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \tag{2.1}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.3}$$

$$\nabla \cdot \mathbf{E} = 0. \tag{2.4}$$

A single-mode field satisfying Maxwell's equations and the boundary conditions is given by

$$E_x(z, t) = \left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} q(t) \sin(kz) \tag{2.5}$$

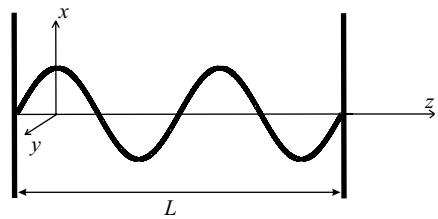


**Fig. 2.1.** Cavity with perfectly conducting walls located at $z = 0$ and $z = L$. The electric field is polarized along the $x$-direction.

where $\omega$ is the frequency of the mode and $k$ is the wave number related to the frequency according to $k = \omega/c$ . The boundary condition at $z = L$ yields the allowed frequencies $\omega_m = c(m\pi/L)$, $m = 1, 2, \ldots$ We assume that $\omega$ in Eq. (2.5) is one of these frequencies and ignore the rest for now. $V$ in Eq. (2.5) is the effective volume of the cavity and $q(t)$ is a time-dependent factor having the dimension of length. As we shall see, $q(t)$ will act as a canonical position. The magnetic field in the cavity, from Eq. (2.5) and Eq. (2.2) is $\mathbf{B}(\mathbf{r}, t) = \mathbf{e}_y B_y(z, t)$ where

$$B_y(z, t) = \left(\frac{\mu_0 \varepsilon_0}{k}\right)\left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} \dot{q}(t)\cos(kz). \tag{2.6}$$

Here, $\dot{q}(t)$ will play the role of a canonical momentum for a "particle" of unit mass, i.e. $p(t) = \dot{q}(t)$.

The classical field energy, or Hamiltonian $H$, of the single-mode field is given by

$$\begin{aligned} H &= \frac{1}{2}\int dV\left[\varepsilon_0 \mathbf{E}^2(\mathbf{r}, t) + \frac{1}{\mu_0}\mathbf{B}^2(\mathbf{r}, t)\right] \\ &= \frac{1}{2}\int dV\left[\varepsilon_0 E_x^2(z, t) + \frac{1}{\mu_0}B_y^2(z, t)\right]. \end{aligned} \tag{2.7}$$

From Eqs. (2.5) and (2.6) it is straightforward to show (and is left as an exercise) that

$$H = \frac{1}{2}(p^2 + \omega^2 q^2), \tag{2.8}$$

from which it is apparent that a single-mode field is formally equivalent to a harmonic oscillator of unit mass, where the electric and magnetic fields, apart from some scale factors, play the roles of canonical position and momentum.

Every elementary textbook on quantum mechanics discusses the quantization of the one-dimensional harmonic oscillator. Here we take the approach that having identified the canonical variables $q$ and $p$ for the classical system, we simply use the correspondence rule to replace them by their operator equivalents $\hat{q}$ and $\hat{p}$ where operators will be distinguished from $c$-numbers by the caret. These operators must satisfy the canonical commutation relation

$$[\hat{q}, \hat{p}] = i\hbar\hat{I}. \tag{2.9}$$

Henceforth we follow custom and drop the identity operator $\hat{I}$ and write $[\hat{q}, \hat{p}] = i\hbar$. Then the electric and magnetic fields of the single mode become the operators

$$\hat{E}_x(z,t) = \left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} \hat{q}(t)\sin(kz), \tag{2.10}$$

and

$$\hat{B}_y(z,t) = \left(\frac{\mu_0\varepsilon_0}{k}\right)\left(\frac{2\omega^2}{V\varepsilon_0}\right)^{1/2} \hat{p}(t)\,\cos(kz), \tag{2.11}$$

respectively. The Hamiltonian becomes

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \omega^2\hat{q}^2). \tag{2.12}$$

The operators $\hat{q}$ and $\hat{p}$ are Hermitian and therefore correspond to observable quantities. However, it is convenient, and traditional, to introduce the non-Hermitian (and therefore non-observable) annihilation ($\hat{a}$) and creation ($\hat{a}^\dagger$) operators through the combinations

$$\hat{a} = (2\hbar\omega)^{-1/2}(\omega\hat{q} + i\hat{p}) \tag{2.13}$$

$$\hat{a}^\dagger = (2\hbar\omega)^{-1/2}(\omega\hat{q} - i\hat{p}). \tag{2.14}$$

The electric and magnetic field operators then become, respectively,

$$\hat{E}_x(z,t) = \mathcal{E}_0(\hat{a} + \hat{a}^\dagger)\sin(kz), \tag{2.15}$$

$$\hat{B}_y(z,t) = \mathcal{B}_0\frac{1}{i}(\hat{a} - \hat{a}^\dagger)\cos(kz), \tag{2.16}$$

where $\mathcal{E}_0 = (\hbar\omega/\varepsilon_0 V)^{1/2}$ and $\mathcal{B}_0 = (\mu_0/k)(\varepsilon_0\hbar\omega^3/V)^{1/2}$ represent respectively the electric and magnetic fields "per photon". The quotation marks indicate that this is not exactly correct since, as we shall show, the average of these fields for a definite number of photons is zero. Nevertheless, they are useful measures of the fluctuations of the quantized field. Operators $\hat{a}$ and $\hat{a}^\dagger$ satisfy the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{2.17}$$

and, as a result, the Hamiltonian operator takes the form

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{2.18}$$

So far, we have said nothing of the time dependence of the operators $\hat{a}$ and $\hat{a}^\dagger$. For an arbitrary operator $\hat{O}$ having no explicit time dependence, Heisenberg's equation reads

$$\frac{d\hat{O}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{O}]. \tag{2.19}$$

For the annihilation operator $\hat{a}$ this becomes

$$\begin{aligned}\frac{d\hat{a}}{dt} &= \frac{i}{\hbar}[\hat{H}, \hat{a}] \\ &= \frac{i}{\hbar}\left[\hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \hat{a}\right] \\ &= i\omega(\hat{a}^\dagger\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{a}) \\ &= i\omega[\hat{a}, \hat{a}^\dagger]\hat{a} = -i\omega\,\hat{a},\end{aligned} \tag{2.20}$$

which has the solution

$$\hat{a}(t) = \hat{a}\,(0)\,e^{-i\omega t}. \tag{2.21}$$

By the same method, or simply by taking the Hermitian conjugate of Eq. (2.21), we have

$$\hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega t}. \tag{2.22}$$

An alternate way of obtaining these solutions is to write the formal solution to Eq. (2.19) in the form

$$\hat{O}(t) = e^{i\hat{H}t/\hbar}\hat{O}(0)e^{-i\hat{H}t/\hbar} \tag{2.23}$$

and then to use the Baker–Hausdorf lemma [1] to obtain

$$\begin{aligned}\hat{O}(t) = \hat{O}(0) &+ \frac{it}{\hbar}[\hat{H}, \hat{O}\,(0)] \\ &+ \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2[\hat{H}, [\hat{H}, \hat{O}\,(0)]] + \cdots \\ &+ \frac{1}{n!}\left(\frac{it}{\hbar}\right)^n[\hat{H}, [\hat{H}, [\hat{H}, \ldots [\hat{H}, \hat{O}(0)]]]] + \cdots\end{aligned} \tag{2.24}$$

For the operator $\hat{a}$ this results in

$$\begin{aligned}\hat{a}(t) = &\ \hat{a}(0)\left[1 - i\omega\,t - \frac{\omega^2t^2}{2!} + i\frac{\omega^3t^3}{3!} + \cdots\right] \\ &= \hat{a}(0)e^{-i\omega\,t}.\end{aligned} \tag{2.25}$$

The use of this method of solution may seem analogous to the use of a sledge-hammer to crack a nut but will turn out to be quite useful later when we take up cases involving nonlinear interactions.

The operator product $\hat{a}^\dagger\hat{a}$ has a special significance and is called the number operator, which we denote as $\hat{n}$. We let $|n\rangle$ denote an energy eigenstate of the single mode field with the energy eigenvalue $E_n$ such that

$$\hat{H}|n\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|n\rangle = E_n|n\rangle. \tag{2.26}$$

If we multiply Eq. (2.26) by $\hat{a}^\dagger$ then we can generate a new eigenvalue equation

$$\hbar\omega\left(\hat{a}^\dagger\hat{a}\hat{a} + \frac{1}{2}\hat{a}^\dagger\right)|n\rangle = E_n\hat{a}^\dagger|n\rangle. \tag{2.27}$$

Using the commutation relations of Eq. (2.17) we can rewrite this as

$$\hbar\omega\left[(\hat{a}^\dagger\hat{a}-\hat{a}^\dagger)+\frac{1}{2}\hat{a}\right]|n\rangle = E_n\hat{a}^\dagger|n\rangle, \tag{2.28}$$

or

$$\hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)(\hat{a}^\dagger|n\rangle) = (E_n+\hbar\omega)(\hat{a}^\dagger|n\rangle), \tag{2.29}$$

which is the eigenvalue problem for the eigenstate $(\hat{a}^\dagger|n\rangle)$ with the energy eigenvalue $E_n+\hbar\omega$. It should be clear now why $\hat{a}^\dagger$ is called the creation operator: it creates a "quantum" of energy $\hbar\omega$. One could also say, rather loosely, that a "photon" of energy $\hbar\omega$ is created by $\hat{a}^\dagger$. Similarly, if we multiply Eq. (2.26) by the operator $\hat{a}$ and use the commutation relation we obtain

$$\hat{H}\,(\hat{a}\,|n\rangle) = (E_n-\hbar\omega)\,(\hat{a}\,|n\rangle) \tag{2.30}$$

where it is evident that the operator $\hat{a}$ destroys or annihilates one quantum of energy or one photon, the eigenstate $(\hat{a}|n\rangle)$ possessing the energy eigenvalue $E_n-\hbar\omega$. Evidently, repeating the procedure on Eq. (2.30) will result in the lowering of the energy eigenvalue by integer multiples of $\hbar\omega$. But the energy of the harmonic oscillator must always be positive so there must be a lowest-energy eigenvalue, $E_0>0$, with the corresponding eigenstate $|0\rangle$ such that

$$\hat{H}(\hat{a}\,|0\rangle) = (E_0-\hbar\omega)\,(\hat{a}\,|0\rangle) = 0 \tag{2.31}$$

because

$$\hat{a}\,|0\rangle = 0. \tag{2.32}$$

Thus, the eigenvalue problem for the ground state is

$$\hat{H}|0\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)|0\rangle = \frac{1}{2}\hbar\omega|0\rangle \tag{2.33}$$

so that the lowest-energy eigenvalue is the so-called zero-point energy $\hbar\omega/2$. Since $E_{n+1}=E_n+\hbar\omega$, the energy eigenvalues are

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right), \qquad n=0,1,2,\ldots \tag{2.34}$$

(These energy levels are pictured, against the harmonic oscillator potential, in Fig. 2.2.)

Thus for the number operator $\hat{n}=\hat{a}^\dagger\hat{a}$ we have

$$\hat{n}\,|n\rangle = n\,|n\rangle\,. \tag{2.35}$$

These number states must be normalized according to $\langle n|n\rangle=1$. For the state $\hat{a}|n\rangle$ we have

$$\hat{a}\,|n\rangle = c_n|n-1\rangle\,, \tag{2.36}$$

where $c_n$ is a constant to be determined. Then the inner product of $\hat{a}|n\rangle$

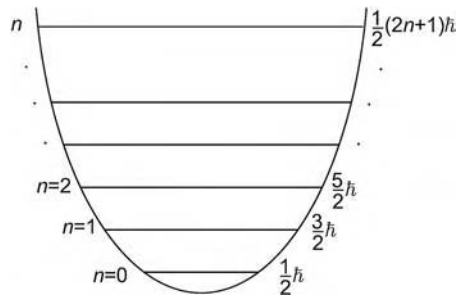


**Fig. 2.2.** The energy levels of a harmonic oscillator of frequency $\omega$.

with itself is

$$\begin{aligned}(\langle n|\,\hat{a}^\dagger)(\hat{a}\,|n\rangle) &= \langle n|\,\hat{a}^\dagger\hat{a}\,|n\rangle = n \\ &= \langle n-1|\,c_n^* c_n\,|n-1\rangle = \left|c_n^2\right|,\end{aligned} \tag{2.37}$$

and thus $|c_n^2| = n$ so we can take $c_n = \sqrt{n}$. Thus

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,. \tag{2.38}$$

Similarly we can show that

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{2.39}$$

From this last result it is straightforward to show that the number states $|n\rangle$ may be generated from the ground state $|0\rangle$ by the repeated action of the creation operator $\hat{a}$:

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\,|0\rangle\,. \tag{2.40}$$

Because $\hat{H}$ and $\hat{n}$ are Hermitian operators, states of different number are orthogonal, i.e. $\langle n'|n\rangle = \delta_{nn'}$ and furthermore, the number states form a complete set, i.e.

$$\sum_{n=0}^{\infty} |n\rangle\,\langle n| = 1. \tag{2.41}$$

The only nonvanishing matrix elements of the annihilation and creation operators are

$$\langle n-1|\,\hat{a}\,|n\rangle = \sqrt{n}\,\langle n-1|n-1\rangle = \sqrt{n} \tag{2.42}$$

$$\langle n+1|\,\hat{a}\,|n\rangle = \sqrt{n+1}\,\langle n+1|n+1\rangle = \sqrt{n+1}\,. \tag{2.43}$$

## 2.2 Quantum fluctuations of a single-mode field

The number state $|n\rangle$ is a state of well-defined energy but it is not a state of well-defined electric field since

$$\langle n|\hat{E}_x(z,t)\,|n\rangle = \mathcal{E}_0 \sin(kz)\,[\langle n|\,\hat{a}\,|n\rangle + \langle n|\,\hat{a}^\dagger\,|n\rangle] = 0, \tag{2.44}$$

i.e. the mean field is zero. But the mean of the square of this field, which contributes to the energy density, is not zero:

$$\begin{aligned}\langle n|\,\hat{E}_x^2(z,t)\,|n\rangle &= \mathcal{E}_0^2\sin^2(kz)\,\langle n|\,\hat{a}^{\dagger 2}+\hat{a}^2+\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger\,|n\rangle \\ &= \mathcal{E}_0^2\sin^2(kz)\,\langle n|\,\hat{a}^{\dagger 2}+\hat{a}^2+2\hat{a}^\dagger\hat{a}+1\,|n\rangle \\ &= 2\mathcal{E}_0^2\sin^2(kz)\left(n+\frac{1}{2}\right).\end{aligned} \tag{2.45}$$

The fluctuations in the electric field may be characterized by the variance

$$\langle(\Delta\hat{E}_x(z,t))^2\rangle = \left\langle\hat{E}_x^2(z,t)\right\rangle - \langle\hat{E}_x(z,t)\rangle^2 \tag{2.46}$$

or by the standard deviation $\Delta E_x = \langle(\Delta E_x(z,t))^2\rangle^{1/2}$, which is sometimes referred to as the uncertainty of the field. For the number state $|n\rangle$ we have

$$\Delta E_x = \sqrt{2\mathcal{E}_0}\,\sin(kz)\left(n+\frac{1}{2}\right)^{1/2}. \tag{2.47}$$

Note that even when $n=0$, the field has fluctuations, the so-called vacuum fluctuations. Now the number states $|n\rangle$ are taken to represent a state of the field containing $n$ photons. Yet as we have seen, the average field is zero. This is all in accordance with the uncertainty principle because the number operator $\hat{n}$ does not commute with the electric field:

$$[\hat{n},\,\hat{E}_x] = \mathcal{E}_0\sin(kz)\,(\hat{a}^\dagger-\hat{a}). \tag{2.48}$$

Thus $\hat{n}$ and $\hat{E}_x$ are complementary quantities for which their respective uncertainties obey the inequality*

$$\Delta n\,\Delta E_x \geq \frac{1}{2}\mathcal{E}_0|\sin(kz)||\langle\hat{a}^\dagger-\hat{a}\rangle|. \tag{2.49}$$

For a number state $|n\rangle$, the right-hand side vanishes but $\Delta n = 0$ as well. If the field were accurately known, then the number of photons would be uncertain. There is a connection here to the notion of the phase of the electric field. In classical physics, the amplitude and phase of a field can be simultaneously well defined. This is not so in quantum mechanics. In fact, the history of the concept of a quantum phase operator is long and contentious and we shall deal with this issue at length later. For now, we simply take a heuristic point of view for which the phase is in some sense complementary to number much in the way that time is complementary to energy. By analogy to the time–energy uncertainty relation there should be a number–phase uncertainty relation of the form

$$\Delta n\,\Delta\phi \geq 1. \tag{2.50}$$

From this, one could argue that for a well-defined (accurately known) phase the photon number is uncertain, whereas for a well-defined photon number, the phase is uncertain and, in fact, the phase is randomly distributed over the range

* Recall that for operators $\hat{A}$ and $\hat{B}$ satisfying $[\hat{A},\,\hat{B}] = \hat{C}$, $\Delta A\,\Delta B \geq \frac{1}{2}|\langle\hat{C}\rangle|$.

$0 < \phi < 2\pi$. We shall examine the issue of the quantum phase in more detail in Section 2.7.

## 2.3 Quadrature operators for a single-mode field

When we explicitly include the time dependence of the electric field operator we have

$$\hat{E}_x = \mathcal{E}_0(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t})\sin(kz) \tag{2.51}$$

where $\hat{a}(0) \equiv \hat{a}$ and $\hat{a}^\dagger(0) \equiv \hat{a}^\dagger$. We now introduce the so-called quadrature operators

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \tag{2.52}$$

$$\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger) \tag{2.53}$$

in terms of which the field operator may be recast as

$$\hat{E}_x(t) = 2\mathcal{E}_0 \sin(kz)[\hat{X}_1 \cos(\omega t) + \hat{X}_2 \sin(\omega t)]. \tag{2.54}$$

It is evident that $\hat{X}_1$ and $\hat{X}_2$ are associated with field amplitudes oscillating out of phase with each other by $90^\circ$ (and hence are in quadrature). Note that $\hat{X}_1$ and $\hat{X}_2$ are essentially the position and momentum operators obtainable from Eqs. (2.13) and (2.14) but scaled to be dimensionless. They satisfy the commutation relation

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2} \tag{2.55}$$

from which it follows that

$$\langle(\Delta\hat{X}_1)^2\rangle\langle(\Delta\hat{X}_2)^2\rangle \geq \frac{1}{16}. \tag{2.56}$$

For the number states, $\langle n|\hat{X}_1|n\rangle = 0 = \langle n|\hat{X}_2|n\rangle$ but

$$\begin{aligned}\langle n|\hat{X}_1^2|n\rangle &= \frac{1}{4}\langle n|\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger|n\rangle \\ &= \frac{1}{4}\langle n|\hat{a}^2 + \hat{a}^\dagger 2 + 2\hat{a}^\dagger\hat{a} + 1|n\rangle \\ &= \frac{1}{4}(2n{+}1)\end{aligned} \tag{2.57}$$

and similarly

$$\langle n|\hat{X}_2^2|n\rangle = \frac{1}{4}(2n{+}1). \tag{2.58}$$

Thus for a number state, the uncertainties in both quadratures are the same and furthermore the vacuum state ($n = 0$) minimizes the uncertainty product since

$$\left\langle(\Delta\hat{X}_1)^2\right\rangle_{\text{vac}} = \frac{1}{4} = \left\langle(\Delta\hat{X}_2)^2\right\rangle_{\text{vac}}. \tag{2.59}$$

Before moving on to multimode fields, we want to stress that the quanta of the single-mode cavity field are the excitations of energy in discrete amounts of $\hbar\omega$. These quanta, universally referred to as photons, are not localized particles (in field theory, there is no position operator for photons) but rather are spread out over the entire mode volume. This is in sharp contrast to the view of photons as "corpuscles" of light as in the old quantum theory.

## 2.4 Multimode fields

The results for the single-mode field confined to a cavity can be generalized to multimode radiation fields. We shall consider these fields to be in free space where it is assumed that there are no sources of radiation and no charges so that Eqs. (2.1–2.4) still hold. The electric and magnetic radiation fields may be given in terms of the vector potential $\mathbf{A}(\mathbf{r}, t)$, which satisfies the wave equation

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = 0 \tag{2.60}$$

and the Coulomb gauge condition

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) = 0 \tag{2.61}$$

where

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \tag{2.62}$$

and

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t). \tag{2.63}$$

The reason for this choice of gauge will be explained in Chapter 4.

We now imagine that free space can be modeled as a cubic cavity of side length $L$ with perfectly reflecting walls. The idea here is that $L$ should be very large compared with the dimensions of anything inside the cube with which the radiation could interact (e.g. atoms). We also assume that $L$ is much larger than the wavelengths of the field. All physical results obtained from such a model should be independent of the size of the cavity as, after all calculations are done, we take $L \to \infty$.

The purpose of the cubical cavity is to allow us to impose periodic boundary conditions on the faces of the cube. For example, in the $x$-direction we shall require that plane waves satisfy the condition

$$e^{ik_x x} = e^{ik_x(x+L)} \tag{2.64}$$

from which it follows that

$$k_x = \left(\frac{2\pi}{L}\right) m_x, \qquad m_x = 0, \pm 1, \pm 2, \ldots \tag{2.65}$$

Similarly for the $y$- and $z$-directions we have

$$k_y = \left(\frac{2\pi}{L}\right) m_y, \qquad m_y = 0, \pm 1, \pm 2, \ldots \tag{2.66}$$

$$k_z = \left(\frac{2\pi}{L}\right) m_z, \qquad m_z = 0, \pm 1, \pm 2, \ldots \tag{2.67}$$

The wave vector is then

$$\mathbf{k} = \frac{2\pi}{L}(m_x, m_y, m_z) \tag{2.68}$$

and its magnitude is related to the frequency $\omega_k$ according to $k = \omega_k/c$. A set of integers $(m_x, m_y, m_z)$ specifies a normal mode of the field (apart from polarization), the number of modes being infinite but denumerable. This is mathematically simpler than dealing with the continuum of modes in free space. The total number of modes in the intervals $\Delta m_x$, $\Delta m_y$, $\Delta m_z$ is

$$\Delta m = \Delta m_x \Delta m_y \Delta m_z = 2\left(\frac{L}{2\pi}\right)^3 \Delta k_x \Delta k_y \Delta k_z, \tag{2.69}$$

where the factor of 2 takes into account the two independent polarizations. In a quasi-continuous limit, wherein we assume that wavelengths are small compared to $L$, we shall have waves densely packed in $k$-space and may therefore approximate $\Delta m$ by the differential

$$dm = 2\left(\frac{V}{8\pi^3}\right) dk_x dk_y dk_z \tag{2.70}$$

where we have set $V = L^3$. In $k$-space spherical polar coordinates

$$\mathbf{k} = k\,(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{2.71}$$

and we have

$$dm = 2\left(\frac{V}{8\pi^3}\right) k^2 dk d\Omega \tag{2.72}$$

where $d\Omega = \sin\theta d\theta d\phi$ is the element of solid angle around the direction of $\mathbf{k}$. By using the relation $k = \omega_k/c$ we can transform Eq. (2.72) into

$$dm = 2\left(\frac{V}{8\pi^3}\right) \frac{\omega^2 k}{c^3} d\omega_k d\Omega. \tag{2.73}$$

Integrating Eq. (2.72) over the solid angle gives us:

$$\left.\begin{array}{l}\text{the numbers of modes}\\ \text{in all directions}\\ \text{in the range } k \text{ to}\\ k + dk\end{array}\right\} = V\frac{k^2}{\pi^2} dk = V\rho_k dk \tag{2.74}$$

where $\rho_k dk$ is the mode density (number of modes per unit volume) and obviously $\rho_k = k^2/\pi^2$. Integrating Eq. (2.73) in the same fashion yields:

$$\left.\begin{array}{l}\text{the numbers of modes}\\ \text{in all directions}\\ \text{in the range } \omega_k \text{ to}\\ \omega_k + d\omega_k\end{array}\right\} = V\frac{\omega_k^2}{\pi^2 c^3}d\omega_k \equiv V\rho(\omega_k)\,d\omega_k \tag{2.75}$$

where $\rho(\omega_k)d\omega_k$ is also the mode density with $\rho(\omega_k) = \omega_k^2/(\pi^2 c^3)$.

The vector potential can be expressed as a superposition of plane waves in the form

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k},s} \mathbf{e}_{\mathbf{k}s}\left[A_{\mathbf{k}s}(t)e^{i\mathbf{k}\cdot\mathbf{r}} + A^*_{\mathbf{k}s}(t)\,e^{-i\mathbf{k}\cdot\mathbf{r}}\right] \tag{2.76}$$

where $A_{\mathbf{k}s}$ is the complex amplitude of the field and where $\mathbf{e}_{\mathbf{k}s}$ is a real polarization vector. The sum over $\mathbf{k}$ simply means the sum over the set of integers $(m_x, m_y, m_z)$ and the sum over $s$ is the sum over the two independent polarizations. These polarizations must be orthogonal

$$\mathbf{e}_{\mathbf{k}s}\cdot\mathbf{e}_{\mathbf{k}s'} = \delta_{ss'} \tag{2.77}$$

and from the gauge condition of Eq. (2.66) must satisfy

$$\mathbf{k}\cdot\mathbf{e}_{\mathbf{k}s} = 0, \tag{2.78}$$

known as the *transversality* condition. The Coulomb gauge is sometimes known as the transverse gauge wherein the polarization is orthogonal to the propagation direction. The polarization vectors $\mathbf{e}_{\mathbf{k}1}$ and $\mathbf{e}_{\mathbf{k}2}$ form a right-handed system such that

$$\mathbf{e}_{\mathbf{k}1}\times\mathbf{e}_{\mathbf{k}2} = \frac{\mathbf{k}}{|\mathbf{k}|} = \boldsymbol{\kappa}. \tag{2.79}$$

In free space, the sum in Eq. (2.76) is replaced by the integral:

$$\sum_k \rightarrow \frac{V}{\pi^2}\int k^2 dk. \tag{2.80}$$

Now from Eqs. (2.60) and (2.61) we obtain for the complex amplitudes $A_{\mathbf{k},s}(t)$ the harmonic oscillator equation

$$\frac{d^2 A_{\mathbf{k}s}}{dt^2} + \omega_k^2 A_{\mathbf{k}s} = 0 \tag{2.81}$$

where $\omega_k = ck$. The solution is

$$A_{\mathbf{k}s}(t) = A_{\mathbf{k}s}e^{-i\omega_k t} \tag{2.82}$$

where we have set $A_{\mathbf{k}s}(0) \equiv A_{\mathbf{k}s}$. From Eqs. (2.62) and (2.63), the electric and magnetic fields respectively are

$$\mathbf{E}(\mathbf{r}, t) = i\sum_{\mathbf{k},s}\omega_k\mathbf{e}_{\mathbf{k}s}\left[A_{\mathbf{k}s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - A^*_{\mathbf{k}s}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right], \tag{2.83}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{i}{c}\sum_{\mathbf{k},s}\omega_k(\boldsymbol{\kappa}\times\hat{\mathbf{e}}_{\mathbf{k}s})\left[A_{\mathbf{k}s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - A^*_{\mathbf{k}s}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right]. \tag{2.84}$$

The energy of the field is given by

$$H = \frac{1}{2}\int_V \left(\varepsilon_0 \mathbf{E}\cdot\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}\right) dV. \tag{2.85}$$

The periodic boundary condition results in

$$\int_0^L e^{\pm ik_x x} dx = \begin{cases} L & k_x = 0 \\ 0 & k_x \neq 0 \end{cases} \tag{2.86}$$

with similar results for the $y$- and $z$-directions. These may collectively be written as

$$\int_V e^{\pm i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} dV = \delta_{\mathbf{k}\mathbf{k}'} V. \tag{2.87}$$

From this we find that the contribution to $H$ from the electric field is

$$\frac{1}{2}\int_V \mathcal{E}_0 \mathbf{E}\cdot\mathbf{E} dV = \varepsilon_0 V \sum_{\mathbf{k}s} \omega_k^2 A_{\mathbf{k}s}(t) A^*_{\mathbf{k}s}(t) - R \tag{2.88}$$

where

$$R = \frac{1}{2}\varepsilon_0 V \sum_{\mathbf{k}ss'} \omega_k^2 \mathbf{e}_{\mathbf{k}s}\cdot\hat{\mathbf{e}}_{-\mathbf{k}s'}[A_{\mathbf{k}s}(t)A_{-\mathbf{k}s'}(t) + A^*_{\mathbf{k}s}(t)A^*_{-\mathbf{k}s'}(t)]. \tag{2.89}$$

To obtain the magnetic contribution we need the vector identity

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}) \tag{2.90}$$

from which we obtain

$$(\mathbf{k}\times\mathbf{e}_{\mathbf{k}s})\cdot(\mathbf{k}\times\mathbf{e}_{\mathbf{k}s'}) = \delta_{ss'} \tag{2.91}$$

$$(\mathbf{k}\times\mathbf{e}_{\mathbf{k}s})\cdot(-\mathbf{k}\times\mathbf{e}_{-\mathbf{k}s'}) = -\mathbf{e}_{\mathbf{k}s}\cdot\mathbf{e}_{-\mathbf{k}s'}. \tag{2.92}$$

Using these results we have

$$\frac{1}{2}\int \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B} dV = \varepsilon_0 V \sum_{\mathbf{k}s} \omega_k^2 A_{\mathbf{k}s}(t) A^*_{\mathbf{k}s}(t) + R. \tag{2.93}$$

Thus adding Eqs. (2.88) and (2.93) we obtain the field energy

$$H = 2\varepsilon_0 V \sum_{\mathbf{k}s} \omega_k^2 A_{\mathbf{k}s}(t) A^*_{\mathbf{k}s}(t) \tag{2.94}$$

$$= 2\varepsilon_0 V \sum_{\mathbf{k}s} \omega_k^2 A_{\mathbf{k}s} A^*_{\mathbf{k}s} \tag{2.95}$$

where we have used Eq. (2.82).

The energy of Eq. (2.95) has a very simple form in terms of the amplitudes $A_{\mathbf{k}s}$. In order to quantize the field, the canonical variables $p_{\mathbf{k}s}$ and $q_{\mathbf{k}s}$ must be introduced. We set

$$A_{\mathbf{k}s} = \frac{1}{2\omega_k(\varepsilon_0 V)^{1/2}}\left[\omega_k q_{\mathbf{k}s} + i p_{\mathbf{k}s}\right], \tag{2.96}$$

$$A^*_{\mathbf{k}s} = \frac{1}{2\omega_k(\varepsilon_0 V)^{1/2}}\left[\omega_k q_{\mathbf{k}s} - i p_{\mathbf{k}s}\right], \tag{2.97}$$

such that upon substitution into Eq. (2.95) we obtain

$$H = \frac{1}{2}\sum_{\mathbf{k}s}\left(p_{\mathbf{k}s}^2 + \omega_k^2 q_{\mathbf{k}s}^2\right), \tag{2.98}$$

each term of which is the energy of a simple harmonic oscillator of unit mass. The quantization of the field proceeds by demanding that the canonical variables become operators satisfying the commutation relations

$$[\hat{q}_{\mathbf{k}s}, \hat{q}_{\mathbf{k}'s'}] = 0 = [\hat{p}_{\mathbf{k}s}, \hat{p}_{\mathbf{k}'s'}] \tag{2.99}$$

$$[\hat{q}_{\mathbf{k}s}, \hat{p}_{\mathbf{k}'s'}] = i\,\hbar\delta_{\mathbf{k}\mathbf{k}'}\delta_{ss'}. \tag{2.100}$$

As for the single-mode field, annihilation and creation operators may be defined as

$$\hat{a}_{\mathbf{k}s} = \frac{1}{(2\,\hbar\omega_k)^{1/2}}\left[\omega_k\hat{q}_{\mathbf{k}s} + i\hat{p}_{\mathbf{k}s}\right], \tag{2.101}$$

$$\hat{a}_{\mathbf{k}s}^\dagger = \frac{1}{(2\,\hbar\omega_k)^{1/2}}\left[\omega_k\hat{q}_{\mathbf{k}s} - i\hat{p}_{\mathbf{k}s}\right], \tag{2.102}$$

which satisfy

$$[\hat{a}_{\mathbf{k}s}, \hat{a}_{\mathbf{k}'s'}] = 0 = \left[\hat{a}_{\mathbf{k}s}^\dagger, \hat{a}_{\mathbf{k}'s'}^\dagger\right] \tag{2.103}$$

$$\left[\hat{a}_{\mathbf{k}s}, \hat{a}_{\mathbf{k}'s'}^\dagger\right] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{ss'}. \tag{2.104}$$

The energy of the field becomes the Hamiltonian operator

$$\hat{H} = \sum_{\mathbf{k}s}\hbar\omega_k\left(\hat{a}_{\mathbf{k}s}^\dagger\hat{a}_{\mathbf{k}s} + \frac{1}{2}\right) \tag{2.105}$$

$$= \sum_{\mathbf{k}s}\hbar\omega_k\left(\hat{n}_{\mathbf{k}s} + \frac{1}{2}\right), \tag{2.106}$$

where

$$\hat{n}_{\mathbf{k}s} = \hat{a}_{\mathbf{k}s}^\dagger\hat{a}_{\mathbf{k}s} \tag{2.107}$$

is the number operator for the mode $\mathbf{k}s$. Each of these modes, being independent of all the others, has an associated set of number eigenstates $|n_{\mathbf{k}s}\rangle$. For the $j$th mode, let $\hat{a}_{\mathbf{k}_j s_j} \equiv \hat{a}_j$, $\hat{a}_{\mathbf{k}_j s_j}^\dagger \equiv \hat{a}_j$ and $\hat{n}_{\mathbf{k}_j s_j} \equiv \hat{n}_j$. The Hamiltonian for the field is then

$$\hat{H} = \sum_j \hbar\omega_j\left(\hat{n}_j + \frac{1}{2}\right) \tag{2.108}$$

and a multimode photon number state is just a product of the number states of all the modes which we write as

$$\begin{aligned} |n_1\rangle\,|n_2\rangle\,|n_3\rangle\ldots &\equiv |n_1, n_2, n_3\ldots\rangle \\ &= |\{n_j\}\rangle. \end{aligned} \tag{2.109}$$

This is an eigenstate of $\hat{H}$ such that

$$\hat{H}|\{n_j\}\rangle = E|\{n_j\}\rangle \tag{2.110}$$

where the eigenvalue $E$ is

$$E = \sum_j \hbar\omega_j \left(n_j + \frac{1}{2}\right). \tag{2.111}$$

Of course, these number states are orthogonal according to

$$\langle n_1, n_2, \ldots | \, n'_1, n'_2, \ldots\rangle = \delta_{n_1 n'_1}\delta_{n_2 n'_2}\ldots \tag{2.112}$$

The action of the annihilation operator of the $j$th mode on the multimode number state is

$$\hat{a}_j|n_1, n_2, \ldots n_j, \ldots\rangle = \sqrt{n_j}\,|n_1, n_2, \ldots n_j - 1, \ldots\rangle. \tag{2.113}$$

Similarly, for the creation operator

$$\hat{a}_j^\dagger|n_1, n_2, \ldots n_j, \ldots\rangle = \sqrt{n_j + 1}\,|n_1, n_2, \ldots n_j + 1, \ldots\rangle. \tag{2.114}$$

The multimode vacuum state is denoted

$$|\{0\}\rangle = |0_1, 0_2, \ldots 0_j, \ldots\rangle \tag{2.115}$$

for which

$$\hat{a}_j|\{0\}\rangle = 0 \tag{2.116}$$

for all $j$. All the number states can be generated from the vacuum according to

$$|\{n_j\}\rangle = \prod_j \frac{\left(\hat{a}_j^\dagger\right)^{n_j}}{\sqrt{n_j!}}\,|\{0\}\rangle\,. \tag{2.117}$$

Upon quantization of the field, the amplitudes $A_{\mathbf{k}s}$ become operators which, from Eqs. (2.98) and (2.101), have the form

$$\hat{A}_{\mathbf{k}s} = \left(\frac{\hbar}{2\omega_k\varepsilon_0 V}\right)^{\frac{1}{2}} \hat{a}_{\mathbf{k}s} \tag{2.118}$$

and thus the quantized vector potential has the form

$$\hat{\mathbf{A}}(\mathbf{r}, t) = \sum_{\mathbf{k}s}\left(\frac{\hbar}{2\omega_k\varepsilon_0 V}\right)^{\frac{1}{2}} \mathbf{e}_{\mathbf{k}s}\left[\hat{a}_{\mathbf{k}s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + \hat{a}_{\mathbf{k}s}^\dagger e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right]. \tag{2.119}$$

The electric field operator is then

$$\hat{\mathbf{E}}(\mathbf{r}, t) = i\sum_{\mathbf{k}s}\left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{\frac{1}{2}} \mathbf{e}_{\mathbf{k}s}\left[\hat{a}_{\mathbf{k}s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - \hat{a}_{\mathbf{k}s}^\dagger e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right] \tag{2.120}$$

while the magnetic field operator is

$$\hat{\mathbf{B}}(\mathbf{r}, t) = \frac{i}{c}\sum_{\mathbf{k}s}(\boldsymbol{\kappa}\times\mathbf{e}_{\mathbf{k}s})\left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{\frac{1}{2}} \mathbf{e}_{\mathbf{k}s}\left[\hat{a}_{\mathbf{k}s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - \hat{a}_{\mathbf{k}s}^\dagger e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right], \tag{2.121}$$

where $\boldsymbol{\kappa} = \mathbf{k}/|\mathbf{k}|$. The annihilation and creation operators appearing in Eqs. (2.119–2.121) are to be understood as Heisenberg picture operators evaluated

at time $t = 0$. As in the single-mode case, the time-dependent annihilation operator for a free field is given by

$$\hat{a}_{\mathbf{k}s}(t) = \hat{a}_{\mathbf{k}s}(0)e^{-i\omega_k t}. \tag{2.122}$$

Thus the electric field, for instance, can be written as

$$\hat{\mathbf{E}}(\mathbf{r}, t) = i \sum_{\mathbf{k}s} \left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{\frac{1}{2}} \mathbf{e}_{\mathbf{k}s}\left(\hat{a}_{\mathbf{k}s}(t)e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{a}^{\dagger}_{\mathbf{k}s}(t)e^{-i\mathbf{k}\cdot\mathbf{r}}\right). \tag{2.123}$$

Sometimes this field is written as

$$\hat{\mathbf{E}}(\mathbf{r}, t) = \hat{\mathbf{E}}^{(+)}(\mathbf{r}, t) + \hat{\mathbf{E}}^{(-)}(\mathbf{r}, t) \tag{2.124}$$

where

$$\hat{\mathbf{E}}^{(+)}(\mathbf{r}, t) = i \sum_{\mathbf{k}s} \left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{\frac{1}{2}} \mathbf{e}_{\mathbf{k}s}\hat{a}_{\mathbf{k}s}(t)e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2.125}$$

and where

$$\hat{\mathbf{E}}^{(-)}(\mathbf{r}, t) = \left[\hat{\mathbf{E}}^{(+)}(\mathbf{r}, t)\right]^{\dagger}. \tag{2.126}$$

$\hat{\mathbf{E}}^{(+)}$ is called the positive frequency part of the field as it contains all terms that oscillate as $e^{-i\omega t}$ for $\omega > 0$, while $\hat{\mathbf{E}}^{(-)}$ is called the negative frequency part. The former is essentially a collective annihilation operator while the latter is a collective creation operator. Similar expressions can be written for the magnetic field and for the vector potential.

