Statistical Computing

Assignment 1

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1 Exersice 1

Question1. Let $Y \sim \mathcal{E}xp(\lambda)$ and let $a > 0$. We consider the variable after restricting its support to be $[a, +\infty)$. That is, let $X = Y \mid Y \geq a$, i.e. X has the law of Y conditionally on being in $[a, +\infty)$. Calculate $F_X(x)$, the cumulative distribution function of X, and $F_X^{-1}(u)$, the quantile function of X. Describe an algorithm to simulate X from $U \sim \mathcal{U}_{[0,1]}$.

Answer. For $x < a$, $F_X(x) = 0$. For $x \geq a$, We have:

$$
\begin{aligned}
&F_X(X \leq x) = \mathbb{P}(Y \leq x | Y \geq a) \\
&= \frac{\mathbb{P}(a \leq Y \leq x)}{\mathbb{P}(Y \geq a)} = \frac{\int_a^x \lambda e^{-\lambda y} \mathrm{d}y}{\int_a^{+\infty} \lambda e^{-\lambda y} \mathrm{d}y} \\
&= \frac{-e^{-\lambda y}|_a^x}{-e^{-\lambda y}|_a^{+\infty}} = \frac{e^{-\lambda a} - e^{-\lambda x}}{e^{-\lambda a}} \\
&= 1 - e^{-\lambda(x-a)}
\end{aligned}
$$

Hence the **cumulative distribution function** $F_X(x)$ is:

$$F_X(x) = (1 - e^{-\lambda(x-a)})\mathbb{1}_{x \geq a}$$

Here, $\mathbb{1}_{x \geq a}$ is the characteristic function, such that:

$$\mathbb{1}_{x \geq a} = \begin{cases} 1, x \geq a \\ 0, x < a \end{cases}$$

Then we solve $u = F_X(x)$, We get: $x = a - \ln(1-u)/\lambda$, hence the **quantile function** $F_X^{-1}(u)$ is:

$$F_X^{-1}(u) = a - \ln(1-u)/\lambda, u \in [0,1)$$

If we want to simulate X from $U \sim \mathcal{U}_{[0,1]}$, we know that if $U \sim \mathcal{U}_{[0,1]}$, $1-U$ is also uniform distribution in $[0,1]$, hence we can design an **algorithm** as follow:

- Simulate $U \sim \mathcal{U}_{[0,1]}$.
- Return $X = a - \ln(1-U)/\lambda$.

Question2. Let a and b be given, with $a < b$. Show that we can simulate $X = Y \mid a \leq Y \leq b$ from $U \sim \mathcal{U}_{[0,1]}$ using

$$X = F_Y^{-1}\left(F_Y(a)(1-U) + F_Y(b)U\right)$$

i.e. show that if X is given by the formula above, then $\Pr(X \leq x) = \Pr(Y \leq x \mid a \leq Y \leq b)$. Apply the formula to simulate an exponential random variable conditioned to be greater than a, as in the previous question.

Answer. Similar to the answer in **Question1**, we have:

$$\begin{aligned}
&\mathbb{P}(Y \leq x | a \leq Y \leq b)\\
&= \frac{\mathbb{P}(a \leq Y \leq x)}{\mathbb{P}(a \leq Y \leq b)} = \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)}\\
&= \mathbb{P}(U \leq \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)})\\
&= \mathbb{P}(U(F_Y(b) - F_Y(a)) \leq F_Y(x) - F_Y(a))\\
&= \mathbb{P}(F_Y(a)(1-U) + F_Y(b)U) \leq F_Y(x))\\
&= \mathbb{P}(F_Y^{-1}(F_Y(a)(1-U) + F_Y(b)U \leq x) = F_X(X \leq x)
\end{aligned}$$

Hence, using $X = F_Y^{-1}\left(F_Y(a)(1-U) + F_Y(b)U\right)$, we can simulate $X = Y \mid a \leq Y \leq b$ from $U \sim \mathcal{U}_{[0,1]}$.

Apply the formula to simulate an **exponential random variable** conditioned to be greater than a. i.e. $Y \sim \mathcal{E}xp(\lambda)$, $b = +\infty$. So we get $F_Y(y) = 1 - e^{-\lambda y}$, $F_Y^{-1}(u) = -\ln(1-u)/\lambda$, Hence:

$$F_Y(a) = 1 - e^{-\lambda a}, F_Y(b) = 1$$

$$
\begin{aligned}
X &= F_Y^{-1}\left(F_Y(a)(1-U) + F_Y(b)U\right) \\
&= F_Y^{-1}((1-e^{-\lambda a})(1-U) + U) \\
&= F_Y^{-1}(1 - e^{-\lambda a} + Ue^{-\lambda a}) \\
&= a - \ln(1-U)/\lambda
\end{aligned}
$$

So we apply the formula above to get the result in **Question1**.

