Degradation Modeling with Bayesian Updating

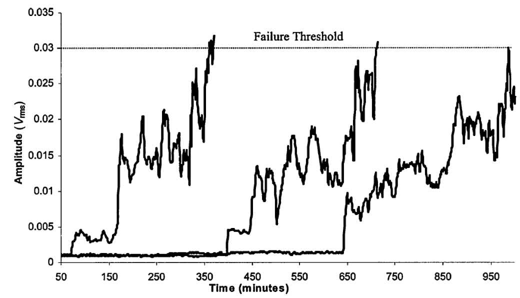
Prof. Nagi Gebraeel
Industrial and Systems Engineering
Georgia Tech

Bayesian Degradation Modeling

- Bayesian Degradation Models
  - Linear Degradation Model
    - Assumptions and Prior Distributions
    - Derivations of Posterior Distributions of Model Parameters
    - Remaining life predictions.
  - Exponential Degradation Model
    - Derivations of Posterior Distributions of Model Parameters
    - Remaining life predictions

Bayesian Statistics in Degradation Modeling

- Recall that condition monitoring is helpful in designing a degradation signal. Degradation signals provide the basis for developing models that can be used to estimate the residual life of the device.
- Even though the degradation rates of the three components differ significantly, the signals all exhibit similar shapes.
- It is not unusual for a population of "identical" devices to have a common degradation signal form while exhibiting widely different degradation rates and failure times.



3

Recap: Degradation Modeling

- Model parameters are either deterministic or stochastic.
  - **Deterministic parameters** are assumed to be known and constant throughout the device's life. These might represent some feature common to all devices in the population.
  - **Stochastic parameters** are used to model the degradation characteristics that are unique to the individual device, typically the rate of degradation. These parameters are assumed to follow some distributional form across the population of devices, with those of the individual device being an unknown "draw" from the population.
- Error terms are sometimes included in the model.
  - Error terms are used to capture device and environmental noise, signal transients, measurement errors, and variations due to monitoring equipment.

4

Prognostics Degradation Modeling

- Prognostics is a framework where degradation signals are used to predict the remaining useful lifetime of partially degraded components.
- We focus on a recent mathematical framework for prognostics that leverages the degradation modeling framework covered in previous modules.
  - First, we assume that a fleet of components follows a **generalized degradation model** with parameters estimated from a historical database of degradation signals.
  - Next, we focus on **partially degraded components** still operating in the field. Our goal is to estimate the remaining useful life of these components.
  - We assume that fielded components are monitored by sensors and utilize their signals to update/revise the generalized degradation model.

5

Prognostics Degradation Modeling

- The degradation model for an individual device is obtained by updating the stochastic parameters of general model based on the unique degradation characteristics captured by the partial degradation signal observed from the fielded component .
- To do this, we use Bayesian updating to combine two sources of information:
  - The distribution of the parameters across the population of devices,
  - The real-time sensor information collected from the device through condition monitoring, that is, the degradation signal from the individual device.

6

Prognostic Degradation Modeling Example #1

- **Example: Single parameter linear model with independent random errors.**
- Consider a linear degradation model where $S(t)$ denote the level of degradation (also known as degradation signal) as a function of t.

$$S(t) = \phi + \theta t + \epsilon(t)$$

- Where,
  - $\phi = S(0)$, the initial value of the signal
  - Generally, θ is not known in advance. We assume that it follows some distribution across the population of components, where the value of θ for an individual component is an unknown "draw" from the population.
  - For now, let $\pi(\theta)$ denote the prior distribution on θ where $\theta \sim N(\mu_0, \sigma_0^2)$.
  - $\epsilon(t),\ \ t > 0$ is a r.v. with mean 0 and variance σ^2. For $\forall\ t_1, \ldots, t_k$, and $\alpha = 0$, then $\epsilon(t_1), \ldots, \epsilon(t_2)$ are independent identically distributed (iid) Normal r.v.'s, and they are also independent of θ.

7

Linear Degradation Model with IID Error Term

- Next, assume that we observed a signal from a component that is still functioning in the field and similar to the components used to estimate the model parameters.
- Let $S_j = S(t_j)$ denote the signal value at time t_j.
- Now suppose that we observed $S_1, \ldots, S_k$, we are interested in using this partial signal to update the model and update our predictions.
- We use a Bayesian approach to estimate θ for the component given the observed (partial) degradation signal.