In most quantum optical situations, the coupling of the field to matter is through the electric field interacting with a dipole moment or through some nonlinear type of interaction involving powers of the electric field. Thus we shall be mostly interested in the electric field throughout the rest of the book. Furthermore, note that the magnetic field is "weaker" than the electric field by a factor of $1/c$. The field couples to the spin magnetic moment of the electrons and this interaction is negligible for essentially all the aspects of quantum optics that we are concerned with.

For a single-mode plane wave field the electric field is

$$\hat{\mathbf{E}}(\mathbf{r}, t) = i\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{\frac{1}{2}} \mathbf{e}_x[\hat{a}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} - \hat{a}^{\dagger}e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}]. \tag{2.127}$$

In much of quantum optics, the spatial variation of the field over the dimensions of the atomic system may be negligible. For optical radiation, $\lambda$ is on the order of several thousand ångströms so that

$$\frac{\lambda}{2\pi} = \frac{1}{|\mathbf{k}|} \gg |\mathbf{r}_{\text{atom}}|, \tag{2.128}$$

where $|\mathbf{r}_{\text{atom}}|$ is a length characteristic of the size of an atom. Under this condition

$$e^{\pm i\mathbf{k}\cdot\mathbf{r}} \approx 1 \pm i\mathbf{k}\cdot\mathbf{r} \tag{2.129}$$

and we can replace the exponential by unity to obtain

$$\begin{aligned}\hat{\mathbf{E}}(\mathbf{r},t) &\approx \hat{\mathbf{E}}(t)\\ &= i\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{\frac{1}{2}} \mathbf{e}_x[\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}].\end{aligned} \tag{2.130}$$

This approximation, which will be discussed again in Chapter 4, is called the "dipole" approximation.

## 2.5 Thermal fields

As is well known, quantum theory originated with Planck's discovery of the radiation law that now bears his name. We refer, of course, to the law describing the radiation emitted by an ideal object known as a black body – a perfect emitter and absorber of radiation. A black body can be modeled as a cavity (or actually a small hole in the cavity) containing radiation at thermal equilibrium with its walls. The radiation is thus coupled to a heat bath and so is not, unlike in the preceding sections of this chapter, a truly free field. But assuming the coupling is weak, we can, according to the theory of statistical mechanics, treat the field as if it were an isolated system that can be described as a microcanical ensemble.

We consider then, for the moment, a single-mode field in thermal equilibrium with the walls of a cavity at temperature $T$. According to statistical mechanics, the probability $P_n$ that the mode is thermally excited in the $n$th level is

$$P_n = \frac{\exp(-E_n/k_\mathrm{B}T)}{\sum_n \exp(-E_n/k_\mathrm{B}T)} \tag{2.131}$$

where the $E_n$ are given in Eq. (2.131) and where $k_\mathrm{B}$ is the Boltzmann constant ($k_\mathrm{B} = 1.38 \times 10^{-23}$J/K). We introduce here the density operator (whose general properties are described in Appendix A) for the thermal field:

$$\hat{\rho}_\mathrm{Th} = \frac{\exp(-\hat{H}/k_\mathrm{B}T)}{\mathrm{Tr}[\exp(-\hat{H}/k_\mathrm{B}T)]} \tag{2.132}$$

where $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$ and where

$$\begin{aligned}\mathrm{Tr}[\exp(-\hat{H}/k_\mathrm{B}T)] &= \sum_{n=0}^{\infty} \langle n|\exp(-\hat{H}/k_\mathrm{B}T)|n\rangle\\ &= \sum_{n=0}^{\infty} \exp(-E_n/k_\mathrm{B}T) \equiv Z\end{aligned} \tag{2.133}$$

is the partition function. With $E_n = \hbar\omega(n + \frac{1}{2})$

$$Z = \exp(-\hbar\omega/2k_\mathrm{B}T)\sum_{n=0}^{\infty}\exp(-\hbar\omega n/k_\mathrm{B}T). \tag{2.134}$$

Since $\exp(-\hbar\omega/k_{\mathrm{B}}T) < 1$, the sum is a geometric series and thus

$$\sum_{n=0}^{\infty} \exp(-\hbar\omega n/k_{\mathrm{B}}T) = \frac{1}{1-\exp(-\hbar\omega/k_{\mathrm{B}}T)} \tag{2.135}$$

so that

$$Z = \frac{\exp(-\hbar\omega/2k_{\mathrm{B}}T)}{1-\exp(-\hbar\omega/k_{\mathrm{B}}T)}. \tag{2.136}$$

Evidently

$$P_n = \langle n\,|\hat{\rho}_{\mathrm{Th}}|\,n\rangle = \frac{1}{Z}\exp(-E_n/k_{\mathrm{B}}T). \tag{2.137}$$

Also note the density operator itself can be written as

$$\begin{aligned}\hat{\rho}_{\mathrm{Th}} &= \sum_{n'=0}^{\infty}\sum_{n=0}^{\infty} |n'\rangle\langle n'|\hat{\rho}_{\mathrm{Th}}\,|n\rangle\,\langle n| \\ &= \frac{1}{Z}\sum_{n=0}^{\infty}\exp(-E_n/k_{\mathrm{B}}T)\,|n\rangle\,\langle n| \\ &= \sum_{n=0}^{\infty} P_n\,|n\rangle\,\langle n|\,.\end{aligned} \tag{2.138}$$

The average photon number of the thermal field is calculated as

$$\begin{aligned}\bar{n} &= \langle\hat{n}\rangle = \mathrm{Tr}\,(\hat{n}\hat{\rho}_{\mathrm{Th}}) = \sum_{n=0}^{\infty}\langle n\,|\hat{n}\hat{\rho}_{\mathrm{Th}}|\,n\rangle \\ &= \sum_{n=0}^{\infty} n P_n = \exp(-\hbar\omega/2k_{\mathrm{B}}T)\frac{1}{Z}\sum_{n=0}^{\infty} n\,\exp(-\hbar\omega n/k_{\mathrm{B}}T).\end{aligned} \tag{2.139}$$

Noting that with $x = \hbar\omega/k_{\mathrm{B}}T$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} n e^{-nx} &= -\frac{d}{dx}\sum_{n=0}^{\infty} e^{-nx} \\ &= -\frac{d}{dx}\left(\frac{1}{1-e^{-x}}\right) \\ &= \frac{e^{-x}}{(1-e^{-x})^2}.\end{aligned} \tag{2.140}$$

Thus we have

$$\begin{aligned}\bar{n} &= \frac{\exp(-\hbar\omega/k_{\mathrm{B}}T)}{1-\exp(-\hbar\omega/k_{\mathrm{B}}T)} \\ &= \frac{1}{\exp(\hbar\omega/k_{\mathrm{B}}T)-1}.\end{aligned} \tag{2.141}$$

Evidently

$$\bar{n} \approx \begin{cases} \dfrac{k_{\mathrm{B}}T}{\hbar\omega} & (k_{\mathrm{B}}T \gg \hbar\omega) \\ \dfrac{\hbar\omega}{k_{\mathrm{B}}T} & (k_{\mathrm{B}}T \ll \hbar\omega) \end{cases}. \tag{2.142}$$

At room temperatures, the average number of photons at optical frequencies is very small (on the order of $10^{-40}$) . At the surface temperature of the sun (6000 K) and at the frequency of yellow light ($6 \times 10^{14}$ Hz, $\lambda = 500$ nm) the average photon number is about $10^{-2}$. On the other hand, the average photon number rapidly increases with increasing wavelength. Again at room temperature, $\bar{n} \simeq 1$ for $\lambda$ in the range $\lambda$ =10–100 μm. In the microwave part of the spectrum, $\bar{n} \gg 1$.

From Eq. (2.141) it follows that

$$\exp(-\hbar\omega/k_{\mathrm{B}}T) = \frac{\bar{n}}{1+\bar{n}} \tag{2.143}$$

and from Eqs. (2.137) and (2.138) it follows that $\hat{\rho}_{\mathrm{Th}}$ can be written in terms of $\bar{n}$ as

$$\hat{\rho}_{\mathrm{Th}} = \frac{1}{1+\bar{n}} \sum_{n=0}^{\infty} \left( \frac{\bar{n}}{1+\bar{n}} \right)^n |n\rangle \langle n| . \tag{2.144}$$

The probability of finding $n$ photons in the field is given in terms of $\bar{n}$ as

$$P_n = \frac{\bar{n}^n}{(1+\bar{n})^{n+1}}. \tag{2.145}$$

In Fig. 2.3 we plot $P_n$ versus $n$ for two different values of $\bar{n}$. It is clear in both cases that the most probable photon number is the vacuum, $P_n$ decreasing monotonically with $n$. There is obviously nothing special about $P_n$ for $n$ near or at $\bar{n}$ (which need not be integer).

The fluctuations in the average photon number are given as

$$\langle (\Delta n)^2 \rangle = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 . \tag{2.146}$$

It can be shown, in a manner similar to the derivation of, $\bar{n}$ that

$$\begin{aligned} \langle \hat{n}^2 \rangle &= \mathrm{Tr}\left( \bar{n}^2 \hat{\rho}_{\mathrm{Th}} \right) \\ &= \bar{n} + 2\bar{n}^2 \end{aligned} \tag{2.147}$$

so that

$$\langle (\Delta n)^2 \rangle = \bar{n} + \bar{n}^2 \tag{2.148}$$

from which it is apparent that the *fluctuations* of $\hat{n}$ are larger than the *average* $\bar{n}$. The root-mean-square (r.m.s.) deviation is

$$\Delta n = (\bar{n} + \bar{n}^2)^{1/2} \tag{2.149}$$

which for $\bar{n} \gg 1$ is approximately

$$\Delta n \approx \bar{n} + \frac{1}{2}. \tag{2.150}$$

The relative uncertainty is given by the ratio $\Delta n/\bar{n}$, which is approximately 1 for $\bar{n} \gg 1$ and is approximately $1/\sqrt{\bar{n}}$ for $\bar{n} \ll 1$. Obviously, $\Delta n/\bar{n} \to \infty$ as $\bar{n} \to 0$.

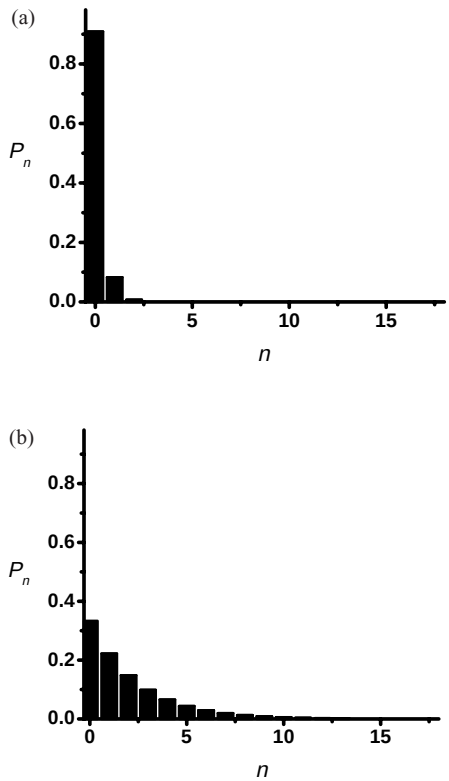


**Fig. 2.3.** Thermal photon number distributions for (a) $\bar{n} = 0.1$, and (b) $\bar{n} = 2$.

The average energy of the photons in the cavity is $\hbar\omega\,\bar{n}$. Planck's radiation law is obtained by multiplying the average energy of the photons by the density of modes per unit interval in $\omega$ in a unit volume, $\rho(\omega) = \omega^2/\pi^2c^3$ (where the two independent polarization directions have been taken into account), to obtain the average energy density per unit interval in $\omega$ as

$$\begin{aligned}\bar{U}(\omega) &= \hbar\omega\,\bar{n}\,\rho(\omega) \\ &= \frac{\hbar\omega^3}{\pi^2c^3}\,\frac{1}{\exp(\hbar\omega/k_{\mathrm{B}}T)-1}.\end{aligned} \tag{2.151}$$

For $k_{\mathrm{B}}T \gg \hbar\omega$, this takes the simpler form

$$\bar{U}(\omega) \approx \frac{\omega^3 k_{\mathrm{B}}T}{\pi^2c^3} \qquad (k_{\mathrm{B}}T \gg \hbar\omega) \tag{2.152}$$

which is known as Rayleigh's law. This is sometimes called the "classical limit" obtained from Planck's law of Eq. (2.150) by setting $\hbar \to 0$. (We note, however, that setting $\hbar$ to zero is not a well-defined limit: $\hbar$ contains dimensional information.) On the other hand, for low temperatures where $k_{\mathrm{B}}T \ll \hbar\omega$ we obtain

$$\bar{U}(\omega) \approx \frac{\hbar\omega^3}{\pi^2c^3}\exp\left(-\frac{\hbar\omega}{k_{\mathrm{B}}T}\right) \qquad (k_{\mathrm{B}}T \ll \hbar\omega)\,. \tag{2.153}$$

This is Wien's law. By differentiation, it follows that $\bar{U}(\omega)$ has a maximum at

$$\omega_{\max} = \frac{2.8 k_{\mathrm{B}} T}{\hbar} = \frac{2\pi c}{\lambda_{\max}} \tag{2.154}$$

which is Wien's displacement law.

The average energy per unit volume is obtained by integrating over all frequencies:

$$\begin{aligned} \bar{U} &= \int_0^\infty \bar{U}(\omega) d\omega \\ &= \frac{\hbar^2}{\pi^2 c^3} \int_0^\infty \frac{\omega^3}{\exp(\hbar\omega/k_{\mathrm{B}} T) - 1}\, d\omega \\ &= \frac{\pi^2 k_{\mathrm{B}}^4 T^4}{15 c^3 \hbar^3}. \end{aligned} \tag{2.155}$$

This is the Stefan–Boltzmann law.

## 2.6 Vacuum fluctuations and the zero-point energy

We have seen that the quantized radiation field fluctuates. For a single-mode field the fluctuations in the electric field strength are given by Eq. (2.47). With the field mode in the vacuum state $|0\rangle$, the r.m.s. fluctuation of the field strength is

$$\Delta E_x = \mathcal{E}_0 \sin(kz). \tag{2.156}$$

These vacuum fluctuations and the zero-point energy have a common origin in the noncommutability of the operators $\hat{a}$ and $\hat{a}^\dagger$. The zero-point energy and the vacuum fluctuations actually present severe problems in quantum field theory. The most glaring of these comes about as follows: the universe contains an infinite number of radiation modes, each with a finite zero-point energy, $\hbar\omega/2$. The total zero-point energy (ZPE) of the universe then is

$$E_{\mathrm{ZPE}} = \frac{\hbar}{2} \sum_\omega \omega \to \infty \tag{2.157}$$

unless somehow the high-frequency modes are excluded. It is frequently said that energy differences only are important but this cannot quite be the whole story because according to general relativity, it is the total energy that counts, not just energy differences [2]. Other "infinites" appearing in the theory of quantum electrodynamics have been "swept under the rug" through the renormalization procedure, but this particular one still sticks out like a sore thumb. In fact, the vacuum energy and fluctuations actually give rise to observable effects. For example, spontaneous emission, which generates most of the visible light around us as thermal radiation, is a direct result of the vacuum fluctuations as we will show in Chapter 4. The ZPE gives rise to at least two other effects, one being the Lamb shift and the other being the Casimir effect.

The Lamb shift is a discrepancy between experiment and the Dirac relativistic theory of the hydrogen atom. The theory predicts that the $2^2S_{1/2}$ and $2^2P_{1/2}$ levels should be degenerate. Early optical work suggested that these states were not degenerate but separated by about 0.033 cm. By using an elegant combination of atomic beam and microwave techniques, Lamb and Retherford [3] showed that the $2^2S_{1/2}$ state has a higher energy than the $2^2P_{1/2}$ state by the equivalent of about 1000 MHz. In 1947, Bethe [4] explained this splitting as being caused by the interaction of the bound electron with the ZPE. Here we present a simple intuitive interpretation originally given by Welton in 1948 [5]. For this calculation only, we follow the lead of Welton who uses cgs units.

Each mode contains ZPE $h\nu/2$, where $\nu = \omega/2\pi$. The number of modes in a cavity of volume $V$ with frequency between $\nu$ and $\nu + d\nu$ is $(8\pi/c^3)\nu^2 d\nu$. Thus the ZPE field energy is

$$\begin{aligned}\left(\frac{8\pi}{c^3}\nu^2 d\nu V\right)\frac{1}{2}h\nu &= \frac{1}{8\pi}\int_V \left(E_\nu^2 + B_\nu^2\right) dV \\ &= \frac{1}{8\pi}E_\nu^2 V\end{aligned} \tag{2.158}$$

where $E_\nu$ is the amplitude of the electric field component of frequency $\nu$. Thus

$$E_\nu^2 = \frac{32\pi^2}{c^3}h\nu^3 d\nu. \tag{2.159}$$

The electron bound in the hydrogen atom interacts with the fluctuating zero-point electric field and with the Coulomb potential of the proton $-e^2/r$. If $r$ represents the electron's "standard orbit" and $\Delta r$ represents the fluctuations from this orbit, then the change in the potential energy is $\Delta V = V(r + \Delta r) - V(r)$ which by Taylor's theorem gives

$$\begin{aligned}\Delta V = {} & \Delta x\frac{\partial V}{\partial x} + \Delta y\frac{\partial V}{\partial y} + \Delta z\frac{\partial V}{\partial z} \\ & + \frac{1}{2}(\Delta x)^2\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}(\Delta y)^2\frac{\partial^2 V}{\partial y^2} + \frac{1}{2}(\Delta z)^2\frac{\partial^2 V}{\partial z^2} + \cdots\end{aligned} \tag{2.160}$$

Since the fluctuations are isotropic $\langle\Delta x\rangle = \langle\Delta y\rangle = \langle\Delta z\rangle = 0$ and $\langle(\Delta x)^2\rangle = \langle(\Delta y)^2\rangle = \langle(\Delta z)^2\rangle = \langle(\Delta r)^2\rangle/3$. Then

$$\langle\Delta V\rangle = \frac{1}{6}\langle(\Delta r)^2\rangle\nabla^2 V. \tag{2.161}$$

For the atomic state $|nlm_l\rangle$ the energy shift to first order is

$$\begin{aligned}\Delta E &= \langle nlm_l|\,\langle\Delta V\rangle\,|nlm_l\rangle \\ &= \frac{1}{6}\langle(\Delta r)^2\rangle\,\langle nlm_l|\,\nabla^2 V\,|nlm_l\rangle\,.\end{aligned} \tag{2.162}$$

With $V = -e^2/r$ and $\nabla^2(1/r) = -4\pi\delta(r)$ we obtain

$$\langle nlm_l|\,\nabla^2 V\,|nlm_l\rangle = 4\pi e^2\left|\psi_{nml_l}(r=0)\right|^2. \tag{2.163}$$

All the atomic wave functions of nonrelativistic quantum theory vanish at the origin except for the s-states with $l = 0$, where

$$|\psi_{n00}(r = 0)^2| = \frac{1}{\pi n^3 a_0^3} \tag{2.164}$$

where $a_0$ is the Bohr radius. For p-states the wave function vanishes at the origin and therefore so does the energy shift. To obtain $\langle(\Delta r)^2\rangle$ we assume that the important field frequencies greatly exceed the atomic resonance frequencies, the lower frequencies being shielded by the atomic binding and unable to influence the motion of the electrons. The displacement $\Delta r_\nu$ induced with frequency between $\nu$ and $\nu + d\nu$ is determined by

$$\frac{d^2 \Delta r_\nu}{dt^2} = \frac{eE_\nu}{m} \exp(2\pi\nu i t). \tag{2.165}$$

The solution is

$$\Delta r_\nu = -\frac{e^2}{m} \frac{E_\nu}{4\pi^2\nu^2} \exp(2\pi\nu i t). \tag{2.166}$$

The mean square displacement induced by these modes is

$$\langle(\Delta r_\nu)^2\rangle = -\frac{e^2}{m^2} \frac{E_\nu^2}{32\pi^4\nu^4} = \frac{e^2 h}{\pi^2 m^2 c^3} \frac{d\nu}{\nu}. \tag{2.167}$$

The s-state energy shift obtained by summing over all frequencies is

$$\Delta E = \frac{2}{3} \left(\frac{e^2}{\hbar c}\right)^2 \left(\frac{\hbar}{mc}\right) \frac{hc}{\pi^2 n^3 a_0^3} \int \frac{d\nu}{\nu} \tag{2.168}$$

where $e^2/\hbar c$ is the fine structure constant, and $\hbar/mc$ is the Compton wavelength of the electron. The integral is divergent but may be cut off at both high and low frequencies. At low frequencies, the atom does not respond to the fluctuating field, the frequency of the electron's orbit $\nu_0 = e^2/\hbar a_0^3 n^3$ being the natural cut-off. At high frequencies, relativistic effects show up in the electron's motion. But the preceding analysis is nonrelativistic so that

$$\frac{v}{c} = \left(\frac{p/m}{c}\right) = \frac{pc}{mc^2} = \frac{\hbar k}{mc} < 1 \tag{2.169}$$

which restricts $k$ to less than $(mc/\hbar)$ and angular frequencies to less than $mc^2/\hbar$ in the integral of Eq. (2.168). Thus for the $2^2S_{1/2}$ state of hydrogen, with $a_0 = \hbar^2/me^2$, the energy shift is

$$\Delta E = \frac{1}{6\pi} \left(\frac{e^2}{\hbar c}\right)^3 \frac{me^4}{\hbar^2} \log\left(\frac{mc^2}{h\nu_0}\right) \tag{2.170}$$

which gives $\Delta E/h \sim 1000$ MHz. The $2^2P_{1/2}$ state is unaffected to this order.

The Casimir effect [6], in the simplest version, is the occurrence of a force between two parallel perfectly conducting plates owing to a change in the ZPE

resulting from the boundary conditions on the plates. We will follow the discussion of Milonni and Shih [7] to show how this force arises.

Consider a parallelepiped with perfectly conducting walls of lengths $L_x = L_y = L$ and $L_z = d$. The boundary conditions on the walls restrict the allowed frequencies to those given by

$$\omega_{lmn} = \pi c \left( \frac{l^2}{L^2} + \frac{m^2}{L^2} + \frac{n^2}{d^2} \right)^{\frac{1}{2}} \tag{2.171}$$

where $l, m, n$ take on non-negative integer values. If there is no box, all frequencies are allowed. The ZPE in the box is

$$E_0(d) = \sum\nolimits'_{l,m,n} (2) \frac{1}{2} \hbar \omega_{lmn} \tag{2.172}$$

where the factor of two accounts for two independent polarizations and where the prime on the summation sign means that the two is to be removed if one of the integers $l$, $m$, $n$ is zero, as there is only one independent polarization for that case. We shall be interested only in the case where $L \gg d$ so that we can replace the sums of $l$ and $m$ by integrals to write

$$E_0(d) = \frac{\hbar c L^2}{\pi} \sum_{n=0}^{\infty} \int_0^{\infty} dx \int_0^{\infty} dy \left( x^2 + y^2 + \frac{\pi^2 n^2}{d^2} \right)^{\frac{1}{2}}. \tag{2.173}$$

On the other hand, if $d$ is arbitrarily large, the sum over $n$ can be replaced by an integral so that

$$E_0(\infty) = \frac{\hbar c L^2}{\pi^2} \frac{d}{\pi} \int_0^{\infty} dx \int_0^{\infty} dy \int_0^{\infty} dz (x^2 + y^2 + z^2)^{\frac{1}{2}}. \tag{2.174}$$

When the plates are separated by the distance $d$, the potential energy of the system is just $U(d) = E_0(d) - E_0(\infty)$, which is the energy required to bring the plates from infinity to a distance $d$. Thus

$$\begin{aligned} U(d) = \frac{L^2 \hbar c}{\pi} \Bigg[ & \sum_{n=0}^{\infty} \int_0^{\infty} dx \int_0^{\infty} dy \left( x^2 + y^2 + \frac{\pi^2 n^2}{d^2} \right)^{\frac{1}{2}} \\ & - \frac{d}{\pi} \int_0^{\infty} dx \int_0^{\infty} dy \int_0^{\infty} dz (x^2 + y^2 + z^2)^{\frac{1}{2}} \Bigg]. \end{aligned} \tag{2.175}$$

Transforming to polar coordinates in the $x$–$y$ plane we have

$$\begin{aligned} U(d) = \frac{L^2 \hbar c}{\pi^2} \frac{\pi}{2} \Bigg[ & \sum_{n=0}^{\infty} \int_0^{\infty} dr\, r \left( r^2 + \frac{n^2 \pi^2}{d^2} \right)^{\frac{1}{2}} \\ & - \frac{d}{\pi} \int_0^{\infty} dz \int_0^{\infty} dr\, r (r^2 + z^2)^{\frac{1}{2}} \Bigg]. \end{aligned} \tag{2.176}$$

Making the change of variable $w = r^2$

$$U(d) = \frac{L^2\hbar c}{4\pi^2}\frac{\pi^3}{d^3}\left[\sum_{n=0}^{\infty}\int_0^{\infty} dw\ (w+n^2)^{\frac{1}{2}} - \int_0^{\infty} dz \int_0^{\infty} dw (w+z^2)^{\frac{1}{2}}\right]. \tag{2.177}$$

Both ZPEs in Eq. (2.177) are infinite but the difference is finite. We can write this as

$$U(d) = \frac{\pi^2\hbar c}{4d^3}L^2\left[\frac{1}{2}F(0) + \sum_{n=1}^{\infty}F(n) - \int_0^{\infty} dz F(n)\right] \tag{2.178}$$

where

$$F(u) \equiv \int_0^{\infty} dw (w+u^2)^{\frac{1}{2}} \tag{2.179}$$

with $u = n$ or $z$. The difference can be estimated by the Euler–Maclaurin formula:

$$\sum_0^{\infty} F(n) - \int_0^{\infty} dz F(z) = -\frac{1}{2}F(0) - \frac{1}{12}F'(0) + \frac{1}{720}F'''(0)\ldots \tag{2.180}$$

since $F'(z) = 2z^2$, $F'(0) = 0$, $F'''(0) = -4$, and all higher-order derivations vanish. Thus

$$U(d) = \frac{\pi^2\hbar c}{4d^3}L^3\left(-\frac{4}{720}\right) = -\frac{\pi^2\hbar c}{720d^3}L^2. \tag{2.181}$$

This implies that the force per unit area between the plates is given by

$$F(d) = U'(d) = -\frac{\pi^2\hbar c}{240d^4} \tag{2.182}$$

which is the Casimir force. The existence of this force was confirmed by experiments carried out by Sparnaay [8] in 1957.

## 2.7 The quantum phase

Consider now a light wave as pictured in the classical electromagnetic theory. The electric field of a single mode can be written as

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \mathbf{e}_x E_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)\\ &= \mathbf{e}_x \frac{1}{2}E_0\{\exp[i\mathbf{k}\cdot\mathbf{r} - \omega t + \phi] + \exp[-i\mathbf{k}\cdot\mathbf{r} - \omega t + \phi]\}\end{aligned} \tag{2.183}$$

where $E_0$ is the amplitude of the field and $\phi$ is its phase. Compare this with Eq. (2.127). The equations would be quite similar if the operator $\hat{a}$ could be

factored into polar form. The earliest attempt at such a decomposition appears to be due to Dirac [9] who factored the annihilation and creation operators according to

$$\hat{a} = e^{i\hat{\phi}}\sqrt{\hat{n}} \tag{2.184}$$

$$\hat{a}^\dagger = \sqrt{\hat{n}}\, e^{-i\hat{\phi}} \tag{2.185}$$

where $\hat{\phi}$ was to be interpreted as a Hermitian operator for phase. From the fundamental commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ it follows that

$$e^{i\hat{\phi}}\hat{n}e^{-i\hat{\phi}} - \hat{n} = 1 \tag{2.186}$$

or

$$e^{i\hat{\phi}}\hat{n} - \hat{n}e^{i\hat{\phi}} = e^{i\hat{\phi}}. \tag{2.187}$$

By expanding the exponentials one can see that Eq. (2.186) is satisfied as long as

$$[\hat{n}, \hat{\phi}] = i. \tag{2.188}$$

It thus appears that number and phase are complementary observables and therefore the fluctuations in these quantities should satisfy the uncertainty relation $\Delta n\, \Delta\phi \geq \frac{1}{2}$.

Unfortunately, things are not so simple. To see that something is quite wrong with the above, consider the matrix element of the commutator for the arbitrary number states $|n\rangle$ and $|n'\rangle$:

$$\langle n'|[\hat{n}, \hat{\phi}]|n\rangle = i\delta_{nn'}. \tag{2.189}$$

Expanding the left side results in

$$(n' - n)\langle n'|\hat{\phi}|n\rangle = i\delta_{nn'} \tag{2.190}$$

which contains an obvious contradiction in the case when $n' = n$ (giving $0 = i$). The Dirac approach fails because of the underlying assumption that a Hermitian phase operator $\hat{\phi}$ actually exists. There are in fact two reasons for the failure of the Dirac approach. If $\hat{\phi}$ exists as a Hermitian operator, then $\exp(i\hat{\phi})$ should be a unitary operator. From Eqs. (2.184) and (2.185) we should have

$$e^{i\phi} = \hat{a}(\hat{n})^{-\frac{1}{2}} \tag{2.191}$$

$$e^{-i\hat{\phi}} = (\hat{n})^{-\frac{1}{2}}\hat{a}^\dagger = (e^{i\hat{\phi}})^\dagger. \tag{2.192}$$

Now

$$(e^{i\hat{\phi}})^\dagger(e^{i\hat{\phi}}) = 1 \tag{2.193}$$

but

$$(e^{i\hat{\phi}})(e^{i\hat{\phi}})^\dagger = \hat{a}\frac{1}{\hat{n}}\hat{a}^\dagger \neq 1. \tag{2.194}$$

So, in fact, $\exp(i\hat{\phi})$ is not a unitary operator from which it follows that $\hat{\phi}$ is not Hermitian. The root of the problem is that the operator $\hat{n}$ has a spectrum bounded from below; it does not include the negative integers. One way to fix this is simply

to include negative integers into the spectrum. The negative number states are of course nonphysical but, as Barnett and Pegg have shown [10], it is possible to construct unitary operators of the form

$$e^{i\hat{\phi}} \equiv \sum_{n=-\infty}^{\infty} |n\rangle\langle n+1| \tag{2.195}$$

$$(e^{i\hat{\phi}})^{\dagger} = e^{-i\hat{\phi}} = \sum_{n=-\infty}^{\infty} |n+1\rangle\,\langle n| \tag{2.196}$$

where it is easy to see that

$$e^{i\hat{\phi}}(e^{i\hat{\phi}})^{\dagger} = (e^{i\hat{\phi}})^{\dagger}e^{i\hat{\phi}} = 1. \tag{2.197}$$

It must be understood that the introduction of the negative number states is only formal and that they are decoupled from the positive (physical) number states. No new predictions arise from including these negative number states. But by using Eq. (2.194) it is easy to show that Eq. (2.186) still holds from which Eq. (2.187) follows and we are still back to the contradiction encountered in Eq. (2.189).

This brings us to the second problem, which is connected with the fact that $\hat{\phi}$ is supposed to be an angle operator. The situation is very similar to the more familiar problem of angular momentum of a particle moving in the $x$–$y$ plane with angular momentum about the $z$-axis. If $\phi$ is the azimuthal angle defined as

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{2.198}$$

(modulo $2\pi$), then the orbital angular momentum is

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = i\,\hbar\frac{\partial}{\partial\phi} \tag{2.199}$$

from which it follows that

$$[\phi, \hat{L}_z] = i. \tag{2.200}$$

But $\hat{L}_z$ is Hermitian only in a space of periodic functions, i.e. wave functions of the form

$$\psi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}, \qquad m = 0, \pm1, \pm2, \ldots \tag{2.201}$$

where

$$\hat{L}_z\psi_m(\phi) = \hbar m\psi_m(\phi). \tag{2.202}$$

(Note that the spectrum of $\hat{L}_z$ contains negative integers.) But $\phi$ itself is not a periodic function, having the range $-\infty < \phi < \infty$. In fact the fluctuations $\Delta\phi$ can be greater than $2\pi$ and for a number state, Eq. (2.199) would seem to imply that $\Delta\phi \to \infty$, a nonsensical behavior. A possible solution to this problem is to introduce a periodic coordinate $\Phi(\phi)$ [10] behaving in a discontinuous fashion according to Fig. 2.4.

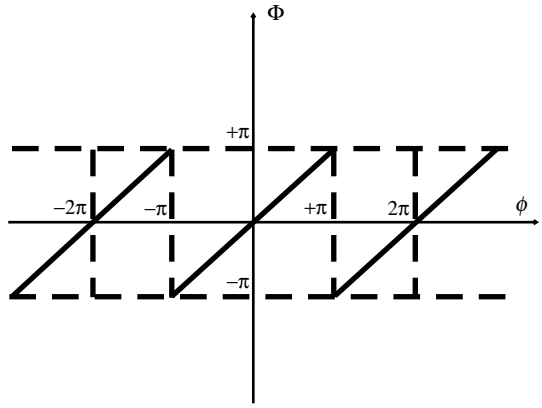


**Fig. 2.4.** The periodic function Φ, arbitrarily chosen to be discontinuous.

The problem with this is that the commutation relation is no longer of the canonical form but rather is

$$[\Phi, \hat{L}_z] = i \left\{ 1 - 2\pi \sum_{n=-\infty}^{\infty} \delta \left[ \phi - (2n+1)\pi \right] \right\}. \tag{2.203}$$

The delta functions arise from the discontinuities of $\Phi$ and they will occur no matter where the discontinuities are placed. One of the delta functions would always be present in any particular $2\pi$ interval so no sensible uncertainty relation can be formulated.