Question3. Here is a simple algorithm to simulate $X = Y \mid Y > a$ for $Y \sim \mathcal{E}xp(\lambda)$:

(a) Let $Y \sim \mathcal{E}xp(\lambda)$. Simulate $Y = y$.

(b) If $Y > a$ then stop and return $X = y$, and otherwise, start again at step (a).

Show that this is just a rejection algorithm, by writing the proposal and target densities q and π, as well as the bound $M = \max_x \pi(x)/q(x)$. Calculate the expected number of trials to the first acceptance. Why is inversion to be preferred for $a \gg 1/\lambda$?

Answer. The **proposal density** is $q(x)$:

$$q(x) = \lambda e^{-\lambda x} \mathbb{1}_{x \geq 0}$$

The **target density** is $\pi(x)$:

$$\pi(x) = \lambda e^{-\lambda(x-a)} \mathbb{1}_{x \geq a}$$

And the **bound** M is:

$$M = \max_x \frac{\pi(x)}{q(x)} = \max_x \frac{\lambda e^{-\lambda(x-a)} \mathbb{1}_{x \geq a}}{\lambda e^{-\lambda x} \mathbb{1}_{x \geq 0}} = \max_x e^{\lambda a} \mathbb{1}_{x \geq a} = e^{\lambda a}$$

Hence, the **acceptance probability** α is:

$$\alpha = \frac{\pi(x)}{Mq(x)} = \frac{\lambda e^{-\lambda(x-a)} \mathbb{1}_{x \geq a}}{e^{\lambda a}(\lambda e^{-\lambda x} \mathbb{1}_{x \geq 0})} = \mathbb{1}_{x \geq a}$$

As the procedure above, we know that the algorithm in this question is just a **rejection algorithm**.

In order to calculate the expected number of trials to the first acceptance N, we have the lemma that N is a geometrically distribution with success probability $p = e^{-\lambda a}$, hence the **expected number** of N is:

$$\mathbb{E}(N) = \frac{1}{p} = e^{\lambda a}$$

In our application, we prefer $a \gg 1/\lambda$, becauce we know that $X = a - \ln(1-U)/\lambda$ from **Question1**. Though $\mathbb{E}(N) = e^{\lambda a}$ is large, we can reduce the relative error of X by $a \gg 1/\lambda$.

2 Exersice 2

Question1. Let X_1, X_2 be two independent random variables with $X_1 \sim \mathcal{G}amma(a,1)$ and $X_2 \sim \mathcal{G}amma(b,1)$. Show that $R = X_1/(X_1+X_2)$ and $S = X_1+X_2$ are independent and that $R \sim \mathcal{B}eta(a,b), S \sim \mathcal{G}amma(a+b,1)$. Recall that Gamma and Beta densities are

$$f_\Gamma(x;\alpha,\beta) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}, x \in (0,\infty)$$

and

$$f_B(x;a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, x \in (0,1)$$

Answer. The PDF of X_1 is:

$$f_1(x_1) = \frac{x_1^{a-1} e^{-x_1}}{\Gamma(a)}, x_1 \in [0,+\infty)$$

The PDF of X_2 is:

$$f_2(x_2) = \frac{x_2^{a-1} e^{-x_2}}{\Gamma(b)}, x_2 \in [0,+\infty)$$

X_i $(i=1,2)$ are independent. Hence, the joint probability density function is:

$$f_{1,2}(x_1,x_2) = f_1(x_1)f_2(x_2) = \frac{x_1^{a-1} x_2^{b-1} e^{-(x_1+x_2)}}{\Gamma(a)\Gamma(b)}, x_1 \times x_2 \in [0,+\infty) \times [0,+\infty)$$

And $r = \dfrac{x_1}{x_1+x_2} \in [0,1]$, $s = x_1 + x_2 \in [0,+\infty)$.
Cauculate the inversion: $x_1 = rs$, $x_2 = s(1-r)$.

