Bayesian Statistics

$\pi(\theta)$: prior distribution of θ

$f(\mathbf{x}|\theta)$: sampling distribution given θ

$$\pi(\theta|\mathbf{x}) = \frac{f(\mathbf{x}|\theta)\,\pi(\theta)}{\int f(\mathbf{x}|\theta)\,\pi(\theta)\,d\theta}$$

11 — 8 failures, 3 successes

Model a Bernoulli $\sim (\pi)$

Classical:

$$\pi^3 (1-\pi)^8$$

$$\hat{\pi} = \frac{y}{n} = 3/11$$

$$se(\hat{\pi}) = \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$$

Bayesian: Create an informative prior for π

Prior is Beta: $$P(\pi|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma\alpha\,\Gamma\beta}\,\pi^{\alpha-1}(1-\pi)^{\beta-1}$$

Corresponding likelihood fn:

$$f(y|\pi,n) = \binom{n}{y}\pi^y (1-\pi)^{n-y}$$

Posterior: $$P(\pi|y) \propto f(y|\pi,n)\cdot P(\pi|\alpha,\beta)$$

Case 1 : $\alpha = \beta = 1$

2 : $\alpha = 2.4$, $\beta = 2$

Case 1: $P(\pi|y) \propto \pi^3 (1-\pi)^8 \cdot \frac{\Gamma(1+1)}{\Gamma(1)\Gamma(1)} \pi^{1-1}(1-\pi)^{1-1}$

$$\propto \pi^{4-1}(1-\pi)^{9-1}$$

$\therefore P(\pi|y) \sim \text{Beta}(4,9)$ $\qquad \frac{\Gamma(13)}{\Gamma(4)\Gamma(9)}$ proportionality constant

2: $P(\pi|y) \propto \pi^{5.4-1}(1-\pi)^{10-1} \sim \text{Beta}(5.4, 10)$

Eg. Let $X \sim N(\theta, \sigma^2)$, σ^2 is known, prior of $\theta \sim N(\mu, \tau^2)$

The posterior of θ is also normal:

$$E(\theta|x) = \left(\frac{\tau^2}{\tau^2+\sigma^2}\right) x^{\text{obs}} + \left(\frac{\sigma^2}{\tau^2+\sigma^2}\right)\mu^{\text{prior}} \qquad Var(\theta|x) = \frac{\sigma^2\tau^2}{\sigma^2+\tau^2}$$

$$* \underset{P(\theta|x)}{\pi(\theta|x)} = \frac{f(x|\theta)\pi(\theta)}{\int_\theta f(x|\theta)\pi(\theta)\,d\theta} = \frac{h(x,\theta)}{m(x)}$$

$$* h(x,\theta) = \underbrace{\frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{1}{2}\frac{(\theta-\mu)^2}{\tau^2}\right\}}_{\text{prior}} * \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}\left\{-\frac{1}{2}\frac{(x-\theta)^2}{\sigma^2}\right\}}_{\text{sampling dist}}$$

$$= \frac{1}{\sqrt{2\pi\tau^2\sigma^2}} \exp\left\{-\frac{1}{2}\underbrace{\left[\frac{(\theta-\mu)^2}{\tau^2} + \frac{(x-\theta)^2}{\sigma^2}\right]}_{\text{complete the squares}}\right\} \leftarrow \bigcirc\theta^2 + \theta + C \rightarrow \frac{(\theta - \tilde{\mu}(x))^2}{\tilde{\sigma}(x)^2}$$

$$\frac{1}{2}\left[\frac{(\theta-\mu)^2}{\tau^2} + \frac{(x-\theta)^2}{\sigma^2}\right] = \frac{1}{2}\left[\theta^2\left(\frac{1}{\tau^2}+\frac{1}{\sigma^2}\right) - 2\theta\left(\frac{\mu}{\tau^2}+\frac{x}{\sigma^2}\right) + \left(\frac{\mu^2}{\tau^2}+\frac{x^2}{\sigma^2}\right)\right]$$

If we define: $\ell = \frac{1}{\tau^2} + \frac{1}{\sigma^2} = \frac{\tau^2+\sigma^2}{\tau^2\sigma^2}$

$$\Rightarrow \frac{1}{2}\ell\left[\theta^2 - \frac{2\theta}{\ell}\left(\frac{\mu}{\tau^2}+\frac{x}{\sigma^2}\right)\right] + \frac{1}{2}\left(\frac{\mu^2}{\tau^2}+\frac{x^2}{\sigma^2}\right)$$

$$\Rightarrow \frac{1}{2}\ell\left[\theta - \frac{1}{\ell}\left(\frac{\mu}{\tau^2}+\frac{x}{\sigma^2}\right)\right]^2 + \frac{(\mu-x)^2}{2(\sigma^2+\tau^2)}$$

$$\therefore\ h(x,\theta) = \frac{1}{2\pi\sigma\tau} \exp\left\{-\frac{1}{2}\,\rho\left[\theta - \frac{1}{\rho}\left(\frac{\mu}{\tau^2} + \frac{x}{\sigma^2}\right)\right]^2\right\}$$
$$* \exp\left\{-\frac{1}{2}\,\frac{(\mu - x)^2}{2(\sigma^2 + \tau^2)}\right\}$$

$$m(x) = \int_{-\infty}^{\infty} h(x,\theta) = \frac{1}{\sqrt{2\pi\rho}\,\sigma\tau} \exp\left\{\frac{(\mu - x)^2}{2(\sigma^2 + \tau^2)}\right\}$$

$$\pi(\theta|x) = \frac{h(x,\theta)}{m(x)} = \sqrt{\frac{\rho}{2\pi}} \exp\left\{-\frac{1}{2}\,\rho\left[\theta - \frac{1}{\rho}\left(\frac{\mu}{\tau^2} + \frac{x}{\sigma^2}\right)\right]\right\}$$

- the marginal distribution of $X \sim N(\mu, \sigma^2 + \tau^2)$

  posterior of $\theta|x \sim N(\mu(x), \rho^{-1})$

  $$\mu(x) = \frac{1}{\rho}\left(\frac{\mu}{\tau^2} + \frac{x}{\sigma^2}\right)$$