Over the years, there have been many attempts to create a formalism for the description of the quantum phase that in one way or another overcomes the obstacles discussed above. These schemes have been reviewed extensively in the recent literature. For pedagogical reasons, we first introduce the approach due to Susskind and Glogover [11] and its improvements by Carruthers and Nieto [12] wherein a kind of one-sided "unitary" phase operator (actually an operator analogous to an exponential phase factor) is introduced. Eigenstates of this operator, the phase eigenstates, are introduced and then used to construct a phase distribution for an arbitrary state of the field [13]. Various averages over the phase may be calculated from this distribution. Results obtained from this procedure are identical to those obtained by the scheme of Pegg and Barnett [14] who factor the annihilation operator à la Dirac but in a truncated Hilbert space. After the calculation of expectation values in this truncated space the dimension of the space is allowed to go to infinity. By using the phase eigenstates we shall avoid such calculational complications.

The Susskind–Glogower (SG) operators are defined by the relations

$$\hat{E} \equiv (\hat{n}+1)^{-\frac{1}{2}} \hat{a} = (\hat{a}\hat{a}^\dagger)^{-\frac{1}{2}} \hat{a} \tag{2.204}$$

$$\hat{E}^\dagger = \hat{a}^\dagger (\hat{n}+1)^{-\frac{1}{2}} = \hat{a}^\dagger (\hat{a}\hat{a}^\dagger)^{-\frac{1}{2}} \tag{2.205}$$

where $\hat{E}$ and $\hat{E}^\dagger$ are to be the analogs of the phase factors $\exp(\pm i\phi)$. These $\hat{E}$ operators are sometimes called "exponential" operators and, from context, should not be confused with the field operators. When applied to the number states $|n\rangle$ they yield

$$\begin{aligned} \hat{E}\,|n\rangle &= |n-1\rangle \qquad \text{for } n \neq 0 \\ &= 0 \qquad\qquad\ \ \text{for } n = 0, \end{aligned} \tag{2.206}$$

$$\hat{E}^\dagger|n\rangle = |n+1\rangle. \tag{2.207}$$

From this, it is easy to see that useful and equivalent expressions for these exponential operators are

$$\hat{E} = \sum_{n=0}^{\infty} |n\rangle\langle n+1| \qquad \hat{E}^\dagger = \sum_{n=0}^{\infty} |n+1\rangle\langle n|. \tag{2.208}$$

It is easy to show that

$$\hat{E}\hat{E}^\dagger = \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} |n\rangle\langle n+1|n'+1\rangle\langle n'| = \sum_{n=0}^{\infty} |n\rangle\langle n| = 1 \tag{2.209}$$

but that

$$\hat{E}^\dagger\hat{E} = \sum_{n=0}^{\infty}\sum_{n=0}^{\infty} |n+1\rangle\langle n|n'\rangle\langle n'+1| = \sum_{n=0}^{\infty} |n+1\rangle\langle n+1| = 1 - |0\rangle\langle 0|. \tag{2.210}$$

The presence of the projection operator $|0\rangle\langle 0|$ spoils the unitarity of $\hat{E}$. But for a state with average photon number $\bar{n} \geq 1$, the contribution from the vacuum state will be small and $\hat{E}$ will be approximately unitary.

Of course, $\hat{E}$ and $\hat{E}^\dagger$ themselves are not observables but the operators

$$\hat{C} \equiv \frac{1}{2}(\hat{E} + \hat{E}^\dagger), \qquad \hat{S} \equiv \frac{1}{2i}(\hat{E} - \hat{E}^\dagger) \tag{2.211}$$

are the obvious analogs of $\cos\phi$ and $\sin\phi$. These operators are Hermitian and satisfy the commutation relation

$$[\hat{C}, \hat{S}] = \frac{1}{2} i\,|0\rangle\langle 0| \tag{2.212}$$

and thus commute for all states but the vacuum. Furthermore, the quantum form of the familiar trigonometric identity becomes

$$\hat{C}^2 + \hat{S}^2 = 1 - \frac{1}{2}|0\rangle\langle 0| \tag{2.213}$$

where once again we see the spoiling effect of the vacuum. It can also be shown that

$$[\hat{C}, \hat{n}] = i\hat{S} \quad \text{and} \quad [\hat{S}, \hat{n}] = -i\hat{C} \tag{2.214}$$

the first of which is similar to Eq. (2.48). The uncertainty relations obeyed in these cases respectively are

$$(\Delta n)(\Delta C) \geq \frac{1}{2}|\langle \hat{S} \rangle| \tag{2.215}$$

and

$$(\Delta n)(\Delta S) \geq \frac{1}{2}|\langle \hat{C} \rangle| \tag{2.216}$$

where the $\Delta$ means the root-mean-square deviation.

In the case of number states $|n\rangle$, $\Delta n = 0$,

$$\langle n|\, \hat{C}\, |n\rangle = \langle n|\, \hat{S}\, |n\rangle = 0 \tag{2.217}$$

and

$$\langle n|\, \hat{C}^2\, |n\rangle = \langle n|\, \hat{S}^2\, |n\rangle = \begin{cases} \frac{1}{2}, & n \geq 1 \\ \frac{1}{4}, & n = 0 \end{cases}. \tag{2.218}$$

Thus for $n \geq 1$ the uncertainties in $\hat{C}$ and $\hat{S}$ are

$$\Delta C = \Delta S = \frac{1}{\sqrt{2}} \tag{2.219}$$

which would seem to correspond to a phase angle equally likely to have any value in the range 0 to $2\pi$. (Note that this will not be true for the vacuum state!) In any case, the right-hand sides of Eqs. (2.214) and (2.215) are zero for a number state as required in order that the uncertainty relations be satisfied.

The eigenstates $|\phi\rangle$ of the exponential operator satisfying the eigenvalue equation

$$\hat{E}\, |\phi\rangle = e^{i\phi}\, |\phi\rangle \tag{2.220}$$

are given by

$$|\phi\rangle = \sum_{n=0}^{\infty} e^{in\phi}\, |n\rangle\,. \tag{2.221}$$

These states are not normalizable nor are they orthogonal as the scalar product of $|\phi\rangle$ and $|\phi'\rangle$ is not the delta function $\delta(\phi - \phi')$. By virtue of the fact that

$$\int_0^{2\pi} e^{i(n-n')\phi} d\phi = 2\pi\, \delta_{nn'} \tag{2.222}$$

it is easy to show that the phase eigenstates resolve to unity according to

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi\, |\phi\rangle\, \langle\phi| = 1. \tag{2.223}$$

The expectation values of the $\hat{C}$ and $\hat{S}$ operators obviously take the form of $\cos\phi$ and $\sin\phi$ respectively.

Now an arbitrary state $|\psi\rangle$ of the field will be given as a superposition of all the number states, i.e.

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n |n\rangle \tag{2.224}$$

where the coefficients $C_n$ must satisfy

$$\sum_{n=0}^{\infty} |C_n|^2 = 1 \tag{2.225}$$

in order that $|\psi\rangle$ be normalized. We may associate a phase distribution $\mathcal{P}(\phi)$ with the state $|\psi\rangle$ according to the prescription

$$\begin{aligned} \mathcal{P}(\phi) &\equiv \frac{1}{2\pi} |\langle \phi \mid \psi \rangle|^2 \\ &= \frac{1}{2\pi} \left| \sum_{n=0}^{\infty} e^{-in\phi} C_n \right|^2 . \end{aligned} \tag{2.226}$$

Clearly, $\mathcal{P}(\phi)$ is always positive and can be shown to be normalized as follows: writing

$$\mathcal{P}(\phi) \equiv \frac{1}{2\pi} |\langle \phi \mid \psi \rangle|^2 = \frac{1}{2\pi} \langle \phi \mid \psi \rangle \langle \psi \mid \phi \rangle \tag{2.227}$$

and then integrating over $\phi$ using the resolution of unity as given by Eq. (2.223) we have

$$\int_0^{2\pi} \mathcal{P}(\phi)\, d\phi = \langle \psi \mid \psi \rangle = 1. \tag{2.228}$$

More generally, for a state described by a density operator $\hat{\rho}$, we would have

$$\mathcal{P}(\phi) = \frac{1}{2\pi} \langle \phi | \hat{\rho} | \phi \rangle. \tag{2.229}$$

The distribution can be used to calculate the average of any function of $\phi$, $f(\phi)$ according to

$$\langle f(\phi) \rangle = \int_0^{2\pi} f(\phi) \mathcal{P}(\phi)\, d\phi. \tag{2.230}$$

This is essentially the same as would be obtained from the Pegg–Barnett formalism of Reference [14].

For the present we consider only a number state $|n\rangle$ for which all the coefficients in Eq. (2.224) vanish but for $C_n = 1$. This leads to

$$\mathcal{P}(\phi) = \frac{1}{2\pi} \tag{2.231}$$

which is uniform, as expected. Furthermore, the average of $\phi$ is $\pi$ and the average over $\phi^2$ is $4\pi^2/3$ and thus the fluctuations in $\phi$ are

$$\Delta\phi = \sqrt{\langle \phi^2 \rangle - \langle \phi \rangle^2} = \frac{\pi}{\sqrt{3}} \tag{2.232}$$

as we would expect for a uniform probability distribution over the range 0 to $2\pi$. Note that this applies to all number states, including the vacuum. We shall

apply this formalism to other field states, in particular the coherent states, in later chapters.

One difficulty with the attempt to find a sensible quantum description of phase is that it is not obvious how to connect the various formalisms for phase with realistic experiments on the measurement of phase. The approach taken here is supported by the work of Shapiro and Shepard [15] who have shown, by using quantum estimation theory [16], that phase states $|\phi\rangle$ generate the probability operator measure for maximum likelihood phase estimation. On the other hand, Mandel and co-workers [17] have taken an operational approach to phase operators based on classical phase measurements. They show that phase measurements are difficult even in classical optics and that these difficulties carry over into quantum optics and, further, that different measurement procedures lead to different phase operators. It does not appear, at least from the operational point of view, that there exists one correct phase operator. Henceforth we shall use the eigenstates of the SG operator under the assumption that the corresponding phase distribution, at the very least, qualitively, if not quantitatively, describes a quantum phase variable.

## Problems

1. For the single-mode field given by Eq. (2.5), use Maxwell's equations to obtain the corresponding magnetic field given by Eq. (2.6).
2. For the single-mode field of the previous problem, obtain the Hamiltonian and show that it has the form of a simple harmonic oscillator.
3. Give a proof of the Baker–Hausdorf lemma: for any two operators $\hat{A}$ and $\hat{B}$,

$$e^{i\lambda\hat{A}}\hat{B}\,e^{-i\lambda\hat{A}} = \hat{B} + i\lambda[\hat{A},\hat{B}] + \frac{(i\lambda)^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \cdots$$

4. In the special case where $[\hat{A},\hat{B}] \neq 0$, but where $[\hat{A},[\hat{A},\hat{B}]] = 0 = [\hat{B},[\hat{A},\hat{B}]]$, show that

$$e^{\hat{A}+\hat{B}} = \exp\left(-\frac{1}{2}[\hat{A},\hat{B}]\right)e^{\hat{A}}e^{\hat{B}} = \exp\left(\frac{1}{2}[\hat{A},\hat{B}]\right)e^{\hat{B}}e^{\hat{A}}.$$

   This is known as the Baker–Hausdorf–Campbell theorem.
5. Suppose the state of a single-mode cavity field is given at time $t = 0$ by

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|n\rangle + e^{i\varphi}\,|n+1\rangle)$$

   where $\varphi$ is some phase. Find the state $|\psi(t)\rangle$ at times $t > 0$. For this time-evolved state, verify the uncertainty relation of Eq. (2.49).
6. Consider the superposition state $|\psi_{01}\rangle = \alpha|0\rangle + \beta|1\rangle$ where $\alpha$ and $\beta$ are complex and satisfy $|\alpha|^2 + |\beta|^2 = 1$. Calculate the variances of the quadrature operators $\hat{X}_1$ and $\hat{X}_2$. Are there any values of the parameters $\alpha$ and $\beta$ for which either of the quadrature

variances become *less* than for a vacuum state? If so, check to see if the uncertainty principle is violated. Repeat with the state $|\psi_{02}\rangle = \alpha|0\rangle + \beta|2\rangle$.

7. Many processes involve the absorption of single photons from a quantum field state, the process of absorption being represented by the action of the annihilation operator $\hat{a}$. For an arbitrary field state $|\psi\rangle$, the absorption of a single photon yields the state $|\psi'\rangle \sim \hat{a}|\psi\rangle$. Normalize this state. Compare the average photon numbers $\bar{n}$ of the state $|\psi\rangle$ and $\bar{n}'$ of $|\psi'\rangle$. Do you find that $\bar{n}' = \bar{n} - 1$?
8. Consider the superposition of the vacuum and 10 photon number state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |10\rangle\right).$$

Calculate the average photon number for this state. Next assume that a single photon is absorbed and recalculate the average photon number. Does your result seem sensible in comparison with your answer to the previous question?

9. Consider the multimode expressions for the electric and magnetic fields of Eqs. (2.83) and (2.84) respectively. (a) Show that they satisfy the free-space Maxwell equations. (b) Use these fields and follow through the derivation of the field energy given by Eq. (2.95).
10. It is sometimes useful to characterize the photon number probability distribution $P_n$ by its factorial moments. The $r$th factorial moment is defined as

$$\begin{aligned}&\langle \hat{n}\,(\hat{n}-1)\,(\hat{n}-2)\ldots(\hat{n}-r+1)\rangle \\ &\quad = \sum_n n\,(n-1)\,(n-2)\ldots(n-r+1)\,P_n.\end{aligned}$$

Show that for a thermal field, the right-hand side has the value $r!\bar{n}^r$.

11. Work out the commutator of the cosine and sine operators, $\hat{C}$ and $\hat{S}$ respectively. Calculate the matrix elements of the commutator and show that only the diagonal ones are nonzero.
12. Consider the mixed state (see Appendix A) described by the density operator

$$\hat{\rho} = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

and the pure superposition state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle).$$

Calculate the corresponding phase distributions $\mathcal{P}(\phi)$ and compare them.

13. Show that for the thermal state, $\mathcal{P}(\phi) = 1/2\pi$.

## References

[1] See, for example, J. J. Sakurai, *Modern Quantum Mechanics*, revised edition (Reading: Addison-Wesley, 1994), p. 96.
[2] See the discussion in I. J. K. Aitchison, *Contemp. Phys.*, **26** (1985), 333.
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# Chapter 3
# Coherent states

At the end of the preceding chapter, we showed that the photon number states $|n\rangle$ have a uniform phase distribution over the range 0 to $2\pi$. Essentially, then, there is no well-defined phase for these states and, as we have already shown, the expectation value of the field operator for a number state vanishes. It is frequently suggested (see, for example, Sakurai [1]) that the classical limit of the quantized field is the limit in which the number of photons becomes very large such that the number operator becomes a continuous variable. However, this cannot be the whole story since the mean field $\langle n|\hat{E}_x|n\rangle = 0$ no matter how large the value of $n$. We know that at a fixed point in space a classical field oscillates sinusoidally in time. Clearly this does not happen for the expectation value of the field operator for a number state. In this chapter we present a set of states, the coherent states [2], which do give rise to a sensible classical limit; and, in fact, these states are the "most classical" quantum states of a harmonic oscillator, as we shall see.

## 3.1 Eigenstates of the annihilation operator and minimum uncertainty states

In order to have a non-zero expectation value of the electric field operator or, equivalently, of the annihilation and creation operators, we are required to have a superposition of number states differing only by $\pm 1$. For example, this could contain only the states $|n\rangle$ and $|n \pm 1\rangle$:

$$|\psi\rangle = C_n|n\rangle + C_{n+1}|n \pm 1\rangle \tag{3.1}$$

(where $|C_n|^2 + |C_{n+1}|^2 = 1$). Generally, a superposition of all the number states will have the property that the expectation value of say, $\hat{a}$, will not vanish. Clearly, by inspecting Eqs. (2.15) and (2.16), the replacement of $\hat{a}$ and $\hat{a}^\dagger$ by continuous variables produces a classical field. A unique way to make this replacement is to seek eigenstates of the annihilation operator. These states are denoted as $|\alpha\rangle$ and satisfy the relation

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle \tag{3.2}$$

where $\alpha$ is a complex number, otherwise arbitrary. (Note that complex eigenvalues are allowed here as $\hat{a}$ is non-Hermitian.) The states $|\alpha\rangle$ are "right" eigenstates of $\hat{a}$ and $\langle\alpha|$ are "left" eigenstates of $\hat{a}^\dagger$ with eigenvalue $\alpha^*$:

$$\langle\alpha|\,\hat{a}^\dagger = \alpha^*\,\langle\alpha|\,. \tag{3.3}$$

Since the number states $|n\rangle$ form a complete set we may expand $|\alpha\rangle$ according to

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n\,|n\rangle\,. \tag{3.4}$$

Acting with $\hat{a}$ on each term of the expansion, Eq. (3.2) becomes

$$\hat{a}\,|\alpha\rangle = \sum_{n=1}^{\infty} C_n\sqrt{n}\,|n-1\rangle = \alpha\sum_{n=0}^{\infty} C_n\,|n\rangle\,. \tag{3.5}$$

Equating coefficients of $|n\rangle$ on both sides leads to

$$C_n\sqrt{n} = \alpha C_{n-1} \tag{3.6}$$

or

$$\begin{aligned} C_n &= \frac{2}{\sqrt{n}}C_{n-1} = \frac{\alpha^2}{\sqrt{n(n-1)}}C_{n-2} = \ldots \\ &= \frac{\alpha^n}{\sqrt{n!}}C_0 \end{aligned} \tag{3.7}$$

and thus

$$|\alpha\rangle = C_0\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,. \tag{3.8}$$

From the normalization requirement we determine $C_0$:

$$\begin{aligned} \langle\alpha\mid\alpha\rangle = 1 &= |C_0|^2\sum_n\sum_{n'}\frac{\alpha^{*n}\alpha^{n'}}{\sqrt{n!n'!}}\,\langle n\mid n'\rangle \\ &= |C_0|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} = |C_0|^2\,e^{|\alpha|^2} \end{aligned} \tag{3.9}$$

which implies that $C_0 = \exp(-\frac{1}{2}|\alpha|^2)$. Thus our normalized coherent states are

$$|\alpha\rangle = \exp\left(-\tfrac{1}{2}\,|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,. \tag{3.10}$$

Let us now consider the expectation value of the electric field operator

$$\hat{E}_x(\mathbf{r},t) = i\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{\frac{1}{2}}\left[\hat{a}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} - \hat{a}^\dagger e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]. \tag{3.11}$$

We obtain

$$\langle\alpha|\,\hat{E}_x(\mathbf{r},t)\,|\alpha\rangle = i\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{\frac{1}{2}}\left[\alpha e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} - \alpha^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]. \tag{3.12}$$

Writing $\alpha$ in polar form, $\alpha = |\alpha| e^{i\theta}$, we have

$$\langle \alpha | \hat{E}_x(\mathbf{r}, t) | \alpha \rangle = 2 |\alpha| \left( \frac{\hbar\omega}{2\varepsilon_0 V} \right)^{\frac{1}{2}} \sin(\omega t - \mathbf{k} \cdot \mathbf{r} - \theta) \tag{3.13}$$

which looks like a classical field. Furthermore, we can show that

$$\langle \alpha | \hat{E}_x^2(\mathbf{r}, t) | \alpha \rangle = \frac{\hbar\omega}{2\varepsilon_0 V} [1 + 4 |\alpha|^2 \sin^2 (\omega t - \mathbf{k} \cdot \mathbf{r} - \theta)] . \tag{3.14}$$

Thus the fluctuations in $\hat{E}_x(\mathbf{r}, t)$,

$$\Delta E_x \equiv \langle (\Delta \hat{E}_x)^2 \rangle^{\frac{1}{2}} = \left( \frac{\hbar\omega}{2\varepsilon_0 V} \right)^{\frac{1}{2}} , \tag{3.15}$$

are identical to those for a vacuum state. The coherent state is nearly a classical-like state because it not only yields the correct form for the field expectation values but contains only the noise of the vacuum. Indeed, using the quadrature operators of Eqs. (2.52) and (2.53), it is an easy exercise to show that for the coherent states

$$\langle (\Delta \hat{X}_1)^2 \rangle_\alpha = \frac{1}{4} = \langle (\Delta \hat{X}_2)^2 \rangle_\alpha \tag{3.16}$$

which again shows that these states have the fluctuations of the vacuum. Thus with respect to the field quadrature operators, the coherent states both minimize (actually equalize) the uncertainty product and exhibit equal uncertainties, those of the vacuum, in each quadrature. As a matter of fact, this property can be used as an alternate definition for the coherent states. Consider three Hermitian operators $\hat{A}$, $\hat{B}$, and $\hat{C}$, satisfying

$$[\hat{A}, \hat{B}] = i\hat{C} \tag{3.17}$$

which implies the uncertainty relation

$$\langle (\Delta \hat{A})^2 \rangle \langle (\Delta \hat{B})^2 \rangle \geq \frac{1}{4} \langle (\hat{C})^2 \rangle . \tag{3.18}$$

States that equalize this relation are those that satisfy the eigenvalue equation [3]

$$\left[ \hat{A} + \frac{i\langle \hat{C} \rangle}{2\langle (\Delta \hat{B})^2 \rangle} \hat{B} \right] |\psi\rangle = \left[ \langle \hat{A} \rangle + \frac{i\langle \hat{C} \rangle}{2\langle (\Delta \hat{B})^2 \rangle} \langle \hat{B} \rangle \right] |\psi\rangle . \tag{3.19}$$

The states $|\psi\rangle$ satisfying Eq. (3.19) sometimes go by the name "intelligent" states [4]. For the case where

$$\langle (\Delta \hat{A})^2 \rangle = \langle (\Delta \hat{B})^2 \rangle = \frac{1}{4} \langle \hat{C} \rangle , \tag{3.20}$$

the eigenvalue equation becomes

$$[\hat{A} + i\hat{B}] |\psi\rangle = [\langle \hat{A} \rangle + i \langle \hat{B} \rangle] \, |\psi\rangle . \tag{3.21}$$

For $\hat{A} = \hat{X}_1$ and $\hat{B} = \hat{X}_2$ this is equivalent to Eq. (3.2) with $\alpha = \langle \hat{X}_1 \rangle + i \langle \hat{X}_2 \rangle$.

What is the physical meaning of the complex parameter $\alpha$? From Eq. (3.13) it is apparent that $|\alpha|$ is related to the amplitude of the field. The expectation value

of the photon number operator $\hat{n} = \hat{a}^\dagger \hat{a}$ is

$$\bar{n} = \langle \alpha | \, \hat{n} \, | \alpha \rangle = |\alpha|^2 \tag{3.22}$$

and thus $|\alpha|^2$ is just the average photon number of the field. To calculate the fluctuations of the photon number we need to calculate

$$\begin{aligned} \langle \alpha | \, \hat{n}^2 \, | \alpha \rangle &= \langle \alpha | \, \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \, | \alpha \rangle \\ &= \langle \alpha | \, \left( \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} + \hat{a}^\dagger \hat{a} \right) | \alpha \rangle \\ &= |\alpha|^4 + |\alpha|^2 = \bar{n}^2 + \bar{n} \end{aligned} \tag{3.23}$$

and thus

$$\Delta n = \sqrt{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2} = \bar{n}^{1/2} \tag{3.24}$$

which is characteristic of a Poisson process. In fact, for a measurement of the number of photons in the field, the probability of detecting $n$ photons is

$$\begin{aligned} P_n &= |\langle n \mid \alpha \rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \\ &= e^{-\bar{n}} \frac{\bar{n}^n}{n!} \end{aligned} \tag{3.25}$$

which is a Poisson distribution with a mean of $\bar{n}$. Note that the fractional uncertainty in the photon number is

$$\frac{\Delta n}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}} \tag{3.26}$$

which decreases with increasing $\bar{n}$. In Fig. 3.1 we plot a couple of examples of the photon number probability distribution for different $\bar{n}$.

Let us now look at the phase distribution of the coherent states. For a coherent state $|\alpha\rangle$, with $\alpha = |\alpha| \, e^{i\theta}$, the corresponding phase distribution is

$$\begin{aligned} \mathcal{P}(\varphi) &= \frac{1}{2\pi} \, |\langle \varphi \mid \alpha \rangle|^2 \\ &= \frac{1}{2\pi} \, e^{-|\alpha|^2} \left| \sum_{n=0}^{\infty} e^{in(\varphi - \theta)} \frac{|\alpha|^n}{\sqrt{n}\,!} \right|^2 . \end{aligned} \tag{3.27}$$

For large $|\alpha|^2$, the Poisson distribution may be approximated as a Gaussian [5]:

$$e^{-|\alpha|^2/2} \frac{|\alpha|^{2n}}{n!} \, e^{-|\alpha|^2} \approx (2\pi \, |\alpha|^2)^{-\frac{1}{2}} \exp\left[ -\frac{(n - |\alpha|^2)^2}{2 \, |\alpha|^2} \right] \tag{3.28}$$

so that the sum in Eq. (3.27) may be evaluated to obtain an approximate form for $\mathcal{P}(\varphi)$ as

$$\mathcal{P}(\varphi) \approx \left( \frac{2 \, |\alpha|^2}{\pi} \right)^{\frac{1}{2}} \exp[-2 \, |\alpha|^2 \, (\varphi - \theta)^2] \, . \tag{3.29}$$

This is a Gaussian peaked at $\varphi = \theta$. Furthermore, the peak becomes narrower with increasing $\bar{n} = |\alpha|^2$, as illustrated in Fig. 3.2.

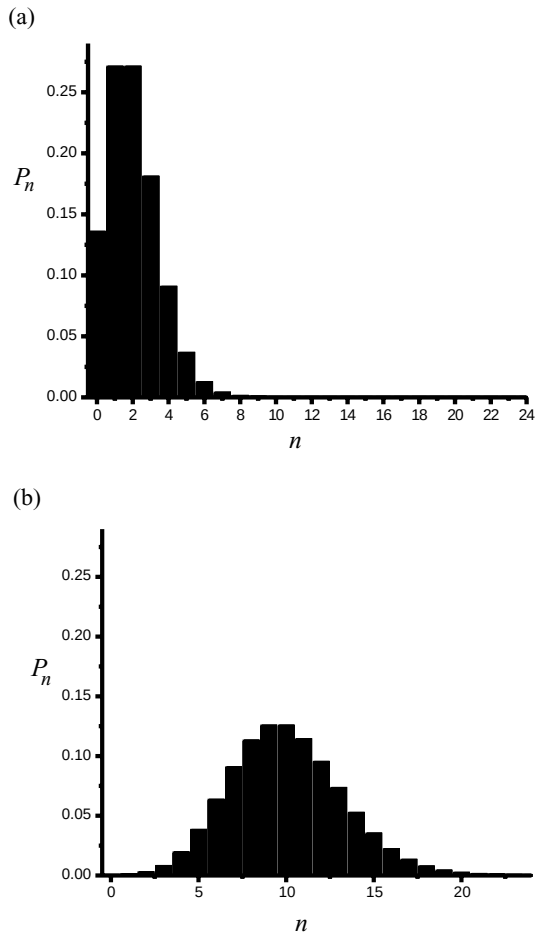


**Fig. 3.1.** Coherent state photon number probability distributions for (a) $\bar{n} = 2$ and (b) $\bar{n} = 10$.

The coherent states $|\alpha\rangle$ are quantum states very close to classical states because (i) the expectation value of the electric field has the form of the classical expression, (ii) the fluctuations in the electric field variables are the same as for a vacuum, (iii) the fluctuations in the fractional uncertainty for the photon number decrease with the increasing average photon number, and (iv) the states become well localized in phase with increasing average photon number. However, in spite of their near-classical properties, they are still quantum states. In Fig. 3.3 we illustrate this with a sketch where the expectation of the field operator and its fluctuations are plotted against time. The field clearly has the classical form of a sine wave but with the quantum fluctuations superimposed, indicating that it does have quantum features. In fact, *all* states of light must have some quantum features as the quantum theory of light is more fundamental than the classical theory. However, the quantum features of light are generally difficult to observe. We shall deal with some of these features in Chapter 7.

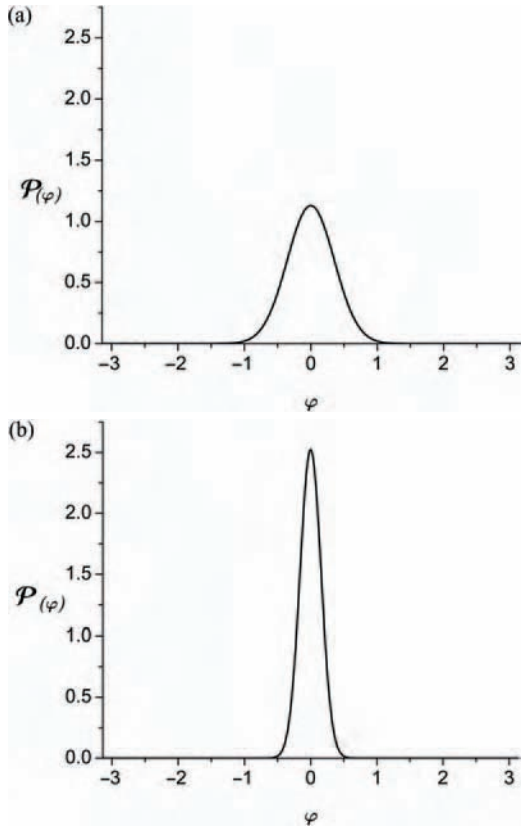


**Fig. 3.2.** Phase distributions for coherent states with $\theta = 0$ for (a) $\bar{n} = 2$ and (b) $\bar{n} = 10$.

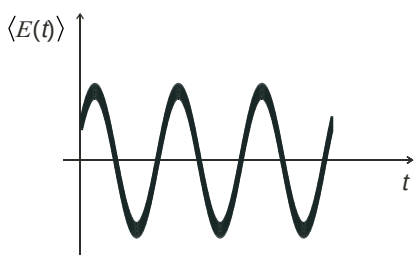


**Fig. 3.3.** Coherent state expectation value of the electric field as a function of time for a fixed position showing the quantum fluctuations. The fluctuations of the field are the same at all times such that the field is as close to a classical field as is possible for any quantum state.

## 3.2 Displaced vacuum states

We have discussed two ways in which the coherent states may be defined: as right eigenstates of the annihilation operator and as states that minimize the uncertainty relation for the two orthogonal field quadratures with equal uncertainties (identical to those of a vacuum state) in each quadrature. There is, in fact, a third

definition that leads to equivalent states. This involves the displacement of the vacuum. As we will show, this is closely related to a mechanism for generating the coherent states from classical currents.

The displacement operator $\hat{D}(\alpha)$ is defined as [2]

$$\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) \tag{3.30}$$

and the coherent states are given as

$$|\alpha\rangle = \hat{D}(\alpha)\,|0\rangle\,. \tag{3.31}$$

To see this, consider the identity (the disentangling theorem)

$$\begin{aligned} e^{\hat{A}+\hat{B}} &= e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]} \\ &= e^{\hat{B}} e^{\hat{A}} e^{\frac{1}{2}[\hat{A},\hat{B}]} \end{aligned} \tag{3.32}$$

valid if $[\hat{A}, \hat{B}] \neq 0$ but where also

$$[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0. \tag{3.33}$$

With $\hat{A} = \alpha\,\hat{a}^\dagger$ and $\hat{B} = -\alpha^* \hat{a}$, $[\hat{A}, \hat{B}] = |\alpha|^2$ and Eq. (3.33) holds. Thus

$$\hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}}\,. \tag{3.34}$$

Expanding $\exp(-\alpha^* \hat{a})$ makes it clear that

$$e^{-\alpha^* \hat{a}}\,|0\rangle = \sum_{l=0}^{\infty} \frac{(-\alpha^* \hat{a})^l}{l!}\,|0\rangle = |0\rangle \tag{3.35}$$

since $\hat{a}^l |0\rangle = 0$ except for $l = 0$. But

$$\begin{aligned} e^{\alpha \hat{a}^\dagger}\,|0\rangle &= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} (\hat{a}^\dagger)^n\,|0\rangle \\ &= \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle \end{aligned} \tag{3.36}$$

where we have used the fact that $(\hat{a}^\dagger)^n |0\rangle = \sqrt{n!}|n\rangle$. Thus we have

$$\begin{aligned} |\alpha\rangle &= \hat{D}(\alpha)\,|0\rangle \\ &= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,, \end{aligned} \tag{3.37}$$

in agreement with our previous definitions.

The displacement operator $\hat{D}$ is, of course, a unitary operator. It can be shown that

$$\begin{aligned} \hat{D}^\dagger(\alpha) &= \hat{D}(-\alpha) \\ &= e^{-\frac{1}{2}|\alpha|^2} e^{-\alpha\,\hat{a}^\dagger} e^{\alpha^* \hat{a}}. \end{aligned} \tag{3.38}$$

An alternate representation of $\hat{D}(\alpha)$ is

$$\hat{D}(\alpha) = e^{\frac{1}{2}|\alpha|^2} e^{-\alpha^* \hat{a}} e^{\alpha\,\hat{a}^\dagger}. \tag{3.39}$$

Evidently

$$\hat{D}(\alpha)\hat{D}^{\dagger}(\alpha) = \hat{D}^{\dagger}(\alpha)\hat{D}(\alpha) = 1 \tag{3.40}$$

as required for unitarity.

The displacement operator obeys the semigroup relation: the product of two displacement operators, say of $\hat{D}(\alpha)$ and $\hat{D}(\beta)$, is, up to an overall phase factor, the displacement operator $\hat{D}(\alpha+\beta)$. To see this let $\hat{A} = \alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}$, $\hat{B} = \beta\hat{a}^{\dagger} - \beta^{*}\hat{a}$ where

$$[\hat{A}, \hat{B}] = \alpha\beta^{*} - \alpha^{*}\beta = 2i\,\mathrm{Im}(\alpha\beta^{*}). \tag{3.41}$$

Then, by using Eq. (3.32),

$$\begin{aligned}\hat{D}(\alpha)\,\hat{D}(\beta) &= e^{\hat{A}}e^{\hat{B}} \\ &= \exp\left[i\,\mathrm{Im}\,(\alpha\beta^{*})\right] \\ &\quad \times \exp[(\alpha+\beta)\,\hat{a}^{\dagger} - (\alpha^{*}+\beta^{*})\,\hat{a}^{\dagger}] \\ &= \exp[i\,\mathrm{Im}(\alpha\beta^{*})]\hat{D}(\alpha+\beta)\ .\end{aligned} \tag{3.42}$$

Thus, applied to the vacuum

$$\begin{aligned}\hat{D}(\alpha)\,\hat{D}(\beta) &= \hat{D}(\alpha)\,|\beta\rangle \\ &= \exp[i\,\mathrm{Im}\,(\alpha\beta^{*})]\ |\alpha+\beta\rangle\ .\end{aligned} \tag{3.43}$$

The phase factor $\exp[i\,\mathrm{Im}(\alpha\beta^{*})]$ is an overall phase factor and so is physically irrelevant.

