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Residual-life distributions from component degradation signals: A Bayesian approach

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Real-time condition monitoring is becoming an important tool in maintenance decision-making. Condition monitoring is the process of collecting real-time sensor information from a functioning device in order to reason about the health of the device. To make effective use of condition information, it is useful to characterize a device degradation signal, a quantity computed from condition information that captures the current state of the device and provides information on how that condition is likely to evolve in the future. If properly modeled, the degradation signal can be used to compute a residual-life distribution for the device being monitored, which can then be used in decision models. In this work, we develop Bayesian updating methods that use real-time condition monitoring information to update the stochastic parameters of exponential degradation models. We use these degradation models to develop a closed-form residual-life distribution for the monitored device. Finally, we apply these degradation and residual-life models to degradation signals obtained through the accelerated testing of bearings.

1. Introduction

Condition monitoring is the process of collecting real-time sensor information from a functioning device in order to reason about the health of the device. The promise of condition monitoring is twofold. First, since it provides knowledge of a device's health, it eliminates maintenance that is not really necessary. Second, by making failures more predictable, it promotes failure avoidance, which leads to less unscheduled downtime, fewer emergencies, and less scrap. Condition monitoring technology has evolved rapidly over the past decade, becoming cheaper, more powerful, and more user-friendly. Emerging wireless technologies coupled with satellite links and the internet enable the monitoring of geographically remote devices from a centralized location (Lee and Schneeman, 1999; Arnon, 2000; Feijs and Manders, 2000). Indeed, the potential of condition monitoring to support high-level decision-making such as device replacement, maintenance scheduling, and spare parts management has never been greater and will only increase in the future. The objective of this work is to develop methods that use condition-based information to predict the residual life of devices being monitored.

To make the most effective use of condition information, it is helpful to identify a degradation signal, a quantity computed from sensor information that captures the current state of the device and provides information on how that condition is likely to evolve in the future (Nelson, 1990). The degradation signal provides the basis for developing models that can be used to estimate the residual life of the device. For example, Fig. 1 illustrates a vibration-based degradation signal for three of the bearings tested in this work. As the bearings degrade, the vibration they exhibit tends to increase. When the vibration reaches a standard threshold (which typically depends on the required precision of the specific application), the bearing is considered to have failed. Note that even though the degradation rates of the three bearings differ significantly, the signals all exhibit similar shapes. Indeed, it is not unusual for a population of "identical" devices to have a common degradation signal form while exhibiting widely different degradation rates and failure times.

The functional form associated with the degradation process is driven by the underlying physical phenomenon. The objective of this paper is to model the functional form of the degradation process. Our approach is to develop a parameterized model of the degradation signal of a population of devices. We use the phrase "degradation signal" to refer to the "real" condition of the device as detected through condition monitoring, whereas we use the phrase "degradation model" to refer to our attempt to model the degradation signal mathematically for predictive purposes. This degradation model must capture the functional form of the degradation signal for the population of devices. That is, we expect every device in the population to have a degradation

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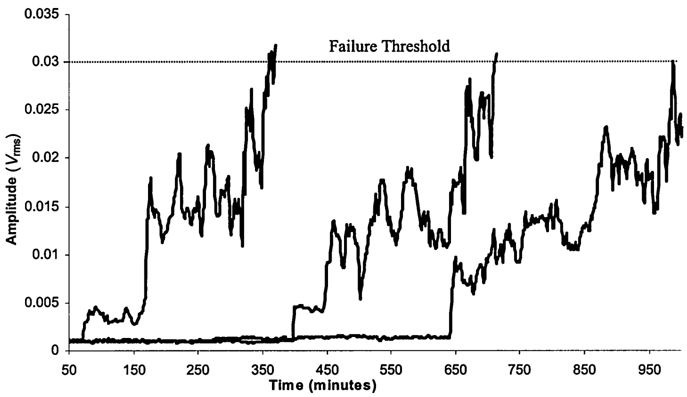


Fig. 1. Bearing degradation signals.

signal that exhibits the form expressed by this model. Model parameters are either deterministic or stochastic. Deterministic parameters are assumed to be known and constant throughout the device's life. These might represent some feature common to all devices in the population. Stochastic parameters are used to model the degradation characteristics that are unique to the individual device, typically the rate of degradation. These parameters are assumed to follow some distributional form across the population of devices, with those of the individual device being an unknown "draw" from the population. Error terms are included in the model to capture device and environmental noise, signal transients, measurement errors, and variations due to monitoring equipment.

The degradation model for an individual device is obtained by estimating the stochastic parameters of that device. To do this, we use Bayesian updating to combine two sources of information: (i) the distribution of the parameters across the population of devices; and (ii) the real-time sensor information collected from the device through condition monitoring, that is, the degradation signal from the individual device. Our objective is to use this estimated model of the device's degradation signal to make reasonable predictions about the residual life of the device. We do this by using the device's degradation model to develop its residual-life distribution.

Specifically, this paper develops a Bayesian updating procedure and residual-life distributions for two different exponential degradation signal models. The first model assumes that the degradation signal exhibits independent random fluctuations about an exponential signal trajectory, whereas the second model assumes that the error fluctuations follow a Brownian motion process. In both cases, the purpose of the Bayesian updating is to improve the estimation of the stochastic parameters in the exponential model, that is, to improve our estimate of the true signal trajectory.

The paper is organized as follows. Section 2 provides a brief literature review, and Section 3 develops Bayesian updating and residual-life distributions for the two-parameter exponential model with random error terms and the two-parameter exponential model with Brownian error terms. Section 4 then applies these degradation and residual-life models to data obtained through the accelerated testing of bearings, thus illustrating their application and usefulness. Section 5 provides a conclusion and a discussion of future research directions.

2. Literature review

This section reviews some of the relevant literature related to condition monitoring and degradation modeling. The goal of much condition monitoring research is to identify measurable quantities that allow successful diagnosis of a device's health. For example, in power generators, the integrity of insulating materials is critical, and thus identifying measurable quantities that capture insulator degradation, designing and integrating sensing capability, and using monitoring information to diagnose insulation health are active areas of research (Feser *et al.*, 1995). Martin (1994), Fararooy and Allan (1995), Dimla (1999), and Thorsen and Dalva (1999), provide surveys of similar research in other application areas, including cutting tools, high-voltage induction motors, railway equipment, and machine tools.

If devices exhibit a predictable degradation pattern, then in addition to diagnosing current health, analysts are able to predict future health. In this case, a random coefficients model can be used to capture the degradation form common to a population of devices, with the coefficients being used to model the differences between individual devices. For example, Lu and Meeker (1993) consider the case in which the life distribution of a population of devices is to be computed using degradation information obtained from a randomly selected set of devices. The authors present several random coefficient models and illustrate various methods for computing life distributions with these models.

Wang (2000) enumerates the underlying assumptions of the random coefficients model, which are: (i) the condition of the device deteriorates with operating time and the level of deterioration can be observed at any time; (ii) the mean and variance of device deterioration can be increasing in time; (iii) device failure occurs when the degradation signal reaches a well-defined threshold; (iv) the device being monitored comes from a population of devices, each of which exhibits the same degradation form; and (v) the distribution of the stochastic parameters across the population of devices is known. We note that both Lu and Meeker (1993) and Wang (2000) assume that the error in the degradation signal is independent and identically distributed (iid) $N(0, \sigma^2)$ across the population of devices.

Yang and Yang (1998) develop a random-coefficient-based approach that uses the life times of failed devices plus degradation information from un-failed devices to obtain better estimates of life parameters. The authors experimentally demonstrate that this approach provides better estimators than traditional life testing in which only failure times are recorded. Yang and Jeang (1994) use a random coefficients model to study the effect of cutting tool flank wear on the surface roughness in metal cutting. The degradation signal considered is the surface roughness of a machined part. The authors use their model to develop an inspection scheme for deciding when tools require changing.

Tseng *et al.* (1995) combine random coefficient models for luminosity degradation with experimental design to identify manufacturing settings that provide slow rates of luminous degradation of fluorescent lamps. Their experimental results indicate that the mean life can be improved by up to 67%. Goode *et al.* (1998) use an exponential degradation model to develop predictions for the condition of a hot strip mill. The authors use case studies based on actual hot strip mill failures to compare the effectiveness of degradation-based life prediction with time-to-failure predictions based on a Weibull model. They conclude that the degradation model provides a much greater accuracy than the reliability model. Chinnam (1999) presents a neural-network-based model for online estimation of component reliability. Degradation signals that capture the physical characteristics of the component are viewed as a time series and a multi-layer perceptron model is used to perform a 1-step prediction of the degradation measurement. The model is used to estimate the in-process reliability of drill-bits in a drilling process.

Doksum and Hoyland (1992) develop inverse Guassian life models and maximum likelihood estimators for units subject to accelerated stress testing. Accumulated decay is modeled as a Wiener process with drift and diffusion dependent on and changing with the stress level. They illustrate how to use test data to estimate the mean life under normal stress levels. Whitmore (1995) models degradation as a Wiener process and illustrates how to account for measurement errors. Similarly, Whitmore and Schenkelberg (1997) use a Wiener process to model degradation data collected from accelerated testing and develop methods for estimating the parameters of time and stress transformations. The authors provide a case study on self-regulating heating cables.

