Question 4

1. $E(X) = \int x f(x)\,dx = \int x \beta(1-x)^{\beta-1} dx = \int_0^1 x \, d\,{-(1-x)^{\beta}}$

$= -x(1-x)^{\beta}\Big|_0^1 - \int_0^1 -(1-x)^{\beta} dx$

$= -\int_0^1 d \frac{(1-x)^{\beta+1}}{\beta+1}$

$= -\frac{1}{\beta+1} \int_0^1 d(1-x)^{\beta+1} = \frac{1}{\beta+1}$

$\hat{\beta}_M = \frac{1}{E(X)} - 1$

2. $E(X^2) = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2 \beta(1-x)^{\beta-1} dx = \frac{2}{\beta+1} - \frac{1}{\beta+2}$

$\mathrm{var}(X) = E(X^2) - E^2(X) = \frac{2}{\beta+1} - \frac{1}{\beta+2} - \frac{1}{(\beta+1)^2}$

according to CLT

$\bar{X} - E(X) \to N(0, \frac{\mathrm{var}(X)}{n}) \Rightarrow \sqrt{n}(\bar{x} - E(X)) \to N(0, \mathrm{var}(X))$

$(\frac{1}{\bar{X}} - 1) - (\frac{1}{E(X)} - 1) \to N(0, \frac{\mathrm{var}(X)}{n} \times (E(X)^{-1})'^{2})$ Delta method

$\Rightarrow N(0, \frac{\mathrm{var}(X)}{n} \times E(X)^{-4})$

$\sqrt{n}(\hat{\beta}_M - \beta) \to N(0, \frac{\mathrm{Var}(X)}{E(X)^4})$ according to delta method

$$\to N\left(0, \frac{\frac{2}{\hat{\beta}_M+1} - \frac{1}{\hat{\beta}_M+2} - \frac{1}{(\hat{\beta}_M+1)^2}}{\left(\frac{1}{\hat{\beta}_M+1}\right)^4}\right)$$

3. MLE estimator of β

$L = \prod_{i=1}^{n} f_\beta(X_i)$ $l = \log L = \sum_{i=1}^{n} \log \beta(1-X)^{\beta-1} = \sum_{i=1}^{n} \log\beta + (\beta-1)\log(1-X_i)$

$= n\log\beta + (\beta-1)\sum_{i=1}^{n}\log(1-X_i)$

$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log(1-X_i) = 0$ $\hat{\beta}_{MLE} = \frac{-n}{\sum_{i=1}^{n}\log(1-X_i)}$

4. $Var(\hat{\beta}_{MLE}) \approx \frac{I^{-1}(\beta)}{n}$

$I(\beta) = var\left(\frac{\partial L}{\partial \beta}\right) = -E\left(\frac{\partial^2 L\beta}{\partial \beta^2}\right) = -E(-n\beta^{-2}) = nE(\beta^{-2})$
$= \frac{n}{\beta^2}$

$\beta_{MLE} \xrightarrow{n\to\infty} N\left(\beta, \frac{\beta^2}{n^2}\right)$

$var(\beta_{MLE}) \approx \frac{\beta^2}{n^2}$

5.

$$\frac{MSE(\beta_{MLE})}{MSE(\beta_m)} \approx \frac{Var(\beta_{MLE})}{Var(\beta_m)} = \frac{\frac{\beta^2}{n^2}}{2(\beta+1)^3 - \frac{(\beta+1)^4}{\beta+2} - (\beta+1)^2}$$

among all unbiased estimators. MLE estimator have the lowest Variance. $var(\beta_m)$ is $O(\beta)$ which will be large when β is large. However. if β is very small. $var\ \beta_m$ can be small as β_{MLE} as well

6. $f_\eta(x) = \begin{cases} \frac{1}{\eta}(1-x)^{\frac{1}{\eta}-1} & 0<x<1. \\ 0 & \text{others.} \end{cases}$

$L = \prod_{i=1}^{n} f_\eta(x_i) = \prod_{i=1}^{n} \frac{1}{\eta}(1-x_i)^{\frac{1}{\eta}-1}$

$l = \sum_{i=1}^{n} -\log\eta + (\frac{1}{\eta}-1)\log(1-x_i).$

$\frac{\partial l}{\partial \eta} = -\frac{n}{\eta} + \left(\sum_{i=1}^{n}\log(1-x_i)\right) \times \frac{1}{\eta} \times (-\eta^{-2}) = 0$

$\frac{\partial l}{\partial \eta} = 0 \qquad \hat{\eta}_{MLE} = \sqrt{\frac{-\sum_{i=1}^{n}\log(1-x_i)}{n}}$

$-\beta^{-2}$

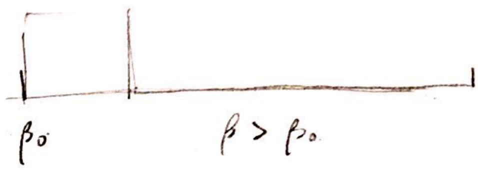
7. $H_0: \beta = \beta_0$

$H_1: \beta > \beta_0$

$\hat{\beta} \sim (\beta, \mathrm{var}(\hat{\beta}))$

test statistics $\qquad$ Z test

$$\frac{\hat{\beta} - \beta_0}{\mathrm{var}(\hat{\beta})}$$



8. $$\frac{\hat{\beta} - \beta_0}{\mathrm{var}(\hat{\beta})} > Z_\alpha$$

$$\hat{\beta} > Z_\alpha \mathrm{Var}(\hat{\beta}) + \beta_0$$

critical value $\quad Z_\alpha \mathrm{Var}(\hat{\beta}) + \beta_0$

9. $$\hat{\beta}_{MLE} \sim N\left(\beta, \frac{\hat{\beta}^2}{n}\right)$$

$$\frac{\hat{\beta}_{MLE} - \beta}{\sqrt{\frac{\hat{\beta}^2}{n}}} \sim N(0, 1)$$

$$\beta \in \left(\hat{\beta} - 1.96 \times \frac{\hat{\beta}}{\sqrt{n}}, \ \hat{\beta} + 1.96 \times \frac{\hat{\beta}}{\sqrt{n}}\right)$$

with 95% confidence