## 3.3 Wave packets and time evolution

From Eqs. (2.13) and (2.14) we obtain the "position" operator

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a} + \hat{a}^{\dagger}) = \sqrt{\frac{2\,\hbar}{\omega}}\,\hat{X}_1, \tag{3.44}$$

where $\hat{X}_1$ is the quadrature operator of Eq. (2.52) The eigenstates of the operator $\hat{q}$ we denote as $|q\rangle$ where $\hat{q}\,|q\rangle = q\,|q\rangle$. The corresponding wave functions for the number states are [6]

$$\psi_n(q) = \langle q \mid n\rangle = (2^n n!)^{-1/2}\left(\frac{\omega}{\pi\,\hbar}\right)^{1/4}\exp(-\xi^2/2)H_n(\xi) \tag{3.45}$$

where $\xi = q\sqrt{\omega/\hbar}$ and where the $H_n(\xi)$ are Hermite polynomials. The corresponding wave function for the coherent state is then

$$\psi_{\alpha}(q) \equiv \langle q \mid \alpha\rangle = \left(\frac{\omega}{\pi\,\hbar}\right)^{1/4} e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha/\sqrt{2})^n}{n!}H_n(\xi). \tag{3.46}$$

From the generating function for Hermite polynomials, we can obtain the closed-form expression for the coherent-state wave function

$$\psi_{\alpha}(q) = \left(\frac{\omega}{\pi\,\hbar}\right)^{1/4} e^{-|\alpha|^2/2}e^{\xi^2/2}e^{-(\xi-\alpha/\sqrt{2})^2}, \tag{3.47}$$

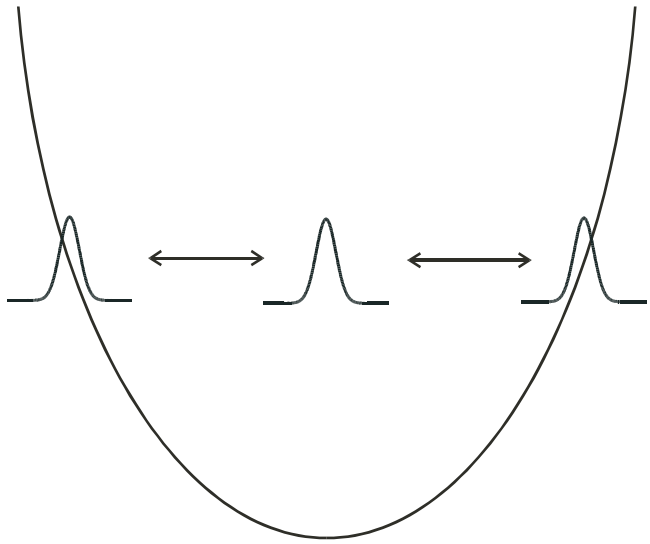

**Fig. 3.4.** A coherent-state wave function moves through the harmonic oscillator potential, between the classical turning points, without dispersion.

a Gaussian wave function. Of course, the probability distribution over the "position" variable $q$,

$$P(q) = |\psi_\alpha(q)|^2 , \tag{3.48}$$

is also Gaussian.

We now consider the time evolution of a coherent state for a single-mode free field where the Hamiltonian is given by Eq. (2.18). The time-evolving coherent state is given by

$$\begin{aligned} |\alpha, t\rangle &\equiv \exp(-i\hat{H}t/\hbar)\,|\alpha\rangle = e^{-i\omega t/2} e^{-i\omega t\,\hat{n}}\,|\alpha\rangle \\ &= e^{-i\omega t/2} |\alpha\, e^{-i\,\omega t}\rangle, \end{aligned} \tag{3.49}$$

and so the coherent state remains a coherent state under free field evolution. The corresponding time-evolving wave function is

$$\psi_\alpha(q,t) = \left(\frac{\omega}{\pi\,\hbar}\right)^{1/4} e^{-|\alpha|^2/2} e^{\xi^2/2} e^{-(\xi - \alpha\, e^{-i\,\omega t}/\sqrt{2})^2}, \tag{3.50}$$

a Gaussian whose shape does not change with time and whose centroid follows the motion of a classical point particle in a harmonic oscillator potential (the demonstration of this is left as an exercise, e.g. Problem 2 at the end of this chapter). In Fig. 3.4 we illustrate the motion of a coherent-state wave packet in the harmonic oscillator potential. The motional states of laser-cooled trapped atoms or ions can be engineered to have this minimum uncertainty character as we shall see in Chapter 10.

Ultimately, the stability of the wave packet results from the fact that, for the harmonic oscillator, the energy levels are integer spaced. For quantum systems where the energy levels are not integer spaced, such as for the Coulomb problem, formulating coherent states is a bit of a challenge and no such states are truly

stable for all times. Consideration of such states is beyond the scope of this book, but see Reference [7].

## 3.4 Generation of coherent states

A coherent state may be generated by a classical oscillating current. Let the quantum electromagnetic vector potential be $\hat{\mathbf{A}}(\mathbf{r}, t)$ for a field interacting with a classical current described by the current density $\mathbf{j}(\mathbf{r}, t)$. According to classical electromagnetic theory, the interaction energy $\hat{V}(t)$ is given by

$$\hat{V}(t) = \int d^3\mathbf{r}\, \mathbf{j}\,(\mathbf{r}, t) \cdot \hat{\mathbf{A}}\,(\mathbf{r}, t)\,. \tag{3.51}$$

For a single-mode field, $\hat{\mathbf{A}}(\mathbf{r}, \mathrm{t})$ is given, in the interaction picture, by

$$\hat{\mathbf{A}}(\mathbf{r}, t) = \mathbf{e}\left(\frac{\hbar}{2\omega_0\varepsilon_0 V}\right)^{\frac{1}{2}} \left[\hat{a}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \hat{a}^\dagger e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] \tag{3.52}$$

where $\hat{a}$ means $\hat{a}(0)$. Substituting this into Eq. (3.51), we have

$$\hat{V}(t) = -\left(\frac{\hbar}{2\omega\varepsilon_0 V}\right)^{\frac{1}{2}} \left[\hat{a}\mathbf{e}\cdot\mathbf{J}\,(\mathbf{k}, t)\, e^{-i\omega t} + \hat{a}^\dagger \mathbf{e}\cdot\mathbf{J}^*\,(\mathbf{k}, t)\; e^{i\omega t}\right] \tag{3.53}$$

where

$$\mathbf{J}\,(\mathbf{k}, t) = \int d^3\mathbf{r}\, \mathbf{j}(\mathbf{r}, t) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{3.54}$$

Since $\hat{V}(t)$ depends on time, the associated evolution operator is a time-ordered product [8]. But for an infinitesimally short time interval from $t$ to $t + \delta t$, the evolution operator is

$$\begin{aligned} \hat{U}(t + \delta t, t) &\cong \exp[-i\,\hat{V}\,(t)\,\delta t/\hbar] \\ &= \exp\{-\delta t[u(t)\hat{a} - u^*(t)\hat{a}^\dagger]\} \\ &= \hat{D}\,[u(t)\delta t]\,, \end{aligned} \tag{3.55}$$

where

$$u(t) = -\left(\frac{\hbar}{2\omega\mathcal{E}_0 V}\right)^{\frac{1}{2}} \mathbf{e}\cdot\mathbf{J}(\mathbf{k}, t)\, e^{-i\omega t}. \tag{3.56}$$

For a finite time interval, say from 0 to $T$, the evolution operator may be written as

$$\hat{U}(T, 0) = \lim_{\delta t\to 0} \hat{\mathcal{T}} \prod_{l=0}^{T/\delta t} \hat{D}\,[u(t_l)\delta t] \tag{3.57}$$

where $\hat{\mathcal{T}}$ is the time-ordering operator and where $t_l = l\,\delta t$. Equation (3.57) becomes, by using Eq. (3.42),

$$\begin{aligned} \hat{U}(T, 0) &= \lim_{\delta t\to 0} e^{i\Phi}\hat{D}\left[\sum_{l=0}^{T/\delta t} u\,(t_l)\,\delta t\right] \\ &= e^{i\Phi}\hat{D}\,[\alpha\,(T)]\,, \end{aligned} \tag{3.58}$$

where

$$\alpha(T) = \lim_{\delta t \to 0} \sum_{l=0}^{T/\delta t} u(t_l)\,\delta t = \int_0^T u(t')dt' \tag{3.59}$$

and where $\Phi$ is the accumulated overall phase. With the initial state the vacuum, the state at time $T$ is just the coherent state $|\alpha(T)\rangle$ with $\alpha(T)$ given by Eq. (3.59), apart from the irrelevant overall phase.

## 3.5 More on the properties of coherent states

The number states are orthonormal, $\langle n|n'\rangle = \delta_{nn'}$, and complete, $\sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{I}$, and as such can be used as a basis for the expansion of an arbitrary state vector of the field, i.e. for a given $|\psi\rangle$

$$|\psi\rangle = \sum_n C_n |n\rangle \tag{3.60}$$

where $C_n = \langle n \mid \psi\rangle$, the coherent states being a particular example. But the coherent states themselves are not orthogonal: for $|\alpha\rangle$ and $|\beta\rangle$

$$\begin{aligned}
\langle \beta \mid \alpha\rangle &= e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2} \\
&\quad \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\beta^{*n}\alpha^m}{\sqrt{n!m!}} \langle n \mid m\rangle \\
&= e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2} \sum_{n=0}^{\infty} \frac{(\beta^*\alpha)^n}{n!} \\
&= e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \beta^*\alpha} \\
&= \exp\left[\frac{1}{2}(\beta^*\alpha - \beta\alpha^*)\right] \exp\left[-\frac{1}{2}|\beta - \alpha|^2\right].
\end{aligned} \tag{3.61}$$

The first term is just a complex phase so that

$$|\langle \beta \mid \alpha\rangle|^2 = e^{-|\beta-\alpha|^2} \neq 0. \tag{3.62}$$

Thus the coherent states are not orthogonal. Of course, if $|\beta - \alpha|^2$ is large, they are nearly orthogonal.

The completeness relation for the coherent states is given as an integral over the complex $\alpha$-plane according to

$$\int |\alpha\rangle\langle\alpha| \frac{d^2\alpha}{\pi} = 1 \tag{3.63}$$

where $d^2\alpha = d\mathrm{Re}(\alpha)d\mathrm{Im}(\alpha)$. The proof of this goes as follows: writing

$$\int |\alpha\rangle\langle\alpha|\, d^2\alpha = \int e^{-|\alpha|^2} \sum_n \sum_m \frac{\alpha^n \alpha^{*m}}{\sqrt{n!m!}} |n\rangle\langle m|\, d^2\alpha, \tag{3.64}$$

we transform to polar coordinates setting $\alpha = re^{i\theta}$, $d^2\alpha = r\,dr\,d\theta$ so that

$$\int |\alpha\rangle\langle\alpha|\,d^2\alpha = \sum_n \sum_m \frac{|n\rangle\langle m|}{\sqrt{n!m!}} \int_0^\infty dr e^{-r^2} r^{n+m+1} \int_0^{2\pi} d\theta e^{i(n-m)\theta}. \tag{3.65}$$

But

$$\int_0^{2\pi} d\theta e^{i(n-m)\theta} = 2\pi\delta_{nm}, \tag{3.66}$$

and, with a further change of variables, $r^2 = y$, $2r\,dr = dy$, we have

$$\int |\alpha\rangle\langle\alpha|\,d^2\alpha = \pi \sum_{n=0}^\infty \frac{|n\rangle\langle n|}{n!} \int_0^\infty dy e^{-y} y^n. \tag{3.67}$$

Since

$$\int_0^\infty dy e^{-y} y^n = n! \tag{3.68}$$

we have

$$\int |\alpha\rangle\langle\alpha|\,d^2\alpha = \pi \sum_{n=0}^\infty |n\rangle\langle n| = \pi, \tag{3.69}$$

which completes the proof.

Any state vector $|\psi\rangle$ in the Hilbert space of the quantized single-mode field can be expressed in terms of the coherent states as

$$|\psi\rangle = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha \mid \psi\rangle. \tag{3.70}$$

But suppose the state $|\psi\rangle$ itself is the coherent state $|\beta\rangle$. Then

$$\begin{aligned} |\beta\rangle &= \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha \mid \beta\rangle \\ &= \int \frac{d^2\alpha}{\pi} |\alpha\rangle \exp\left[-\frac{1}{2}|\alpha| - \frac{1}{2}|\beta|^2 + \alpha^*\beta\right]. \end{aligned} \tag{3.71}$$

This last equation shows that the coherent states are not linearly independent. The coherent states are said to be "overcomplete", there being more than enough states available to express any state in terms of the coherent states. Note that in Eq. (3.71) the quantity $\langle\alpha \mid \beta\rangle = \exp(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta)$ plays the role of a Dirac delta function. It is often referred to as a reproducing kernel.

For arbitrary state $|\psi\rangle$ we may write

$$\begin{aligned} \langle\alpha \mid \psi\rangle &= \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{n=0}^\infty \psi_n \frac{(\alpha^*)^n}{\sqrt{n!}} \\ &= \exp\left(-\frac{1}{2}|\alpha|^2\right) \psi(\alpha^*) \end{aligned} \tag{3.72}$$

where $\psi_n = \langle n|\psi\rangle$ and

$$\psi(z) = \sum_{n=0}^{\infty} \psi_n \frac{z^n}{\sqrt{n!}} \tag{3.73}$$

is absolutely convergent anywhere on the complex $z$-plane, i.e. $\psi(z)$ is an entire function, since $\langle\psi \mid \psi\rangle = \sum_n |\langle n \mid \psi\rangle|^2 = 1$. The functions $\psi(z)$ constitute the Segal–Bargmann [9] space of entire functions. If $|\psi\rangle$ is a number state, $|\psi\rangle = |n\rangle$, then $\psi_n(z) = z^n/\sqrt{n!}$. These functions form an orthonormal basis on the Segal–Bargmann space.

Let $\hat{F}$ be an operator given as a function of $\hat{a}$, and $\hat{a}^\dagger$, $\hat{F} = F(\hat{a}, \hat{a}^\dagger)$. In terms of the number states, $\hat{F}$ may be decomposed as

$$\begin{aligned} \hat{F} &= \sum_n \sum_m |m\rangle \langle m| \hat{F} |n\rangle \langle n| \\ &= \sum_n \sum_m |m\rangle \hat{F}_{mn} \langle n| \end{aligned} \tag{3.74}$$

where the $F_{mn}$ are the matrix elements $\langle m|\hat{F}|n\rangle$.

With coherent states

$$\hat{F} = \frac{1}{\pi^2} \int d^2\beta \int d^2\alpha \, |\beta\rangle \langle\beta| \hat{F} |\alpha\rangle \langle\alpha| \,. \tag{3.75}$$

But

$$\begin{aligned} \langle\beta| \hat{F} |\alpha\rangle &= \sum_n \sum_m F_{mn} \langle\beta \mid m\rangle \langle n \mid \alpha\rangle \\ &= \exp\left[-\frac{1}{2}(|\beta|^2 + |\alpha|^2)\right] F(\beta^*, \alpha) \,, \end{aligned} \tag{3.76}$$

where

$$F(\beta^*, \alpha) = \sum_m \sum_n F_{mn} \frac{(\beta^*)^m (\alpha)^n}{\sqrt{m!n!}} \,. \tag{3.77}$$

Thus

$$\begin{aligned} \hat{F} = \frac{1}{\pi^2} \int d^2\beta \int d^2\alpha \exp\left[-\frac{1}{2}(|\beta|^2 + |\alpha|^2)\right] \\ \times F(\beta^*, \alpha) |\beta\rangle \langle\alpha| \,. \end{aligned} \tag{3.78}$$

Suppose now that $\hat{F}$ is a Hermitian operator with eigenstates $|\lambda\rangle$ such that

$$\hat{F} = \sum_\lambda \lambda \, |\lambda\rangle \langle\lambda| \,. \tag{3.79}$$

Then

$$\langle m| \hat{F} |n\rangle = \sum_\lambda \lambda \langle m \mid \lambda\rangle \langle\lambda \mid n\rangle \,. \tag{3.80}$$

But

$$|\langle m|\hat{F}|n\rangle| \leq \sum_{\lambda} \lambda\, |\langle m \mid \lambda\rangle\, \langle \lambda \mid n\rangle| \leq \sum_{\lambda} \lambda = \mathrm{Tr}\,\hat{F} \tag{3.81}$$

which implies that $|\langle m|\hat{F}|n\rangle|$ has an upper bound. This being the case, it follows that the function $F(\beta^*, \alpha)$ is an entire function in both $\beta^*$ and $\alpha$.

The diagonal elements of an operator $\hat{F}$ in a coherent state basis completely determine the operator. From Eqs. (3.76) and (3.77) we have

$$\langle \alpha|\, \hat{F}\, |\alpha\rangle\, e^{\alpha^*\alpha} = \sum_n \sum_m \frac{\alpha^{*m}\alpha^n}{\sqrt{m!n!}}\, \langle m|\, \hat{F}\, |n\rangle\,. \tag{3.82}$$

Treating $\alpha$ and $\alpha^*$ as independent variables it is apparent that

$$\frac{1}{\sqrt{m!n!}} \left[\frac{\partial^{n+m}(\langle\alpha|\hat{F}|\alpha\rangle e^{\alpha^*\alpha})}{\partial\alpha^{*m}\,\partial\alpha^n}\right]\Bigg|_{\substack{\alpha^*=0\\ \alpha=0}} = \langle m|\, \hat{F}\, |n\rangle\ . \tag{3.83}$$

Thus from "diagonal" coherent-state matrix elements of $\hat{F}$ we can obtain all the matrix elements of the operator in the number basis.

## 3.6 Phase-space pictures of coherent states

It is well known that the concept of phase space in quantum mechanics is problematic owing to the fact that the canonical variables $\hat{x}$ and $\hat{p}$ are incompatible, i.e. they do not commute. Thus the state of a system is not well localized as a point in phase space as it is in classical mechanics. Nevertheless, we have shown that the coherent states minimize the uncertainty relation for the two orthogonal quadrature operators and that the uncertainties of the two quadratures are equal. Recall that $\hat{X}_1 = (\hat{a} + \hat{a}^\dagger)/2$ and $\hat{X}_2 = (\hat{a} - \hat{a}^\dagger)/2i$. These operators are dimensionless scaled position and momentum operators, respectively. Their coherent-state expectation values are $\langle \hat{X}_1\rangle_\alpha = \frac{1}{2}(\alpha + \alpha^*) = \mathrm{Re}\alpha$, $\langle \hat{X}_2\rangle_\alpha = \frac{1}{2i}(\alpha - \alpha^*) = \mathrm{Im}\alpha$. Thus the complex $\alpha$-plane plays the role of phase space where, up to scale factors, the real and imaginary parts of $\alpha$ are position and momentum variables respectively. A coherent state $|\alpha\rangle$with $\alpha = |\alpha|e^{i\theta}$ may be represented pictorially then as in Fig. 3.5, where the shaded circle represents "area of uncertainty" of the coherent state, the fluctuations being equal in all directions of phase space, the center of the circle located at distance $|\alpha| = \langle\hat{n}\rangle^{1/2}$ from the origin and at angle $\theta$ above the position axis. Further, $\Delta\theta$, in a qualitative sense, represents the phase uncertainty of the coherent state and it should be clear that $\Delta\theta$ diminishes for increasing $|\alpha|$, the fluctuations in $X_1$ and $X_2$ being independent of $\alpha$ and identical to those of the vacuum. For the vacuum, $|\alpha| = 0$, the phase-space representation is given in Fig. 3.6, where it is evident that uncertainty in the phase is as large as possible, i.e. $\Delta\theta = 2\pi$.

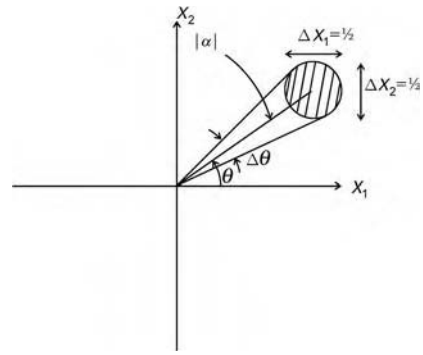


**Fig. 3.5.** Phase-space portrait of a coherent state of amplitude $|\alpha|$ and phase angle $\theta$. Note the error circle is the same for all coherent states. Note that as $|\alpha|$ increases, the phase uncertainty $\Delta\theta$ decreases, as would be expected in the "classical limit".

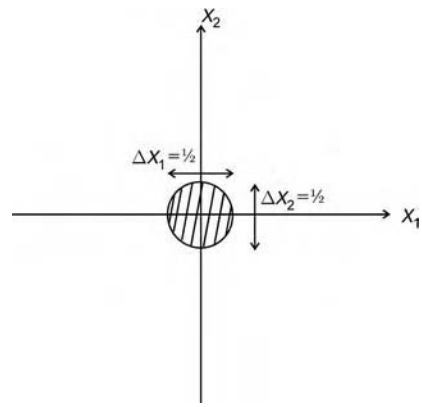


**Fig. 3.6.** Phase-space portrait of the quantum vacuum state.

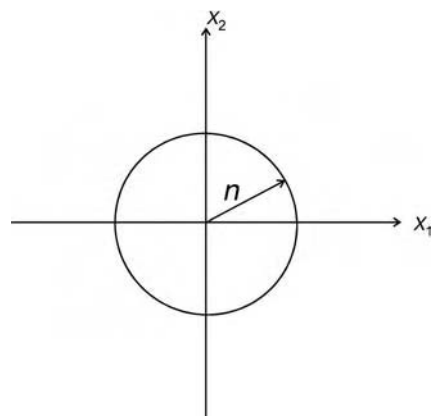


**Fig. 3.7.** Phase-space portrait of the number state $|n\rangle$. The uncertainty in the photon number is $\Delta n = 0$ while the phase is entirely random.

A number state $|n\rangle$ can be represented in phase space as a circle of radius $n$, the uncertainty in $n$ being zero and the uncertainty in phase again being $2\pi$, as in Fig. 3.7. It must be understood that these pictures are qualitative in nature but are useful as a graphical way of visualizing the distribution of noise in various quantum states of the field. As most quantum states of the field have no classical

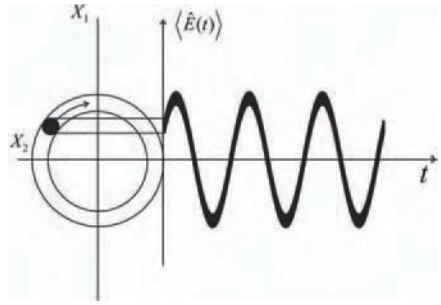


**Fig. 3.8.** The error circle of a coherent state (the black dot) revolves about the origin of phase space at the oscillator angular frequency $\omega$ and the expectation value of the electric field is the projection onto an axis parallel with $X_1$.

analog, the corresponding phase-space portraits should not be taken too literally. Yet these representations will be quite useful when we discuss the nature of the squeezed states of light in Chapter 7.

Finally in this section, we make one more use of the phase-space diagrams, namely to illustrate the time evolution of quantum states for a non-interacting field. We have seen that, for a noninteracting field, a coherent state $|\alpha\rangle$ evolves to the coherent state $|\alpha e^{-i\omega t}\rangle$. This can be pictured as a clockwise rotation of the error circle in phase since $\langle \alpha e^{-i\omega t}|\hat{X}_1|\alpha e^{-i\omega t}\rangle = \alpha \cos\omega t$, $\langle \alpha e^{-i\omega t}|\hat{X}_2|\alpha e^{-i\omega t}\rangle = -\alpha \sin\omega t$, assuming $\alpha$ real. Because in the Schrödinger picture the electric field operator is given, from Eq. (2.15) and (2.52), as

$$\hat{E}_x = 2\mathcal{E}_0 \sin(kz)\hat{X}_1, \tag{3.84}$$

the expectation value for the coherent state $|\alpha e^{-i\omega t}\rangle$ is

$$\langle \alpha e^{-i\omega t}|\hat{E}_x|\alpha e^{-i\omega t}\rangle = 2\mathcal{E}_0 \sin(kz)\alpha \cos\omega t. \tag{3.85}$$

Thus apart from the scale factor $2\mathcal{E}_0 \sin(kz)$, the time evolution of the electric field and its fluctuations, is given by the projection of the $\langle \hat{X}_1 \rangle$ axis as a function of time as indicated in Fig. 3.8.

The evolution of points within the error circle are shown indicating the uncertainty of the electric field – the "quantum flesh" on the "classical bones", so to speak. Note that the greater the excitation of the field, i.e. $\alpha$, the more classical the field appears since the fluctuations are independent of $\alpha$. But the coherent state is the most classical of all the quantum states so it is apparent that for the field in something other than a coherent state, the expectation value of the field may in no way appear classical-like. A number state is a very nonclassical state and, by using representative points from its phase-space portrait, it is easy to see, from Fig. 3.7, that the expectation value of the field is zero. But it is possible to imagine other kinds of states having no vanishing expectation values of the field but where fluctuations may be less than those of a coherent state in one part of the field. These are the squeezed states to be taken up in Chapter 7.

## 3.7 Density operators and phase-space probability distributions

Recall (see also Appendix A) that for a mixture (or ensemble) of quantum states $|\psi_1\rangle, |\psi_2\rangle, \ldots$, the density operator is given as

$$\hat{\rho} = \sum_i p_i \, |\psi_i\rangle \, \langle\psi_i| \tag{3.86}$$

where the $p_i$ are the probabilities of finding the system in the $i$th member of the ensemble and

$$\mathrm{Tr}\,(\hat{\rho}) = \sum_i p_i = 1. \tag{3.87}$$

(For a pure state, $\hat{\rho} = |\psi\rangle\langle\psi|$.) The expectation value of an operator $\hat{O}$ is given by

$$\langle\hat{O}\rangle = \mathrm{Tr}(\hat{O}\hat{\rho}) = \sum_i p_i \, \langle\psi_i| \, \hat{O} \, |\psi_i\rangle \, . \tag{3.88}$$

As in Eq. (3.74), we can resolve unity in terms of the number states on both sides of the density operator to obtain

$$\hat{\rho} = \sum_n \sum_m |m\rangle \, \rho_{mn} \, \langle n| \, . \tag{3.89}$$

All the matrix elements $\rho_{mn} = \langle m|\hat{\rho}|n\rangle$ are required completely to determine the operator $\hat{\rho}$. The diagonal elements, $P_n = \rho_{nn}$, are just the probabilities of finding $n$ photons in the field.

On the other hand, resolving unity with coherent states on both sides of $\hat{\rho}$, as in Eq. (3.75), results in

$$\hat{\rho} = \iint \langle\alpha'|\hat{\rho}|\alpha''\rangle|\alpha'\rangle\langle\alpha''| \frac{d^2\alpha' d^2\alpha''}{\pi^2}. \tag{3.90}$$

But there is yet another way to represent $\hat{\rho}$ in terms of coherent states, namely

$$\hat{\rho} = \int P\,(\alpha) \, |\alpha\rangle \, \langle\alpha| \, d^2\alpha, \tag{3.91}$$

where $P(\alpha)$ is a weight function sometimes known as the Glauber–Sudarshan $P$ function [2,10]. The right-hand side of Eq. (3.91) is the "diagonal" form of the density operator and the $P$ function is analogous to the phase-space distributions of statistical mechanics. Here, the real and imaginary parts of $\alpha$ are the variables of the phase space. $P(\alpha)$ must be real as $\hat{\rho}$ is a Hermitian operator. Also

$$\begin{aligned}
\mathrm{Tr}\hat{\rho} &= \mathrm{Tr} \int P\,(\alpha) \, |\alpha\rangle \, \langle\alpha| \, d^2\alpha \\
&= \int \sum_n P\,(\alpha) \, \langle n|\alpha\rangle \, \langle\alpha|n\rangle \, d^2\alpha \\
&= \int P\,(\alpha) \sum_n \langle\alpha \, |n\rangle \, \langle n| \, \alpha\rangle \, d^2\alpha \\
&= \int P\,(\alpha) \, \langle\alpha \mid \alpha\rangle \, d^2\alpha = \int P\,(\alpha) \, d^2\alpha = 1,
\end{aligned} \tag{3.92}$$

just as we would expect of a phase-space probability distribution.

But for some quantum states of the field, $P(\alpha)$ can have properties quite unlike those of any *true* probability distribution where one would expect to have $P(\alpha) \geq 0$. There are quantum states for which $P(\alpha)$ is negative or highly singular. In these cases, the corresponding quantum states are called "nonclassical". In fact, we may define a nonclassical state as a state for which the corresponding $P(\alpha)$ is negative in some regions of phase space (the $\alpha$-plane) or is more singular than a delta function (the reason for the second criterion will become clear shortly). To speak of nonclassical states of light might seem an oxymoron. After all, aren't *all* states of light quantum mechanical? Well, yes, *all* states of light *are* quantum mechanical, but, waxing Orwellian, it turns out that some states are more quantum mechanical than others. States for which $P(\alpha)$ is positive or no more singular than a delta function are, in this sense, classical. Coherent states, which we have already shown to be quasi-classical in that they describe states of the field having properties close to what we would expect for classical oscillating coherent fields, have $P$ functions given as delta functions, as we shall show, and are therefore classical in the sense described here. Certain effects, among them being quadrature and amplitude (or number) squeezing, can occur only for states for which the $P$ functions are negative or highly singular. For that reason, the various forms of squeezing are known as distinctly non-classical effects. These effects will be discussed in Chapter 7.

But how do we calculate $P(\alpha)$? A rather general procedure, which we follow here, has been given by Mehta [11]. Starting with Eq. (3.91) and using coherent state $|u\rangle$ and $|-u\rangle$, we have

$$\begin{aligned}\langle -u\,|\hat{\rho}|\,u\rangle &= \int P(\alpha)\,\langle -u\,|\alpha\rangle\,\langle\alpha|\,u\rangle\,d^2\alpha \\ &= \int P(\alpha)\exp\left[-\frac{1}{2}\,|u|^2-\frac{1}{2}\,|\alpha|^2-u^*\alpha\right] \\ &\quad\times\exp\left[-\frac{1}{2}\,|\alpha|^2-\frac{1}{2}\,|u|^2+a^*u\right]d^2\alpha \\ &= e^{-|u|^2}\int P(\alpha)\,e^{-|\alpha|^2}e^{\alpha^*u-\alpha u^*}d^2\alpha. \end{aligned} \tag{3.93}$$

Now let $\alpha = x + iy$ and $u = x' + iy'$ so that $\alpha^*u - \alpha u^* = 2\,i(x'y - xy')$. We then define the Fourier transforms in the complex plane:

$$g(u) = \int f\,(\alpha)\,e^{\alpha^*u-\alpha u^*}d^2\alpha \tag{3.94a}$$

$$f\,(\alpha) = \frac{1}{\pi^2}\int g\,(u)\,e^{u^*\alpha-u\alpha^*}d^2\alpha. \tag{3.94b}$$

With the identifications

$$\begin{aligned} g\,(u) &= e^{|u|^2}\,\langle -u|\,\hat{\rho}\,|u\rangle \\ f\,(\alpha) &= P(\alpha)\,e^{-|u|^2} \end{aligned} \tag{3.95}$$

we then obtain from Eq. (3.94b)

$$P(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2} \int e^{|u|^2} \langle -u| \rho |u\rangle e^{u^*\alpha - u\alpha^*} d^2u. \tag{3.96}$$

Care must be taken in regard to the convergence of the integral since $e^{|u|^2} \to \infty$ as $|u| \to \infty$.

Now let us consider some examples. First we consider the pure coherent state $|\beta\rangle$, where $\hat{\rho} = |\beta\rangle\langle\beta|$. With

$$\begin{aligned} \langle -u| \hat{\rho} |u\rangle &= \langle -u \mid \beta\rangle \langle \beta \mid u\rangle \\ &= e^{-|\beta|^2} e^{-|u|^2} e^{-u^*\beta + \beta^* u}, \end{aligned} \tag{3.97}$$

then

$$P(\alpha) = e^{|\alpha|^2} e^{-|\beta|^2} \frac{1}{\pi^2} \int e^{u^*(\alpha-\beta) - u(\alpha^* - \beta^*)} d^2u. \tag{3.98}$$

But the Fourier integral is just the two-dimensional form of the Dirac delta function:

$$\begin{aligned} \delta^2(\alpha - \beta) &= \delta[\mathrm{Re}(\alpha) - \mathrm{Re}(\beta)]\, \delta[\mathrm{Im}(\alpha) - \mathrm{Im}(\beta)] \\ &= \frac{1}{\pi^2} \int e^{u^*(\alpha-\beta) - u(\alpha^* - \beta^*)} d^2u \end{aligned} \tag{3.99}$$

so that

$$P(\alpha) = \delta^2(\alpha - \beta). \tag{3.100}$$

This is the same as the distribution for a classical harmonic oscillator.

But if a coherent state is a classical-like state, the number state $|n\rangle$ is at the other extreme, a state that cannot at all be described classically. For a pure number state $|n\rangle$, $\hat{\rho} = |n\rangle\langle n|$ and

$$\langle -u| \hat{\rho} |u\rangle = \langle -u \mid n\rangle \langle n \mid u\rangle = e^{-|u|^2} \frac{(-u^*u)^n}{n!}. \tag{3.101}$$

Thus

$$P(\alpha) = \frac{e^{|\alpha|^2}}{n!} \frac{1}{\pi^2} \int (-u^*u)^n e^{u^*\alpha - u\alpha^*} d^2u. \tag{3.102}$$

The integral does not exist in terms of ordinary functions. Formally, we can write

$$\begin{aligned} P(\alpha) &= \frac{e^{|\alpha|^2}}{n!} \frac{\partial^{2n}}{\partial\alpha^n \partial\alpha^{*n}} \frac{1}{\pi^2} \int e^{u^*\alpha - u\alpha^*} d^2u \\ &= \frac{e^{|\alpha|^2}}{n!} \frac{\partial^{2n}}{\partial\alpha^n \partial\alpha^{*n}} \delta^{(2)}(\alpha). \end{aligned} \tag{3.103}$$

The derivative of the delta function, called a tempered distribution, is more singular than a delta function, and has meaning only under the integral sign,

e.g. for some function $F(\alpha, \alpha^*)$

$$\int F(\alpha, \alpha^*) \frac{\partial^{2n}}{\partial\alpha^n \partial\alpha^{*n}} \delta^{(2)}(\alpha) d^2\alpha = \left[ \frac{\partial^{2n} F(\alpha, \alpha^*)}{\partial\alpha^n \partial\alpha^{*n}} \right]_{\substack{\alpha^*=0 \\ \alpha=0}} . \tag{3.104}$$

At this point we introduce the optical equivalence theorem of Sudarshan [10]. Suppose we have a "normally ordered" function of the operators $\hat{a}$ and $\hat{a}^\dagger$, $G^{(\mathrm{N})}(\hat{a}, \hat{a}^\dagger)$, where the annihilation operators stand to the right of the creation operators:

$$\hat{G}^{(\mathrm{N})}(\hat{a}, \hat{a}^\dagger) = \sum_n \sum_m C_{nm} (\hat{a}^\dagger)^n \hat{a}^m. \tag{3.105}$$

(It should not be hard to guess what an *antinormally* ordered operator looks like!) The average of this function is

$$\begin{aligned} \langle G^{(\mathrm{N})}(\hat{a}, \hat{a}^\dagger) \rangle &= \mathrm{Tr}\left[ \hat{G}^{(\mathrm{N})}(\hat{a}, \hat{a}^\dagger) \hat{\rho} \right] \\ &= \mathrm{Tr} \int P(\alpha) \sum_n \sum_m C_{nm} (\hat{a}^\dagger)^n a^m |\alpha\rangle \langle\alpha| d^2\alpha \\ &= \int P(\alpha) \sum_n \sum_m C_{nm} \langle\alpha| (\hat{a}^\dagger)^n a^m |\alpha\rangle d^2\alpha \\ &= \int P(\alpha) \sum_n \sum_m C_{nm} \alpha^{*n} \alpha^m d^2\alpha \\ &= \int P(\alpha) G^{(\mathrm{N})}(\alpha, \alpha^*) \, d^2\alpha. \end{aligned} \tag{3.106}$$

The last line is the **optical equivalence theorem**: the expectation value of a normally ordered operator is just the $P$ function weighted average of the function obtained from the operator by the replacements $a \to \alpha$ and $a^\dagger \to \alpha^*$.

Normally ordered operators will be important later, particularly in the discussion of the photoelectric detection of light resulting from the absorption of photons. It is useful to introduce the normal ordering operator denoted : :. For an arbitrary function of $\hat{a}$ and $\hat{a}^\dagger$, $O(\hat{a}, \hat{a}^\dagger)$, we have

$$: O(\hat{a}, \hat{a}^\dagger) : \equiv O^{(\mathrm{N})}(\hat{a}, \hat{a}^\dagger) \tag{3.107}$$

where the commutation relations are to be *disregarded*. The number operator $\hat{n} = \hat{a}^\dagger \hat{a}$ is already normally ordered so

$$\langle \hat{n} \rangle = \langle \hat{a}^\dagger \hat{a} \rangle = \int P(\alpha) |\alpha|^2 d^2\alpha. \tag{3.108}$$

But $\hat{n}^2 = \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a}$ is not normally ordered. Thus

$$: \hat{n}^2 : = (\hat{a}^\dagger)^2 \hat{a}^2$$

and

$$\langle : \hat{n}^2 : \rangle = \langle (\hat{a}^\dagger)^2 \hat{a}^2 \rangle = \int P(\alpha) |\alpha|^4 d^2\alpha. \tag{3.109}$$

The utility of the normal ordering operator and the optical equivalence theorem will be evident in Chapter 7.