Lu *et al.* (2001) present methods for forecasting system performance reliability for systems with multiple failure modes. Time series forecasting is used to develop a joint density function for the performance measures. This joint density is then integrated to obtain a reliability function that can be used to assess the system reliability. The authors illustrate their work on an x-y positioning system with two failure points. Lu *et al.* (2001) develop a model for estimating the conditional performance reliability in real-time for an individual component while in operation. Sampled measurements are treated as a realization of a stochastic process, and exponential smoothing is used to develop a conditional distribution of the performance variable. As an example, the authors determine the conditional reliability of a tool given its predicted tool wear.

The models developed in this paper are similar to the closed-form models presented by Lu and Meeker (1993), but with some key differences. Whereas Lu and Meeker (1993) develop methods to compute life distributions for a population of components, we focus on computing a residual-life distribution for a single operating device. That is, we use the distributions of the stochastic parameters across the population of devices, which we refer to as the prior distributions, together with monitoring information collected from the device in question, to compute a residual life distribution for that individual device. Furthermore, whereas Lu and Meeker (1993) assume the error terms to be independent normal random variables, our work uses a degradation model that assumes a Brownian motion (Wiener) error process. Finally, because we are computing the residual life, we must explicitly consider the signal error terms in our life distribution, which significantly complicates the derivations.

We now proceed to Section 3, which presents our degradation models, our Bayesian updating approach for estimating model parameters, and our derivations of the residual-life distributions.

3. Bayesian degradation signal models

This section develops methods that combine two sources of information, the reliability characteristics of a device's population and real-time sensor information from a functioning device, to periodically update the distribution of the device's residual life. Specifically, we develop two exponential degradation models, one with random error terms (Section 3.1) and one with a Brownian motion error process (Section 3.2). In the first case, error is modeled as a stochastic process with iid components. In other words, for observation times, $t_1, t_2, \ldots, t_k$, $t_1 < t_2 < \cdots < t_k$, the error terms observed at those times, $\epsilon(t_1), \ldots, \epsilon(t_k)$, are iid random variables. As previously discussed, this type of error term is widely used in the existing literature, see for example Lu and Meeker (1993), Goode *et al.* (1998), Yang and Yang (1998) and Wang (2000). In the second case, the error term is modeled as a modified Brownian motion with independently and identically distributed increments. That is, the error increments, $\epsilon(t_1), \epsilon(t_2) - \epsilon(t_1), \ldots, \epsilon(t_k) - \epsilon(t_{k-1})$, are assumed to be independent random variables. This type of error term has been used by Doksum and Hoyland (1992), Whitmore (1995) and Whitmore and Schenkelberg (1997). We use these degradation models, along with assumptions regarding the distributional forms of the stochastic parameters and error terms, to develop Bayesian updating methods for continually re-estimating a functioning device's stochastic parameters. Finally, we use these updated parameters to estimate the residual-life distribution of the monitored device.

The models presented in this work are appropriate for specific degradation phenomena that have an exponentially increasing or decreasing form. While the results presented in this paper apply only to this specific class of models, we note that similar techniques can be applied to a wide variety of other models. Gebraeel *et al.* (2001) present residual-life derivations for a variety of degradation models that capture a diversity of degradation forms including linear, polynomial, and other types of exponential models beyond those presented here. Furthermore, Gebraeel (2003) addresses models with dependent stochastic parameters as well as other computational approaches for deriving empirical residual-life distributions.

3.1. *Two-parameter exponential model with multiplicative random error terms*

In this section we first model the degradation signal for a population of components with an exponential model assuming error terms from an iid random error process. A Bayesian updating method is used to estimate the unknown stochastic parameters of the model for an individual component. Once we have determined the posterior distribution for these unknown parameters, we derive the residual-life distribution for the individual component.

3.1.1. *The degradation signal model*

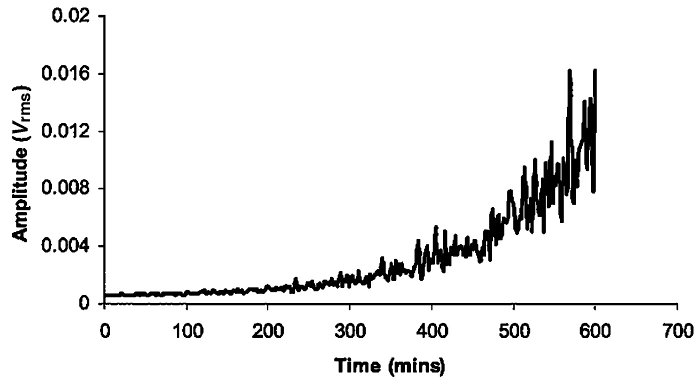
Let $S(t)$ denote the degradation signal as a continuous stochastic process, continuous with respect to time t. We observe the degradation signal at some discrete points in time, $t_1, t_2, \ldots$, where $t_i \geq 0$. Therefore, we can model the degradation signal at time t_i, $i = 1, 2, \ldots$, as follows:

$$S(t_i) = \phi + \theta \exp\left(\beta t_i + \epsilon(t_i) - \frac{\sigma^2}{2}\right)$$
$$= \phi + \theta \exp(\beta t_i) \exp\left(\epsilon(t_i) - \frac{\sigma^2}{2}\right),$$

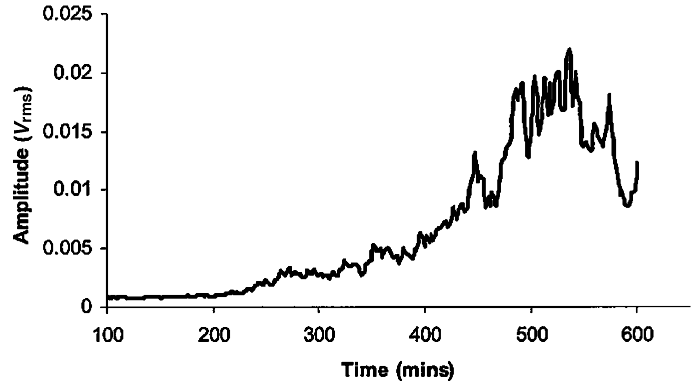
for $i = 1, 2, \ldots$, where ϕ is a known constant, θ is a lognormal random variable, where $\ln\theta$ has mean μ_0 and variance σ_0^2, β is a normal random variable with mean μ_1 and variance σ_1^2, and $\epsilon(t_i)$ is a random error term that follows a normal distribution with mean zero and variance σ^2. We assume $\epsilon(0) = 0$. We assume that θ, β and $\epsilon(t_i)$ are mutually independent, and that $\epsilon(t_1), \epsilon(t_2), \ldots$ are iid random variables. Under these assumptions, it is easy to show that $E[\exp(\epsilon(t_i) - (\sigma^2/2))] = 1$, and thus $E[S(t_i)|\theta, \beta] = \phi + \theta \exp(\beta t_i)$. Figure 2(a) illustrates the type of degradation signal this model is intended to represent. For additional information on this and other models typically used to model degradation, see Nelson (1990) and Shao and Nezu (2000).

For this exponential model, it will be convenient to work with the logged signal at time t_i, which we will denote by $L(t_i)$. We can then define the logged signal at time t_i as follows:

$$L(t_i) = \ln(S(t_i) - \phi) = \ln\theta + \beta\, t_i + \epsilon(t_i) - \frac{\sigma^2}{2}. \quad (1)$$



(a)



(b)

Fig. 2. Typical degradation signal for the exponential model: (a) with iid errors; and (b) with Brownian errors.

Finally, we let $L_i = L(t_i)$ and we define $\theta' = \ln\theta - (\sigma^2/2)$. Note that θ' is a normal random variable with mean $\mu_0 - (\sigma^2/2)$ and variance σ_0^2. Then we can write:

$$L_i = \theta' + \beta t_i + \epsilon(t_i). \quad (2)$$

We will use the observations $L_1, L_2, \ldots$, obtained at times $t_1, t_2, \ldots$, as our data.

Next, suppose we have observed $L_1, \ldots, L_k$ at times $t_1, \ldots, t_k$. Since the error terms, $\epsilon(t_i)$, $i = 1, \ldots, k$, are iid normal random variables, if we know θ' and β, then the conditional joint density function of $L_1, \ldots, L_k$, given θ' and β, is:

$$f(L_1, \ldots, L_k|\theta', \beta) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^k \times \exp\left(-\sum_{i=1}^{k}\left(\frac{(L_i - \theta' - \beta t_i)^2}{2\sigma^2}\right)\right).$$

Generally, however, θ' and β will be unknown. We let $\pi_1(\theta')$ and $\pi_2(\beta)$ denote the prior distributions on θ' and β, respectively. As described above, we assume that these prior distributions are known and represent our knowledge of the reliability characteristics of the population of devices. Here we assume that the prior distributions for θ' and β are normal with mean $\mu_0' = \mu_0 - (\sigma^2/2)$ and variance σ_0^2 and normal with mean μ_1 and variance σ_1^2, respectively. Then, given the observed data, $L_1, \ldots, L_k$, the joint posterior distribution of (θ', β) is given as follows:

