$Y(t): t \geq 0$ BMD μ, σ^2

show:

$$E[X(t) \mid X(u); 0 \leq u \leq s] = X(s)\, e^{(t-s)(\mu + \sigma^2/2)}$$

Since $X(t)$ is a Geometric Brownian Motion process, which is memoryless, the conditional expectation of all subsequent values depends only on the most recent value $[X(s)]$, and is not affected by all prior values $[X(u); 0 \leq u < s]$.

$$\therefore\ X(t) \mid X(u); 0 \leq u \leq s$$

$$\parallel$$

$$X(t) \mid X(s)$$

$$\Downarrow \text{ implies}$$

$$X(t) \mid X(s) = X(s)\, e^{Y(t-s)}$$

$X(t)$ has a lognormal distribution with parameters $(\ln(X(s) + \mu(t-s))$ and $\sigma\sqrt{(t-s)}$

$$E[X(t) \mid X(s)] = E[X(s)\, e^{Y(t-s)}]$$

$$= X(s)\, E[e^{Y(t-s)}]$$, where $Y(t-s)$ has param: μ, σ^2

using $E[e^{aY}] = e^{aE[Y] + a^2 Var(Y)/2}$

$$E[e^{Y(t-s)}] = e^{(t-s)(\mu + \sigma^2/2)}$$

$$E[X(t) \mid X(s)] = X(s)\, e^{(t-s)(\mu + \sigma^2/2)}$$