We can, under suitable conditions, represent other operators besides $\hat{\rho}$ in the "diagonal" coherent state form – sometimes called the $P$-representation. For operator $\hat{B}$, the $P$-representation is

$$\hat{B} = \int B_p(\alpha, \alpha^*) \, |\alpha\rangle \langle\alpha| \, d^2\alpha. \tag{3.110}$$

The average of $\hat{B}$ is given by

$$\begin{aligned} \langle \hat{B} \rangle &= \mathrm{Tr}(\hat{B}\hat{\rho}) \\ &= \sum_n \langle n| \int B_p(\alpha, \alpha^*) \, |\alpha\rangle\langle\alpha|\hat{\rho}|n\rangle d^2\alpha \\ &= \int B_p(\alpha, \alpha^*) \, \langle\alpha|\hat{\rho}|\alpha\rangle d^2\alpha. \end{aligned} \tag{3.111}$$

Evidently, the expectation value of the density operator with respect to the coherent state also plays the role of a phase-space probability distribution. This is usually called the $Q$, or Husimi, function [12]:

$$Q(\alpha) = \langle\alpha| \, \hat{\rho} \, |\alpha\rangle/\pi. \tag{3.112}$$

For $\hat{B} = \hat{I}$ we obtain the normalization condition

$$\int Q(\alpha) d^2\alpha = 1. \tag{3.113}$$

Unlike the $P$ function, the $Q$ function is positive for all quantum states. Of course, we can always define the corresponding $Q$-representation of the operator $\hat{B}$ as just the expectation value with respect to the coherent state:

$$\begin{aligned} B_Q(\alpha, \alpha^*) &\equiv \langle\alpha| \, \hat{B} \, |\alpha\rangle \\ &= e^{-|\alpha|^2} \sum_n \sum_m \frac{B_{nm}}{(n!m!)^{1/2}} \, (\alpha^*)^n \, (\alpha)^m \end{aligned} \tag{3.114}$$

where $B_{nm} = \langle n|\hat{B}|m\rangle$. Once again, let us calculate $\langle \hat{B} \rangle$, but now we write $\hat{\rho}$ in the $P$-representation:

$$\begin{aligned} \langle \hat{B} \rangle = \mathrm{Tr}(\hat{B}\hat{\rho}) &= \mathrm{Tr} \int \hat{B} P(\alpha) |\alpha\rangle\langle\alpha| d^2\alpha \\ &= \int P(\alpha) \langle\alpha|\hat{B}|\alpha\rangle d^2\alpha \\ &= \int P(\alpha) B_Q(\alpha, \alpha^*) d^2\alpha. \end{aligned} \tag{3.115}$$

Thus if we use the $P$-representation of $\hat{\rho}$ we need the $Q$-representation of $\hat{B}$, or, from Eq. (3.111), if we use the $P$-representation of $\hat{B}$ we need the $Q$-representation of $\hat{\rho}$.

The $Q$ function has the character of a probability distribution in the sense of positivity whereas the $P$ function is really a quasi-probability distribution.

There is, in fact, one other important quasi-probability distribution over phase space, namely, the Wigner function. The Wigner function seems to be the earliest introduced of the phase-space quasi-probability distributions, making its debut in 1932 [13]. It is defined, for an arbitrary density operator $\hat{\rho}$, as

$$W(q,p) \equiv \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\left\langle q+\frac{1}{2}x\left|\hat{\rho}\right|q-\frac{1}{2}x\right\rangle e^{ipx/\hbar}dx \tag{3.116}$$

where $|q \pm \frac{1}{2}x\rangle$ are the eigenkets of the position operator. If the state in question is a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$ then

$$W(q,p) \equiv \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\psi^*\left(q-\frac{1}{2}x\right)\psi\left(q+\frac{1}{2}x\right)e^{ipx/\hbar}dx \tag{3.117}$$

where $\langle q+\frac{1}{2}x \mid \psi\rangle = \psi(q+\frac{1}{2}x)$, etc. Integrating over the momentum we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}W(q,p)\,dp &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\psi^*\left(q-\frac{1}{2}x\right)\psi\left(q+\frac{1}{2}x\right)\int_{-\infty}^{\infty}e^{ipx/\hbar}dpdx\\ &= \int_{-\infty}^{\infty}\psi^*\left(q-\frac{1}{2}x\right)\psi\left(q+\frac{1}{2}x\right)\delta(x)dx\\ &= |\psi(q)|^2\end{aligned} \tag{3.118}$$

which is the probability density for position variable $q$. Likewise, integrating over $q$ we obtain

$$\int_{-\infty}^{\infty}W(q,p)\,dq = |\varphi(p)|^2 \tag{3.119}$$

where $\varphi(p)$ is the momentum space wave related to the coordinate space wave function $\psi(q)$ by a Fourier transform. The right-hand side of Eq. (3.119) is, of course, just the probability density in momentum space. But $W(q,p)$ itself is not a true probability distribution since it can take on negative values for some non-classical states, as we shall show. Like the other distributions, the Wigner function can be used to calculate averages. However, functions of the operators $\hat{q}$ and $\hat{p}$ to be averaged must be Weyl, or symmetrically, ordered in terms of these operators. For example, the classical function $qp$ must be replaced by $(\hat{q}\hat{p}+\hat{p}\hat{q})/2$ and

$$\langle\hat{q}\hat{p}+\hat{p}\hat{q}\rangle = \int(qp+pq)\,W(q,p)dqdp\,. \tag{3.120}$$

In general, if $\{G(\hat{q},\hat{p})\}_{\mathrm{W}}$ is a Weyl-ordered function, where the bracket $\{\ \}_{\mathrm{W}}$ means Weyl, or symmetric, ordering, then

$$\langle\{G(\hat{q},\hat{p})\}_{\mathrm{W}}\rangle = \int\{G(\hat{q},\hat{p})\}_{\mathrm{W}}\,W(q,p)dqdp \tag{3.121}$$

is the corresponding phase-space average.

## 3.8 Characteristic functions

Consider for a moment a classical random variable that we denote $x$. Suppose that $\rho(x)$ is a classical probability density associated with the variable $X$. Then it follows that

$$\rho(x) \geq 0 \tag{3.122}$$

and

$$\int \rho(x)dx = 1. \tag{3.123}$$

The $n$th moment of $x$ is defined as

$$\langle x^n \rangle = \int dx\, x^n \rho(x). \tag{3.124}$$

If all moments $\langle x^n \rangle$ are known then $\rho(x)$ is completely specified. This can be seen by introducing the characteristic function

$$\begin{aligned} C(k) &= \langle e^{ikx} \rangle = \int dx\, e^{ikx} \rho(x) \\ &= \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle x^n \rangle . \end{aligned} \tag{3.125}$$

Evidently, the probability density is just the Fourier transform of the characteristic function:

$$\rho(x) = \frac{1}{2\pi} \int dk e^{-ikx} C(k). \tag{3.126}$$

Thus, if all the moments $\langle x^n \rangle$ are known, $C(k)$ is known and thus $\rho(x)$ is known. On the other hand if we are given the characteristic function, we can calculate the moments according to

$$\langle x^n \rangle = \frac{1}{i^n} \left. \frac{d^n C(k)}{dk^n} \right|_{k=0}. \tag{3.127}$$

We now proceed to introduce quantum mechanical characteristic functions. There are, in fact, three such functions, namely

$$C_{\mathrm{W}}(\lambda) = \mathrm{Tr}[\hat{\rho}\, e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}}] = \mathrm{Tr}[\hat{\rho}\, \hat{D}(\lambda)] \quad \text{(Wigner)} \tag{3.128a}$$

$$C_{\mathrm{N}}(\lambda) = \mathrm{Tr}[\hat{\rho} e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}}] \qquad \text{(normally ordered)} \tag{3.128b}$$

$$C_{\mathrm{A}}(\lambda) = \mathrm{Tr}[\hat{\rho} e^{-\lambda \hat{a}^\dagger} e^{\lambda^* \hat{a}}] \qquad \text{(antinormally ordered).} \tag{3.128c}$$

These functions are related through the disentangling theorem of Eq. (3.32) according to

$$C_{\mathrm{W}}(\lambda) = C_{\mathrm{N}}(\lambda)\, e^{-\frac{1}{2}|\lambda|^2} = C_{\mathrm{A}}(\lambda)\, e^{\frac{1}{2}|\lambda|^2}. \tag{3.129}$$

Furthermore, it is easy to show that

$$\langle(\hat{a}^\dagger)^m\hat{a}^n\rangle = \mathrm{Tr}[\hat{\rho}(\hat{a}^\dagger)^m\hat{a}^n] = \frac{\partial^{(m+n)}}{\partial\lambda^m\partial(-\lambda^*)^n}C_{\mathrm{N}}(\lambda)\bigg|_{\lambda=0} \tag{3.130a}$$

$$\langle\hat{a}^m(\hat{a}^\dagger)^n\rangle = \mathrm{Tr}[\hat{\rho}\,\hat{a}^m(\hat{a}^\dagger)^n] = \frac{\partial^{(m+n)}}{\partial\lambda^n\partial(-\lambda^*)^m}C_{\mathrm{A}}(\lambda)\bigg|_{\lambda=0} \tag{3.130b}$$

$$\langle\{(\hat{a}^\dagger)^m\hat{a}^n\}_{\mathrm{W}}\rangle = \mathrm{Tr}[\hat{\rho}\,\{(\hat{a}^\dagger)^m\hat{a}^n\}_{\mathrm{W}}] = \frac{\partial^{(m+n)}}{\partial\lambda^m\partial(-\lambda^*)^n}C_{\mathrm{W}}(\lambda)\bigg|_{\lambda=0}. \tag{3.130c}$$

Instead of the three different characteristic functions, we can introduce the $s$-parameterized function of Cahill and Glauber [14]:

$$C(\lambda, s) = \mathrm{Tr}[\hat{\rho}\exp(\lambda\hat{a}^\dagger - \lambda^*\hat{a} + s|\lambda|^2/2)] \tag{3.131}$$

such that $C(\lambda, 0) = C_{\mathrm{W}}(\lambda)$, $C(\lambda, 1) = C_{\mathrm{N}}(\lambda)$, and $C(\lambda, -1) = C_{\mathrm{A}}(\lambda)$.

The connections between these characteristic functions and the various quasi-probability distributions will now be made. For example, the antinormally ordered characteristic function may be written as

$$\begin{aligned} C_{\mathrm{A}}(\lambda) &= \mathrm{Tr}[\hat{\rho}e^{-\lambda^*\hat{a}}e^{\lambda\hat{a}^\dagger}] \\ &= \mathrm{Tr}[e^{\lambda\hat{a}^\dagger}\hat{\rho}\,e^{-\lambda^*\hat{a}}] \\ &= \frac{1}{\pi}\int d^2\alpha\,\langle\alpha|e^{\lambda\hat{a}^\dagger}\hat{\rho}\,e^{-\lambda^*\hat{a}}|\alpha\rangle \\ &= \int d^2\alpha\,Q(\alpha)e^{\lambda\alpha^*-\lambda^*\alpha}, \end{aligned} \tag{3.132}$$

which is just a two-dimensional Fourier transform of the $Q$ function. The inverse Fourier transform yields:

$$Q(\alpha) = \frac{1}{\pi^2}\int C_{\mathrm{A}}(\lambda)e^{\lambda^*\alpha-\lambda\alpha^*}d^2\lambda. \tag{3.133}$$

Now consider the normally ordered characteristic function. Writing $\hat{\rho}$ in the $P$-representation we have

$$\begin{aligned} C_{\mathrm{N}}(\lambda) &= \mathrm{Tr}[\hat{\rho}\,e^{\lambda\hat{a}^\dagger}e^{-\lambda^*\hat{a}}] \\ &= \int P(\alpha)\langle\alpha|\,e^{\lambda\hat{a}^\dagger}e^{-\lambda^*\hat{a}}|\alpha\rangle d^2\alpha \\ &= \int P(\alpha)\,e^{\lambda\alpha^*-\lambda^*\alpha}d^2\alpha \end{aligned} \tag{3.134}$$

which is just the Fourier transform of the $P$ function. The inverse transform yields

$$P(\alpha) = \frac{1}{\pi^2}\int e^{\lambda^*\alpha-\lambda\,\alpha^*}C_{\mathrm{N}}(\lambda)d^2\lambda. \tag{3.135}$$

And finally, and perhaps to no great surprise, it turns out that the Wigner function may be obtained as the Fourier transform of the Weyl-ordered

characteristic function:

$$W(\alpha) \equiv \frac{1}{\pi^2}\int \exp(\lambda^*\alpha - \lambda\alpha^*)\, C_{\rm W}(\lambda) d^2\lambda$$
$$= \frac{1}{\pi^2}\int \exp(\lambda^*\alpha - \lambda\alpha^*)\, C_{\rm N}(\lambda) e^{-|\lambda|^2/2}\, d^2\lambda. \tag{3.136}$$

This definition is equivalent to the previous one with the proper interpretation of variables.

As an application of the characteristic function, we now derive the $P$ function corresponding to the thermal, or chaotic, state of the field. Recall that the field in this case is a mixed state given by density operator $\hat{\rho}_{\rm Th}$ of Eq. (2.144). We first calculate the $Q$ function for this density operator according to

$$Q(\alpha) = \langle\alpha|\hat{\rho}_{\rm Th}|\alpha\rangle/\pi$$
$$= \frac{1}{\pi}\, e^{-|\alpha|^2} \sum_n \sum_m \langle m|\hat{\rho}_{\rm Th}\,|n\rangle\, \frac{(\alpha^*)^m\, \alpha^n}{(m!n!)^{1/2}}$$
$$= \frac{e^{-|\alpha|^2}}{\pi\,(1+\bar{n})}\, \sum_m \left(\frac{\bar{n}}{1+\bar{n}}\right)^n \frac{(\alpha^*\alpha)^n}{n!}$$
$$= \frac{1}{\pi\,(1+\bar{n})} \exp\left(-\frac{|\alpha|^2}{1+\bar{n}}\right), \tag{3.137}$$

where $\bar{n}$ for the thermal state is given by Eq. (2.141). Then from Eq. (3.132) we have

$$C_{\rm A}(\lambda) = \frac{1}{\pi\,(1+\bar{n})}\int d^2\alpha \exp\left(-\frac{|\alpha|^2}{1+\bar{n}}\right) e^{\lambda\alpha^* - \lambda^*\alpha}. \tag{3.138}$$

Now letting $\alpha = (q+ip)/\sqrt{2}$, $\lambda = (x+iy)/\sqrt{2}$ where $d^2\alpha = dqdp/2$ we have

$$C_{\rm A}(x,y) = \frac{1}{2\pi\,(1+\bar{n})}\int \exp\left[-\frac{(q^2+p^2)}{2(1+\bar{n})}\right]$$
$$\times \exp[i(yq - xp)] dqdp. \tag{3.139}$$

Using the standard Gaussian integral

$$\int e^{-as^2} e^{\pm\beta s} ds = \sqrt{\frac{\pi}{a}}\, e^{\beta^2/4a}, \tag{3.140}$$

we straightforwardly obtain

$$C_{\rm A}(\lambda) = \exp[-(1+\bar{n})|\lambda|^2]. \tag{3.141}$$

But from Eq. (3.129) we have $C_{\rm N}(\lambda) = C_{\rm A}(\lambda)\exp(|\lambda|^2)$ and thus from Eq. (3.135) we finally obtain

$$P(\alpha) = \frac{1}{\pi^2}\int \exp(-\bar{n}|\lambda|^2)\, e^{\lambda^*\alpha - \lambda\alpha^*} d^2\lambda$$
$$= \frac{1}{\pi\bar{n}}\, \exp\left(-\frac{|\alpha|^2}{\bar{n}}\right). \tag{3.142}$$

This is Gaussian so it may be interpreted as a true probability distribution.

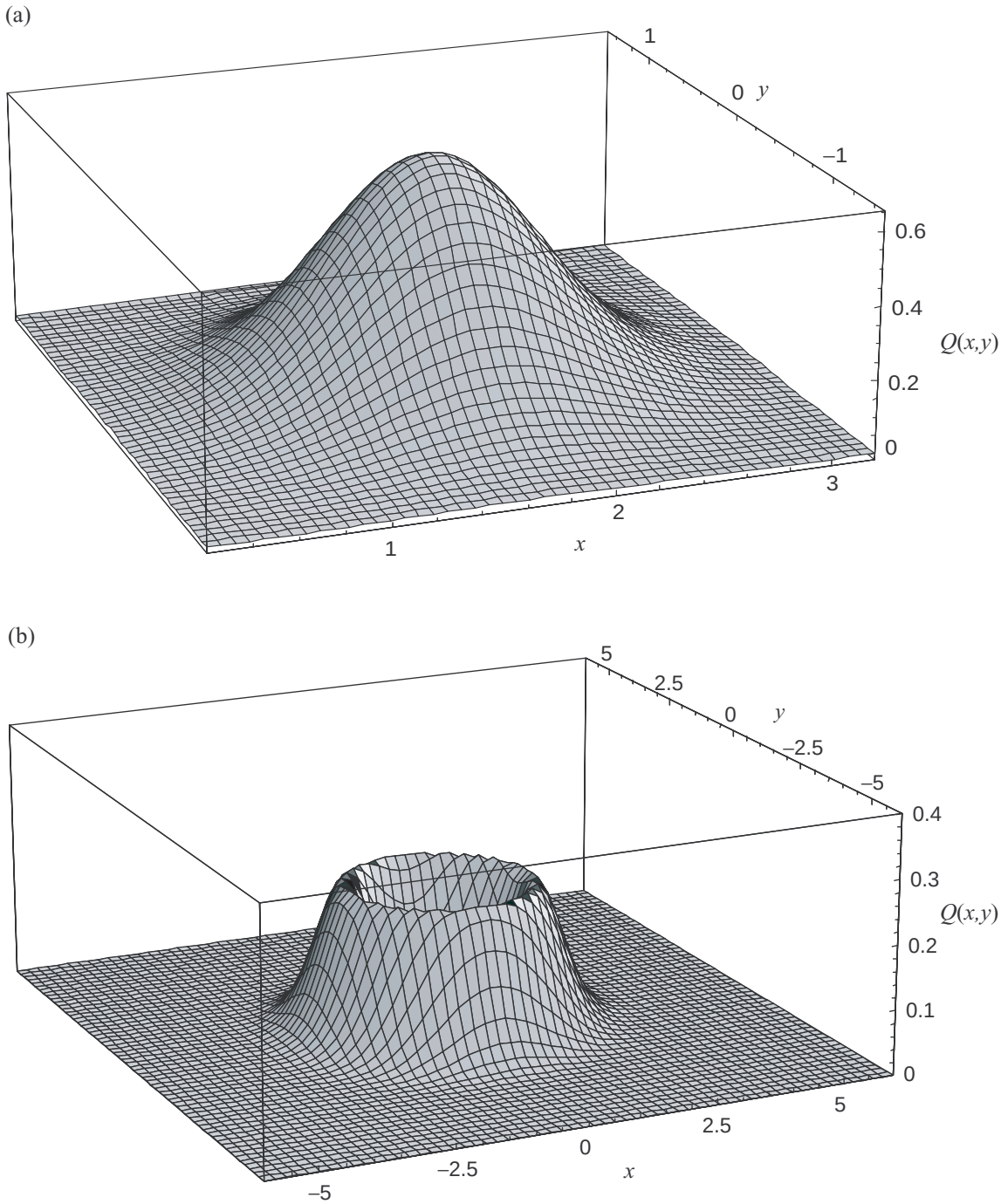


**Fig. 3.9.** $Q$ function for (a) a coherent state with $\bar{n} = 10$, (b) a number state with $n = 3$.

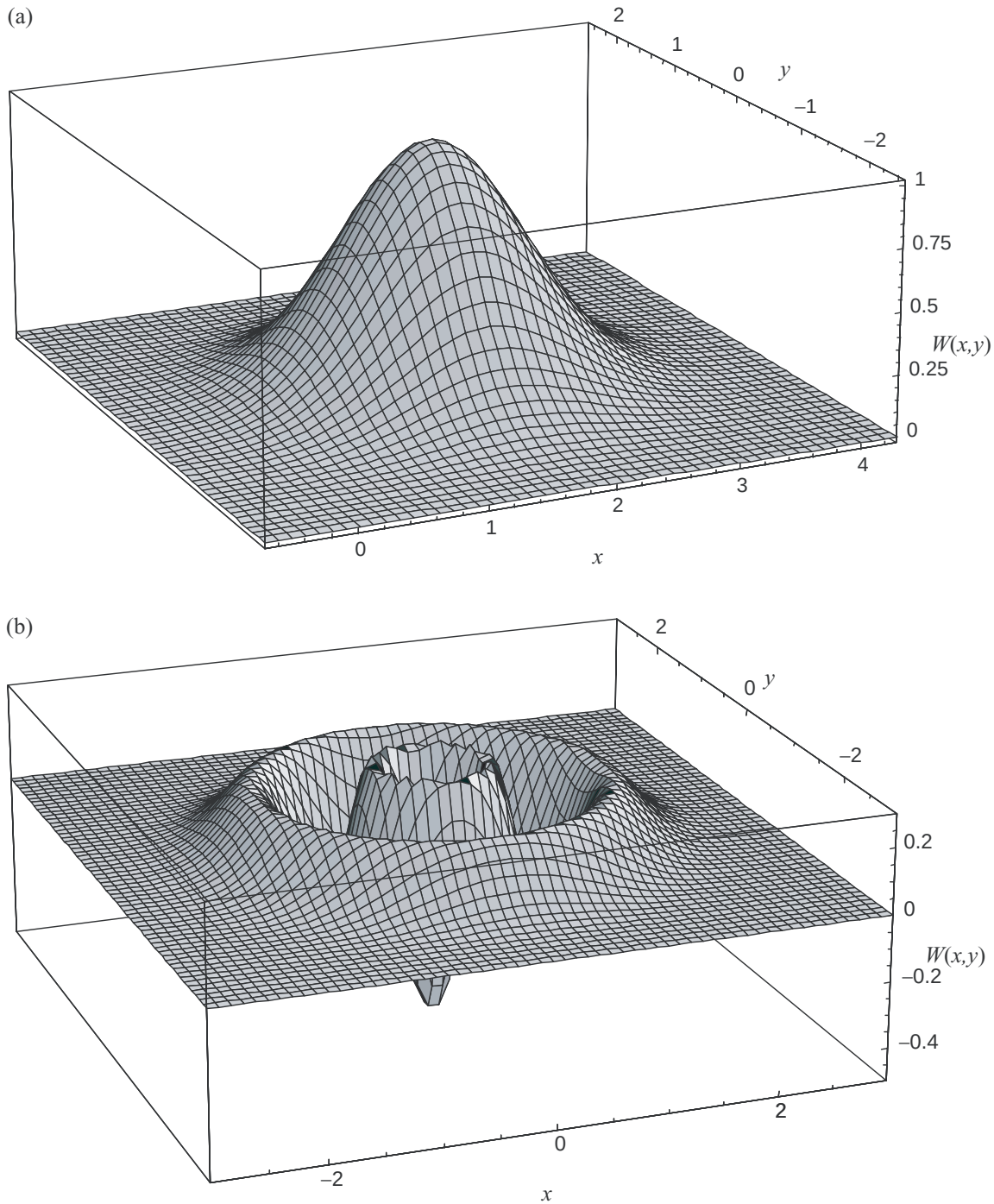


**Fig. 3.10.** Wigner function for (a) a coherent state with $\bar{n} = 10$, (b) a number state with $n = 3$.

Finally, to conclude this chapter, we examine the $Q$ and Wigner functions for the most classical of the quantum states, the coherent state, and for the most quantum-mechanical state, the number state. For the coherent state $\hat{\rho} = |\beta\rangle\langle\beta|$ one easily has the $Q$ function

$$Q(\alpha) = \frac{1}{\pi}|\langle\alpha \mid \beta\rangle|^2 = \frac{1}{\pi}\exp(-|\alpha - \beta|^2), \tag{3.143}$$

whereas for the number state $\hat{\rho} = |n\rangle\langle n|$,

$$Q(\alpha) = \frac{1}{\pi}|\langle\alpha \mid n\rangle|^2 = \frac{1}{\pi}\exp(-|\alpha|^2)\frac{|\alpha|^{2n}}{n!}. \tag{3.144}$$

Setting $\alpha = x + iy$, we plot these as functions of $x$ and $y$ in Fig. 3.9.

The $Q$ function of the coherent state is just a Gaussian centered at $\beta$ while the state for a number state is rather annular of center radius $r \sim n$. Note how these functions correlate with the phase-space figures introduced in Section 3.6. The corresponding Wigner functions, obtained from Eq. (3.136), are

$$W(\alpha) = \frac{2}{\pi}\exp(-2|\alpha - \beta|^2) \tag{3.145}$$

for the coherent state $|\beta\rangle$, and for a number state $|n\rangle$

$$W(\alpha) = \frac{2}{\pi}(-1)^n L_n(4|\alpha|^2)\exp(-2|\alpha|^2), \tag{3.146}$$

where $L_n(\zeta)$ is a Laguerre polynomial. (The derivation of these functions is left as an exercise. See Problem 3.12.) We plot these functions in Fig. 3.10, again with $\alpha = x + iy$. Evidently, the $Q$ and Wigner functions for the coherent state are identical apart from an overall scale factor. But for the number state we see that the Wigner function oscillates and becomes negative over a wide region of phase space. The $Q$ function, of course, can never become negative. It is always a probability distribution. But the Wigner function is not always positive, the Wigner function of the number state being a case in point, in which case it is not a probability distribution. A state whose Wigner function takes on negative values over some region of phase space is *nonclassical.* However, the converse is not necessarily true. A state can be nonclassical yet have a non-negative Wigner function. As we said earlier, for nonclassical states, the $P$ function becomes negative or more singular than a delta function over some region of phase space. The squeezed states are strongly nonclassical in this sense, as we shall discuss in Chapter 7, yet their Wigner functions are always positive. Nevertheless, for these and other nonclassical states, the Wigner function is still of primary interest as the $P$ function generally cannot be written down as a function in the usual sense, whereas this can always be done for the Wigner function. Furthermore, the Wigner function can be more sensitive to the quantum nature of some states than can the $Q$ function, as we have seen for the number state. More importantly, as we shall see in Chapter 7 (and below in Problem 12), the Wigner function can display the *totality* of interference effects associated with a quantum state [15]. Finally, it happens that it is possible to reconstruct the Wigner from experimental

data, a procedure known as quantum-state tomography [16]. A discussion of this procedure would take us beyond the scope of this book, but we refer the reader to the references and bibliography for this chapter.

## Problems

1. Investigate the possible existence of *right* eigenstates of the creation operator $\hat{a}^\dagger$.
2. Show, perhaps with the help of phase-space graphics, for a coherent state $|\alpha\rangle$, of average photon number $\bar{n} = |\alpha|^2$, that the phase uncertainty for the state is $\Delta\phi = 1/(2\bar{n})^{1/2}$ if $\bar{n} \gg 1$.
3. Complete the derivation of the wave function for the coherent state given by Eq. (3.47).
4. Prove the following identities:

$$\hat{a}^\dagger |\alpha\rangle \langle\alpha| = \left(\alpha^* + \frac{\partial}{\partial\alpha}\right) |\alpha\rangle \langle\alpha| ,$$
$$|\alpha\rangle \langle\alpha| \hat{a} = \left(\alpha + \frac{\partial}{\partial\alpha^*}\right) |\alpha\rangle \langle\alpha| .$$

5. Verify Eq. (3.16), that the quantum fluctuations of the field quadrature operators are the same as for the vacuum when the field is in a coherent state.
6. For a coherent state $|\alpha\rangle$, evaluate the factorial moments

$$\langle \hat{n}\,(\hat{n}-1)\,(\hat{n}-2)\ldots(\hat{n}-r+1)\rangle .$$

7. For a coherent state $|\alpha\rangle$ evaluate the sine and cosine operators of Eqs. (2.211) and their squares. (You will not get closed forms.) Examine the limit where the average photon number $\bar{n} = |\alpha|^2 \gg 1$. Then examine the uncertainty products of Eqs. (2.215) and (2.216) in this limit.
8. Consider again the exponential phase operator $\hat{E} = (\hat{n}+1)^{-1/2}\hat{a}$. (a) Seek normalized right eigenstates $|z\rangle$ of this operator satisfying the equation $\hat{E}|z\rangle = z|z\rangle$ where $z$ is a complex number. Is there a restriction on the range of $|z|$? (b) Can one resolve unity with these states? (c) Examine the states for the case where $|z| \to 1$. How is this special case related to the phase states $|\phi\rangle$ of Eq. (2.221)? Is there a conflict with Eq. (2.223)? (d) Obtain the photon number distribution for this state and express it in terms of the average photon number $\bar{n}$ for the state. Does it look like anything you may have already seen in this book? (e) Obtain the phase distribution for $|z\rangle$.
9. Work out the normally ordered variance of the photon number operator, $\langle : (\Delta\hat{n})^2 : \rangle \equiv \langle : \hat{n}^2 : \rangle - \langle : \hat{n} : \rangle^2$, where $: \hat{n} := \hat{n} = \hat{a}^\dagger\hat{a}$ is already normally ordered, in terms of the $P$ function. Show that for a coherent state $\langle : (\Delta\hat{n})^2 : \rangle = 0$. Suppose now that for some quantum state we find that $\langle : (\Delta\hat{n})^2 : \rangle < 0$. What does this say about the $P$ function for the state? Can such a state be considered a classical state?
10. Work out the normally ordered variances $\langle : (\Delta\hat{X})_i^2 : \rangle$, $i = 1, 2$, of the quadrature operators in terms of the $P$ function. Show that these variances vanish for a coherent state. Examine the conditions under which some quantum state might yield $\langle : (\Delta\hat{X})_i^2 : \rangle < 0$ for either $i = 1$ or 2. Can this condition hold simultaneously for both quadratures?

11. Show that the two expressions given for the Wigner function, Eqs. (3.116) and (3.136), are equivalent.
12. Carry through the derivation of the Wigner functions of Eqs. (3.145) and (3.146) for the coherent state and the number state, respectively.
13. Consider the superposition state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\beta\rangle + |-\beta\rangle\right)$$

where the $|\pm\beta\rangle$ are coherent states. (a) Show that this state is normalized for the case where $|\beta|^2 \gg 1$. (b) Obtain the photon number probability distribution. (c) Obtain the phase distribution. (d) Obtain the $Q$ and Wigner functions for this state and display them as three-dimensional plots. Is $|\psi\rangle$ a classical state?

14. Show that for some state $|\psi\rangle$, the Wigner function can be written in the form

$$W(\alpha) = \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n \langle\psi|\,\hat{D}(\alpha)\,|n\rangle\langle n|\,\hat{D}^{\dagger}(\alpha)\,|\psi\rangle\,.$$

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Chapter 4

# Emission and absorption of radiation by atoms

In this chapter, we first discuss atom–field interactions using quantum mechanical perturbation theory for a classical driving field and then for a quantum mechanical field. In the latter case, spontaneous emission appears as a fully quantum-mechanical phenomenon. We then examine the so-called Rabi model, a model of a "two-level" atom interacting with a strong near-resonant classical field, introduce the rotating wave approximation, and then introduce the fully quantum mechanical Rabi model, better known as the Jaynes–Cummings model. Differences in the evolution predicted by the two-models are drawn out, where we show that the Jaynes–Cummings model predicts behavior that has no semiclassical analog and that depends entirely on the discreteness of photons. Finally, we study an extension of the Jaynes–Cummings model, the dispersive model, for the case where the quantized field is far out-of-resonance with the atomic transition frequency. This will eventually allow us to describe how superpositions of different coherent states of the field, states known as Schrödinger cat states, can be generated from atom–field interactions.

## 4.1 Atom–field interactions

To begin, let us suppose that the Hamiltonian of an electron bound to an atom in the absence of external fields, in the configuration representation, is given by

$$\hat{H}_0 = \frac{1}{2m}\hat{\mathbf{P}}^2 + V(r) \tag{4.1}$$

where $V(r)$ is the usual Coulomb interaction binding the electron to the nucleus and $r = |\mathbf{r}|$. In the configuration space representation $\hat{\mathbf{P}} = -i\nabla$, $\hat{\mathbf{r}}|\mathbf{r}\rangle = \mathbf{r}|\mathbf{r}\rangle$ and the wave functions are given by $\psi(\mathbf{r}) = \langle \mathbf{r} \mid \psi \rangle$. We assume that energy eigenstates $|k\rangle$ of $\hat{H}_0$, satisfying the time-independent Schrödinger equation

$$\hat{H}_0 \psi_k^{(0)}(\mathbf{r}) = E_k \psi_k^{(0)}(\mathbf{r}), \tag{4.2}$$

where $\langle \mathbf{r} \mid k \rangle = \psi_k^{(0)}(\mathbf{r})$, are known. In the presence of external fields the Hamiltonian is modified to

$$\hat{H}(\mathbf{r}, t) = \frac{1}{2m}[\hat{\mathbf{P}} + e\mathbf{A}(\mathbf{r}, t)]^2 - e\,\Phi(\mathbf{r}, t) + V(r) \tag{4.3}$$

where $\mathbf{A}(\mathbf{r}, t)$ and $\Phi(\mathbf{r}, t)$ are the vector and scalar potentials respectively of the external field and where $-e$ is the electron charge, $e$ taken to be positive. The fields themselves are given by

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= -\nabla\Phi(\mathbf{r}, t) - \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}, \\ \mathbf{B}(\mathbf{r}, t) &= \nabla \times \mathbf{A}(\mathbf{r}, t), \end{aligned} \tag{4.4}$$

and are invariant under the gauge transformations

$$\begin{aligned} \Phi'(\mathbf{r}, t) &= \Phi(\mathbf{r}, t) - \frac{\partial \chi(\mathbf{r}, t)}{\partial t}, \\ \mathbf{A}'(\mathbf{r}, t) &= \mathbf{A}(\mathbf{r}, t) + \nabla\chi(\mathbf{r}, t). \end{aligned} \tag{4.5}$$

The time-dependent Schrödinger equation is

$$\hat{H}(\mathbf{r}, t)\Psi(\mathbf{r}, t) = i\,\hbar\frac{\partial \Psi(\mathbf{r}, t)}{\partial t}. \tag{4.6}$$

In order eventually to simplify the form of the atom–field interaction, we define a unitary operator $\hat{R}$ such that $\Psi'(\mathbf{r}, t) \equiv \hat{R}\,\Psi(\mathbf{r}, t)$. We have

$$\hat{H}'\Psi'(\mathbf{r}, t) = i\,\hbar\frac{\partial \Psi'(\mathbf{r}, t)}{\partial t} \tag{4.7}$$

where

$$\hat{H}' = \hat{R}\hat{H}\hat{R}^{\dagger} + i\,\hbar\frac{\partial \hat{R}}{\partial t}\hat{R}^{\dagger}. \tag{4.8}$$

We now choose $\hat{R} = \exp(-ie\chi(\mathbf{r}, t)/\hbar)$ so that (using $\hat{\mathbf{P}} = -i\,\hbar\nabla$)

$$\hat{H}' = \frac{1}{2m}[\hat{\mathbf{P}} + e\mathbf{A}']^2 - e\,\Phi' + V(r) \tag{4.9}$$

where $\mathbf{A}'$ and $\Phi'$ are given by Eq. (4.5). At this point we make a definite choice of gauge, namely the Coulomb (or radiation) gauge, for which $\Phi = 0$ and $\mathbf{A}$ satisfies the transversality condition $\nabla \cdot \mathbf{A} = 0$. The vector potential $\mathbf{A}$, for no sources near the atom, satisfies the wave equation

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \tag{4.10}$$

This choice of gauge is not relativistically invariant, in contrast to the Lorentz gauge, but the domain of quantum optics is, for the most part, nonrelativistic so that no inconsistency will be introduced. The Coulomb gauge has the advantage that the radiation field is completely described by the vector potential, as is obvious from Eq. (4.3), which, in this gauge, reads

$$\begin{aligned} \hat{H}(\mathbf{r}, t) &= \frac{1}{2m}[\hat{\mathbf{P}} + e\mathbf{A}(\mathbf{r}, t)]^2 + V(r) \\ &= \frac{\hat{\mathbf{P}}^2}{2m} + \frac{e}{m}\,\mathbf{A}\cdot\hat{\mathbf{P}} + \frac{e^2}{2m}\mathbf{A}^2 + V(r). \end{aligned} \tag{4.11}$$

Equation (4.9) now reads

$$\hat{H}'(\mathbf{r},t) = \frac{1}{2m}[\hat{\mathbf{P}} + e(\mathbf{A} + \nabla\chi)]^2 + e\frac{\partial\chi}{\partial t} + V(r). \tag{4.12}$$

The solution of the wave equation (4.10) has the form

$$\mathbf{A} = \mathbf{A}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + c.c. \tag{4.13}$$

where $|\mathbf{k}| = 2\pi/\lambda$ is the wave vector of the radiation. For $|\mathbf{r}|$ of typical atomic dimensions (a few Ångströms) and $\lambda$ of typical optical wavelengths (a few hundred nanometers in the range 400–700 nm), $\mathbf{k}\cdot\mathbf{r} \ll 1$ so that over the extent of an atom, the vector potential is spatially uniform, $\mathbf{A}(\mathbf{r},t) \simeq \mathbf{A}(t)$. This is the so-called dipole approximation. We now choose the gauge function $\chi(\mathbf{r},t) = -\mathbf{A}(t)\cdot\mathbf{r}$. With this choice,

$$\begin{aligned} \nabla\chi(\mathbf{r},t) &= -\mathbf{A}(t), \\ \frac{\partial\chi}{\partial t}(\mathbf{r},t) &= -\mathbf{r}\cdot\frac{\partial\mathbf{A}}{\partial t} = -\mathbf{r}\cdot\mathbf{E}(t), \end{aligned} \tag{4.14}$$

and thus

$$\hat{H}' = \frac{\hat{\mathbf{P}}^2}{2m} + V(r) + e\,\mathbf{r}\cdot\mathbf{E}(t). \tag{4.15}$$

This equation contains only one interaction term (within the dipole approximation) as opposed to the two terms of Eq. (4.11). We shall work with $\hat{H}'$ in all that follows, the interaction being in what is sometimes called the "length" gauge. The quantity $-e\mathbf{r}$ is the dipole moment: $\mathbf{d} = -e\mathbf{r}$. In general, i.e. for an unspecified representation, the dipole moment is an operator, $\hat{\mathbf{d}}$. We shall denote it as such in what follows. Thus we write

$$\hat{H}' = \hat{H}_0 - \hat{\mathbf{d}}\cdot\mathbf{E}(t) \tag{4.16}$$

where $\hat{H}_0$ is given by Eq. (4.1).