Proposition 1. *Given the observed data, $L_1, \ldots, L_k$, the joint posterior distribution of (θ', β) is a bivariate normal distribution with mean $(\mu_{\theta'}, \mu_\beta)$, variance $(\sigma_{\theta'}^2, \sigma_\beta^2)$ and correlation coefficient ρ, where:*

$$\mu_{\theta'} = \frac{\left(\sum_{i=1}^k L_i\sigma_0^2 + \mu_0'\sigma^2\right)\left(\sum_{i=1}^k t_i^2\sigma_1^2 + \sigma^2\right) - \left(\sum_{i=1}^k t_i\sigma_0^2\right)\left(\sum_{i=1}^k L_it_i\sigma_1^2 + \mu_1\sigma^2\right)}{\left(k\sigma_0^2 + \sigma^2\right)\left(\sum_{i=1}^k t_i^2\sigma_1^2 + \sigma^2\right) - \left(\sum_{i=1}^k t_i\sigma_1^2\right)\left(\sum_{i=1}^k t_i\sigma_0^2\right)},$$

$$\mu_\beta = \frac{\left(k\sigma_0^2 + \sigma^2\right)\left(\sum_{i=1}^k L_it_i\sigma_1^2 + \mu_1\sigma^2\right) - \left(\sum_{i=1}^k t_i\sigma_1^2\right)\left(\sum_{i=1}^k L_i\sigma_0^2 + \mu_0'\sigma^2\right)}{\left(k\sigma_0^2 + \sigma^2\right)\left(\sum_{i=1}^k t_i^2\sigma_1^2 + \sigma^2\right) - \left(\sum_{i=1}^k t_i\sigma_1^2\right)\left(\sum_{i=1}^k t_i\sigma_0^2\right)},$$

$$\sigma_{\theta'}^2 = \frac{\bar{\sigma}^2}{\sigma_1^2}\frac{\sum_{i=1}^k t_i^2\sigma_1^2 + \sigma^2}{\left(k\sigma_0^2 + \sigma^2\right)\left(\sum_{i=1}^k t_i^2\sigma_1^2 + \sigma^2\right) - \left(\sum_{i=1}^k t_i\right)^2\sigma_0^2\sigma_1^2},$$

$$\sigma_\beta^2 = \frac{\bar{\sigma}^2}{\sigma_0^2}\frac{k\sigma_0^2 + \sigma^2}{\left(k\sigma_0^2 + \sigma^2\right)\left(\sum_{i=1}^k t_i^2\sigma_1^2 + \sigma^2\right) - \left(\sum_{i=1}^k t_i\right)^2\sigma_0^2\sigma_1^2}, \qquad \rho = \frac{-\sigma_0\sigma_1\sum_{i=1}^k t_i}{\sqrt{k\sigma_0^2 + \sigma^2}\sqrt{\sigma_1^2\sum_{i=1}^k t_i^2 + \sigma^2}},$$

where $\bar{\sigma}^2 = \sigma^2\sigma_0^2\sigma_1^2$.

Proof. Given the prior distributions on θ' and β, we can find the posterior distribution of (θ', β), denoted $p(\theta', \beta \mid L_1, \ldots, L_k)$, as follows:

$$\begin{aligned}
p(\theta', \beta \mid L_1, \ldots, L_k) &\propto f(L_1, \ldots, L_k|\theta', \beta)\pi_1(\theta')\pi_2(\beta) \\
&\propto \exp\left\{-\sum_{i=1}^k \frac{(L_i - \theta' - \beta t_i)^2}{2\sigma^2}\right\}\exp\left\{\frac{-1}{2\sigma_0^2}(\theta' - \mu_0')^2\right\}\exp\left\{\frac{-1}{2\sigma_1^2}(\beta - \mu_1)^2\right\}, \\
&\propto \exp\left\{-\frac{1}{2}\left[\frac{\sigma_0^2\sigma_1^2}{\sigma^2\sigma_0^2\sigma_1^2}\left(k\theta'^2 + \beta^2\sum_{i=1}^k t_i^2 - 2\theta'\sum_{i=1}^k L_i - 2\beta\sum_{i=1}^k L_it_i + 2\theta'\beta\sum_{i=1}^k t_i\right)\right.\right. \\
&\quad \left.\left. + \frac{\sigma^2\sigma_1^2}{\sigma^2\sigma_0^2\sigma_1^2}(\theta'^2 - 2\mu_0'\theta') + \frac{\sigma^2\sigma_0^2}{\sigma^2\sigma_0^2\sigma_1^2}(\beta^2 - 2\mu_1\beta)\right]\right\}, \\
&\propto \exp\left\{-\frac{1}{2}\left[\theta'^2\left(\frac{k\sigma_0^2\sigma_1^2 + \sigma^2\sigma_1^2}{\bar{\sigma}^2}\right) + \beta^2\left(\frac{\sum_{i=1}^k t_i^2\sigma_0^2\sigma_1^2 + \sigma^2\sigma_0^2}{\bar{\sigma}^2}\right) - 2\theta'\left(\frac{\sum_{i=1}^k L_i\sigma_0^2\sigma_1^2 + \mu_0'\sigma^2\sigma_1^2}{\bar{\sigma}^2}\right)\right.\right. \\
&\quad \left.\left. - 2\beta\left(\frac{\sum_{i=1}^k L_it_i\sigma_0^2\sigma_1^2 + \mu_1\sigma^2\sigma_0^2}{\bar{\sigma}^2}\right) + 2\theta'\beta\left(\frac{\sum_{i=1}^k t_i\sigma_0^2\sigma_1^2}{\bar{\sigma}^2}\right)\right]\right\}, \qquad (3) \\
&\propto \exp\left\{-\frac{1}{2}\left[\theta'^2\left(\frac{1}{\sigma_{\theta'}^2(1-\rho^2)}\right) + \beta^2\left(\frac{1}{\sigma_\beta^2(1-\rho^2)}\right) - 2\theta'\left(\frac{\mu_{\theta'}}{\sigma_{\theta'}^2(1-\rho^2)} - \frac{\mu_\beta\rho}{\sigma_{\theta'}\sigma_\beta(1-\rho^2)}\right)\right.\right. \\
&\quad \left.\left. - 2\beta\left(\frac{\mu_\beta}{\sigma_\beta^2(1-\rho^2)} - \frac{\mu_{\theta'}\rho}{\sigma_{\theta'}\sigma_\beta(1-\rho^2)}\right) + 2\theta'\beta\left(\frac{-\rho}{\sigma_{\theta'}\sigma_\beta(1-\rho^2)}\right)\right]\right\}, \\
&\propto \frac{1}{2\pi\sigma_{\theta'}\sigma_\beta\sqrt{1-\rho^2}}\exp\left\{-\left[\frac{\sigma_\beta^2(\theta' - \mu_{\theta'})^2 - 2\sigma_{\theta'}\sigma_\beta\rho(\theta' - \mu_{\theta'})(\beta - \mu_\beta) + \sigma_{\theta'}^2(\beta - \mu_\beta)^2}{2\sigma_{\theta'}^2\sigma_\beta^2(1-\rho^2)}\right]\right\}, \qquad (4)
\end{aligned}$$

where $\bar{\sigma}^2 = \sigma^2\sigma_0^2\sigma_1^2$ and $\mu_{\theta'}$, $\sigma_{\theta'}^2$, μ_β, σ_β^2 and ρ are as defined in the proposition. Notice that this last equation is the bivariate normal density function (see, for example, Larsen and Marx (1986). Thus, (θ', β) given $L_1, \ldots, L_k$ has a bivariate normal posterior distribution with mean $(\mu_{\theta'}, \mu_\beta)$, variance $(\sigma_{\theta'}^2, \sigma_\beta^2)$, and correlation ρ. ■

Finally, we note that it is easy to show that $-1 \le \rho < 0$. Notice that, while our model assumes that the unknown model parameters (i.e., the random variables) θ' and β are independent, there is a correlation coefficient in the posterior distribution of θ' and β. This is due to the fact that, in the posterior distribution for θ' and β, our beliefs about

θ' and β are updated from a single set of data $(L_1, \ldots, L_k)$. Thus, it is natural that these posterior beliefs about θ' and β would be correlated.

3.1.2. *Determining the residual-life distribution*

Once we have updated the posterior distribution of (θ', β), our goal is to determine the distribution of the residual life of the functioning device. In other words, we would like to determine the distribution of the time until failure for the component. For this purpose we will assume that failure occurs when the degradation signal reaches some given failure threshold, D, and thus our objective is to estimate the distribution of the time until the signal reaches D. In this paper, we take the threshold value, D, as fixed and known. In general, such failure thresholds are not always clearly defined and developing one for a given application requires knowledge of industrial standards, application precision, and engineering judgment.

At a given time, t_k, our objective is to determine the distribution of the time until the signal reaches the failure threshold, D. To do this, we first compute the posterior distribution for (θ', β) as described above. We then define the random variable $L(t+t_k)$ to be the logged degradation signal value observed at time $t+t_k$, $t>0$, given $L_1, \ldots, L_k$ observed at times $t_1, \ldots, t_k$. We next determine the distribution of $L(t+t_k)$ given $L_1, \ldots, L_k$. Since $L(t+t_k) = \theta' + \beta(t+t_k) + \epsilon(t+t_k) - (\sigma^2/2)$, the distribution of $L(t+t_k)$, given $L_1, \ldots, L_k$, is normal with mean:

$$\tilde{\mu}(t+t_k) \stackrel{\triangle}{=} \mu_{\theta'} + \mu_\beta(t+t_k) - \frac{\sigma^2}{2},$$

and variance

$$\tilde{\sigma}^2(t+t_k) \stackrel{\triangle}{=} \sigma_{\theta'}^2 + (t+t_k)^2\sigma_\beta^2 + \sigma^2 + 2\rho(t+t_k)\sigma_{\theta'}\sigma_\beta.$$

Next, we let T denote the residual life of the component at time t_k and we note that T satisfies $L(T+t_k) = D$. We can then find the conditional cumulative distribution function (cdf) of T given $L_1, \ldots, L_k$, $F_{T|L_1,\ldots,L_k}(t) = P\{T \le t|L_1, \ldots, L_k\}$, as follows:

$$\begin{aligned}
P\{T \le t|L_1, \ldots, L_k\} &= P\{L(t+t_k) \ge D|L_1, \ldots, L_k\},\\
&= 1 - P\{L(t+t_k) \le D|L_1, \ldots, L_k\},\\
&= 1 - P\left\{Z < \frac{D - \tilde{\mu}(t+t_k)}{\sqrt{\tilde{\sigma}^2(t+t_k)}}\right\},\\
&= P\left\{Z \ge \frac{D - \tilde{\mu}(t+t_k)}{\sqrt{\tilde{\sigma}^2(t+t_k)}}\right\},\\
&= \Phi(g(t)),
\end{aligned}$$

where Z is a standard normal random variable, $\Phi(\cdot)$ is the cdf of a standard normal random variable, and $g(t) = (\tilde{\mu}(t+t_k) - D)/\sqrt{\tilde{\sigma}^2(t+t_k)}$.