The Jacobi determinant of this transformation is:

$$J = \begin{vmatrix} s & r \\ -s & 1-r \end{vmatrix} = s$$

Hence, the PDF of (R, S) is:

$$\begin{aligned} f(r,s) &= \frac{(rs)^{a-1}s^{b-1}(1-r)^{b-1}e^{-(rs+s-rs)}}{\Gamma(a)\Gamma(b)}|J| \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}r^{a-1}(1-r)^{b-1} \times \frac{s^{a+b-1}e^{-s}}{\Gamma(a+b)}, (r,s) \in [0,1] \times [0,+\infty) \end{aligned}$$

The definition field of (R, S) is a **rectangle**, and the probability density function of (R, S) **can be written as the product of two probability density function**.

Hence, the **PDF** of R is:

$$f_R(r) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}r^{a-1}(1-r)^{b-1}, r \in [0,1]$$

The **PDF** of S is:

$$f_S(s) = \frac{s^{a+b-1}e^{-s}}{\Gamma(a+b)}, s \in [0,+\infty)$$

Above all, R and S are independent and $R \sim \mathcal{B}eta(a,b), S \sim \mathcal{G}amma(a+b,1)$

Question2. Let $U \sim \mathcal{U}_{[0,1]}$ and $a > 0$. Show that $X = U^{1/a} \sim \mathcal{B}eta(a,1)$.

Answer. The PDF of U is: $f_1(u) = 1, u \in [0,1]$. And the transformation is: $x = g(u) = u^{1/a}$. So the inversion of the transformation is: $u = g^{-1}(x) = x^a$, and $(g^{-1}(x))' = ax^{a-1}$. Hence, the **PDF** of X is:

$$f_X(x) = f_1(g^{-1}(x))\left|(g^{-1}(x))'\right| = ax^{a-1} = \frac{\Gamma(a+1)}{\Gamma(a)\Gamma(1)}x^{a-1}, x \in [0,1]$$

That is $X = U^{1/a} \sim \mathcal{B}eta(a,1)$.

Question3. Let $U \sim \mathcal{U}_{[0,1]}$ and $V \sim \mathcal{U}_{[0,1]}$ be independent and $a \in (0,1)$. For $Y = U^{1/a}$ and $Z = V^{1/(1-a)}$, calculate

$$\mathbb{P}\left(\frac{Y}{Y+Z} \leq t, Y+Z \leq 1\right)$$

for any $t \in (0,1)$ and deduce that the conditional distribution of $W = Y/(Y+Z)$ given $Y + Z < 1$ is $\mathcal{B}eta(a, 1-a)$. (*Hint: writing both inequalities as constraints on Z could ease the calculation*).

Answer. We know that $y = u^{1/a}$ and $z = v^{1/(1-a)}$. So the inversion are: $u = y^a$, $v = z^{1-a}$. The Jacobi determinant of the transformation $(U, V) \to (Y, Z)$ is:

$$J = \begin{vmatrix} ay^{a-1} & 0 \\ 0 & (1-a)z^{-a} \end{vmatrix} = a(1-a)y^{a-1}z^{-a}$$

Hence, the PDF of (Y, Z) is:

$$\begin{aligned} &f_{Y,Z}(y,z) = f_{U,V}(u,v)\,|J| \\ &= f_U(u)f_V(v)a(1-a)y^{a-1}z^{-a} = 1 \times 1 \times a(1-a)y^{a-1}z^{-a} \\ &= a(1-a)y^{a-1}z^{-a}, (y,z) \in [0,1] \times [0,1] \end{aligned}$$

Cauculate:

$$\begin{aligned} &\mathbb{P}\left(\frac{Y}{Y+Z} \le t, Y+Z \le 1\right) = \mathbb{P}\left(\frac{Y(1-t)}{t} \le Z \le 1-Y\right) \\ &= \int_0^t \int_{y(1-t)/t}^{1-y} a(1-a)y^{a-1}z^{-a}\mathrm{d}z\mathrm{d}y = \int_0^t a\, y^{a-1}z^{1-a}\Big|_{y(1-t)/t}^{1-y} \mathrm{d}y \\ &= \int_0^t ay^{a-1}((1-y)^{1-a} - y^{1-a}(1-t)^{1-a}t^{a-1})\mathrm{d}y \\ &= \int_0^t ay^{a-1}(1-y)^{1-a}\mathrm{d}y - a(1-t)^{1-a}t^a \end{aligned}$$