## 4.2 Interaction of an atom with a classical field

So far, we have not specified the nature of the interacting field and have not even stated whether we consider the field to be classical or quantum mechanical. The derivation leading to Eq. (4.16) is valid for both classical and quantum fields. But eventually we want to demonstrate differences in the way an atom behaves when interacting with classical or quantum fields. With that in mind, we first turn to the case when an atom is driven by a classical sinusoidal electric field.

We assume that the field has the form $\mathbf{E}(t) = \mathbf{E}_0\cos(\omega t)$, $\omega$ being the frequency of the radiation, and that this field is abruptly turned on at $t = 0$. The

dipole approximation, where we assume that $\mathbf{k}\cdot\mathbf{r} \ll 1$ over the atom, has already been made. We further assume that the initial state of the atom is $|i\rangle$ where $\hat{H}_0|i\rangle = E_i|i\rangle$. For times $t > 0$ we expand the state vector $|\psi(t)\rangle$ in terms of the complete set of uncoupled atomic states $|k\rangle$:

$$|\psi(t)\rangle = \sum_k C_k(t) e^{-iE_k t/\hbar}|k\rangle, \tag{4.17}$$

where the time-dependent amplitudes $C_k(t)$ satisfy the normalization requirement

$$\sum_k |C_k(t)|^2 = 1. \tag{4.18}$$

Substituting this expansion into the time-dependent Schrödinger equation

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \left(\hat{H}_0 + \hat{H}^{(\mathrm{I})}\right)|\psi(t)\rangle, \tag{4.19}$$

where $H^{(\mathrm{I})} = -\hat{\mathbf{d}}\cdot\mathbf{E}(t)$, then multiplying from the left by $\langle l|e^{iE_l t/\hbar}$ leads to the set of coupled first-order differential equations for the amplitudes

$$\dot{C}_l(t) = -\frac{i}{\hbar}\sum_k C_k(t)\langle l|\hat{H}^{(\mathrm{I})}|k\rangle e^{i\omega_{lk}t} \tag{4.20}$$

where the $\omega_{lk} = (E_l - E_k)/\hbar$ are the transition frequencies between levels $l$ and $k$. These equations are, so far, exact and need to be solved subject to the initial condition $C_i(0) = 1$, only state $|i\rangle$ being initially populated. As time goes forward, population will be lost from state $|i\rangle$ and increased in some initially unpopulated state $|f\rangle$, i.e. the amplitude $C_f(t)$ increases. The probability for the atom to make a transition from state $|i\rangle$ to state $|f\rangle$ in time $t$ is given by

$$P_{i\to f}(t) = C_f^*(t)C_f(t) = |C_f(t)|^2. \tag{4.21}$$

The equations for the amplitudes are solvable only for very simple cases. These days, of course, one might solve the set of differential equations numerically, but in the case of a driving field that in some sense is "weak", we can use a time-dependent perturbation theory [1] approach to the problem. "Weak" in this case means that $|\mathbf{E}_0|$ is small, or actually, that $|\langle f|\hat{\mathbf{d}}\cdot\mathbf{E}_0|i\rangle|$ is small. As a matter of bookkeeping, we write the interaction Hamiltonian as $\lambda\hat{H}^{(\mathrm{I})}$, where $\lambda$ is treated as a number in the range $0 \le \lambda \le 1$. (At the end of the calculations we will always take $\lambda \to 1$.) We then expand the probability amplitude for, say, state $|l\rangle$ in the power series

$$C_l(t) = C_l^{(0)}(t) + \lambda C_l^{(1)}(t) + \lambda^2 C_l^{(2)}(t) + \cdots \tag{4.22}$$

Inserting such expansions into Eqs. (4.20) and equating like powers of $\lambda$ we obtain, up to second order,

$$\dot{C}_l^{(0)} = 0, \tag{4.23}$$

$$\dot{C}_l^{(1)} = -\frac{i}{\hbar}\sum_k C_k^{(0)} H_{lk}^{(\mathrm{I})}(t) e^{i\omega_{lk}t}, \tag{4.24}$$

$$\dot{C}_l^{(2)} = -\frac{i}{\hbar}\sum_k C_k^{(1)} H_{lk}^{(\mathrm{I})}(t) e^{i\omega_{lk}t}, \tag{4.25}$$

where $H_{lk}^{(\mathrm{I})}(t) \equiv \langle l|\hat{H}^{(\mathrm{I})}(t)|k\rangle$. Note the general pattern that relates the $n$th order to the $(n-1)$th order:

$$\dot{C}_l^{(n)} = -\frac{i}{\hbar}\sum_k C_k^{(n-1)}(t) H_{lk}^{(\mathrm{I})}(t) e^{i\omega_{lk}t}. \tag{4.26}$$

The essential assumption underlying the perturbation-theory approach is that the driving field is so weak that the atomic populations change very little. That is, if $C_i(0) = 1,\ C_f(0) = 0\ (f \neq i)$ then for $t > 0$, to a good approximation $C_i(t) \approx 1, |C_f(t)| \ll 1\ (f \neq i)$. Thus, in the first-order equation (4.24), the only term surviving the sum on the right-hand side is for $k = i$ yielding

$$\dot{C}_f^{(1)}(t) = -\frac{i}{\hbar} H_{fi}^{(\mathrm{I})}(t) e^{i\omega_{fi}t} C_i^{(0)}(t) \tag{4.27}$$

or

$$C_f^{(1)}(t) = -\frac{i}{\hbar}\int_0^t dt' H_{fi}^{(\mathrm{I})}(t') e^{i\omega_{fi}t'} C_i^{(0)}(t'). \tag{4.28}$$

Inserting this result into the second-order equation, Eq. (4.25), we obtain

$$\begin{aligned} C_f^{(2)}(t) &= -\frac{i}{\hbar}\sum_l \int_0^t dt' H_{fl}^{(\mathrm{I})}(t') e^{i\omega_{fl}t'} C_l^{(1)}(t') \\ &= \left(-\frac{i}{\hbar}\right)^2 \sum_l \int_0^t dt' \int_0^{t'} dt'' H_{fl}^{(\mathrm{I})}(t') e^{i\omega_{fl}t'} \\ &\quad \times H_{li}^{(\mathrm{I})}(t'') e^{i\omega_{li}t''} C_i^{(0)}(t''). \end{aligned} \tag{4.29}$$

Equation (4.28) gives the amplitude for a transition from state $|i\rangle$ to state $|f\rangle$ while Eq. (4.29) gives the amplitude for a transition from state $|i\rangle$ to states $\{|l\rangle\}$ then to state $|f\rangle$. The total transition probability from state $|i\rangle$ to state $|f\rangle$ is

$$P_{i\to f}(t) = \left|C_f^{(0)}(t) + C_f^{(1)}(t) + C_f^{(2)}(t) + \cdots\right|^2. \tag{4.30}$$

Now the dipole moment operator $\hat{\mathbf{d}}$ has nonvanishing matrix elements only between states of opposite parity. Thus the first-order correction to the amplitude

of the initial state vanishes:

$$C_i^{(1)}(t) = -\frac{i}{\hbar}\int_0^t dt' H_{ii}^{(\mathrm{I})}(t') C_i^{(0)}(t') = 0 \tag{4.31}$$

because $H_{ii}^{(\mathrm{I})}(t) = 0$. Therefore, to first order $C_i(t) = C_i^{(0)}(t) = 1$ so that

$$C_f^{(1)}(t) = -\frac{i}{\hbar}\int_0^t dt' H_{fi}^{(\mathrm{I})}(t')\, e^{i\omega_{fi}t'}. \tag{4.32}$$

With $H^{(\mathrm{I})} = -\hat{\mathbf{d}}\cdot\mathbf{E}_0\cos\omega t$, and by expanding the cosine in terms of exponentials, this integrates to

$$C_f^{(1)}(t) = \frac{1}{2\hbar}(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi} \times\left\{\frac{\left(e^{i(\omega+\omega_{fi})t}-1\right)}{(\omega+\omega_{fi})} - \frac{\left(e^{-i(\omega-\omega_{fi})t}-1\right)}{(\omega-\omega_{fi})}\right\}, \tag{4.33}$$

where $(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi} = \langle f|\hat{\mathbf{d}}\cdot\mathbf{E}_0|i\rangle$. If the frequency of the radiation, $\omega$, is near resonance with the atomic transition frequency $\omega_{fi}$, the second term clearly dominates the first. Therefore we may drop the "antiresonant" first term, making the so-called "rotating wave approximation" (RWA), familiar in the context of magnetic resonance [2]. Thus we have to first order the transition probability

$$P_{i\to f}^{(1)}(t) = \left|C_f^{(1)}(t)\right|^2 = \frac{|(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi}|^2}{\hbar^2}\frac{\sin^2(\Delta t/2)}{\Delta^2} \tag{4.34}$$

where $\Delta = \omega - \omega_{fi}$ is the "detuning" between the radiation field and the atomic transition. When $\Delta \neq 0$, $P_{i\to f}^{(1)}(t)$ maximizes at

$$\left(P_{i\to f}^{(1)}\right)_{\max} = \frac{|(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi}|^2}{\hbar^2}\frac{1}{\Delta^2}. \tag{4.35}$$

For the case of exact resonance, $\Delta = 0$,

$$\left(P_{i\to f}^{(1)}\right)_{\max} = \frac{|(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi}|^2}{4\hbar^2}t^2\,. \tag{4.36}$$

For the perturbation expansion to be valid we must have $(P_{i\to f}^{(1)})_{\max} \ll 1$. For the off-resonance case, this places conditions on both $|(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi}|$ and $\Delta$. For the resonant case, Eq. (4.36) is valid only for very short times. In Fig. 4.1 we plot the

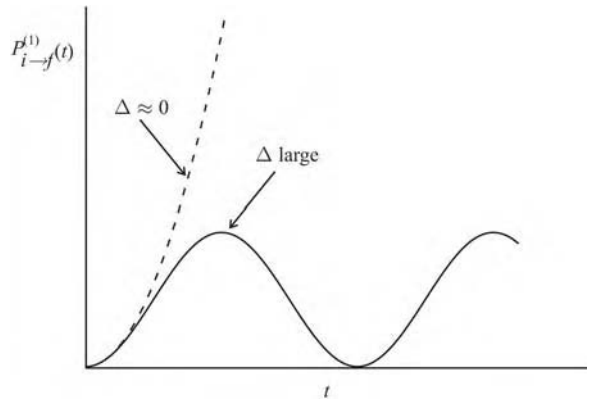



**Fig. 4.1.** A plot of $P^{(1)}_{i\to f}(t)$ versus time for small and large detunings $\Delta$.

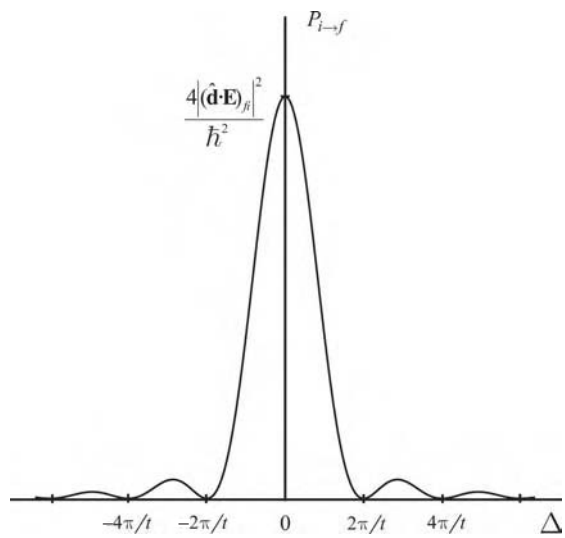



**Fig. 4.2.** The transition probability $P^{(1)}_{i\to f}(t)$ as a function of $\Delta$.

evolution of the probability distribution $P^{(1)}_{i\to f}(t)$ for both small detuning ($\Delta \approx 0$) and large detuning. The latter is periodic. The transition probability $P^{(1)}_{i\to f}(t)$ is a sharply peaked function of the detuning at $\Delta = 0$ as shown in Fig. 4.2. The width of the peak is proportional to $t^{-1}$ while the height is proportional to $t^2$. Thus the area under the peak is proportional to $t$. In fact

$$\int_{-\infty}^{\infty} \frac{\sin^2(\Delta t/2)}{\Delta^2} d(\Delta) = \frac{\pi}{2} t. \tag{4.37}$$

Furthermore, in the limit when $\Delta \approx 0$ and $t \gg 2\pi/\omega_{fi}$, the function in the integrand of Eq. (4.37) may be approximated by a Dirac delta function:

$$\lim_{t\to\infty} \frac{\sin^2(\Delta t/2)}{\Delta^2} = \frac{\pi}{2} t\, \delta(\Delta), \tag{4.38}$$

although the limit $t \to \infty$ is, in fact, actually constrained by the requirement that the right-hand side of Eq. (4.35) be $\ll 1$. In this case the transition probability is

$$P^{(1)}_{i\to f}(t) = \frac{\pi}{2}\frac{|(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi}|^2}{\hbar^2} t\,\delta(\omega-\omega_{fi}). \tag{4.39}$$

We define the time-independent transition probability *rate* as

$$W_{i\to f} = \frac{P^{(1)}_{i\to f}}{t} = \frac{\pi}{2}\frac{|(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi}|^2}{\hbar^2}\,\delta(\omega-\omega_{fi}). \tag{4.40}$$

In practice, there will be a broad range of final states $|f\rangle$ accessible from the initial state, and the driving field will not be monochromatic so that a range of frequencies will have to be summed or integrated over to obtain the total transition rate. If $[f]$ represents a set of accessible final states, then the transition rate for a monochromatic field is

$$W_{i\to[f]} = \frac{\pi}{2}\sum_{[f]}\frac{|(\hat{\mathbf{d}}\cdot\mathbf{E}_0)_{fi}|^2}{\hbar^2}\,\delta(\omega-\omega_{fi}). \tag{4.41}$$

This expression is often famously called Fermi's Golden Rule [3].

Now suppose that the light irradiating the atom is from a lamp emitting a broad range of frequencies where there is no phase relationship between the different frequency components. The amplitude of the light will now be frequency dependent so that the transition probability rate induced by all the frequency components must be

$$\frac{P^{(1)}_{i\to f}(t)}{t} = \frac{1}{\hbar^2}\int d\omega \frac{\sin^2(\Delta t/2)}{\Delta^2}F(\omega) \tag{4.42}$$

where

$$F(\omega) \equiv |\langle f|\hat{\mathbf{d}}\cdot\mathbf{E}_0(\omega)|i\rangle|^2. \tag{4.43}$$

If the function $F(\omega)$ is broadband and varies slowly with $\omega$ in comparison to $(\sin^2(\Delta t/2)/\Delta^2)$, then $F(\omega)$ can be replaced by its resonance value $F(\omega_{fi})$ and taken outside the integral so that

$$P^{(1)}_{i\to f}(t) = \frac{\pi}{2\hbar^2}F(\omega_{fi})\,t. \tag{4.44}$$

Thus the transition rate is

$$W_{i\to f} = \frac{\pi}{2\hbar^2}F(\omega_{fi}). \tag{4.45}$$

The spread of frequencies results in the washing out, or dephasing, of the oscillations seen in Fig. 4.1. This happens because of the lack of phase relations between the various frequency components – the light is incoherent. If the atom is driven by a coherent light field, such as from a laser, dephasing does not occur and

the perturbative time-independent transition rates above generally do not adequately describe the dynamics. We postpone a discussion of driving the atom by a coherent laser field until Section 4.4.

## 4.3 Interaction of an atom with a quantized field

In the preceding discussion, we made no assumption regarding the relative positions of the energy levels $i$ and $f$. Transitions between the levels occur with some nonzero probability as long as $\mathbf{E}_0 \neq 0$ whether $E_i < E_f$ or $E_i > E_f$. As we will show, when the field is quantized, transitions will occur for the case $E_i > E_f$ even when no photons are present – the so-called spontaneous emission. This is only one of several differences that will appear in the atom–field dynamics in the comparison between cases when the field is quantized and when it is not.

We consider a single free-space field mode of the form given by Eq. (2.130)

$$\hat{\mathbf{E}}(t) = i\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2} \mathbf{e}[\hat{a}e^{-i\omega t} - \hat{a}e^{i\omega t}] \tag{4.46}$$

where the dipole approximation has been made. This operator is in the Heisenberg picture but we are going to be working in the Schrödinger picture where the field operator is

$$\hat{\mathbf{E}} = i\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2} \mathbf{e}(\hat{a} - \hat{a}^\dagger). \tag{4.47}$$

The free Hamiltonian $\hat{H}_0$ now must be

$$\hat{H}_0 = \hat{H}_{\text{atom}} + \hat{H}_{\text{field}} \tag{4.48}$$

where $\hat{H}_{\text{atom}}$ is just the free-atom Hamiltonian as before and $\hat{H}_{\text{field}}$ is the free-field Hamiltonian $\hbar\omega\,\hat{a}^\dagger\hat{a}$, where the zero-point term has been dropped as it does not contribute to the dynamics. The interaction Hamiltonian becomes

$$\begin{aligned}\hat{H}^{(\mathrm{I})} &= -\hat{\mathbf{d}}\cdot\hat{\mathbf{E}} = -i\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right)^{1/2}(\hat{\mathbf{d}}\cdot\mathbf{e})(\hat{a} - \hat{a}^\dagger) \\ &= -\hat{\mathbf{d}}\cdot\boldsymbol{\mathcal{E}}_0(\hat{a} - \hat{a}^\dagger)\end{aligned} \tag{4.49}$$

where $\boldsymbol{\mathcal{E}}_0 = i(\hbar\omega/2\varepsilon_0 V)^{1/2}\mathbf{e}$.

Because both atomic and field systems are now quantized, the states of the combined system will involve products of states of both systems. Suppose the initial state of the atom–field system is $|i\rangle = |a\rangle|n\rangle$ where $|a\rangle$ is the initial state of the atom and the field contains $n$ photons. The perturbation interaction of the quantized field causes a transition to the state $|f_1\rangle = |b\rangle|n-1\rangle$, where $|b\rangle$ is another atomic state, by the absorption of a photon or to the state

$|f_2\rangle = |b\rangle|n+1\rangle$ by the emission of a photon. The energies of these states are,

$$\text{for} \quad |i\rangle = |a\rangle|n\rangle, \qquad E_i = E_a + n\,\hbar\omega, \tag{4.50a}$$

$$\text{for} \quad |f_1\rangle = |b\rangle|n-1\rangle, \qquad E_{f_1} = E_b + (n-1)\,\hbar\omega, \tag{4.50b}$$

$$\text{for} \quad |f_2\rangle = |b\rangle|n+1\rangle, \qquad E_{f_2} = E_b + (n+1)\,\hbar\omega, \tag{4.50c}$$

where $E_a$ and $E_b$ are the energies of the atomic states $|a\rangle$ and $|b\rangle$ respectively.

The perturbation given by Eq. (4.49) is now time independent. The matrix elements of the interaction are as follows:

$$\begin{aligned}\langle f_1|\hat{H}^{(\mathrm{I})}|i\rangle &= \langle b, n-1|\hat{H}^{(\mathrm{I})}|a, n\rangle \\ &= -(\hat{\mathbf{d}}\cdot\boldsymbol{\mathcal{E}}_0)_{ba}\sqrt{n}, \qquad \text{(absorption)}\end{aligned} \tag{4.51}$$

and

$$\begin{aligned}\langle f_2|\hat{H}^{(\mathrm{I})}|i\rangle &= \langle b, n+1|\hat{H}^{(\mathrm{I})}|a, n\rangle \\ &= (\hat{\mathbf{d}}\cdot\boldsymbol{\mathcal{E}}_0)_{ba}\sqrt{n+1}, \qquad \text{(emission)}\end{aligned} \tag{4.52}$$

where

$$(\hat{\mathbf{d}}\cdot\boldsymbol{\mathcal{E}}_0)_{ab} = \langle a|\hat{\mathbf{d}}|b\rangle\cdot\boldsymbol{\mathcal{E}}_0 \equiv \mathbf{d}_{ab}\cdot\boldsymbol{\mathcal{E}}_0 \tag{4.53}$$

the factor $\langle a|\hat{\mathbf{d}}|b\rangle = \mathbf{d}_{ab}$ being the dipole matrix element between states $|a\rangle$ and $|b\rangle$. In comparison with the semiclassical case, two things are noteworthy here. The absence of photons ($n = 0$) precludes absorption, just as one might expect. This is obviously in agreement with the case of a classical driving field – no field, no transitions. But in the case of emission, according to Eq. (4.52), transitions may occur even when no photons are present. This is spontaneous emission and it has no semiclassical counterpart. If $n > 0$, the emission of an additional photon is called stimulated emission, this process being the essential one for the operation of the laser (or LASER: Light Amplification by Stimulated Emission of Radiation). The rates of emission and absorption are proportional to the moduli squared of the above matrix elements for the respective processes. The ratio of these rates is

$$\frac{\left|\langle f_2|\hat{H}^{(\mathrm{I})}|i\rangle\right|^2}{\left|\langle f_1|\hat{H}^{(\mathrm{I})}|i\rangle\right|^2} = \frac{n+1}{n}, \tag{4.54}$$

a result to be used shortly.

The perturbation method developed previously can still be used with appropriate modifications to accommodate the fact that the field is now quantized. The Schrödinger equation still has the form of Eq. (4.19) but where now $\hat{H}_0$ is given by Eq. (4.48) and $\hat{H}^{(\mathrm{I})}$ by Eq. (4.49). Ignoring all other atomic states except $|a\rangle$

and $|b\rangle$, the state vector can be written as

$$\begin{aligned}|\psi(t)\rangle = & C_i(t)|a\rangle|n\rangle e^{-iE_at/\hbar}e^{-in\omega t} \\ & + C_{f_1}(t)|b\rangle|n-1\rangle e^{-iE_bt/\hbar}e^{-i(n-1)\omega t} \\ & + C_{f_2}(t)|b\rangle|n+1\rangle e^{-iE_bt/\hbar}e^{-i(n+1)\omega t}\end{aligned} \tag{4.55}$$

where, assuming that $|\psi(0)\rangle = |a\rangle|n\rangle$, $C_i(0) = 1$ and $C_{f1}(0) = C_{f2}(0) = 0$. Following the perturbative method used before, we obtain the first-order correction for the amplitudes $C_{f1}$ and $C_{f2}$ associated with the atom being in state $|b\rangle$:

$$\begin{aligned}C_{f1}^{(1)}(t) &= -\frac{i}{\hbar}\int_0^t dt'\langle f_1|\hat{H}^{(\mathrm{I})}|i\rangle e^{i(E_{f1}-E_i)t/\hbar}, \\ C_{f2}^{(1)}(t) &= -\frac{i}{\hbar}\int_0^t dt'\langle f_2|\hat{H}^{(\mathrm{I})}|i\rangle e^{i(E_{f2}-E_i)t/\hbar},\end{aligned} \tag{4.56}$$

where the former is associated with absorption and the latter with emission. The amplitude associated with the atom being in the state $|b\rangle$, regardless of how it got there, is just the sum of the amplitudes in Eqs. (4.56), i.e. $C_f^{(1)} = C_{f1}^{(1)} + C_{f2}^{(1)}$. From Eqs. (4.50) we obtain

$$C_f^{(1)}(t) = \frac{i}{\hbar}(\hat{\mathbf{d}}\cdot\boldsymbol{\mathcal{E}}_0)_{ab}\left\{(n+1)^{1/2}\frac{\left[e^{i(\omega+\omega_{ba})t}-1\right]}{(\omega+\omega_{ba})} - n^{1/2}\frac{\left[e^{i(\omega-\omega_{ba})t}-1\right]}{(\omega-\omega_{ba})}\right\} \tag{4.57}$$

where $\omega_{ba} = (E_b - E_a)/\hbar$ and where the first term is due to emission and the second to absorption. If the number of photons is large, $n \gg 1$, then we can replace $\sqrt{n+1}$ by $\sqrt{n}$ in the first term and then Eqs. (4.57) and (4.33) are essentially the same, there being the correspondence between the classical and quantum field amplitudes $(\mathbf{E}_0)_{\mathrm{cl}} \leftrightarrow (2i\mathcal{E}_0\sqrt{n})_{\mathrm{quantum}}$. This correspondence between quantum and classical fields has its limits, one being the case when $n = 0$, as already discussed.

If $|b\rangle$ is the excited state then $\omega_{ba} > 0$ so if $\omega \sim \omega_{ba}$ then the first term of Eq. (4.51) can be dropped, which is again the rotating wave approximation. Of course, if $|a\rangle$ is the excited state, then $\omega_{ba} < 0$ and if $\omega \sim -\omega_{ba}$, the second term of Eq. (4.57) can be dropped, and we notice that the remaining term does not vanish even when $n = 0$, the transition between $|a\rangle$ and $|b\rangle$ taking place by spontaneous emission. Thus the rotating wave approximation carries over into the case where the field and the atom are quantized. It can be shown that Fermi's Golden Rule carries over in a similar fashion.

We conclude this section with a field-theoretic derivation of the Planck distribution law. Suppose we have a collection of atoms interacting resonantly with a quantized field of frequency $\omega = (E_a - E_b)/\hbar$, where $|a\rangle$ and $|b\rangle$ are the atomic states with $E_a > E_b$. We let $N_a$ and $N_b$ represent the populations of atoms in states $|a\rangle$ and $|b\rangle$ respectively. Further, we let $W_{\mathrm{emis}}$ represent the transition rate due to photon emission and $W_{\mathrm{abs}}$ the transition rate due to photon absorption.

Because the atoms are constantly emitting and absorbing photons, the atomic populations change with time according to

$$\begin{aligned} \frac{dN_a}{dt} &= -N_a W_{\text{emis}} + N_b W_{\text{abs}} \\ \frac{dN_b}{dt} &= -N_b W_{\text{abs}} + N_a W_{\text{emis}}. \end{aligned} \tag{4.58}$$

At thermal equilibrium, we have

$$\frac{dN_a}{dt} = 0 = \frac{dN_b}{dt} \tag{4.59}$$

and thus we obtain

$$N_a W_{\text{emis}} = N_b W_{\text{abs}}. \tag{4.60}$$

But, according to Boltzmann,

$$\frac{N_b}{N_a} = \exp[(E_a - E_b)/kT] = \exp(\hbar\omega/kT), \tag{4.61}$$

and from Eq. (4.57) we have that

$$\frac{N_b}{N_a} = \frac{W_{\text{emis}}}{W_{\text{abs}}} = \frac{n+1}{n}. \tag{4.62}$$

Thus from Eqs. (4.61) and (4.62) it follows that

$$n = \frac{1}{\exp(\hbar\omega/kT) - 1}, \tag{4.63}$$

in agreement with Eq. (2.141) if we replace $n$ by $\bar{n}$ in Eq. (4.63) to incorporate the fact that we cannot assume a definite number of photons.

Let us compare this derivation to the one given by Einstein [4] before quantum electrodynamics was even invented. His derivation is similar to the above but explicitly makes the distinction between spontaneous and stimulated emission. He introduced the coefficients $A$, $B$, and $C$ having the following meanings: $AN_a$ is the growth rate of the population in state $|b\rangle$ owing to spontaneous emission from state $|b\rangle$ ($A$ being the rate of spontaneous emission); $BU(\omega)N_a$ is the growth rate of the population of state $|b\rangle$ owing to stimulated emission from $|a\rangle$, $U(\omega)$ being the spectral energy density of the field; and $CU(\omega)N_b$ is the rate of growth of state $|a\rangle$ as the result of absorption by atoms in state $|b\rangle$. Note that the spontaneous

emission term is independent of $U(\omega)$. The rate equations for the populations are now

$$\begin{aligned} \frac{dN_a}{dt} &= -[A + BU(\omega)]N_a + CU(\omega)N_b \\ \frac{dN_b}{dt} &= -CU(\omega)N_b + [A + BU(\omega)]N_a. \end{aligned} \tag{4.64}$$

At long times, the populations reach a steady state and the derivatives on the left-hand sides vanish to yield

$$[A + BU(\omega)]N_a = CU(\omega)N_b. \tag{4.65}$$

Using once again the relation in Eq. (4.61), we are led to

$$U(\omega) = \frac{A}{C\exp(\hbar\omega/kT) - B}. \tag{4.66}$$

But for a thermal field, we must have, by comparison to Eq. (2.151), that $C = B$ and that

$$\frac{A}{B} = \frac{\hbar\omega^3}{\pi^2 c^3}. \tag{4.67}$$

It is worth comparing the rates of spontaneous emission $A$ and stimulated emission $BU(\omega)$:

$$\frac{A}{BU(\omega)} = \exp(\hbar\omega/kT - 1). \tag{4.68}$$

For a natural source such as the sun, which we approximate as a black body with surface temperature $T \approx 6000$ K, this ratio is about 400 for $\lambda = 400$ nm and about 30 for $\lambda = 700$ nm [5]. Thus at both ends of the visible spectrum spontaneous emission dominates stimulated emission. In the range of visible light, it is only in "unnatural" sources, where there exists a population inversion (i.e. more atoms in the excited state than in the ground), such as in lasers, where stimulated emission dominates spontaneous emission.

Spontaneous emission is a complex phenomenon and we have discussed only its most essential features. For atoms in free space there is an infinity of modes into which the atoms may radiate. Spontaneous emission in this case is well described by the Weisskopf–Wigner theory [6] as an irreversible decay process. A discussion of that theory is outside the scope of this book. However, under certain circumstances, where there may be only one mode into which an atom can radiate, such as in the case of an atom in a cavity, spontaneous emission can

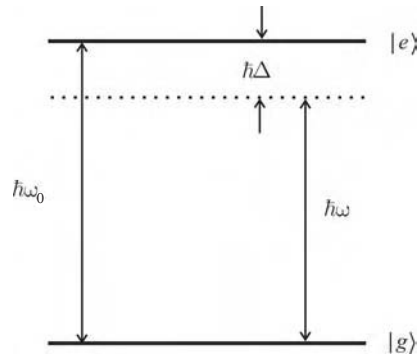



**Fig. 4.3.** Energy level diagram for a two-level atom acting with a near resonant classical driving field of frequency $\omega$. The resonant frequency between the two atomic levels is $\omega_0$ and the detuning $\Delta = \omega_0 - \omega$.

be reversible, that is, the atom can re-absorb the emitted photon. Such behavior is discussed in Section 4.5.

## 4.4 The Rabi model

The perturbation theory approach to atom–field interactions assumes that the initial atomic state population is essentially unchanged, that is, the probability amplitude for the atom being in any other state remains small. On the other hand, a strong laser field of frequency near resonance with a pair of atomic levels (assumed of opposite parity) will cause a large population transfer to the near-resonant state but not to any other. In such a case, perturbation theory must be abandoned. Only the two dominant states will be retained and the problem will be solved more "exactly". This is the Rabi model [7], so named because of its original setting in magnetic resonance as studied by Rabi long ago. We study the semiclassical case first.

For definiteness and to follow convention, we label our two atomic states $|g\rangle$ (for ground) and $|e\rangle$ (for excited). The energy difference between these states is characterized by the transition frequency $\omega_0 = (E_e - E_g)/\hbar$. This frequency is close to the frequency $\omega$ of the driving laser field as shown in Fig. 4.3. The interaction Hamiltonian we write as

$$\hat{H}^{(\mathrm{I})}(t) = \hat{V}_0 \cos\omega t \tag{4.69}$$

where $\hat{V}_0 = -\hat{\mathbf{d}} \cdot \mathbf{E}_0$. We write the state vector as

$$|\psi(t)\rangle = C_g(t)e^{-iE_g t/\hbar}|g\rangle + C_e(t)e^{-iE_e t/\hbar}|e\rangle. \tag{4.70}$$

From the Schrödinger equation

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}(t)\,|\psi(t)\rangle, \tag{4.71}$$

where

$$\hat{H} = \hat{H}_0 + \hat{V}_0 \cos\omega t, \tag{4.72}$$

we arrive at the coupled set of equations for the amplitudes $C_g$ and $C_e$:

$$\begin{aligned} \dot{C}_g &= -\frac{i}{\hbar}\mathcal{V}\cos\omega t e^{-i\omega_0 t}\, C_e \\ \dot{C}_e &= -\frac{i}{\hbar}\mathcal{V}\cos\omega t e^{i\omega_0 t}\, C_g \end{aligned} \tag{4.73}$$

where $\mathcal{V} = \langle e|\hat{V}_0|g\rangle = -\hat{\mathbf{d}}_{eg}\cdot\mathbf{E}_0$, which we have taken to be real. As an initial condition we take all the population to be in the ground state: $C_g(0) = 1$ and $C_e(0) = 0$. In Eqs. (4.73) we expand $\cos\omega t$ in exponentials and retain only those terms oscillating at the frequency $\omega_0 - \omega$ to obtain

$$\begin{aligned} \dot{C}_g &= -\frac{i}{2\hbar}\mathcal{V}\exp[i(\omega-\omega_0)t]C_e \\ \dot{C}_e &= -\frac{i}{2\hbar}\mathcal{V}\exp[-i(\omega-\omega_0)t]C_g. \end{aligned} \tag{4.74}$$

Dropping the terms oscillating at $\omega_0 + \omega$, of course, constitutes the RWA. Eliminating $C_g$, we have for $C_e$:

$$\ddot{C}_e + i(\omega-\omega_0)\dot{C}_e + \frac{1}{4}\,\frac{\mathcal{V}^2}{\hbar^2}C_e = 0. \tag{4.75}$$

As a trial solution we set

$$C_e(t) = e^{i\lambda t} \tag{4.76}$$

which leads to the two roots

$$\lambda_\pm = \frac{1}{2}\{\Delta \pm [\Delta^2 + \mathcal{V}^2/\hbar^2]^{1/2}\} \tag{4.77}$$

where $\Delta = \omega_0 - \omega$ is the detuning of the atomic transition frequency and the laser field. Thus the general solution is of the form

$$C_e(t) = A_+ e^{i\lambda_+ t} + A_- e^{i\lambda_- t}, \tag{4.78}$$

where from the initial conditions we must have

$$A_\pm = \mp\frac{1}{2\hbar}\,\mathcal{V}[\Delta^2 + \mathcal{V}^2/\hbar^2]^{-1/2}. \tag{4.79}$$

Finally, then, our solution is

$$\begin{aligned} C_e(t) &= i\frac{\mathcal{V}}{\Omega_R\hbar}\, e^{i\Delta t/2}\sin(\Omega_R t/2) \\ C_g(t) &= e^{i\Delta t/2}\left\{\cos(\Omega_R t/2) - i\frac{\Delta}{\Omega_R}\sin(\Omega_R t/2)\right\}, \end{aligned} \tag{4.80}$$

where

$$\Omega_R = [\Delta^2 + \mathcal{V}^2/\hbar^2]^{1/2} \tag{4.81}$$

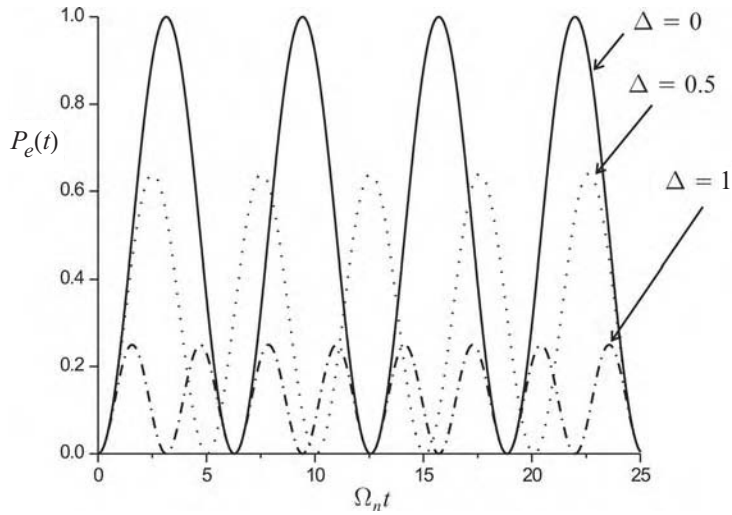



**Fig. 4.4.** Plots of $P_e(t)$ versus $t$ for various detunings $\Delta$.

is the so-called Rabi frequency. The probability that the atom is in state $|e\rangle$ is

$$\begin{aligned} P_e(t) &= |C_e(t)|^2 \\ &= \frac{\mathcal{V}^2}{\Omega_{\mathrm{R}}^2 \hbar^2} \sin^2(\Omega_{\mathrm{R}} t/2), \end{aligned} \tag{4.82}$$

which is plotted in Fig. 4.4 for various values of $\Delta$. For the case of exact resonance, $\Delta = 0$, we have

$$P_e(t) = \sin^2\left(\frac{\mathcal{V}t}{2\hbar}\right), \tag{4.83}$$

and at the time $t = \pi\hbar/\mathcal{V}$ all the atomic population has been transferred to the excited state.