Note that we have determined the residual-life cdf by finding the probability that the value of the signal at time $t+t_k$, $L(t+t_k)$, is greater than D. This procedure provides an approximation for the residual life cdf. In reality, to determine the residual-life cdf we should find $P(T \le t) = 1 - P(T > t) = 1 - P(L(s+t_k) < D, \forall s \le t)$. Thus, the residual-life distributions found in this paper are approximations that will work well when the variance of the error terms, $\epsilon(t)$, is small.

Notice that $\lim_{t\to-\infty} g(t) = -\mu_\beta/\sigma_\beta$, which implies that the domain of the residual life, T, is $(-\infty, \infty)$. Therefore, it is mathematically possible to have $T < 0$. Clearly, however, from a practical point of view we must have $T \ge 0$. Therefore, we use the truncated conditional cdf for T with the constraint $T \ge 0$. This truncated cdf of T is:

$$\begin{aligned}
&P\{T \le t|L_1, \ldots, L_k, T \ge 0\}\\
&\quad = \frac{P\{0 \le T \le t|L_1, \ldots, L_k\}}{P\{T \ge 0|L_1, \ldots, L_k\}} = \frac{\Phi(g(t)) - \Phi(g(0))}{1 - \Phi(g(0))},
\end{aligned}$$

for $t \ge 0$.

We have shown how to find the cdf of the residual life of the component, T, given that we are at time t_k with observed logged signal values, $L_1, \ldots, L_k$. We note that T is not a normal random variable since the variance term in $g(t)$ above is a function of t. Rather, T has a distribution that is similar to the Bernstein distribution (Ahmad and Sheikh, 1984). Given the truncated conditional cdf for T, we can derive the truncated conditional probability density function (pdf) for T by differentiating $F_{T|L_1,\ldots,L_k,T\ge 0}(t)$ with respect to t. Thus, we have:

$$f_{T|L_1,\ldots,L_k,T\ge 0}(t) = \frac{\phi(g(t))g'(t)}{1 - \Phi(g(0))},$$

where $\phi(\cdot)$ is the pdf of a standard normal random variable.

This procedure for updating the distribution of the residual life can be performed each time new sensor information is obtained. In other words, each time we collect a new signal observation, we can recalculate the posterior distribution for (θ', β) and obtain new estimates of $\mu_{\theta'}, \mu_\beta, \sigma_{\theta'}^2, \sigma_\beta^2$ and ρ. Then, given these updated parameters, we can update the distribution of the residual life for the device by updating the values of $\tilde{\mu}(t+t_k)$ and $\tilde{\sigma}^2(t+t_k)$ in the function $g(t)$. Because this procedure only requires the computation of the cdf and pdf for a standard normal random variable, the procedure can be easily implemented on a spreadsheet, as will be seen in Section 4 of this paper.

3.2. *Two-parameter exponential model with multiplicative Brownian motion error*

In this section, we develop an exponential degradation model with a Brownian motion error process. This model is more appropriate for applications where successive error fluctuations in sensor readings are correlated. We present a Bayesian updating procedure, similar to the procedure described above, for determining the distribution of the residual life under the assumption of a Brownian error process. We start by reviewing the definition of a Brownian motion process.

Definition 1. A one-dimensional standard Brownian motion is a real-valued process $W(t)$, $t \geq 0$, that has the following properties:

1. If $t_0 < t_1 < \cdots < t_n$, then $W(t_0), W(t_1) - W(t_0), \ldots, W(t_n) - W(t_{n-1})$ are mutually independent.
2. If $s, t \geq 0$, then
$$P(W(s+t) - W(s) \in A) = \int_A (2\pi t)^{-1/2} e^{-x^2/2t} \mathrm{d}x.$$
3. With probability one, $t \to W(t)$ is continuous.

Part 1 of this definition says that the process $W(t)$ has independent increments. Part 2 of this definition says that the increment $W(s+t) - W(s)$ has a normal distribution with mean zero and variance t. Part 3 of this definition says that $W(t)$, $t \geq 0$, almost surely has continuous paths. For a further discussion of the Brownian motion process and its properties, see Durrett (1991).

3.2.1. *The degradation signal model*

Let $S(t)$ denote the degradation signal as a continuous stochastic process, continuous with respect to time t. We assume that $S(t)$ has the following functional form:

$$\begin{aligned} S(t) &= \phi + \theta \exp\left(\beta t + \epsilon(t) - \frac{\sigma^2 t}{2}\right) \\ &= \phi + \theta \exp(\beta t) \exp\left(\epsilon(t) - \frac{\sigma^2 t}{2}\right), \end{aligned}$$

where ϕ is a constant, θ is a lognormal random variable, where $\ln\theta$ has mean μ_0 and variance σ_0^2, β is a normal random variable with mean μ_1 and variance σ_1^2, and $\epsilon(t) = \sigma W(t)$ is a centered Brownian motion such that the mean of $\epsilon(t)$ is zero and the variance of $\epsilon(t)$ is $\sigma^2 t$. We assume θ, β and $\epsilon(t)$ are mutually independent. Under these assumptions, it is easy to show that $E[\exp(\epsilon(t) - (\sigma^2 t/2))] = 1$, and thus $E[S(t)|\theta, \beta] = \phi + \theta\exp(\beta t)$. Figure 2(b) illustrates the type of degradation signal this model is intended to represent.

For this model, as for the previous model, we find it more convenient to work with the logged degradation signal. Thus, we define $L(t)$ as follows:

$$L(t) = \ln(S(t) - \phi) = \theta' + \beta t + \epsilon(t) - \frac{\sigma^2 t}{2}, \quad (5)$$

where $\theta' = \ln\theta$ is a normal random variable with mean μ_0 and variance σ_0^2. By defining $\beta' = \beta - (\sigma^2/2)$, we can further simplify $L(t)$ as follows:

$$L(t) = \theta' + \beta' t + \epsilon(t). \quad (6)$$

Finally, we let $L_i = L(t_i) - L(t_{i-1})$ denote the difference between the observed value of the logged signal at times t_i and t_{i-1}, for $i = 2, 3, \ldots$, with $L_1 = L(t_1)$.

Next, suppose we have observed $L_1, \ldots, L_k$ at times $t_1, \ldots, t_k$. Since the error increments, $\epsilon(t_i) - \epsilon(t_{i-1})$, $i = 1, \ldots, k$, are independent normal random variables, if we know θ' and β', the conditional joint density function of $L_1, \ldots, L_k$, given θ' and β', is:

$$\begin{aligned} f(L_1, \ldots, L_k|\theta', \beta') = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^k \exp\Bigg(&-\frac{(L_1 - \theta' - \beta' t_1)^2}{2\sigma^2 t_1} \\ &- \sum_{i=2}^{k}\left(\frac{(L_i - \beta'(t_i - t_{i-1}))^2}{2\sigma^2(t_i - t_{i-1})}\right)\Bigg). \end{aligned}$$

Generally, however, θ' and β' will be unknown. We let $\pi_1(\theta')$ and $\pi_2(\beta')$ denote the prior distributions on θ' and β' respectively, which are assumed to be normal with mean μ_0 and variance σ_0^2, and normal with mean $\mu_1' = \mu_1 - (\sigma^2/2)$ and variance σ_1^2, respectively. Then, given the data, $L_1, \ldots, L_k$, observed at times $t_1, \ldots, t_k$, we can find the posterior joint distribution of (θ', β') as follows:

Proposition 2. *Given the observed data, $L_1, \ldots, L_k$, the posterior distribution of (θ', β') is a bivariate normal distribution with mean $(\mu_{\theta'}, \mu_{\beta'})$, variance $(\sigma_{\theta'}^2, \sigma_{\beta'}^2)$ and correlation coefficient ρ, where:*

$$\begin{aligned} \mu_{\theta'} &= \frac{(L_1\sigma_0^2 + \mu_0\sigma^2 t_1)(\sigma_1^2 t_k + \sigma^2) - \sigma_0^2 t_1(\sigma_1^2\sum_{i=1}^k L_i + \mu_1'\sigma^2)}{(\sigma_0^2 + \sigma^2 t_1)(\sigma_1^2 t_k + \sigma^2) - \sigma_0^2\sigma_1^2 t_1}, \\ \mu_{\beta'} &= \frac{(\sigma_1^2\sum_{i=1}^k L_i + \mu_1'\sigma^2)(\sigma_0^2 + \sigma^2 t_1) - \sigma_1^2(L_1\sigma_0^2 + \mu_0\sigma^2 t_1)}{(\sigma_0^2 + \sigma^2 t_1)(\sigma_1^2 t_k + \sigma^2) - \sigma_0^2\sigma_1^2 t_1}, \\ \sigma_{\theta'}^2 &= \frac{\sigma^2\sigma_0^2 t_1(\sigma_1^2 t_k + \sigma^2)}{(\sigma_0^2 + \sigma^2 t_1)(\sigma_1^2 t_k + \sigma^2) - \sigma_0^2\sigma_1^2 t_1}, \\ \sigma_{\beta'}^2 &= \frac{\sigma^2\sigma_1^2(\sigma_0^2 + \sigma^2 t_1)}{(\sigma_0^2 + \sigma^2 t_1)(\sigma_1^2 t_k + \sigma^2) - \sigma_0^2\sigma_1^2 t_1}, \\ \rho &= \frac{-\sigma_0\sigma_1\sqrt{t_1}}{\sqrt{(\sigma_0^2 + \sigma^2 t_1)(\sigma_1^2 t_k + \sigma^2)}}. \end{aligned}$$