For the transformation $(Y, Z) \to (W, M)$, $w = y/(y+z)$, $m = y + z$. The inversion are: $y = mw$, $z = m(1-w)$. The Jacobi determinant of this transformation is:

$$J = \begin{vmatrix} w & m \\ 1-w & -m \end{vmatrix} = -wm - m + wm = -m$$

The PDF of (W, M) is:

$$f_{W,M}(w,m) = a(1-a)(mw)^{a-1}(1-w)^{-a}m^{-a}\,|J| = a(1-a)w^{a-1}(1-w)^{-a}$$

Hence,

$$f_{W|Y+Z\le 1}(w) = \frac{a(1-a)w^{a-1}(1-w)^{-a}}{\mathbb{P}(Y+Z \le 1)}$$

We find that the **kernel** of $f_{W|Y+Z\le 1}(w)$ is $w^{a-1}(1-w)^{-a}$. That means the conditional distribution of $W = Y/(Y+Z)$ given $Y + Z < 1$ is $\mathcal{B}eta(a, 1-a)$.

Question4. In the setting of **Question3**, show that the conditional distribution of TW given $Y + Z \leq 1$, for an independent $T \sim \mathcal{E}xp(1) = \mathcal{G}amma(1, 1)$ random variable, is $\mathcal{G}amma(a, 1)$.

Answer. From **Question3**, we know that $W|Y + Z \leq 1 \sim \mathcal{B}eta(a, 1 - a)$.
$T \sim \mathcal{E}xp(1) = \mathcal{G}amma(1, 1)$, and independent of Y, Z, so we can get $T|Y + Z \leq 1 \sim \mathcal{G}amma(1, 1) = \mathcal{G}amma(a + 1 - a, 1)$.
From the inverse process of **Question1**, we know that: If two independent random variables with $R \sim \mathcal{B}eta(a, b)$, $S \sim \mathcal{G}amma(a + b, 1)$. And $X_1 = RS$, $X_2 = S(1 - R)$ are independent with $X_1 \sim \mathcal{G}amma(a, 1)$, $X_2 \sim \mathcal{G}amma(b, 1)$.
Hence, let $a = a$, $b = 1 - a$ in inverse process of **Question1**, we get:

$$P(TW \leq x|Y + Z \leq 1) = P(A \leq x)$$

where $A \sim \mathcal{G}amma(a, 1)$.

Question5. Let $a \in (0, 1)$.

(a) Simulate two independent $U, V \sim \mathcal{U}_{[0,1]}$

(b) Set $Y = U^{1/a}$ and $Z = V^{1/(1-a)}$

(c) If $(Y + Z) \leq 1$ go to (d), else go to (a).

(d) Simulate an independent $A \sim \mathcal{U}_{[0,1]}$ and set $T = -\log(A)$.

(e) Return $TY/(Y + Z)$.

What is this procedure doing? Explain its relevance for simulations.

Answer. The procedure **generates a random variable from $\mathcal{G}amma(a, 1)$.**
In step (a),(b),(c), we get Y, Z as the random variable in **Question3**.
In step (d), we have the lemma that: $T \sim \mathcal{E}xp(1) = \mathcal{G}amma(1, 1)$.
From the discussion of **Question4**, we know that the result in step (e): $TY/(Y + Z)$ is a random variable from $\mathcal{G}amma(a, 1)$.

Question6. Based on **Question1**, explain how you can generate a $\mathcal{B}eta(a,b)$ random variable from a sequence of $\mathcal{U}_{[0,1]}$ random variables, for any $a > 0$ and $b > 0$. (*Hint: consider* $a \in (0,1)$ *first and use the additivity of Gamma variables to generate* $\mathcal{G}amma(a,1)$ *variables, from which the Beta variables can be constructed*).

Answer. We can generate a random variable from $\mathcal{B}eta(a,b)$ by the procedure below:
(a). For any $x > 0$, cauculate

$$x' = \frac{x}{\lceil x \rceil + 1} \in (0,1)$$

(b). Follow the procedure in **Question5**, we can get $X \sim \mathcal{G}amma(x',1)$.
(c). Run step (b) for $\lceil x \rceil + 1$ times, we get $X_i, i = 1, 2, ..., \lceil x \rceil + 1$.
(d). Cauculate $\sum_{i=1}^{\lceil x \rceil + 1} X_i$, by the lemma we know:

$$\sum_{i=1}^{\lceil x \rceil + 1} X_i \sim \mathcal{G}amma(x,1)$$

(e). set $x = a, b$, run step (a)∼(d), we get two random variables:

$$A = \sum_{i=1}^{\lceil a \rceil + 1} A_i \sim \mathcal{G}amma(a,1)$$

$$B = \sum_{i=1}^{\lceil b \rceil + 1} B_i \sim \mathcal{G}amma(b,1)$$

(f). Let $R = A/(A+B)$, from **Question1**, we get a random variable $R \sim \mathcal{B}eta(a,b)$