8

Linear Prognostic Degradation Model with IID Error Term

- Note that, given θ, $S_1, \dots, S_k$ are independent due to the assumption of IID error terms.
- If we know θ, then the joint density function of $S_1, \dots, S_k$ given θ is

9

Linear Prognostic Degradation Model with IID Error Term

- Given the observed data, $S_1, \dots, S_k$, we can find the posterior distribution of θ as $p(\theta | S_1, \dots, S_k) \propto f(S_1, \dots, S_k | \theta)\, \pi(\theta)$.

$$p(\theta | s_1, \dots, s_k) = \frac{f(s_1, \dots, s_k | \theta)\, \pi(\theta)}{\int f(s_1 \dots s_k | \theta)\, \pi(\theta)\, d\theta}$$

10

Linear Prognostic Degradation Model with IID Error Term

- We can reorganize $P(\theta|S_1, \dots, S_k)$ according to the following Normal distribution

Linear Prognostic Degradation Model with IID Error Term

- By re-expressing some parameters, we can rewrite the posterior parameters of θ as follows:

$$\mu_p = \left(\frac{\sum_{i=1}^k S_i' t_i}{\sum_{i=1}^k t_i^2}\right)\left(\frac{\sigma_0^2}{\sigma_0^2 + \sigma^2(t_1, \dots, t_k)}\right) + \mu_0 \left(\frac{\sigma^2(t_1, \dots, t_k)}{\sigma_0^2 + \sigma^2(t_1, \dots, t_k)}\right)$$

$$\sigma_p^2 = \frac{\sigma_0^2 \sigma^2(t_1, \dots, t_k)}{\sigma_0^2 + \sigma^2(t_1, \dots, t_k)},$$

$$\sigma^2(t_1, \dots, t_k) = \frac{\sigma^2}{\sum_{i=1}^k t_i^2}.$$

- Where $S_i' = \mathrm{S_i} - \phi$

Determining the Predictive Distribution of the Signal

- Given this posterior distribution of θ, we can find the predictive distribution for $S(t)$ for $t > t_k$.
- In this case, since $S(t) = \phi + \theta t + \epsilon(t)$, we prove that the predictive distribution of $S(t)$ given $S_1, \ldots, S_k$ will be Normal with mean $\phi + \mu_p t$ and variance $t^2\sigma_p^2 + \sigma^2$.
- To see this, consider the probability density function of $S(t)$ given $S_1, \ldots, S_k$ can be derived through the formula

$$f(S(t)|S_1, \ldots, S_k) = \int f(S(t)|\theta, S_1, \ldots, S_k) f(\theta|S_1, \ldots, S_k) d\theta$$

13

Determining the Predictive Distribution of the Signal

- Note that $f(S(t)|\theta, S_1, \ldots, S_k) = f(S(t)|\theta)$ because $S(t)$ conditional on θ is independent of $S_1, \ldots, S_k$.
- Therefore, we can obtain the pdf of $S(t)$ given $S_1, \ldots, S_k$ through the updated formula

$$\begin{aligned}
&f(S(t)|S_1, \ldots, S_k) \\
&= \int f(S(t)|\theta) f(\theta|S_1, \ldots, S_k) d\theta \\
&= \int \frac{1}{\sqrt{2\pi\sigma^2 t^{2\alpha}}} \exp\left\{-\frac{(S(t) - \phi - \theta t)^2}{2\sigma^2 t^{2\alpha}}\right\} \frac{1}{\sqrt{2\pi\sigma_p^2}} \exp\left\{-\frac{(\theta - \mu_p)^2}{2\sigma_p^2}\right\} d\theta
\end{aligned}$$

14

Determining the Predictive Distribution of the Signal

- Note that $S(t)$ given $S_1, \ldots, S_k$ is still Normally distributed and it is now easy to know that its mean and variance are $\phi + \mu_p t$ and $t^2\sigma_p^2 + \sigma^2$.