Proof. Given the prior distributions on θ' and β', we can find the posterior distribution of (θ', β'), denoted $p(\theta', \beta'|L_1, \ldots, L_k)$, as follows:

$$\begin{aligned} &p(\theta', \beta'|L_1, \ldots, L_k) \\ &\quad \propto f(L_1, \ldots, L_k|\theta', \beta')\pi_1(\theta')\pi_2(\beta'), \\ &\quad \propto \exp\left\{-\frac{(L_1 - \theta' - \beta' t_1)^2}{2\sigma^2 t_1} - \sum_{i=2}^{k}\frac{(L_i - \beta'(t_i - t_{i-1}))^2}{2\sigma^2(t_i - t_{i-1})}\right\} \\ &\qquad \exp\left\{\frac{-(\theta' - \mu_0)^2}{2\sigma_0^2}\right\}\exp\left\{\frac{-(\beta' - \mu_1')^2}{2\sigma_1^2}\right\}, \end{aligned}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\theta'^2\left(\frac{\sigma_0^2+\sigma^2 t_1}{\sigma^2\sigma_0^2 t_1}\right)+\beta'^2\left(\frac{\sigma_1^2 t_k+\sigma^2}{\sigma^2\sigma_1^2}\right)-2\theta'\left(\frac{L_1\sigma_0^2+\mu_0\sigma^2 t_1}{\sigma^2\sigma_0^2 t_1}\right)\right.\right.$$
$$\left.\left.-2\beta'\left(\frac{\sigma_1^2\sum_{i=1}^k L_i+\mu_1'\sigma^2}{\sigma^2\sigma_1^2}\right)+2\theta'\beta'\left(\frac{1}{\sigma^2}\right)\right]\right\}, \tag{7}$$
$$\propto \exp\left\{-\frac{1}{2}\left[\theta'^2\left(\frac{1}{\sigma_{\theta'}^2(1-\rho^2)}\right)+\beta'^2\left(\frac{1}{\sigma_{\beta'}^2(1-\rho^2)}\right)-2\theta'\left(\frac{\mu_{\theta'}}{\sigma_{\theta'}^2(1-\rho^2)}-\frac{\mu_{\beta'}\rho}{\sigma_{\theta'}\sigma_{\beta'}(1-\rho^2)}\right)\right.\right.$$
$$\left.\left.-2\beta'\left(\frac{\mu_{\beta'}}{\sigma_{\beta'}^2(1-\rho^2)}-\frac{\mu_{\theta'}\rho}{\sigma_{\theta'}\sigma_{\beta'}(1-\rho^2)}\right)+2\theta'\beta'\left(\frac{-\rho}{\sigma_{\theta'}\sigma_{\beta'}(1-\rho^2)}\right)\right]\right\},$$
$$\propto \frac{1}{2\pi\sigma_{\theta'}\sigma_{\beta'}\sqrt{1-\rho^2}}\exp\left\{-\left[\frac{\sigma_{\beta'}^2(\theta'-\mu_{\theta'})^2-2\sigma_{\theta'}\sigma_{\beta'}\rho(\theta'-\mu_{\theta'})(\beta'-\mu_{\beta'})+\sigma_{\theta'}^2(\beta'-\mu_{\beta'})^2}{2\sigma_{\theta'}^2\sigma_{\beta'}^2(1-\rho^2)}\right]\right\}, \tag{8}$$

where $\mu_{\theta'}, \sigma_{\theta'}^2, \mu_{\beta'}, \sigma_{\beta'}^2$ and ρ are as defined in the proposition. Notice that this last equation is the bivariate normal density function (see, for example, Larsen and Marx (1986)). Thus, (θ', β') has a bivariate normal posterior distribution with mean $(\mu_{\theta'}, \mu_{\beta'})$, variance $(\sigma_{\theta'}^2, \sigma_{\beta'}^2)$, and correlation ρ. ■

Finally, we note that it is easy to show that $-1 < \rho < 0$ by observing that $t_1 < t_k$ implies $\rho = -\sqrt{\sigma_0^2\sigma_1^2 t_1}/\sqrt{\sigma_0^2\sigma_1^2 t_k + A} > -1$, where $A = \sigma^2\sigma_1^2 t_1 t_k + \sigma^2\sigma_0^2 + \sigma^4 t_1 \geq 0$.

3.2.2. *Determining the residual-life distribution*

Given the posterior distribution of (θ', β'), we would like to determine the distribution of the time until failure for the component. Thus, similar to the model presented in the previous section, our objective is to determine the distribution of the time until the signal reaches the failure threshold, D. To do this, we first determine the posterior distribution for (θ', β'), as described above. We then define the random variable $L(t + t_k)$ to be the logged degradation signal value observed at time $t + t_k$, $t > 0$, given $L_1, \ldots, L_k$ observed at times $t_1, \ldots, t_k$. We can then determine the distribution of $L(t + t_k)$ given $L_1, \ldots, L_k$ as follows.

Proposition 3. *Given the observed data, $L_1, \ldots, L_k$, $L(t + t_k)$ is a normal random variable with mean $\tilde{\mu}(t + t_k)$ and variance $\tilde{\sigma}^2(t + t_k)$, where:*

$$\tilde{\mu}(t+t_k) \triangleq \sum_{i=1}^k L_i + \mu_{\beta'} t = L(t_k) + \mu_{\beta'} t,$$

and

$$\tilde{\sigma}^2(t+t_k) \triangleq \sigma_{\beta'}^2 t^2 + \sigma^2 t.$$

Proof. First, note that using Equation (6) we can write $L(t + t_k) = L(t_k) + \beta' t + \epsilon(t + t_k) - \epsilon(t_k)$, where $L(t_k) = \sum_{i=1}^k L_i$. Therefore, given $L_1, \ldots, L_k$, $L(t + t_k)$ is a normal random with mean $\tilde{\mu}(t + t_k) = L(t_k) + E[\beta']t = L(t_k) + \mu_{\beta'} t$ and variance $\tilde{\sigma}^2(t + t_k) = t^2 V[\beta'] + V[\epsilon(t + t_k) - \epsilon(t_k)] = \sigma_{\beta'}^2 t^2 + \sigma^2 t$. ■

Next, we let T denote the residual life of the component and we note that T satisfies $L(T + t_k) = D$. Given this proposition, we can then find the conditional cdf of T given $L_1, \ldots, L_k$, $F_{T|L_1,\ldots,L_k}(t) = P\{T \leq t|L_1, \ldots, L_k\}$, as follows:

$$\begin{aligned}
P\{T \leq t|L_1,\ldots,L_k\} &= P\{L(t+t_k) \geq D|L_1,\ldots,L_k\},\\
&= 1 - P\{L(t+t_k) \leq D|L_1,\ldots,L_k\},\\
&= 1 - P\left\{Z < \frac{D-\tilde{\mu}(t+t_k)}{\sqrt{\tilde{\sigma}^2(t+t_k)}}\right\},\\
&= P\left\{Z \geq \frac{D-\tilde{\mu}(t+t_k)}{\sqrt{\tilde{\sigma}^2(t+t_k)}}\right\},\\
&= \Phi(g(t)),
\end{aligned}$$

where Z is a standard normal random variable, $\Phi(\cdot)$ is the cdf of a standard normal random variable, and $g(t) = (\tilde{\mu}(t + t_k) - D)/\sqrt{\tilde{\sigma}^2(t + t_k)}$.

Notice that we predict the distribution of the residual life at time t_k, before the signal value reaches the threshold D, i.e., $\sum_{i=1}^k L_i = L(t_k) < D$. Therefore, we have:

$$\lim_{t\to 0} g(t) = \lim_{t\to 0}\frac{\sum_{i=1}^k L_i + \mu_{\beta'} t - D}{\sqrt{\sigma_{\beta'}^2 t^2 + \sigma^2 t}} = -\infty.$$

Thus, we have $\lim_{t\to 0} F_{T|L_1,\ldots,L_k}(t) = 0$, which implies that the domain of T is $(0, \infty)$. Therefore, unlike in the previous model with independent random errors, this model does not require truncation of the conditional cdf and pdf of T.

We have shown how to find the conditional cdf of the residual life of the component, T, given that we are at time t_k with observed signal values $L_1, \ldots, L_k$. Note that T is not a normal random variable since the variance term in $g(t)$ is a function of t. Rather, as in the previous model, T has a distribution that is similar to the Bernstein distribution, and we can derive its conditional pdf by differentiating $F_{T|L_1,\ldots,L_k}(t)$ with respect to t. In fact, we can write the conditional pdf of T, given $L_1, \ldots, L_k$, as:

$$f_{T|L_1,\ldots,L_k,T\geq 0}(t) = \phi(g(t))g'(t),$$

where $\phi(\cdot)$ is the pdf of the standard normal random variable.