It is frequently convenient to consider the quantity known as the atomic inversion $W(t)$ defined as the difference in the excited- and ground-state populations:

$$W(t) = P_e(t) - P_g(t), \tag{4.84}$$

which, for the resonant case and with the atom initially in the ground state, is

$$\begin{aligned} W(t) &= \sin^2\left(\frac{\mathcal{V}t}{2\hbar}\right) - \cos^2\left(\frac{\mathcal{V}t}{2\hbar}\right) \\ &= -\cos(\mathcal{V}t/\hbar). \end{aligned} \tag{4.85}$$

Note that for $\Delta = 0$ the Rabi frequency is just $\Omega_{\mathrm{R}} = \mathcal{V}/\hbar$, the oscillation frequency of the atomic inversion. Again, for $t = \pi\hbar/\mathcal{V}$ all the population is transferred to the excited state: $W(\pi\hbar/\mathcal{V}) = 1$. In the parlance of NMR experiments

[8], such transfers are called $\pi$-pulses. On the other hand, if $t = \pi\hbar/2\mathcal{V}$ then $W(\pi\hbar/2\mathcal{V}) = 0$ and the population is shared coherently between the excited and ground states with

$$\begin{aligned} C_e(\pi\hbar/2\mathcal{V}) &= \frac{i}{\sqrt{2}}, \\ C_g(\pi\hbar/2\mathcal{V}) &= \frac{1}{\sqrt{2}}, \end{aligned} \tag{4.86}$$

so that

$$|\psi(\pi\hbar/2\mathcal{V})\rangle = \frac{1}{\sqrt{2}}(|g\rangle + i|e\rangle). \tag{4.87}$$

For obvious reasons, the transfer of population from the ground state to that state of Eq. (4.87) is called a $\pi/2$-pulse. Such transfers of populations by $\pi$- or $\pi/2$-pulses are standard procedures for manipulating not only spin states in NMR experiments [2] but have become routine for manipulating atomic or ionic states in laser spectroscopy experiments [9].

The results of the pertubation theory may be recovered from the Rabi model either when the size of $\mathcal{V}/2\hbar$ is so small compared with the detuning $\Delta$ that it can be neglected in $\Omega_R$ or if the radiation field acts only for such a short time that the term $\sin^2(\mathcal{V}t/2\hbar)$ can legitimately be represented by the first term in its expansion. In both cases the depletion of the initial atomic population is small and the perturbation-theory approach remains valid.

## 4.5 Fully quantum-mechanical model; the Jaynes–Cummings model

We now turn to the quantum electrodynamic version of the Rabi model. In our previous pertubation discussion of an atom interacting with a quantized electromagnetic field, we assumed the field to be a single-mode free field (plane wave). As we just discussed above, a free atom interacts with an infinite number of modes and thus the dynamics is not well described assuming only a single-mode field. On the other hand, it has recently become possible to manufacture environments where the density of modes is significantly different than in free space. We have in mind here small microwave cavities, or in some cases, optical cavities, capable of supporting only a single mode or maybe a few widely spaced (in frequency) modes. Thus in some cases, the ideal single-mode interaction can be realized in the laboratory. We shall discuss some specific examples in Chapter 10 but for now we consider an atom, with levels $|g\rangle$ and $|e\rangle$ as before, interacting with a single-mode cavity field of the form

$$\hat{\mathbf{E}} = \mathbf{e}\left(\frac{\hbar\omega}{\varepsilon_0 V}\right)^{1/2}(\hat{a} + \hat{a}^\dagger)\sin(kz) \tag{4.88}$$

where $\mathbf{e}$ is an arbitrarily oriented polarization vector.

The interaction Hamiltonian is now

$$\begin{aligned}\hat{H}^{(\mathrm{I})} &= -\hat{\mathbf{d}}\cdot\hat{\mathbf{E}} \\ &= \hat{d}g(\hat{a}+\hat{a}^{\dagger})\end{aligned} \tag{4.89}$$

where

$$g = -\left(\frac{\hbar\omega}{\varepsilon_0 V}\right)^{1/2}\sin(kz) \tag{4.90}$$

and where $\hat{d} = \hat{\mathbf{d}}\cdot\mathbf{e}$.

At this point it is convenient to introduce the so-called atomic transition operators

$$\hat{\sigma}_+ = |e\rangle\langle g|, \qquad \hat{\sigma}_- = |g\rangle\langle e| = \hat{\sigma}_+^{\dagger}, \tag{4.91}$$

and the inversion operator

$$\hat{\sigma}_3 = |e\rangle\langle e| - |g\rangle\langle g|. \tag{4.92}$$

These operators obey the Pauli spin algebra

$$\begin{aligned}[\hat{\sigma}_+, \hat{\sigma}_-] &= \hat{\sigma}_3 \\ [\hat{\sigma}_3, \hat{\sigma}_\pm] &= 2\hat{\sigma}_\pm.\end{aligned} \tag{4.93}$$

Only the off-diagonal elements of the dipole operator are nonzero, since by parity consideration $\langle e|\hat{d}|e\rangle = 0 = \langle g|\hat{d}|g\rangle$, so that we may write

$$\begin{aligned}\hat{d} &= d|g\rangle\langle e| + d^*|e\rangle\langle g| \\ &= d\hat{\sigma}_- + d^*\hat{\sigma}_+ = d(\hat{\sigma}_+ + \hat{\sigma}_-)\end{aligned} \tag{4.94}$$

where we have set $\langle e|\hat{d}|g\rangle = d$ and have assumed, without loss of generality, that $d$ is real. Thus the interaction Hamiltonian is

$$\hat{H}^{(\mathrm{I})} = \hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a}+\hat{a}^{\dagger}) \tag{4.95}$$

where $\lambda = dg/\hbar$.

If we define the level of the energy to be zero halfway between the states $|g\rangle$ and $|e\rangle$ as in Fig. 4.5, then the free atomic Hamiltonian may be written as

$$\hat{H}_{\mathrm{A}} = \frac{1}{2}(E_e - E_g)\hat{\sigma}_3 = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3, \tag{4.96}$$

where $E_e = -E_g = \frac{1}{2}\hbar\omega_0$. The free-field Hamiltonian is, after dropping the zero-point energy term,

$$\hat{H}_{\mathrm{F}} = \hbar\omega\hat{a}^{\dagger}\hat{a}. \tag{4.97}$$

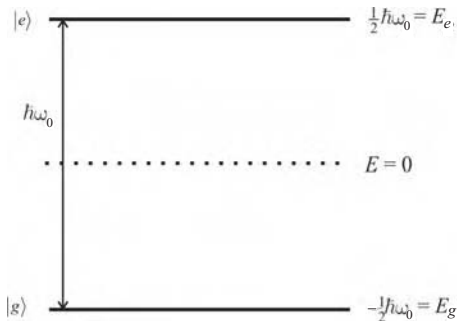



**Fig. 4.5.** Atomic energy level diagram where the $E = 0$ level is taken halfway between the two levels.

Thus the total Hamiltonian is

$$\begin{aligned}\hat{H} &= \hat{H}_{\mathrm{A}} + \hat{H}_{\mathrm{F}} + \hat{H}^{(\mathrm{I})} \\ &= \frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger).\end{aligned} \tag{4.98}$$

In the free-field case, as we have already shown, the operators $\hat{a}$ and $\hat{a}^+$ evolve as

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t}, \quad \hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega t}. \tag{4.99}$$

One can show similarly that for the free-atomic case

$$\hat{\sigma}_\pm(t) = \hat{\sigma}_\pm(0)e^{\pm i\omega_0 t}. \tag{4.100}$$

Thus we can see that the approximate time dependences of the operator products in Eq. (4.98) are as follows:

$$\begin{aligned}\hat{\sigma}_+\hat{a} &\sim e^{i(\omega_0-\omega)t} \\ \hat{\sigma}_-\hat{a}^\dagger &\sim e^{-i(\omega_0-\omega)t} \\ \hat{\sigma}_+\hat{a}^\dagger &\sim e^{i(\omega+\omega_0)t} \\ \hat{\sigma}_-\hat{a} &\sim e^{-i(\omega+\omega_0)t}.\end{aligned} \tag{4.101}$$

For $\omega_0 \approx \omega$ the last two terms vary much more rapidly than the first two. Furthermore, the last two terms do not conserve energy in contrast to the first two. The term $\hat{\sigma}_+\hat{a}^\dagger$ corresponds to the emission of a photon as the atom goes from the ground to the excited state, whereas $\hat{\sigma}_-\hat{a}$ corresponds to the absorption of a photon as the atom goes from the excited to the ground state. Integrating the time-dependent Schrödinger equation, as in the perturbative case, will lead, for the last two terms, to denominators containing $\omega_0 + \omega$ as compared with $\omega_0 - \omega$ for the first two terms. The reader will not be surprised to learn that we are going to drop the non-energy conserving terms, making the RWA again, so that our Hamiltonian in this approximation is

$$\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger). \tag{4.102}$$

The interaction described by this Hamiltonian is widely referred to as the Jaynes–Cummings model [10].

Before solving for the dynamics for any specific cases we take note of certain constants of the motion. An obvious one is the electron "number"

$$\hat{P}_{\mathrm{E}} = |e\rangle\langle e| + |g\rangle\langle g| = 1, \quad [\hat{H}, \hat{P}_{\mathrm{E}}] = 0 \tag{4.103}$$

valid when no other atomic states can become populated. Another is the excitation number

$$\hat{N}_{\mathrm{e}} = \hat{a}^{\dagger}\hat{a} + |e\rangle\langle e|, \quad [\hat{H}, \hat{N}_{\mathrm{e}}] = 0. \tag{4.104}$$

Using these constants of the motion we may break the Hamiltonian Eq. (4.102) into two commuting parts:

$$\hat{H} = \hat{H}_{\mathrm{I}} + \hat{H}_{\mathrm{II}} \tag{4.105}$$

where

$$\begin{aligned} \hat{H}_{\mathrm{I}} &= \hbar\omega\hat{N}_{\mathrm{e}} + \hbar\left(\frac{\omega_0}{2} - \omega\right)\hat{P}_{\mathrm{E}}, \\ \hat{H}_{\mathrm{II}} &= -\hbar\Delta + \hbar\lambda(\hat{\sigma}_{+}\hat{a} + \hat{\sigma}_{-}\hat{a}^{\dagger}), \end{aligned} \tag{4.106}$$

such that $[\hat{H}_{\mathrm{I}}, \hat{H}_{\mathrm{II}}] = 0$. Clearly, all the essential dynamics is contained in $\hat{H}_{\mathrm{II}}$ whereas $\hat{H}_{\mathrm{I}}$ contributes only overall irrelevant phase factors.

Let us now consider a simple example, with $\Delta = 0$, where the atom is initially in the excited state $|e\rangle$ and the field is initially in the number state $|n\rangle$. The initial state of the atom–field system is then $|i\rangle = |e\rangle|n\rangle$ and is of energy $E_i = \frac{1}{2}\hbar\omega + n\hbar\omega$. State $|i\rangle$ is coupled to (and only to) the state $|f\rangle = |g\rangle|n+1\rangle$ with energy $E_f = -\frac{1}{2}\hbar\omega + (n+1)\hbar\omega$. Note that $E_i = E_f$. We write the state vector as

$$|\psi(t)\rangle = C_i(t)|i\rangle + C_f(t)|f\rangle \tag{4.107}$$

where $C_i(0) = 1$ and $C_f(0) = 0$. Following standard procedures we obtain, from the interaction picture Schrödinger equation $i\hbar d|\psi(t)\rangle/dt = \hat{H}_{\mathrm{II}}|\psi(t)\rangle$ the equations for the coefficients

$$\begin{aligned} \dot{C}_i &= -i\lambda\sqrt{n+1}\,C_f, \\ \dot{C}_f &= -i\lambda\sqrt{n+1}\,C_i. \end{aligned} \tag{4.108}$$

Eliminating $C_f$ we obtain

$$\ddot{C}_i + \lambda^2(n+1)\,C_i = 0. \tag{4.109}$$

The solution matching the initial conditions is

$$C_i(t) = \cos(\lambda t\sqrt{n+1}). \tag{4.110}$$

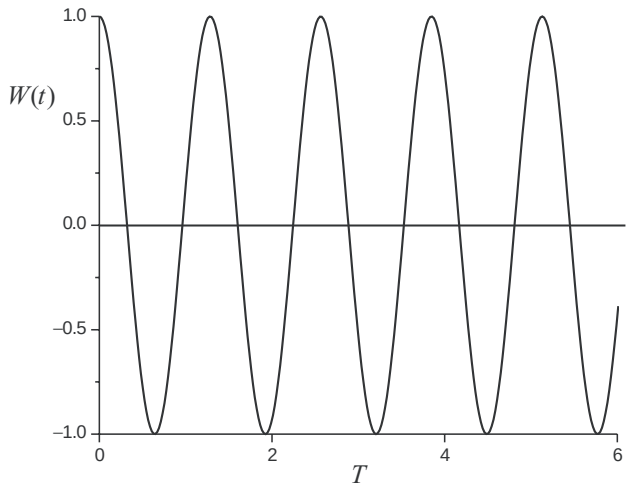



**Fig. 4.6.** Periodic atomic inversion with the field initially in a number state $|n\rangle$ with $n = 5$ photons.

From Eq. (4.108) we obtain

$$C_f(t) = -i \sin(\lambda t\sqrt{n+1}). \tag{4.111}$$

Thus our solution is

$$\begin{aligned} |\psi(t)\rangle &= \cos(\lambda t\sqrt{n+1})\,|e\rangle|n\rangle \\ &\quad -i \sin(\lambda t\sqrt{n+1})\,|g\rangle|n+1\rangle. \end{aligned} \tag{4.112}$$

The probability that the system remains in the initial state is

$$P_i(t) = |C_i(t)|^2 = \cos^2(\lambda t\sqrt{n+1}) \tag{4.113}$$

while the probability that it makes a transition to the state $|f\rangle$ is

$$P_f(t) = |C_f(t)|^2 = \sin^2(\lambda t\sqrt{n+1}). \tag{4.114}$$

The atomic inversion is given by

$$\begin{aligned} W(t) &= \langle\psi(t)|\hat{\sigma}_3|\psi(t)\rangle \\ &= P_i(t) - P_f(t) \\ &= \cos(2\lambda t\sqrt{n+1}). \end{aligned} \tag{4.115}$$

We may define a quantum electrodynamic Rabi frequency $\Omega(n) = 2\lambda\sqrt{n+1}$ so that

$$W(t) = \cos[\Omega(n)t]. \tag{4.116}$$

Clearly, the atomic inversion for the field initially in a number state is strictly periodic (Fig. 4.6), just as in the semiclassical case of Eq. (4.85) (apart from the minus sign due only to a different initial atomic state) and except for the fact that in

the classical case there must always be a field present initially. But in the quantum-mechanical case there are Rabi oscillations even for the case when $n = 0$. These are the vacuum-field Rabi oscillations [11] and, of course, they have no classical counterpart. They are the result of the atom spontaneously emitting a photon then re-absorbing it, re-emitting it, etc.: an example of reversible spontaneous emission. Such effects can be observed if atoms interact with fields in very high $Q$ cavities. But aside from this, overall, the behavior of the atomic dynamics for a definite number of photons is very much like the semiclassical Rabi model, i.e. it is periodic and regular. Perhaps this is a bit counterintuitive since a number state is the most nonclassical of all the field states. Intuition might suggest that when the field is initially in a coherent state, we should recover the semiclassical, periodic and regular, Rabi oscillations. As we are about to demonstrate, intuition, in this case, fails.

Let us now consider a more general (pure state) solution of the dynamics. We assume the atom is initially in a superposition of states $|e\rangle$ and $|g\rangle$:

$$|\psi(0)\rangle_{\text{atom}} = C_g|g\rangle + C_e|e\rangle, \tag{4.117}$$

and the field is initially in the state

$$|\psi(0)\rangle_{\text{field}} = \sum_{n=0}^{\infty} C_n|n\rangle, \tag{4.118}$$

such that the initial atom–field state is

$$|\psi(0)\rangle = |\psi(0)\rangle_{\text{atom}} \otimes |\psi(0)\rangle_{\text{field}}. \tag{4.119}$$

The solution of the Schrödinger equation is now

$$\begin{aligned} |\psi(t)\rangle = \sum_{n=0}^{\infty} \{&[C_e C_n \cos(\lambda t\sqrt{n+1}) - iC_g C_{n+1}\sin(\lambda t\sqrt{n+1})]\,|e\rangle \\ &+ [-iC_e C_{n-1}\sin(\lambda t\sqrt{n}) + C_g C_n \cos(\lambda t\sqrt{n})]|g\rangle\}\,|n\rangle. \end{aligned} \tag{4.120}$$

In general, this is an entangled state.

For the case of the atom initially in the excited state, where $C_e = 1$ and $C_g = 0$, we may write the solution as

$$|\psi(t)\rangle = |\psi_g(t)\rangle|g\rangle + |\psi_e(t)\rangle|e\rangle \tag{4.121}$$

where $|\psi_g(t)\rangle$ and $|\psi_e(t)\rangle$ are the field components of $|\psi(t)\rangle$ given by

$$\begin{aligned} |\psi_g(t)\rangle &= -i\sum_{n=0}^{\infty} C_n \sin(\lambda t\sqrt{n+1})|n+1\rangle, \\ |\psi_e(t)\rangle &= \sum_{n=0}^{\infty} C_n \cos(\lambda t\sqrt{n+1})|n\rangle. \end{aligned} \tag{4.122}$$

The atomic inversion is

$$\begin{aligned} W(t) &= \langle\psi(t)|\hat{\sigma}_3|\psi(t)\rangle \\ &= \langle\psi_e(t)|\psi_e(t)\rangle - \langle\psi_g(t)|\psi_g(t)\rangle \\ &= \sum_{n=0}^{\infty} |C_n|^2 \cos(2\lambda t\sqrt{n+1}). \end{aligned} \tag{4.123}$$

The result is just the sum of $n$-photon inversions of Eq. (4.115) weighted with the photon number distribution of the initial field state.

For the coherent state, again that most classical of all quantum states, we have

$$C_n = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}} \tag{4.124}$$

and the inversion is

$$W(t) = e^{-\bar{n}}\sum_{n=0}^{\infty}\frac{\bar{n}^n}{n!}\cos(2\lambda t\sqrt{n+1}). \tag{4.125}$$

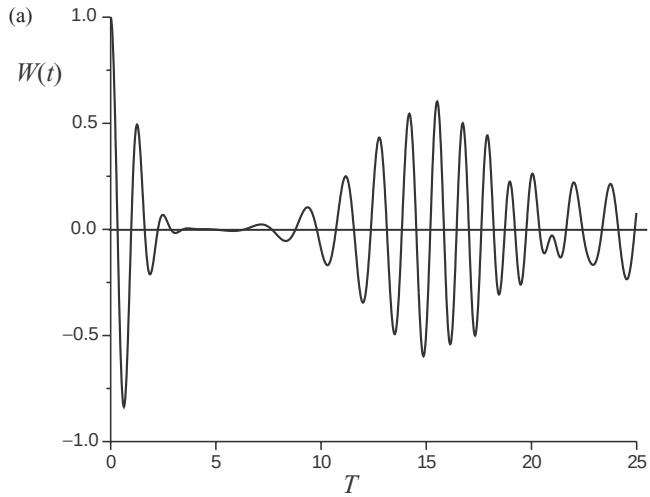
A plot of $W(t)$ versus the scaled time $T = \lambda\, t$ in Fig. 4.7 reveals significant discrepancies between the fully quantized and semiclassical Rabi oscillations. We note first that the Rabi oscillations initially appear to damp out, or collapse. The collapse of the Rabi oscillations was noted fairly early in the study of this "idealized" model interaction [12]. Several years later, perhaps by executing longer runs of a computer program, it was found that after a period of quiescence following the collapse, the Rabi oscillations start to revive [13], although not completely. At longer times one finds a sequence of collapses and revivals, the revivals becoming less distinct as time increases. This collapse and revival behavior of the Rabi oscillations in the fully quantized model is strikingly different than in the semiclassical case where the oscillations have constant amplitude. We must now explain this difference.

First we consider the collapse. The average photon number is $\bar{n} = |\alpha|^2$ so the dominant Rabi frequency is

$$\Omega(\bar{n}) = 2\lambda\sqrt{\bar{n}+1} \approx 2\lambda\sqrt{\bar{n}}, \quad \bar{n} \gg 1. \tag{4.126}$$

But there will be a range of "important" frequencies as a result of the spread of the probabilities $|C_n|^2$ about $\bar{n}$ for photon numbers in the range $\bar{n} \pm \Delta n$; i.e. the frequencies in the range $\Omega(\bar{n} - \Delta n)$ to $\Omega(\bar{n} + \Delta n)$. The collapse time $t_c$ may be estimated from the time-frequency "uncertainty" relation

$$t_c[\Omega(\bar{n} + \Delta n) - \Omega(\bar{n} - \Delta n)] \simeq 1 \tag{4.127}$$



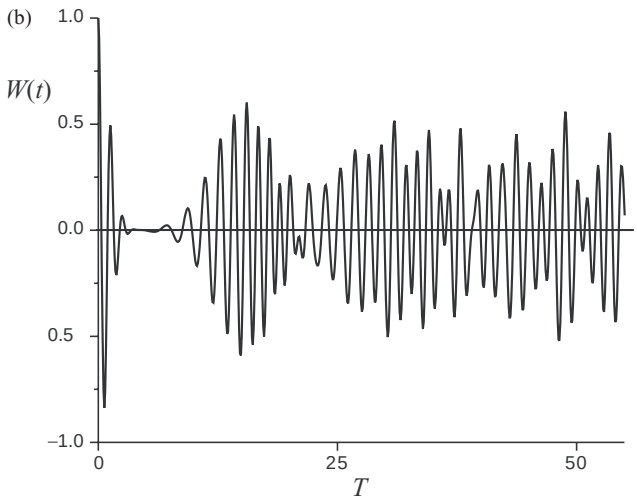




**Fig. 4.7.** (a) Atomic inversion with the field initially in a coherent state $\bar{n} = 5$. (b) Same as (a) but showing the evolution for a longer time, beyond the first revival. Here, $T$ is the scaled time $\lambda t$.

where the spread of frequencies is responsible for the "dephasing" of the Rabi oscillations. For the coherent state, $\Delta n = \bar{n}^{1/2}$, and with

$$\begin{aligned}\Omega(\bar{n} \pm \bar{n}^{1/2}) &\simeq 2\lambda[\bar{n} \pm \bar{n}^{1/2}]^{1/2} \\ &= 2\lambda\bar{n}^{1/2}\left[1 \pm \frac{1}{\bar{n}^{1/2}}\right]^{1/2} \\ &\simeq 2\lambda\bar{n}^{1/2}\left(1 \pm \frac{1}{2\bar{n}^{1/2}}\right) \\ &= 2\lambda\bar{n}^{1/2} \pm \lambda \end{aligned} \tag{4.128}$$

it follows that

$$t_c[\Omega(\bar{n}+\bar{n}^{1/2})-\Omega(\bar{n}-\bar{n}^{1/2})] \simeq t_c 2\lambda \simeq 1 \tag{4.129}$$

and thus $t_c \simeq (2\lambda)^{-1}$, which is independent of $\bar{n}$.

The preceding "derivation" of the collapse time is not very rigorous. We shall now give a more rigorous derivation. We expand $[n+1]^{1/2}$ about $\bar{n}$ as

$$[n+1]^{1/2} = [\bar{n}+1]^{1/2} + \frac{1}{2(\bar{n}+1)^{1/2}}(n-\bar{n}) + \cdots \tag{4.130}$$

so that we may approximate the inversion as

$$W(t) \simeq \frac{1}{2}e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} \Big[ e^{2i\lambda t(\bar{n}+1)^{1/2}} e^{i\lambda n t/(\bar{n}+1)^{1/2}} e^{-i\lambda t\bar{n}/(\bar{n}+1)^{1/2}} + e^{-2i\lambda t(\bar{n}+1)^{1/2}} e^{-i\lambda n t/(\bar{n}+1)^{1/2}} e^{i\lambda t\bar{n}/(\bar{n}+1)^{1/2}} \Big]. \tag{4.131}$$

Note that

$$\sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} e^{in\lambda t/(\bar{n}+1)^{1/2}} = \exp\left[\bar{n}\, e^{i\lambda t/(\bar{n}+1)^{1/2}}\right]. \tag{4.132}$$

For short times $t$

$$e^{i\lambda t/(\bar{n}+1)^{1/2}} \simeq 1 + i\lambda t/(\bar{n}+1)^{1/2} - \frac{\lambda^2 t^2}{2(\bar{n}+1)} \tag{4.133}$$

and thus

$$e^{\bar{n}e^{i\lambda t/(\bar{n}+1)^{1/2}}} \simeq e^{\bar{n}} e^{i\lambda t\bar{n}/(\bar{n}+1)} e^{-\frac{1}{2}\frac{\lambda^2 t^2}{(\bar{n}+1)}\bar{n}}. \tag{4.134}$$

Putting all of this together we arrive at

$$W(t) \simeq \cos\{2\lambda t(\bar{n}+1)^{1/2}\} \exp\left\{-\frac{1}{2}\frac{\lambda^2 t^2 \bar{n}}{\bar{n}+1}\right\}, \tag{4.135}$$

valid for a short time. The inversion evidently exhibits a Gaussian decay law with a decay time given by

$$t_c = \frac{\sqrt{2}}{\lambda}\sqrt{\frac{\bar{n}+1}{\bar{n}}} \simeq \frac{\sqrt{2}}{\lambda}, \qquad \bar{n} \gg 1 \tag{4.136}$$

which, apart from a numerical constant of the order of unity, agrees with our previous estimate.

Now let us examine the phenomenon of the revivals. $W(t)$ obviously consists of a sum of oscillating terms, each term oscillating at a particular Rabi frequency $\Omega(n) = 2\lambda\sqrt{n+1}$. If two neighboring terms are oscillating 180° out of phase with each other we expect at least an approximate cancellation of these terms. On the other hand, if the neighboring terms are in phase with each other we expect a constructive interference. In fact, this should be so whenever neighboring phases

differ by some multiple of $2\pi$. Since only those important frequencies around $\bar{n}$ will contribute, revivals should occur for times $t = t_R$ such that

$$[\Omega(\bar{n}+1) - \Omega(\bar{n})]\, t_R = 2\pi k, \quad k = 0, 1, 2, \ldots \tag{4.137}$$

holds. Expanding $\Omega(\bar{n})$ and $\Omega(\bar{n}+1)$ we easily arrive at $t_R = (2\pi/\lambda)\bar{n}^{1/2}k$, $(\bar{n} \gg 1)$. More rigorously, using Eq. (4.118) we have

$$W(t) \simeq \cos\left[2\lambda t(\bar{n}+1)^{1/2} + \lambda t\bar{n}/(\bar{n}+1)^{-1/2} - \bar{n}\sin\left(\frac{\lambda t}{(\bar{n}+1)^{1/2}}\right)\right]$$
$$\times \exp\left\{-\bar{n}\left[1 - \cos\left(\frac{\lambda t}{(\bar{n}+1)^{1/2}}\right)\right]\right\}. \tag{4.138}$$

Obviously, the amplitude will be a maximum wherever the time $t = t_R = (2\pi/\lambda)(\bar{n}+1)^{1/2}k \approx (2\pi/\lambda)^{1/2}\bar{n}^{1/2}k$, $(\bar{n} \gg 1)$, in agreement with the previous analysis. In Chapter 10 we discuss two experiments, one in the context of cavity quantum electrodynamics and the other in the context of the center-of-mass motion of a trapped ion, where the predicted collapse and revival of the Rabi oscillations have been observed.

## 4.6 The dressed states

There are many ways to solve for the dynamics of the Jaynes–Cummings model (see the reviews [14]). In Section 4.5 we have solved the time-dependent Schrödinger equation first for a field containing $n$ photons and then, by simple extrapolation, for the case of a field in a superposition of the number states. Another important way to obtain the dynamics is first to find the stationary states of the Jaynes–Cummings Hamiltonian. For reasons that should become clear shortly, these eigenstates are called the "dressed" states [15].

Consider once again the Jaynes–Cummings model Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-) \tag{4.139}$$

where we have *not* assumed the resonance condition $\omega = \omega_0$ at this point. In terms of the field number states, the interaction term in $\hat{H}$ causes only transitions of the type

$$|e\rangle|n\rangle \leftrightarrow |g\rangle|n+1\rangle \tag{4.140}$$

or

$$|e\rangle|n-1\rangle \leftrightarrow |g\rangle|n\rangle. \tag{4.141}$$

The product states $|e\rangle|n-1\rangle$, $|g\rangle|n\rangle$, etc., are sometimes referred to as the "bare" states of the Jaynes–Cummings model; they are product states of the unperturbed atom and field. For a fixed $n$, the dynamics is completely confined to the

two-dimensional space of product states, either ($|e\rangle|n-1\rangle$, $|g\rangle|n\rangle$) or ($|e\rangle|n\rangle$, $|g\rangle|n-1\rangle$). We define the following product states for a given $n$:

$$\begin{aligned} |\psi_{1n}\rangle &= |e\rangle|n\rangle \\ |\psi_{2n}\rangle &= |g\rangle|n+1\rangle. \end{aligned} \tag{4.142}$$

Obviously $\langle\psi_{1n} \mid \psi_{2n}\rangle = 0$. Using this basis we obtain the matrix elements of $\hat{H}$, $H_{ij}^{(n)} = \langle\psi_{in}|\hat{H}|\psi_{jn}\rangle$, which are

$$\begin{aligned} H_{11}^{(n)} &= \hbar\left[n\omega + \tfrac{1}{2}\omega_0\right], \\ H_{22}^{(n)} &= \hbar\left[(n+1)\omega - \tfrac{1}{2}\omega_0\right], \\ H_{12}^{(n)} &= \hbar\lambda\sqrt{n+1} = H_{21}^{(n)}. \end{aligned} \tag{4.143}$$

Thus in the $2 \times 2$ subspace of Eq. (4.142) we obtain the matrix representation of $\hat{H}$:

$$\boldsymbol{H}^{(n)} = \begin{bmatrix} n\omega + \frac{1}{2}\hbar\omega_0 & \hbar\lambda\sqrt{n+1} \\ \hbar\lambda\sqrt{n+1} & (n+1)\omega - \frac{1}{2}\omega_0 \end{bmatrix}. \tag{4.144}$$

This matrix is "self-contained" since, as we have said, the dynamics connects only those states for which the photon number changes by $\pm 1$. For a given $n$, the energy eigenvalues of $\boldsymbol{H}^{(n)}$ are as follows:

$$E_{\pm}(n) = \left(n + \frac{1}{2}\right)\hbar\omega \pm \hbar\Omega_n(\Delta) \tag{4.145}$$

where

$$\Omega_n(\Delta) = [\Delta^2 + 4\lambda^2(n+1)]^{1/2} \quad (\Delta = \omega_0 - \omega) \tag{4.146}$$

is the Rabi frequency which now includes the effects of the detuning $\Delta$. Obviously, for $\Delta = 0$ we obtain $\Omega_n(0) = 2\lambda\sqrt{n+1}$, the same quantum electrodynamic Rabi frequencies seen earlier. The eigenstates $|n, \pm\rangle$ associated with the energy eigenvalues are given by

$$\begin{aligned} |n,+\rangle &= \cos(\Phi_n/2)|\psi_{1n}\rangle + \sin(\Phi_n/2)|\psi_{2n}\rangle \\ |n,-\rangle &= -\sin(\Phi_n/2)|\psi_{1n}\rangle + \cos(\Phi_n/2)|\psi_{2n}\rangle \end{aligned} \tag{4.147}$$

where the angle $\Phi_n$ is defined through

$$\Phi_n = \tan^{-1}\left(\frac{2\lambda\sqrt{n+1}}{\Delta}\right) = \tan^{-1}\left(\frac{\Omega_n(0)}{\Delta}\right) \tag{4.148}$$

and where

$$\begin{aligned} \sin(\Phi_n/2) &= \frac{1}{\sqrt{2}}\left[\frac{\Omega_n(\Delta) - \Delta}{\Omega_n(\Delta)}\right]^{1/2} \\ \cos(\Phi_n/2) &= \frac{1}{\sqrt{2}}\left[\frac{\Omega_n(\Delta) + \Delta}{\Omega_n(\Delta)}\right]^{1/2}. \end{aligned} \tag{4.149}$$

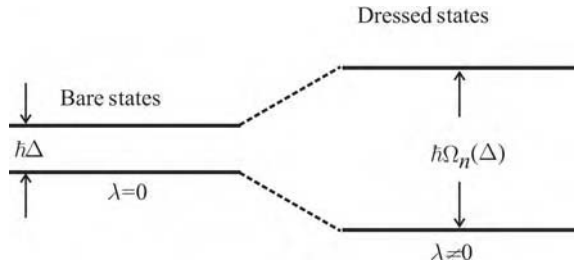



**Fig. 4.8.** Energy-level splitting due to the interaction of the atom with a quantized field. The split levels on the right are the energy levels of the dressed states.