15

Determining the Distribution of Remaining Life

- Once the predictive distribution of $S(t)$ given $S_1, \ldots, S_k$ has been determined, we can estimate the time to failure for the component.
- We will assume that failure occurs when the degradation signal hits some given failure threshold, D_2, and thus our objective is to estimate the distribution of the time until the signal reaches D_2.
  - We assume that the threshold value, D_2, is fixed and known.
- We let the r.v., T, denote the distribution of the component's remaining life, where T satisfies $S(T + t_k) = D_2$. Thus,

$$S(T + t_k) = \phi + \theta(T + t_k) + \epsilon(T + t_k) = D_2$$

  - where $\epsilon(T + t_k)$ is the random error term at time $T + t_k$

16

Determining the Distribution of Remaining Life

- We need to find the conditional CDF of T given $S_1, \ldots, S_k$;

$$F_{T|S_1,\ldots,S_k}(t)$$
$$= P\{T \leq t | S_1, \ldots, S_k\} = 1 - P\{\theta(t+t_k) + \epsilon(t+t_k) < D_2 - \phi | S_1, \ldots, S_k\}$$
$$= P\left\{Z \geq \frac{D_2 - \phi - \mu_p(t+t_k)}{\sqrt{(t+t_k)^2\sigma_p^2 + \sigma^2}}\right\} = P(Z \leq g(t)) = \Phi(g(t)),$$

- Where $\Phi(.)$ is the cdf of the standard normal random variable.
- Note that $\lim_{t\to\infty} g(t) = -\left(\frac{\mu_p}{\sigma_p}\right)$, thus the domain of the remaining useful life is $(-\infty, \infty)$. To preclude negative values of the remaining life, we compute the truncated cdf.

$$P(T \leq t | S_1, \ldots, S_k) = \frac{\Phi\big(g(t)\big) - \Phi\big(g(0)\big)}{1 - \Phi\big(g(0)\big)}$$

17

Prognostic Degradation Modeling Example #2

- **Two Parameter Exponential Model with Multiplicative Brownian Motion Errors.**
- We develop an exponential degradation model with a Brownian motion error process.
  - This model is more appropriate for applications where successive error fluctuations in sensor readings are correlated.
  - We present a Bayesian updating procedure, similar to the procedure described earlier.
- We start by reviewing the definition of a Brownian motion process.
  - If $t_0 < t_1 < \cdots < t_n$, then $W(t_0), W(t_1) - W(t_0), \ldots, W(t_n) - W(t_{n-1})$ are mutually independent.
  - If $s, t \geq 0$, then $P(W(s+t) - W(s) \in A) = \int_A (2\pi t)^{-1/2} e^{-x^2/2t} \mathrm{d}x.$
  - With probability one, $t \to W(t)$ is continuous.

18

Exponential Degradation Model with Brownian Error Term

- Let $S(t)$ denote the degradation signal as a continuous stochastic process, continuous with respect to time t. We assume that $S(t)$ has the following functional form:

$$S(t) = \phi + \theta \exp\left(\beta t + \epsilon(t) - \frac{\sigma^2 t}{2}\right)$$
$$= \phi + \theta \exp(\beta t) \exp\left(\epsilon(t) - \frac{\sigma^2 t}{2}\right),$$

- where ϕ is a constant, θ is a lognormal random variable, where $\ln\theta$ has mean μ_0 and variance σ_0^2.
- β is a normal random variable with mean μ_1 and variance σ_1^2 , and
- $\epsilon(t) = \sigma W(t)$ is a centered Brownian motion with mean 0 and variance of $\sigma^2 t$.

19

Exponential Degradation Model with Brownian Error Term

- We assume θ, β and $\epsilon(t)$ are mutually independent.
- Under these assumptions, it can be shown that $E\left[\exp\left(\epsilon(t) - \left(\frac{\sigma^2 t}{2}\right)\right)\right] = 1$,
- Thus, $E[S(t)|\theta, \beta] = \phi + \theta \exp(\beta t)$
- For this model, we find it more convenient to work with the logged degradation signal. Thus, we define $L(t)$ as follows:

$$L(t) = \ln(S(t) - \phi) = \theta' + \beta t + \epsilon(t) - \frac{\sigma^2 t}{2},$$