As with the previous model, the residual-life distributions found in this section are approximations that will work well when the variance of the error terms, $\epsilon(t)$, is small. In order to find an exact expression for the residual-life distribution for the model with a Brownian error process, we would need to apply the concept of first passage times (see, for example, Cox and Miller (1965)). This is an important topic for future research.

As in the previous model, this procedure for determining the distribution of the residual component life can be performed each time we obtain a new signal observation. This procedure for periodically updating the distribution of the residual life as new signal observations are obtained can be easily implemented on a spreadsheet, as will be seen in the next section.

4. Application of the exponential degradation-signal models to data obtained on bearings

In this section, we apply the degradation models presented above to the degradation signals of 34 bearings that we have run to failure under accelerated testing conditions. We first briefly describe our experimental data. We next describe how we used these data to test the assumptions required for the exponential degradation-signal models. We then describe how we implemented the Bayesian updating methods for the exponential models, as developed in Section 3. Finally, we evaluate the predictive ability of these models and discuss the results of these experiments.

4.1. *The degradation signals of the bearings*

Recall that Fig. 1 illustrated the degradation signals of three test bearings from our degradation database. These degradation signals are based on vibration monitoring and track the evolution of the vibration level with respect to time. An experimental setup has been designed to perform accelerated testing on a set of identical thrust ball bearings. An accelerometer is used to capture the vibration signal resulting from bearing degradation. Signals are transmitted to a data acquisition system for computer display and storage. The failure threshold for these experiments, $D = 0.03$ MV, was computed from published industrial standards (Blake and Mitchel, 1972). For a given bearing, vibration data were collected and processed every 2 minutes, i.e., $t_{i+1} - t_i = 2$ for all $i \geq 0$.

As seen in Fig. 1, the degradation signal consists of two distinct parts. The first part of the signal is stable and represents the vibration level before a bearing defect occurs. The second phase begins when a fatigue defect occurs in the bearing, and the vibration level increases as the defect worsens. Note that the functional form resembles an exponential. Indeed, exponential degradation models have been used for bearing degradation (Shao and Nezu, 2000). In the work that follows, we only use degradation data associated with this second phase.

4.2. *Testing of the exponential model assumptions*

To implement the degradation models, we first logged the signal data and then fitted the models presented in Equations (2) and (6) for each individual bearing. In other words, for each model and for each individual bearing we obtained estimates of the model parameters. In order to simplify the analysis, we took $\phi = 0$. In general, however, ϕ may be any known constant.

For the exponential model with random error terms, we used linear regression on the logged signal values, L_i, from a single bearing to estimate θ' and β for that bearing. First, from Equation (2) and the assumption that $\epsilon(0) = 0$, we note that for a given bearing with true parameter value θ', $L(0) = \theta'$. Therefore, for each bearing we set θ' equal to the log of the initial degradation signal. Then, to estimate β for that bearing, we performed a linear regression on the logged signal values, L_i, for that bearing, in which we forced the intercept, $L(0)$, to equal θ'. We then took the estimated slope from this regression as the true value of β for that bearing.

For the exponential models with Brownian motion error terms, the model requires $\epsilon(0) = 0$, and thus $L(0) = \theta'$. Therefore, for each bearing we set θ' equal to the log of the initial degradation signal. In addition, since the error terms for the Brownian motion model are not iid, we could not use linear regression to estimate β'. Instead, because the error increments for a single bearing with true parameter value β' are iid, the random variables:

$$X_k = \frac{L(t_k) - L(t_{k-1})}{t_k - t_{k-1}} = \frac{L(t_k) - L(t_{k-1})}{2}, \quad k = 1, 2, \ldots,$$

are iid with mean β'. Therefore, we used $\bar{X}$ as the value of β' for that bearing.

Thus, for each bearing and for each model (random error and Brownian motion error) we obtained estimates of the true parameter values (θ' and β for the random error model and θ' and β' for the Brownian model). We then used these estimated parameter values to test the assumptions required for our models. Our model assumes that each parameter value has a normal distribution across the population of devices. For example, for the random error model we assume that $\theta' \sim N(\mu_0', \sigma_0^2)$ and $\beta \sim N(\mu_1, \sigma_1^2)$. In order to test the normality of the model parameters, we performed chi-square goodness-of-fit tests. For these tests we used the sample mean and variance of the 34 observed θ' and β (β') values as our estimates of the mean and variance of the normal distributions. In both cases (random and Brownian motion error terms) the chi-square test failed to reject the null hypothesis that θ' and β (β') are normally distributed, with p-values of 0.55 and 0.74, respectively.

In addition to testing the normality assumption for the model parameters, we also tested the assumptions related to the error terms, for both the iid random error term assumption and the Brownian motion error process assumption. The main difference between these two models is the nature of the independence assumption. For the iid error term assumption, for any observation times $t_1, \ldots, t_k$, the error terms $\epsilon(t_1), \ldots, \epsilon(t_k)$ are assumed to be independent and normally distributed with mean zero and variance σ^2. For the Brownian motion assumption, the error terms are assumed to have independent and normally distributed increments. That is, $\epsilon(t_1), \epsilon(t_2) - \epsilon(t_1), \ldots, \epsilon(t_k) - \epsilon(t_{k-1})$ are assumed to be independent and normally distributed with mean zero.

We tested these independence assumptions as follows. For each bearing, we collected the set of residuals for each observation time (where each residual is the difference between the signal and the fitted model at the given observation time). We then computed the sample correlation between the residuals (for the random error model) or the residual increments (for the Brownian model) for each pair of observation times, t_i and t_k, $i < k$. In other words, for the random error model (Brownian error model), for each observation time we calculated 34 residuals (residual increments), one for each bearing. We then calculated the sample correlation between these 34 residuals (residual increments) at times t_i and t_k, for $i < k$. If the model assumptions hold, we would expect the correlations to be small. The Fig. 4 in Gebraeel *et al.* (2003) plots these sample correlations whereas our Fig. 3 (a and b) plots the average residuals (averaged across the 34 bearings) for each observation time t_i. In both Fig. 3(a) and Fig. 3(b) we show the results for $t_i \in \{4, 14, 24, \ldots, 144\}$. Figure 3 (a and b) suggests that the independence assumption required for the Brownian model fits the given data better than the independence assumption required for the iid model. Figure 3 (a and b) also supports the assumption that the true mean residual is zero.

To test the normality assumption, we created normal probability plots for both the error terms and the error increments, for the iid model and the Brownian model, respectively (The Fig. 6 in Gebraeel *et al.* (2003)). These plots suggest that error increments come closer to satisfying the normality assumption than do the error terms. Thus, the Brownian error model seems to provide a better fit for the given data.

Finally, we estimated the variance parameter, σ^2. To estimate σ^2 under the assumption of random error terms, for each bearing we calculated the residuals, i.e., the difference between the fitted model and the actual signal value, for $t_1, t_2, \ldots$. Note that these residuals are sample values for $\epsilon(t_i)$. Thus, to estimate σ^2, we calculate the sample variance. To estimate σ^2 under the assumption of a Brownian motion error process, for each observation time t_k, we estimated σ^2 by taking one-half of the sample variance of the error increments across the test bearings at that observation time. To understand this, note that for each test bearing the error increments, $\epsilon(t_{k+1}) - \epsilon(t_k)$, $k \geq 0$, are independent and normally distributed with mean zero and variance $\sigma^2(t_{k+1} - t_k)$. Recall that in our experiments we have $t_{k+1} - t_k = 2$ for all k. Therefore, the sample variance of the error increments across the 34 test bearings at each time, t_k, is an estimator of $2\sigma^2$.

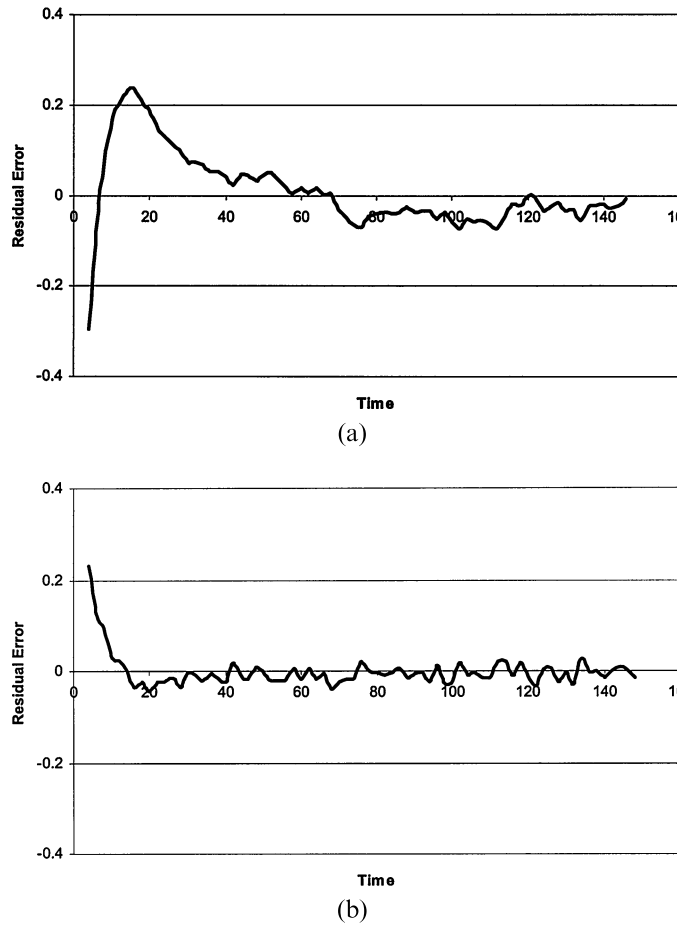


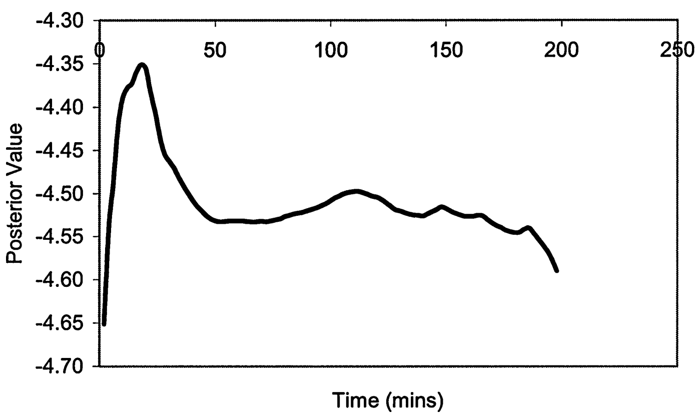
Fig. 3. Average model residuals across 34 bearings: (a) iid and (b) Brownian exponential model.