3 Exersice 3

Consider the following "squeeze" rejection algorithm for sampling from a distribution with density $\pi(x) = \tilde{\pi}(x)/Z_\pi$ on a state space $\mathbb{X}$ such that

$$h(x) \leq \tilde{\pi}(x) \leq M\widetilde{q}(x)$$

where h is a non-negative function, $M > 0$ and $q(x) = \widetilde{q}(x)/Z_q$ is the density of a distribution that we can easily sample from. The algorithm proceeds as follows.

- **(a)** Draw independently $X \sim q, U \sim \mathcal{U}_{[0,1]}$
- **(b)** Accept X if $U \leq h(X)/(M\widetilde{q}(X))$
- **(c)** If X was not accepted in step (b), draw an independent $V \sim \mathcal{U}_{[0,1]}$ and accept X if

$$V \leq \frac{\widetilde{\pi}(X) - h(X)}{M\widetilde{q}(X) - h(X)}$$

Question1. Show that the probability of accepting a proposed $X = x$ in either step (b) or (c) is

$$\frac{\widetilde{\pi}(x)}{M\widetilde{q}(x)}$$

Answer.

$$\mathbb{P}(X = x \text{ accepted in step (b)}) = \int_0^1 \mathbb{1}(u \leq \frac{h(x)}{M\widetilde{q}(x)})\mathrm{d}u = \frac{h(x)}{M\widetilde{q}(x)}$$

$$\mathbb{P}(X = x \text{ accepted in step (c)}) = \int_0^1 \int_0^1 \mathbb{1}(u > \frac{h(x)}{M\widetilde{q}(x)})\mathbb{1}(v \leq \frac{\widetilde{\pi}(x) - h(x)}{M\widetilde{q}(x) - h(x)})\mathrm{d}u\mathrm{d}v$$
$$= \frac{\widetilde{\pi}(x) - h(x)}{M\widetilde{q}(x) - h(x)} \times (1 - \frac{h(x)}{M\widetilde{q}(x)}) = \frac{\widetilde{\pi}(x) - h(x)}{M\widetilde{q}(x) - h(x)} \times \frac{M\widetilde{q}(x) - h(x)}{M\widetilde{q}(x)} = \frac{\widetilde{\pi}(x) - h(x)}{M\widetilde{q}(x)}$$

Hence:

$$\begin{aligned}\mathbb{P}(X = x \text{ accepted}) &= \mathbb{P}(X = x \text{ accepted in step (b)}) + \mathbb{P}(X = x \text{ accepted in step (c)}) \\ &= \frac{h(x)}{M\widetilde{q}(x)} + \frac{\widetilde{\pi}(x) - h(x)}{M\widetilde{q}(x)} = \frac{\widetilde{\pi}(x)}{M\widetilde{q}(x)}\end{aligned}$$

Above all, the **probability of accepting $X = x$ in either step (b) or (c) is** $\frac{\widetilde{\pi}(x)}{M\widetilde{q}(x)}$.

Question2. Deduce from the previous question that the distribution of the samples accepted by the above algorithm is π.

Answer. We have for any measurable set $A \subset \mathbb{X}$:

$$\mathbb{P}(X \in A | X \text{ accepted}) = \frac{\mathbb{P}(X \in A, X \text{ accepted})}{\mathbb{P}(X \text{ accepted})}$$
$$= \frac{\int_{\mathbb{X}} \mathbb{1}_A(x) \frac{\widetilde{\pi}(x)}{M\widetilde{q}(x)} q(x) \mathrm{d}x}{\int_{\mathbb{X}} \frac{\widetilde{\pi}(x)}{M\widetilde{q}(x)} q(x) \mathrm{d}x} = \frac{\int_A \mathbb{1}_A(x) \frac{\pi(x)}{MZ_\pi \widetilde{q}(x)} \frac{\widetilde{q}(x)}{Z_q} \mathrm{d}x}{\int_{\mathbb{X}} \frac{\widetilde{\pi}(x)}{MZ_\pi \widetilde{q}(x)} \frac{\widetilde{q}(x)}{Z_q} \mathrm{d}x}$$
$$= \frac{\int_A \pi(x) \mathrm{d}x}{\int_{\mathbb{X}} \pi(x) \mathrm{d}x} = \int_A \pi(x) \mathrm{d}x$$

Thus, the distribution of the accepted values is precisely π.