- where $\theta = \ln\theta$ is a normal random variable with mean μ_0 and variance σ_0^2. By defining $\beta' = \beta - \left(\frac{\sigma^2}{2}\right)$, we can further simplify $L(t)$as follows:

$$L(t) = \theta' + \beta' t + \epsilon(t)$$

20

Exponential Degradation Model with Brownian Error Term

- Let $L_i = L(t_i) - L(t_{i-1})$ denote the difference between the observed value of the logged signal at times t_i and t_{i-1}, for $i = 2, 3, \dots$, with $L_1 = L(t_1)$.
- Next, suppose we have observed $L_1, \dots, L_k$ at times $t_1, \dots, t_k$.
- Since the error increments, $\epsilon(t_i) - \epsilon(t_{i-1})$, $\; i = 1, \dots, k$ are independent normal random variables , if we know θ' and β', the conditional joint density function of $L_1, \dots, L_k$, given θ' and β', is:

21

Exponential Degradation Model with Brownian Error Term

- Generally, however, θ' and β' will be unknown.
- Let $\pi_1(\theta')$ and $\pi_2(\beta')$ denote the prior distributions on θ' and β', respectively, which are assumed to be normal with mean μ_0 and variance σ_0^2, and normal with mean $\mu_1 = \mu_1 - \left(\frac{\sigma^2}{2}\right)$ and variance σ_1^2,respectively.
- Then, given the data, $L_1, \dots, L_k$, observed at times $t_1, \dots, t_k$, we can find the posterior joint distribution of (θ', β'), denoted as $p(\theta', \beta' | L_1, \dots, L_k)$ as follows:

$$p(\theta', \beta' | L_1, \dots, L_k) \propto f(L_1, \dots, L_k | \theta', \beta') \pi_1(\theta') \pi_2(\beta')$$

22

Exponential Degradation Model with Brownian Error Term

$$p(\theta', \beta'|L_1, \ldots, L_k) \propto f(L_1, \ldots, L_k|\theta', \beta')\pi_1(\theta')\pi_2(\beta')$$

23

Exponential Degradation Model with Brownian Error Term

$$\propto \exp\left\{-\frac{1}{2}\left[\theta'^2\left(\frac{1}{\sigma_{\theta'}^2(1-\rho^2)}\right) + \beta'^2\left(\frac{1}{\sigma_{\beta'}^2(1-\rho^2)}\right) - 2\theta'\left(\frac{\mu_{\theta'}}{\sigma_{\theta'}^2(1-\rho^2)} - \frac{\mu_{\beta'}\rho}{\sigma_{\theta'}\sigma_{\beta'}(1-\rho^2)}\right)\right.\right.$$
$$\left.\left. - 2\beta'\left(\frac{\mu_{\beta'}}{\sigma_{\beta'}^2(1-\rho^2)} - \frac{\mu_{\theta'}\rho}{\sigma_{\theta'}\sigma_{\beta'}(1-\rho^2)}\right) + 2\theta'\beta'\left(\frac{-\rho}{\sigma_{\theta'}\sigma_{\beta'}(1-\rho^2)}\right)\right]\right\},$$
$$\propto \frac{1}{2\pi\sigma_{\theta'}\sigma_{\beta'}\sqrt{1-\rho^2}} \exp\left\{-\left[\frac{\sigma_{\beta'}^2(\theta'-\mu_{\theta'})^2 - 2\sigma_{\theta'}\sigma_{\beta'}\rho(\theta'-\mu_{\theta'})(\beta'-\mu_{\beta'}) + \sigma_{\theta'}^2(\beta'-\mu_{\beta'})^2}{2\sigma_{\theta'}^2\sigma_{\beta'}^2(1-\rho^2)}\right]\right\},$$

24

Exponential Degradation Model with Brownian Error Term

- Given the observed data, $L_1, \dots, L_k$, the posterior distribution of (θ', β') is a bivariate normal distribution with mean $(\mu_{\theta'}, \mu_{\beta'})$, variance $\left(\sigma_{\theta'}^2, \sigma_{\beta'}^2\right)$ and correlation coefficient ρ where:

25

Determining the Remaining Life Distribution

- Given the posterior distribution of (θ', β'), we would like to determine the distribution of the time until failure for the component.
- Our objective is to determine the distribution of the time until the signal reaches the failure threshold, D.
- To do this, we define the random variable $L(t + t_k)$ to be the logged degradation signal value observed at time $t + t_k, t > 0$, given the partial degradation signal $L_1, \dots, L_k$ observed at times $t_1, \dots, t_k$.
- We then determine the distribution of $L(t + t_k)$ given $L_1, \dots, L_k$.