The states $|n,\pm\rangle$ are often referred to as "dressed states" or as the Jaynes–Cummings doublet. The bare states $|\psi_{1n}\rangle$ and $|\psi_{2n}\rangle$, of energies $E_{1n} = \hbar(\omega_0/2 + \hbar\omega)$ and $E_{2n} = \hbar[-\omega_0/2 + (n+1)\omega]$ respectively, are each further split in energy owing to the interaction as indicated in Fig. 4.8. The splitting of the bare states into the dressed states is a kind of Stark shift, often called the AC, or dynamic, Stark shift. Note that in the limit of exact resonance, $\Delta = 0$, the bare states are degenerate but the splitting of the dressed states of course, remains. In this limit, the dressed states are related to the bare states according to

$$\begin{aligned} |n,+\rangle &= \frac{1}{\sqrt{2}}(|e\rangle|n\rangle + |g\rangle|n+1\rangle) \\ |n,-\rangle &= \frac{1}{\sqrt{2}}(-|e\rangle|n\rangle + |g\rangle|n+1\rangle). \end{aligned} \tag{4.150}$$

To see how the dressed states can be used to obtain the dynamics for rather general initial states, let us consider the specific case of a field prepared in some superposition of number states

$$|\psi_f(0)\rangle = \sum_n C_n|n\rangle \tag{4.151}$$

where an atom, prepared in state $|e\rangle$, is injected into the field. Thus the initial state of the atom–field system is

$$\begin{aligned} |\psi_{\rm af}(0)\rangle &= |\psi_f(0)\rangle|e\rangle \\ &= \sum_n C_n|n\rangle|e\rangle = \sum_n C_n|\psi_{1n}\rangle. \end{aligned} \tag{4.152}$$

From Eqs. (4.147), we obtain $|\psi_{1n}\rangle$ in terms of the dressed states $|n,\pm\rangle$ as

$$|\psi_{1n}\rangle = \cos(\Phi_n/2)|n,+\rangle - \sin(\Phi_n/2)|n,-\rangle \tag{4.153}$$

and thus

$$|\psi_{\rm af}(0)\rangle = \sum_n C_n[\cos(\Phi_n/2)|n,+\rangle - \sin(\Phi_n/2)|n,-\rangle]. \tag{4.154}$$

Since the dressed states $|n, \pm\rangle$ are stationary states of the atom–field system, then the state vector for times $t > 0$ is just given by

$$|\psi_{\rm af}(t)\rangle = \exp\left[-\frac{i}{\hbar}\hat{H}t\right]|\psi_{af}(0)\rangle$$
$$= \sum_n C_n\Big[\cos(\Phi_n/2)|n,+\rangle e^{-iE_+(n)t/\hbar} - \sin(\Phi_n/2)|n,-\rangle e^{-iE_-(n)t/\hbar}\Big]. \tag{4.155}$$

Of course, the entire result may now be recast back into the more familiar "bare" state basis by simply substituting $|n, \pm\rangle$ from Eqs. (4.147). In the limit $\Delta = 0$ we will recover the previous result of Eq. (4.120). The demonstration of this is left as an exercise.

## 4.7 Density-operator approach: application to thermal states

So far, we have considered only cases where the field and the atom are initially in pure states. In general though, one or both of the subsystems may initially be in mixed states which requires us to seek a solution in terms of the density operator. For example, the field may initially be in a thermal state described by the density operator of Eq. (2.144). In studying the case of a two-level atom interacting with a thermal state described by a density operator, we are afforded yet another way to solve for the dynamics of the Jaynes–Cummings model.

We shall work in the interaction picture and once again assume the resonance condition so that the dynamics is driven by

$$\hat{H}_{\rm I} = \hbar\lambda(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-). \tag{4.156}$$

If $\hat{\rho}(t)$ is the density operator of the atom–field system at time $t$, the evolution of the system is given by

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_{\rm I}, \hat{\rho}] \tag{4.157}$$

whose solution may be written as

$$\hat{\rho}(t) = \hat{U}_{\rm I}(t)\hat{\rho}(0)\hat{U}_{\rm I}^\dagger(t) \tag{4.158}$$

where

$$\hat{U}_{\rm I}(t) = \exp[-i\hat{H}_{\rm I}t/\hbar]$$
$$= \exp[-i\lambda t(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-)]. \tag{4.159}$$

In the two-dimensional atomic subspace, the operators $\hat{\sigma}_\pm$ and $\hat{\sigma}_3$ have the matrix representations

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.160}$$

where we have used the convention

$$\sigma_j = \begin{pmatrix} \langle e|\hat{\sigma}_j|e\rangle & \langle e|\hat{\sigma}_j|g\rangle \\ \langle g|\hat{\sigma}_j|e\rangle & \langle g|\hat{\sigma}_j|g\rangle \end{pmatrix} \quad j = \pm,\ 3. \tag{4.161}$$

In this two-dimensional subspace the evolution operator $\hat{U}_{\rm I}(t)$ may be expanded as

$$\hat{\boldsymbol{U}}_{\rm I}(t) = \begin{pmatrix} \hat{C}(t) & \hat{S}'(t) \\ \hat{S}(t) & \hat{C}'(t) \end{pmatrix} \tag{4.162}$$

where

$$\hat{C}(t) = \cos(\lambda t\sqrt{\hat{a}\hat{a}^\dagger}) \tag{4.163}$$

$$\hat{S}(t) = -i\hat{a}^\dagger \frac{\sin(\lambda t\sqrt{\hat{a}\hat{a}^+})}{\sqrt{\hat{a}\hat{a}^\dagger}} \tag{4.164}$$

$$\hat{C}'(t) = \cos(\lambda t\sqrt{\hat{a}^\dagger\hat{a}}) \tag{4.165}$$

$$\hat{S}'(t) = -i\hat{a}\frac{\sin(\lambda t\sqrt{\hat{a}^\dagger\hat{a}})}{\sqrt{\hat{a}^\dagger\hat{a}}}. \tag{4.166}$$

(The operators $\hat{C}$, $\hat{S}$, etc., here are not to be confused with the cosine and sine operators of the phase introduced in Chapter 2.) The Hermitian adjoint of $\hat{U}_{\rm I}(t)$ of Eq. (4.162) is just

$$\hat{U}_{\rm I}^\dagger(t) = \hat{U}_{\rm I}(-t) = \begin{pmatrix} \hat{C}(t) & -\hat{S}'(t) \\ -\hat{S}(t) & \hat{C}'(t) \end{pmatrix}. \tag{4.167}$$

We now suppose that at $t = 0$ the density operator for the atom–field system factors into field and atomic parts:

$$\hat{\rho}(0) = \hat{\rho}^{\rm F}(0) \otimes \hat{\rho}^{\rm A}(0). \tag{4.168}$$

We further suppose (to work out a particular example) that the atom is initially in the excited state $|e\rangle$ such that $\hat{\rho}^{\rm A}(0) = |e\rangle\langle e|$. The corresponding density matrix for the atom (using the convention of Eq. (4.161)) is

$$\rho^{\rm A}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.169}$$

and thus we may write for the system

$$\hat{\rho}(0) = \begin{pmatrix} \hat{\rho}^{\rm F}(0) & 0 \\ 0 & 0 \end{pmatrix} = \hat{\rho}^{\rm F}(0) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.170}$$

Using Eqs. (4.162), (4.167) and (4.169) in Eq. (4.158) we find that

$$\hat{\rho}(t) = \begin{pmatrix} \hat{C}(t)\hat{\rho}(0)\hat{C}(t) & -\hat{C}(t)\hat{\rho}^{\rm F}(0)\hat{S}'(t) \\ \hat{S}(t)\hat{\rho}^{\rm F}(0)\hat{C}(t) & -\hat{S}(t)\hat{\rho}^{\rm F}(0)\hat{S}'(t) \end{pmatrix}. \tag{4.171}$$

The reduced density operator of the field is found by tracing over the atomic states and thus

$$\hat{\rho}^{\mathrm{F}}(t) = \mathrm{Tr}_{\mathrm{A}}\hat{\rho}(t) = \hat{C}(t)\hat{\rho}^{\mathrm{F}}(0)\hat{C}(t) - \hat{S}(t)\hat{\rho}^{\mathrm{F}}(0)\hat{S}'(t). \tag{4.172}$$

The density matrix elements for the field are

$$\begin{aligned}\hat{\rho}^{\mathrm{F}}_{nm}(t) &\equiv \langle n|\hat{\rho}^{\mathrm{F}}(t)|m\rangle \\ &= \langle n|\hat{C}(t)\hat{\rho}^{\mathrm{F}}(0)\hat{C}(t)|m\rangle - \langle n|\hat{S}(t)\hat{\rho}^{\mathrm{F}}(0)\hat{S}'(t)|m\rangle.\end{aligned} \tag{4.173}$$

On the other hand, tracing over the field states we obtain the reduced density operator of the atom:

$$\hat{\rho}^{\mathrm{A}}(t) = \mathrm{Tr}_{\mathrm{F}}\hat{\rho}(t) = \sum_{n=0}^{\infty}\langle n|\hat{\rho}(t)|n\rangle. \tag{4.174}$$

The density operator matrix elements are given by

$$\langle i|\hat{\rho}^{\mathrm{A}}(t)|j\rangle = \sum_{n=0}^{\infty}\langle i,n|\hat{\rho}(t)|j,n\rangle = \rho^{\mathrm{A}}_{ij}(t) \tag{4.175}$$

where $i, j = e, g$. The diagonal elements $\rho^{\mathrm{A}}_{ee}(t)$ and $\rho^{\mathrm{A}}_{gg}(t)$ are the populations of the excited and ground states, respectively, and satisfy the condition

$$\rho^{\mathrm{A}}_{gg}(t) + \rho^{\mathrm{A}}_{ee}(t) \ = 1. \tag{4.176}$$

The atomic inversion is given by

$$W(t) = \rho^{\mathrm{A}}_{ee}(t) - \rho^{\mathrm{A}}_{gg}(t) \ = 2\rho^{\mathrm{A}}_{ee}(t) - 1. \tag{4.177}$$

From Eqs. (4.171) and (4.175) we find that

$$\begin{aligned}\rho^{\mathrm{A}}_{ee}(t) &\equiv \sum_{n=0}^{\infty}\langle n|\hat{C}(t)\hat{\rho}^{\mathrm{F}}(0)\hat{C}(t)|n\rangle \\ &= \sum_{n=0}^{\infty}\langle n|\,\hat{\rho}^{\mathrm{F}}(0)|n\rangle\cos^2(\lambda t\sqrt{n+1}).\end{aligned} \tag{4.178}$$

If the field is initially in a pure state

$$|\psi_{\mathrm{F}}\rangle = \sum_{n=0}^{\infty}C_n|n\rangle \tag{4.179}$$

then

$$\hat{\rho}^{\mathrm{F}}(0) = |\psi_{\mathrm{F}}\rangle\langle\psi_{\mathrm{F}}| \tag{4.180}$$

and thus

$$\rho^{\mathrm{A}}_{ee}(t) = \sum_{n=0}^{\infty}|C_n|^2\cos^2(\lambda t\sqrt{n+1}) \tag{4.181}$$

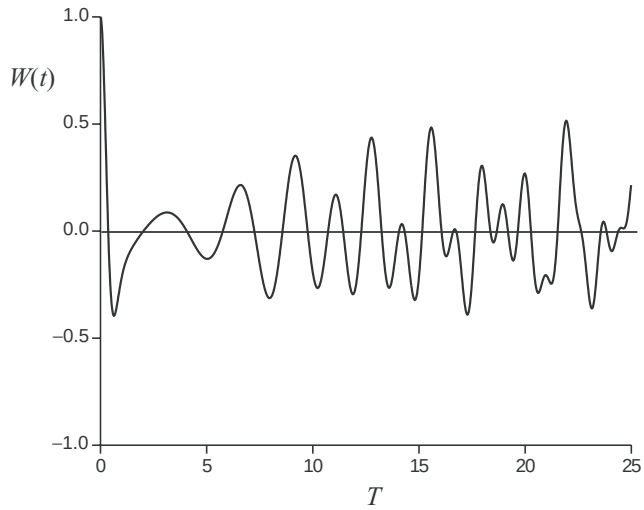



**Fig. 4.9.** The atomic inversion versus time for the atom initially in the excited state and the field initially in a thermal state with $\bar{n} = 2$. Again, $T = \lambda t$.

which, through Eq. (4.177), yields the atomic inversion found in Eq. (4.123). But suppose the field is initially in a thermal state (a mixed state) where

$$\hat{\rho}^{\mathrm{F}}(0) = \hat{\rho}_{\mathrm{Th}} = \sum_n P_n |n\rangle\langle n| \tag{4.182}$$

where $P_n$ is given by Eq. (2.145). From Eq. (4.178) we ultimately obtain the atomic inversion for an atom resonantly interacting with a thermal field as [16]

$$W(t) = \sum_{n=0}^{\infty} P_n \cos(2\lambda t\sqrt{n+1}). \tag{4.183}$$

In Fig. 4.9 we plot $W(t)$ versus $\lambda t$ for a thermal field containing an average photon number of $\bar{n} = 2$. We leave to the reader an analysis of the observed behavior.

## 4.8 The Jaynes–Cummings model with large detuning: a dispersive interaction

In the foregoing we have mostly assumed that the detuning $\Delta = \omega_0 - \omega = 0$. An important variation on the original Jaynes–Cummings model is the situation in which the detuning is large enough such that direct atomic transitions do not occur but where nevertheless "dispersive" interactions between a single atom and a cavity field do occur [17]. This version of the Jaynes–Cummings model is important in a number of applications related to fundamental tests of quantum mechanics, some of which are discussed in Chapter 10.

As is shown in Appendix C, the effective atom–field interaction Hamiltonian in the case of large detuning is given by

$$\hat{H}_{\mathrm{eff}} = \hbar\chi[\hat{\sigma}_+\hat{\sigma}_- + \hat{a}^\dagger\hat{a}\,\hat{\sigma}_3], \tag{4.184}$$

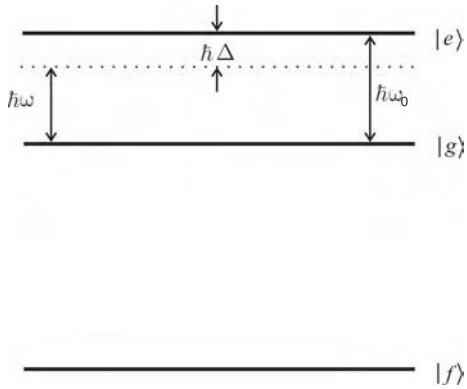



**Fig. 4.10.** The levels $|e\rangle$ and $|g\rangle$ are out of resonance with the field enough so that there are no direct transitions between them and only the dispersive interaction occurs. The state $|f\rangle$ is so far out of resonance with state $|g\rangle$ and the field that not even a dispersive interaction is present.

where $\chi = \lambda^2/\Delta$. Note that $\hat{\sigma}_+\hat{\sigma}_- = |e\rangle\langle e|$. Suppose the initial state of the atom–field system is $|\psi(0)\rangle = |g\rangle|n\rangle$, that is, the atom is in the ground state and the field is in a number state. Then, according to the interaction Hamiltonian of Eq. (4.184), the state at time $t > 0$ is

$$|\psi(t)\rangle = e^{-i\hat{H}_{\text{eff}}t/\hbar}|\psi(0)\rangle = e^{i\chi nt}|g\rangle|n\rangle \tag{4.185}$$

while for the initial state $|\psi(0)\rangle = |e\rangle|n\rangle$ we have

$$|\psi(t)\rangle = e^{-i\hat{H}_{\text{eff}}t/\hbar}|\psi(0)\rangle = e^{i\chi(n+1)t}|e\rangle|n\rangle. \tag{4.186}$$

Evidently, nothing very interesting happens, just the production of unmeasurable phase factors. On the other hand, for initial coherent states of the field we have, for $|\psi(0)\rangle = |g\rangle|\alpha\rangle$,

$$|\psi(t)\rangle = e^{-i\hat{H}_{\text{eff}}t/\hbar}|\psi(0)\rangle = |g\rangle|\alpha\, e^{i\chi t}\rangle, \tag{4.187}$$

and for $|\psi(0)\rangle = |e\rangle|\alpha\rangle$ we have

$$|\psi(t)\rangle = e^{-i\hat{H}_{\text{eff}}t/\hbar}|\psi(0)\rangle = e^{-i\chi t}|e\rangle|\alpha\, e^{-i\chi t}\rangle. \tag{4.188}$$

We notice that the coherent-state amplitude is rotated in phase space by the angle $\chi t$ but that the direction of the rotation depends on the state of the atom. Suppose now that the atom is prepared in a superposition of the ground and excited states. For simplicity we take this to be a "balanced" state of the form $|\psi_{\text{atom}}\rangle = (|g\rangle + e^{i\phi}|e\rangle)/\sqrt{2}$, where $\phi$ is some phase. With the initial state $|\psi(0)\rangle = |\psi_{\text{atom}}\rangle|\alpha\rangle$ we have

$$|\psi(t)\rangle = e^{-i\hat{H}_{\text{eff}}t/\hbar}|\psi(0)\rangle = \frac{1}{\sqrt{2}}\Big(|g\rangle|\alpha\, e^{i\chi t}\rangle + e^{-i(\chi t-\phi)}|e\rangle|\alpha\, e^{-i\chi t}\rangle\Big). \tag{4.189}$$

This state is a bit more interesting, in fact, much more interesting as, in general, there is entanglement between the atom and the field. If we take $\chi t = \pi/2$ we then have

$$\left|\psi\left(\frac{\pi}{2\chi}\right)\right\rangle = \frac{1}{\sqrt{2}}(|g\rangle|i\,\alpha\,\rangle - i\, e^{i\phi}|e\rangle|-i\,\alpha\,\rangle). \tag{4.190}$$

Notice that, in terms of our phase-space pictures, the two coherent states in Eq. (4.190) are separated by 180°, the maximal separation. Coherent states differing in phase by 180° are also maximally *distinguishable* in the sense that there is essentially no overlap between the two states, at least if $|\alpha|$ is large enough. In fact, this is the case even with $|\alpha|$ as low as $\sqrt{2}$. With very large values of $|\alpha|$, the two coherent states are said to be *macroscopically* distinguishable and, for moderate values, *mesoscopically* distinguishable. The entangled state of Eq. (4.190) might bring to mind the tale of Schrödinger's ill-fated cat [18], suspended in a state of limbo, suspended in an entanglement between life and death and a non-decayed or decayed radioactive *microscopic* atom. Symbolically, the entangled state in Schrödinger's famous "paradox" is thus:

$$|\psi_{\text{atom}-\text{cat}}\rangle = \frac{1}{\sqrt{2}}[|\text{atom not decayed}\rangle|\text{cat alive}\rangle + |\text{atom decayed}\rangle|\text{cat dead}\rangle]. \quad (4.191)$$

The parallel between this state and that of Eq. (4.190) is obvious; the states of the two-level atom play the role of the radioactive atom and the two phase-separated coherent field states that of the cat.

Finally, there is another initial atomic state that is often considered. Suppose there is another atomic state, that we denote $|f\rangle$, of energy $E_f \ll E_g$ as pictured in Fig. 4.10, and of parity opposite that of $|g\rangle$. The cavity is assumed to support no mode resonant with the $f \leftrightarrow g$ transition and we further assume that the state $|f\rangle$ is so far out of resonance with the available cavity field mode that there is no discernable dispersive interaction either. Thus with the atom initially prepared in state $|f\rangle$ and with the field initially in the coherent state $|\alpha\rangle$, the initial product state $|f\rangle|\alpha\rangle$ remains as such; i.e. it does not evolve. Now suppose the atom is prepared in a superposition of the form $|\psi_{\text{atom}}\rangle = (|g\rangle + e^{i\phi}|f\rangle)/\sqrt{2}$. The initial-state atom–field state $|\psi(0)\rangle = |\psi_{\text{atom}}\rangle|\alpha\rangle$ is easily seen to evolve into

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(|g\rangle|\alpha\, e^{i\chi t}\rangle + e^{i\phi}|f\rangle|\alpha\rangle). \quad (4.192)$$

In this case, only one component of the initial atomic superposition causes a phase shift in the coherent state, an advantage in certain applications. For $\chi t = \pi/2$ we have

$$\left|\psi\left(\frac{\pi}{2\chi}\right)\right\rangle = \frac{1}{\sqrt{2}}(|g\rangle|-\alpha\rangle + e^{i\phi}|f\rangle|\alpha\rangle), \quad (4.193)$$

another version of the Schrödinger-cat state.

In Chapters 7, 8, and 10 we shall further elaborate on issues related to Schrödinger's cat.

## 4.9 Extensions of the Jaynes–Cummings model

There are many possible extensions of the original Jaynes–Cummings model involving various types of alternative interactions. Among them are models

involving two-photon transitions, multimode and multilevel models, Raman coupled models, two-channel models, etc. We shall not discuss these models here but we do refer to the various review articles that have appeared [14, 19] and references therein. Some of these extensions will appear in the context of homework problems. Further, it turns out that Jaynes–Cummings types of interaction also occur in the context of the vibrational motion of an ion in an electromagnetic trap. The simplest example of these will be discussed in Chapter 10.

## 4.10 Schmidt decomposition and von Neumann entropy for the Jaynes–Cummings model

We finish this chapter with a discussion of the Schmidt decomposition and the related von Neumann entropy as they pertain to the Jaynes–Cummings model. As this system is bipartite, a Schmidt decomposition is assured, as discussed in Appendix A. We have already presented the solution of the time-dependent Schrödinger equation in Eq. (4.120), which we rewrite here as

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} [a_n(t)|g\rangle|n\rangle + b_n(t)|e\rangle|n\rangle], \tag{4.194}$$

where

$$\begin{aligned} a_n(t) &= C_g C_n \cos(\lambda t\sqrt{n}) - iC_e C_{n-1} \sin(\lambda t\sqrt{n}), \\ b_n(t) &= C_e C_n \cos(\lambda t\sqrt{n+1}) - iC_g C_{n+1} \sin(\lambda t\sqrt{n+1}). \end{aligned} \tag{4.195}$$

But according to the Schmidt decomposition, for any instant in time $t$ we can always find bases $\{|u_i(t)\rangle\}$ for the atom and $\{|v_i(t)\rangle\}$ for the field such that the pure state of the system can be written as

$$|\Psi(t)\rangle = g_1(t)|u_1(t)\rangle|v_1(t)\rangle + g_2(t)|u_2(t)\rangle|v_2(t)\rangle. \tag{4.196}$$

The reduced density matrices of the atom and field in these bases are identical:

$$\begin{aligned} \rho_u(t) &= \begin{pmatrix} |g_1(t)|^2 & 0 \\ 0 & |g_2(t)|^2 \end{pmatrix} \\ \rho_v(t) &= \begin{pmatrix} |g_1(t)|^2 & 0 \\ 0 & |g_2(t)|^2 \end{pmatrix}. \end{aligned} \tag{4.197}$$

In order to find the coefficients $g_1(t)$ and $g_2(t)$ and the eigenvectors $|u_i\rangle$ and $|v_i\rangle$ we first calculate the reduced density operator of the atom in the bare basis specified by $|e\rangle$ and $|g\rangle$ and obtain

$$\rho_u = \begin{pmatrix} \sum_{n=0}^{\infty} |a_n(t)|^2 & \sum_{n=0}^{\infty} a_n(t) b_n^*(t) \\ \sum_{n=0}^{\infty} b_n(t) a_n^*(t) & \sum_{n=0}^{\infty} |b_n(t)|^2 \end{pmatrix}. \tag{4.198}$$

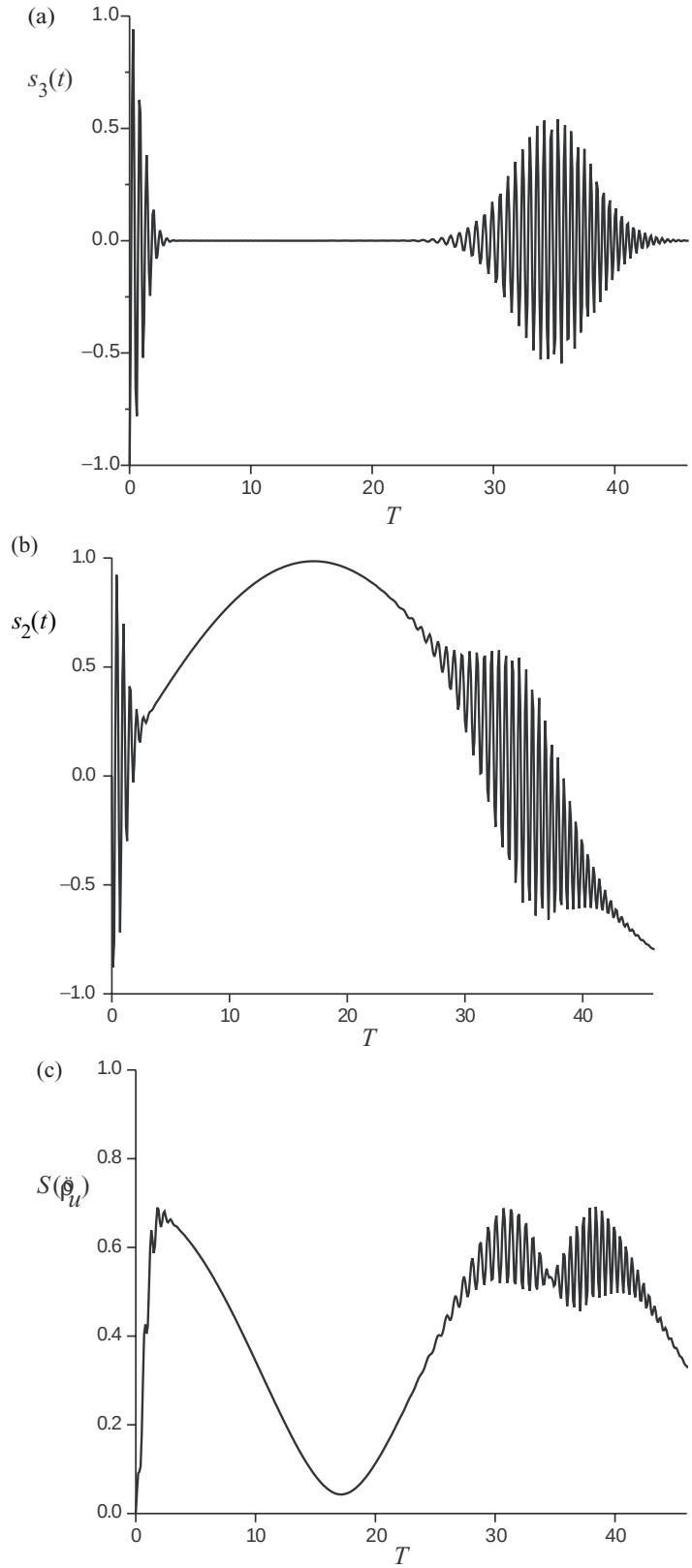



**Fig. 4.11.** Plots of $s_3$, $s_2$ and $S(\hat{\rho}_u)$ against the scaled time $\lambda\, t$. Again $T = \lambda t$.

Then we can use the parametrization of the density matrix in terms of the components of the Bloch vector as in Eq. (A25) to obtain

$$\begin{aligned} s_1(t) &= \sum_{n=0}^{\infty}\Big[a_n(t)b_n^*(t) + b_n(t)a_n^*(t)\Big], \\ s_2(t) &= -i\sum_{n=0}^{\infty}\Big[a_n(t)b_n^*(t) - b_n(t)a_n^*(t)\Big], \\ s_3(t) &= \sum_{n=0}^{\infty}[|a_n(t)|^2 - |b_n(t)|^2]. \end{aligned} \tag{4.199}$$

Then, as discussed in Appendix A, the coefficients $g_1(t)$ and $g_2(t)$ can be expressed in terms of the length of the Bloch vector according to

$$g_1(t) = \frac{1}{2}[1 + |\mathbf{s}(t)|], \quad g_2(t) = \frac{1}{2}[1 - |\mathbf{s}(t)|]. \tag{4.200}$$

Notice that the field mode, previously described in an infinite-dimensional Fock-state basis, is reduced to a "two-level" system in the Schmidt basis. The length of the Bloch vector is a measure of the purity of the atom–field system. For the atom and field in pure states the length of the Bloch vector is unity.

As discussed in Appendix A, the utility of the Schmidt decomposition is that it is easy to obtain an expression for von Neumann entropy, which for each of the subsystems of the Jaynes–Cummings model is

$$S(\hat{\rho}_u) = -g_1(t)\ln g_1(t) - g_2(t)\ln g_2(t) = S(\hat{\rho}_v). \tag{4.201}$$

Whenever $|\mathbf{s}| = 1$ we have $S(\hat{\rho}_u) = 0 = S(\hat{\rho}_v)$. In Fig. 4.11, for an atom initially in the excited state and for the field initially in a coherent state with $\alpha = \sqrt{30}$, we plot (a) the $s_3$ component of the Bloch vector, (b) the $s_2$ component, and (c) the von Neumann entropy $S(\hat{\rho}_u)$, all against the scaled time $T = \lambda t$. With our particular initial condition, $s_1 = 0$ for all times. We notice that $s_3$ undergoes the collapse and revival of the atomic inversion and that $s_2$ goes close to unity at one point during the quiescent period of the collapse of the former, indicating that the atom and field are nearly in pure states at that time. The von Neumann entropy, of course, is a minimum at that point. This result is perhaps a bit surprising and counter-intuitive. This behavior was first noticed, through a different type of calculation, by Gea-Banacloche [20] and was further examined by Phoenix and Knight [21].

## Problems

1. Consider the semiclassical Rabi model (i.e. a two-level atom with a prescribed classical field) in the RWA as described by Eqs. (4.74). Obtain the solution assuming the atom initially in the excited state. Calculate the atomic inversion as a function of time.
2. Using the result of the previous problem, obtain the time-dependent expectation value of the atomic dipole moment operator $\hat{d} = d(\hat{\sigma}_+ + \hat{\sigma}_-)$ for the case of exact resonance. Compare the evolution of the dipole moment with the atomic inversion for the same resonance condition and comment on any similarities or differences.

3. In the fully quantized model of a two-level atom interacting with a quantized field within the RWA, the Jaynes–Cummings model, we have obtained the exact resonance solution for the initial state where the atom is excited and where the field is in a number state $|n\rangle$ (see Eqs. (4.107) to (4.116)). Recall that the Rabi oscillations of the atomic inversion are periodic for this case, just as in the semiclassical model. Use that solution to obtain the expectation value of the atomic dipole moment operator and compare with the result of the previous question. Is the time evolution of the atomic dipole moment in any way similar to that obtained in the semiclassical case? Comment on the result.
4. Obtain the expectation value of the atomic dipole moment as given by the Jaynes–Cummings model in the case where the field is initially in a coherent state. How does the result compare with the two previous cases? You should make a plot of the expectation value of the dipole moment as a function of time.
5. In the text, we obtained the dynamics of the Jaynes–Cummings model assuming exact resonance, $\Delta = 0$. Reconsider the problem for the case where $\Delta \neq 0$. Obtain plots of the atomic inversion and note the effect of the nonzero detuning on the collapse and revivals of the Rabi oscillations. Perform an analysis to obtain the effect of the nonzero detuning on the collapse and revival times.
6. Consider the resonant Jaynes–Cummings model for the initial thermal state as in Section 4.7. Assume the atom is initially in the excited state. Analyse the collapse of the Rabi oscillations and determine the dependence of the collapse time on the average photon number of the thermal field.
7. Consider a simple model of degenerate Raman scattering, pictured in Fig. 4.12 (where $E_g = E_e$), and described by the interaction Hamiltonian $\hat{H}_{\rm I} = \hbar\lambda\hat{a}^\dagger\hat{a}(\sigma_+ + \sigma_-)$, where, as usual, $\sigma_+ = |e\rangle\langle g|$ and $\sigma_- = |g\rangle\langle e|$.
   (a) Obtain the dressed states for this model.
   (b) Assuming initially the field in a coherent state and the atom in the ground state, obtain the atomic inversion and show that the revivals of the Rabi oscillations are regular and complete.
   (c) Obtain the atomic inversion for an initial thermal state.
8. A resonant two-photon extension of the Jaynes–Cummings model is described by the effective Hamiltonian $H_{\rm eff} = \hbar\eta(\hat{a}^2\hat{\sigma}_+ + \hat{a}^{2\dagger}\hat{\sigma}_-)$, where, for the sake of simplicity, a small Stark shift term has been ignored. This Hamiltonian represents two-photon

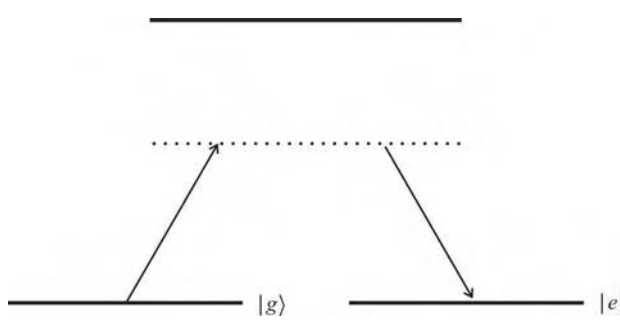



**Fig. 4.12.** Energy-level diagram for the degenerate Raman coupled model. The broken line represents a "virtual" intermediate state, too far off-resonance from a real level, the upper solid line, to become populated.

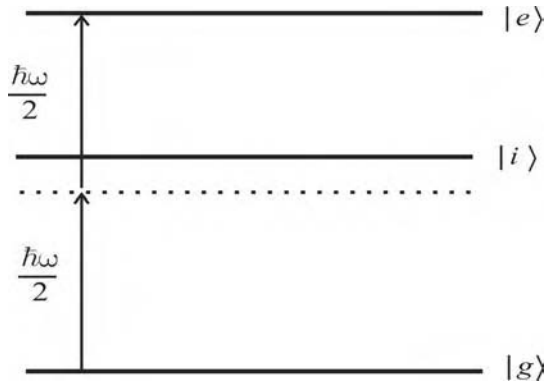



**Fig. 4.13.** Energy-level diagram for the resonant two-photon process. States $|e\rangle$ and $|g\rangle$ are of like parity whereas the intermediate state $|i\rangle$ is of opposite parity. The broken line represents a virtual atomic level, detuned from state $|i\rangle$.

absorption and emission between atomic levels of like parity. The process is represented by Fig. 4.13, where the broken line represents a virtual intermediate state of opposite parity.

(a) Obtain the dressed states for this system.

(b) Obtain the atomic inversion for this model assuming the atom initially in the ground state and that the field is initially in a number state. Repeat for a coherent state. Comment on the nature of the collapse and revival phenomena for these states.

(c) Obtain the atomic inversion for an initial thermal state.

9. A two-mode variation on the two-photon model of the previous problem is described by the Hamiltonian $H_{\text{eff}} = \hbar\eta(\hat{a}\,\hat{b}\hat{\sigma}_+ + \hat{a}^\dagger\hat{b}^\dagger\hat{\sigma}_-)$, that is, a photon from each mode is absorbed or emitted. Obtain the atomic inversion for the case where both modes are initially in coherent states. Analyse the collapse and revival phenomena.

11. Consider the resonant Jaynes–Cummings model with the atom initially in the excited state and the field initially in a coherent state with $\alpha = \sqrt{30}$.

(a) By taking the trace of the density operator over either the atomic or field states and thus obtaining a reduced density operator (see Appendix A), determine the degree to which the atom and field are entangled as a function of time. Pay particular attention to what happens at about midpoint between the initial collapse of the Rabi oscillations and the first revival. Is the behavior surprising? What connection, if any, is there between the result you have obtained and the results discussed in Section 4.10.

(b) Obtain plots (either contour or 3D) of the $Q$ function (Eq. (3.112)) of the field for various times between time $t = 0$ and the time of the first revival of the Rabi oscillations.

12. Consider the atom–field state of Eq. (4.193) and assume that $\phi = 0$. Suppose that there is some way to determine the state of the atom (such is possible via field ionization, as will be discussed in Chapter 10). For example, if the atom is detected in state $|g\rangle$ then the field state will be reduced to $|-\alpha\rangle = \langle g \mid \psi(\pi)\rangle / |\langle g \mid \psi(\pi)\rangle|$ with a similar result for detection of the atomic state $|f\rangle$. (See Appendix D.) But suppose it were somehow

possible to detect the atom in the superposition states $|S\pm\rangle = (|g\rangle \pm |f\rangle)/\sqrt{2}$. What field states are generated in these cases?

13. Reconsider the problem discussed in Section 4.10 but with the initial state of the atom taken to be the "balanced" superposition $|\psi(0)\rangle_{\text{atom}} = (|e\rangle + |g\rangle)/\sqrt{2}$. Obtain relevant plots of the components of the Bloch vector and of the von Neumann entropy. Contrast the results obtained in this case to the case where the atom is initially in the excited state.

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