Question3. Show that the probability that step (c) has to be carried out is

$$1 - \frac{\int_{\mathbb{X}} h(x) \mathrm{d}x}{MZ_q}$$

Answer. From **Question1**, we know that:

$$\mathbb{P}(X = x \text{ accepted in step (b)}) = \frac{h(x)}{M\widetilde{q}(x)}$$

Thus, the **probability that step (c) has to be carried out is:**

$$\mathbb{P}(X \text{ not accepted in step (b)}) = \int_{\mathbb{X}} (1 - \frac{h(x)}{M\widetilde{q}(x)}) q(x) \mathrm{d}x$$
$$= \int_{\mathbb{X}} q(x) \mathrm{d}x - \int_{\mathbb{X}} \frac{h(x)}{MZ_q} \mathrm{d}x = 1 - \frac{\int_{\mathbb{X}} h(x) \mathrm{d}x}{MZ_q}$$

Question4. Let $\widetilde{\pi}(x) = \exp(-x^2/2)$ and $\widetilde{q}(x) = \exp(-|x|)$. Using the fact that

$$\widetilde{\pi}(x) \geq 1 - \frac{x^2}{2}$$

for any $x \in \mathbb{R}$, how could you use the squeeze rejection sampling algorithm to sample from $\pi(x)$. What is the probability of not having to evaluate $\widetilde{\pi}(x)$? Why could it be beneficial to use this algorithm instead of the standard rejection sampling procedure?

Answer. In squeeze rejection sampling above, set:

$$h(x) = \begin{cases} 1 - \frac{x^2}{2}, |x| \leq \sqrt{2} \\ 0, \text{otherwise} \end{cases}$$

We have the bound M:

$$M = \sup_{x\in\mathbb{X}} \frac{\widetilde{\pi}(x)}{\widetilde{q}(x)} = \sup_{x\in\mathbb{X}} \exp(-\frac{x^2}{2} + |x|)$$

Consider $f(x) = -\frac{x^2}{2} + |x|$, $f(x)$ is even function. Thus, when $x = \frac{-1}{-\frac{1}{2}\times 2} = 1$, $\min_{x\in\mathbb{X}} f(x) = -\frac{1}{2}$. Hence, bound $M = \exp(1/2)$. We know that:

$$\int_{-\infty}^{+\infty} \widetilde{q}(x)\mathrm{d}x = \int_{-\infty}^{+\infty} e^{-|x|}\mathrm{d}x = 2\int_{0}^{+\infty} e^{-x}\mathrm{d}x = 2$$

Thus: $Z_q = 2$. The **squeeze rejection sampling algorithm proceeds to sample from $\pi(x)$ as blow**:

- Draw independtly $X \sim \widetilde{q}/Z_q$, $U \sim \mathcal{U}_{[0,1]}$.
- Accept X if $U \leq \frac{1 - x^2/2}{\exp(1/2 - |x|)}$.
- If X was not accepted in step(b), draw an independent $V \sim \mathcal{U}_{[0,1]}$, and accepted X if:

$$V \leq \frac{\exp(-x^2/2) - 1 + x^2/2}{\exp(1/2 - |x|) - 1 + x^2/2}$$

From **Question3**, we know the **probability of not having to evaluate $\widetilde{\pi}(x)$ is**:

$$\frac{\int_{\mathbb{X}} h(x)\mathrm{d}x}{MZ_q} = \frac{\int_{-\sqrt{2}}^{\sqrt{2}} 1 - \frac{x^2}{2}\mathrm{d}x}{\exp(1/2)\times 2} = \frac{2\sqrt{2} - \frac{2\sqrt{2}+2\sqrt{2}}{6}}{2\exp(1/2)} \approx 0.5718$$

In standard rejection sampling procedure, we need to cauculate $\widetilde{\pi}(x) = \exp(-x^2/2)$. However, in squeeze sampling algorithm, it has probability of 0.5718 that we do not need cauculate $\exp(-x^2/2)$. We just need cauculate $h(x) = 1 - x^2/2$, it is cheaper to evaluate than $\widetilde{\pi}(x)$. We can get the results faster by squeeze rejection sampling algorithm than standard rejection sampling.