In Sections 4.3 and 4.4, we discuss how we applied the degradation models and the Bayesian updating methodology to our bearing data.

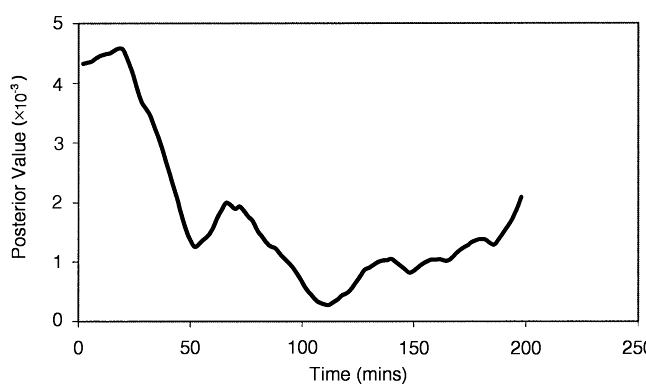
4.3. *Implementing the Bayesian updating models*

To implement the Bayesian updating methods developed in Section 3, we constructed a spreadsheet to compute the distribution of the residual life for a bearing given the degradation signal observations for both the Brownian error and the random error models. In practice, the choice for using either of the two models depends on the characteristics of the actual degradation signals. Initial experimentation is necessary to investigate the degree of correlation between the error terms. In the case of correlated error terms, we suggest the use of the Brownian error model. In contrast, for uncorrelated error terms both models are applicable. However, further testing is required to investigate which of the models provides more accurate estimates of the residual-life distribution.

With these spreadsheets, we can update the distribution of the residual life for an operating bearing any time we obtain a new signal observation, i.e., at any observation time t_k. For each bearing and each t_k, we can calculate the posterior means and variances for the stochastic parameters, θ' and β (β'), as discussed in Section 4.4. (Fig. 4 (a and b) illustrates a representative evolution of the posterior means for θ' and β (β').) Given these posterior means and variances, we can then compute the distribution of the residual life given the degradation signal observations obtained up to that point in time. Thus, we can calculate $F_{T|S_1,\ldots,S_k}(t) = P\{T \leq t | S_1, \ldots, S_k\}$, as a function of $T > 0$. Recall that T is the random variable representing the residual life of the bearing, so $t_k + T$ is the total life of the bearing. While we can update the distribution of residual life at any observation time t_k, for the purpose of evaluating the predictive ability of these models, we focused on three



(a)



(b)

Fig. 4. Evolution of the posterior means for: (a) θ'; and (b) β.

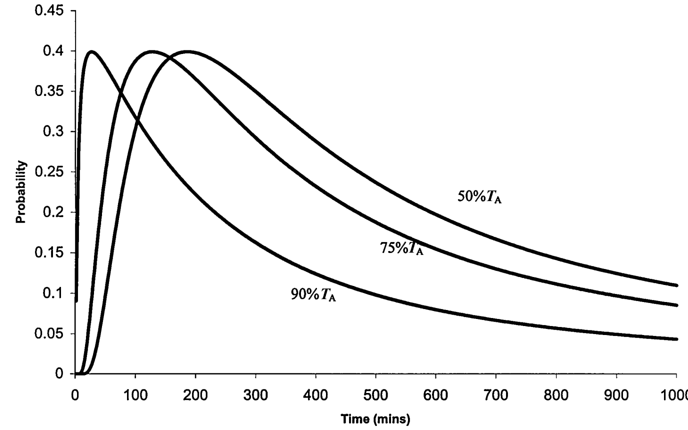


Fig. 5. Residual-life density functions for bearing 3 using the Brownian exponential model.

values of t_k, $t_k \in \{0.5T_A, 0.75T_A, 0.9T_A\}$, where T_A is the actual life of the bearing. Figure 5 illustrates the probability density functions for the residual life obtained at these times for one of our test bearings. As noted earlier, these distributions do not follow any known distributional form, but are similar to the Bernstein distribution (Ahmad and Sheikh, 1984). In addition, it is easy to show that, as is the case for the Bernstein distribution, the moments of these distributions do not exist. Therefore, we use the 5th, 50th and 95th percentiles of the residual-life distribution to evaluate the performance of these distributions, as described in the next section.

4.4. *Evaluating the residual-life distributions with the experimental data*

In this section we compare the performance of the estimated residual-life distributions for the exponential models described in Section 3. To provide a benchmark, at times t_k, $t_k \in \{0.5T_A, 0.75T_A, 0.9T_A\}$, we calculate the conditional distribution of the time until failure minus t_k, given survival up until time t_k, based only on the Bayesian prior distributions. In other words, we take the best available prediction of the residual-life distribution at time t_k, given no condition monitoring. The result is a residual-life distribution that can be used as a benchmark to assess the advantage of the Bayesian updating method developed in this paper. We refer to this distribution as CNU (Conditioned: No Updating).

Recall that we have 34 bearings in our degradation signal database. When we implemented the Bayesian updating for each bearing, we used the set of 33 other bearings to estimate the parameters of the prior distributions for θ' and β (β'), i.e., to estimate $\mu_{\theta'}$, $\sigma^2_{\theta'}$, μ_β ($\mu_{\beta'}$), and σ^2_β ($\sigma^2_{\beta'}$). To estimate these prior means and variances we calculated the sample means and variances of the 33 estimated θ' and β (β') values. We estimated σ^2, the error variance, in

Table 1. Failure times evaluated at three prediction intervals using BM, iid, and CNU residual-life distributions

Bearing	*Actual life*	*Prediction time*	*BM model*			*IID model*			*CNU model*		
1	408		5%	50%	95%	5%	50%	95%	5%	50%	95%
	204	$0.5T_A$	232	346	1610	226	534	>10000	232	422	1890
	306	$0.75T_A$	320	396	1606	330	960	>10000	322	496	2572
	368	$0.9T_A$	373	421	1561	400	1768	>10000	382	562	3592
2	328		5%	50%	95%	5%	50%	95%	5%	50%	95%
	164	$0.5T_A$	184	258	830	188	316	1014	206	414	1934
	246	$0.75T_A$	250	276	744	256	372	2746	266	452	2258
	296	$0.9T_A$	311	383	1109	308	498	>10000	312	492	2710
3	208		5%	50%	95%	5%	50%	95%	5%	50%	95%
	104	$0.5T_A$	156	290	1158	146	212	330	188	410	1802
	156	$0.75T_A$	186	284	1010	184	310	916	202	416	1838
	188	$0.9T_A$	191	215	665	204	334	1438	220	424	1896
4	270		5%	50%	95%	5%	50%	95%	5%	50%	95%
	135	$0.5T_A$	149	213	787	175	297	790	192	408	1896
	204	$0.75T_A$	225	311	1095	212	296	844	230	428	2036
	244	$0.9T_A$	251	301	947	254	386	>10000	264	450	2234
5	230		5%	50%	95%	5%	50%	95%	5%	50%	95%
	116	$0.5T_A$	209	411	1797	360	732	>10000	198	412	1462
	174	$0.75T_A$	187	251	811	226	356	774	216	418	1490
	208	$0.9T_A$	213	247	731	290	290	514	236	428	1534
6	314		5%	50%	95%	5%	50%	95%	5%	50%	95%
	158	$0.5T_A$	193	313	1357	168	234	440	202	410	1912
	236	$0.75T_A$	274	410	1756	248	412	>10000	258	444	2182
	284	$0.9T_A$	293	351	1203	298	528	>10000	300	480	2560
7	508		5%	50%	95%	5%	50%	95%	5%	50%	95%
	254	$0.5T_A$	270	346	1062	284	562	>10000	272	452	2250
	382	$0.75T_A$	399	485	1461	402	774	>10000	396	580	4542
	458	$0.9T_A$	479	589	1861	486	1166	>10000	472	680	>10000
8	148		5%	50%	95%	5%	50%	95%	5%	50%	95%
	74	$0.5T_A$	108	218	1166	142	292	1124	186	404	1840
	112	$0.75T_A$	147	265	1341	244	1862	>10000	190	406	1848
	134	$0.9T_A$	157	249	1175	212	1044	>10000	194	408	1860
9	228		5%	50%	95%	5%	50%	95%	5%	50%	95%
	114	$0.5T_A$	142	254	1656	168	502	>10000	190	402	1724
	172	$0.75T_A$	185	263	1475	202	688	>10000	210	410	1774
	206	$0.9T_A$	221	305	1703	226	630	>10000	232	422	1850
10	286		5%	50%	95%	5%	50%	95%	5%	50%	95%
	144	$0.5T_A$	153	205	727	164	220	322	198	412	1764
	216	$0.75T_A$	233	309	1041	222	274	476	240	434	1904
	258	$0.9T_A$	259	273	779	264	346	860	278	458	2098
11	360		5%	50%	95%	5%	50%	95%	5%	50%	95%
	180	$0.5T_A$	206	296	992	200	330	1180	214	412	1874
	270	$0.75T_A$	290	374	1150	290	610	>10000	288	466	2320
	324	$0.9T_A$	336	398	1140	348	824	>10000	340	516	2944
12	414		5%	50%	95%	5%	50%	95%	5%	50%	95%
	208	$0.5T_A$	245	379	1861	260	1218	>10000	234	420	1746
	312	$0.75T_A$	357	529	2879	388	>10000	>10000	326	496	2334
	374	$0.9T_A$	377	419	1509	420	>10000	>10000	386	562	3168
13	326		5%	50%	95%	5%	50%	95%	5%	50%	95%
	164	$0.5T_A$	205	359	2551	264	>10000	>10000	204	406	1712
	246	$0.75T_A$	273	401	2489	292	676	>10000	264	442	1952
	294	$0.9T_A$	297	331	1505	328	2194	>10000	310	482	2280
14	400		5%	50%	95%	5%	50%	95%	5%	50%	95%
	200	$0.5T_A$	224	308	974	206	272	504	228	420	1946

(Continued on next page)

Table 1. Failure times evaluated at three prediction intervals using BM, iid, and CNU residual-life distributions *(Continued)*

Bearing	*Actual life*	*Prediction time*		*BM model*			*IID model*			*CNU model*	
	300	$0.75T_A$	322	410	1228	316	562	>10000	316	492	2656
	360	$0.9T_A$	380	470	1382	384	413	>10000	374	556	3738
15	360		5%	50%	95%	5%	50%	95%	5%	50%	95%
	180	$0.5T_A$	242	424	2320	294	>10000	>10000	214	412	1778
	270	$0.75T_A$	320	490	2500	332	>10000	>10000	288	462	2164
	324	$0.9T_A$	328	368	1260	376	>10000	>10000	340	512	2692
16	360		5%	50%	95%	5%	50%	95%	5%	50%	95%
	180	$0.5T_A$	234	360	1074	212	290	446	216	414	1714
	270	$0.75T_A$	292	372	956	278	366	730	288	462	2040
	324	$0.9T_A$	336	394	944	332	432	1062	340	510	2492
17	220		5%	50%	95%	5%	50%	95%	5%	50%	95%
	110	$0.5T_A$	144	248	1034	238	524	>10000	188	408	1852
	166	$0.75T_A$	173	215	831	378	378	8166	206	416	1906
	198	$0.9T_A$	204	248	830	338	338	2198	228	426	1984

the manner described above. We then used the degradation signal data for the remaining individual bearing (the 34th bearing) to test our residual-life models by assuming that this 34th bearing is the functioning bearing.

In order to evaluate the performance of the residual-life distribution for these three different models, at each bearing and each model, at each of the three values of t_k, i.e., for $t_k \in \{0.5T_A, 0.75T_A, 0.9T_A\}$, we first calculated an approximate *90th Percentile Interval* (PI) for the residual life, T. We define the 90th PI to be the interval (a, b) where a denotes the fifth percentile and b denotes the 95th percentile of the residual-life distribution. Since for each bearing and each model we consider three different values of t_k, we obtain three different percentile intervals, constructed at different points in the life of the functioning bearing. We let (a_j, b_j) denote these percentile intervals, where j represents the jth chosen prediction time, t_j, $j = 1, 2, 3$. In other words, $j = 1$ refers to prediction time $0.5T_A$, $j = 2$ refers to prediction time $0.75T_A$, etc. At each time, t_j, $j = 1, 2, 3$, to obtain these 90th PIs, we use the residual-life distributions calculated in our spreadsheet to search for the values of a_j and b_j that satisfy the following equalities:

$$P\{T \leq a_j \mid S(t_1), S(t_2), \ldots, S(t_j)\} = 0.05,$$
$$P\{T \leq b_j \mid S(t_1), S(t_2), \ldots, S(t_j)\} = 0.95.$$

Table 1 presents the median, the fifth percentile and 95th percentile of the residual-life distribution plus the current operating time for each of the 15 bearings and each model (Brownian, iid and CNU) at $t_k = 0.5T_A, 0.75T_A, 0.9T_A$. For a complete table for all the 34 bearings, please refer to Table 1 in Gebraeel *et al.* (2003). Figure 6 displays the same information in graphical form for a typical bearing.

For further evaluation, we used the median of each residual-life distribution as a predictor of failure time. Let T_{P1}, T_{P2}, T_{P3} denote the predicted (median) life at times $0.5T_A, 0.75T_A, 0.9T_A$, respectively, for a given model. We used:

$$\sqrt{\left(\frac{\sum_{j=1}^{3}(T_A - T_{Pj})^2}{3}\right)},$$

as a measure of prediction error for that model. Figure 7 plots these errors for each of the three models applied to each of the 34 bearings. The figure shows that the Brownian model outperforms both the iid and CNU models for this predictor. This illustrates the following important point. Condition information in the form of degradation signals can significantly improve our estimates of remaining life. However, accurate modeling of the signal error is essential. Conveniently assuming that error terms in a random coefficients model are iid, as is done in much of the literature, can lead to very poor results if the assumption is incorrect.

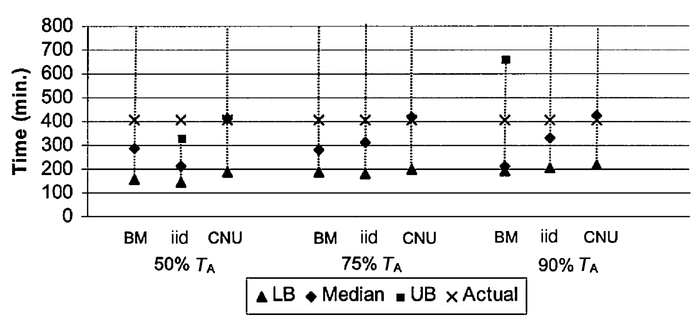


Fig. 6. Prediction intervals for the Brownian Motion (BM), iid and CNU models for bearings 1–3.

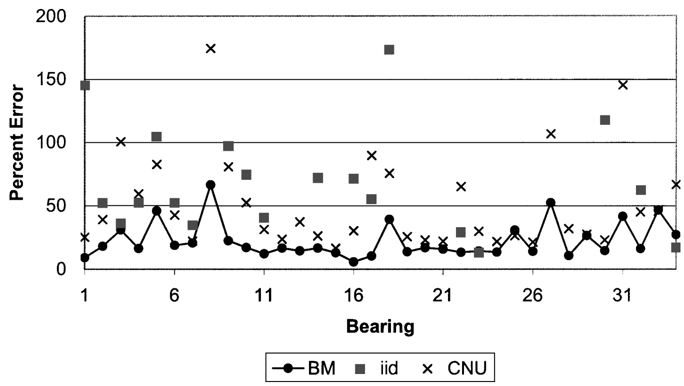


Fig. 7. Prediction errors for the BM, iid and CNU models for the 34 bearings.

5. Conclusions

In this paper, we presented methods that combine two sources of information: (i) the reliability characteristics of a device's population; and (ii) real-time sensor information from the functioning device, to periodically update the distribution of the device's residual life. To do this, we developed a Bayesian approach for updating our estimates of the stochastic parameters in exponential random-coefficient models. We then used these models with their updated parameters to develop residual-life distributions for a partially degraded device. Finally, we applied these models to bearing degradation signals that we collected through accelerated testing.

We believe that the major contributions of this work are: (i) a method to compute residual-life distributions for a functioning component by combining information from a degradation database with real-time condition information; (ii) models that work for components with exponential-like degradation signals that exhibit either iid or Brownian error characteristics; and (iii) an approach to apply these models to real-world data and evaluate their effectiveness. We believe that the results presented in Section 4.4 clearly indicate the value of using the Bayesian approach to incorporate real-time condition information into our remaining life models. Furthermore, as noted above, we believe that our work illustrates the importance of accurately modeling the degradation signal error. Future research directions include sensitivity analysis for the exponential models presented here, developing new types of error models, and formulating condition-based component replacement and spare parts inventory models given these residual-life distributions.

Finally, in this paper we have focused on developing specific degradation signal models for which we can obtain easy-to-compute residual-life distributions. For example, because we have assumed normal or lognormal prior distributions for the unknown parameters in our degradation signal models, all of the models developed in this paper are quite easy to compute using a standard spreadsheet. We note, however, that the Bayesian-updating approach we present in this paper could be applied to much more general models. For example, if we are not worried about obtaining closed-form expressions for the posterior distributions, we can assume any form for the prior distributions on the unknown parameters. This, of course, would require numerical integration in order to compute the posterior distributions. The general approach, however, would remain unchanged. Similarly, we have also applied our Bayesian approach to degradation signal models of other forms, e.g., linear and polynomial models.

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