26

Determining the Remaining Life Distribution

- Given the observed data, $L_1, \dots, L_k,\ L(t+t_k)$ is a normal random variable with mean $\tilde{\mu}(t+t_k)$ and variance $\tilde{\sigma}^2(t+t_k)$ where:

$$\tilde{\mu}(t+t_k) \triangleq \sum_{i=1}^{k} L_i + \mu_{\beta'} t = L(t_k) + \mu_{\beta'} t,$$

$$\tilde{\sigma}^2(t+t_k) \triangleq \sigma_{\beta'}^2 t^2 + \sigma^2 t.$$

Determining the Remaining Life Distribution

- Proof:
- We can write $L(t+t_k) = L(t_k) + \beta' t + \epsilon(t+t_k) - \epsilon(t_k)$, where $L(t_k) = \sum_{i=1}^{k} L_i$
- Therefore, given $L_1, \dots, L_k,\ L(t+t_k)$ is a normal random variable with mean
- And variance $\tilde{\sigma}^2(t+t_k) = t^2 V[\beta'] + V[\epsilon(t+t_k) - \epsilon(t_k)]$

Determining the Remaining Life Distribution

- Next, we let T denote the residual life of the component and we note that T satisfies $L(t + t_k) = D$.
- The conditional cdf of the remaining life, $F_{T|L_1,\ldots,L_k}(t) = P\{T \leq t | L_1, \ldots, L_k\}$

Section Summary

Understand Bayesian Perspective.
Compute posterior Distributions
Degradation Modeling with Bayesian Updating Framework
- Linear model with iid Error
- Exponential model with Brownian error

Linear Degradation Model

$$
\begin{aligned}
&p(\theta|S_1,\ldots,S_k)\\
&\propto exp\left\{\frac{-1}{2\sigma^2}\sum_{i=1}^{k}\left(\frac{S_i^2+\theta^2t_i^2+\phi^2-2\theta t_iS_i-2\phi S_i+2\phi\theta t_i}{t_i^{2\alpha}}\right)\right\}exp\left\{\frac{-1}{2\sigma_0^2}(\theta-\mu_0)^2\right\}\\
&\propto \exp\left\{\frac{-1}{2\sigma^2}\sum_{i=1}^{k}\left(\frac{\theta^2}{t_i^{2(\alpha-1)}}-\frac{2\theta(S_i-\phi)}{t_i^{2\alpha-1}}\right)-\frac{1}{2\sigma_0^2}(\theta^2-2\mu_0\theta)\right\}\\
&=\exp\left\{\frac{-1}{2\sigma^2\sigma_0^2}\left[\sigma_0^2\sum_{i=1}^{k}\left(\frac{\theta^2}{t_i^{2(\alpha-1)}}-\frac{2(\theta/t_i)S_i'}{t_i^{2(\alpha-1)}}\right)+\sigma^2(\theta^2-2\mu_0\theta)\right]\right\}\\
&=\exp\left\{\frac{-1}{2\sigma^2\sigma_0^2}\left[\theta^2\left(\sigma_0^2\sum_{i=1}^{k}\left(\frac{1}{t_i^{2(\alpha-1)}}\right)+\sigma^2\right)-2\theta\left(\sigma_0^2\sum_{i=1}^{k}\left(\frac{(S_i'/t_i)}{t_i^{2(\alpha-1)}}\right)+\mu_0\sigma^2\right)\right]\right\}
\end{aligned}
$$

31

Linear Degradation Model

$$
\begin{aligned}
&=\exp\left\{\frac{-1}{2\sigma^2\sigma_0^2}\left[\theta^2\left(\sigma_0^2\sum_{i=1}^{k}\left(\frac{1}{t_i^{2(\alpha-1)}}\right)+\sigma^2\right)-2\theta\left(\sigma_0^2\sum_{i=1}^{k}\left(\frac{(S_i'/t_i)}{t_i^{2(\alpha-1)}}\right)+\mu_0\sigma^2\right)\right]\right\}\\
&=\exp\left\{\frac{-1}{2\sigma^2\sigma_0^2}\left(\sigma_0^2\sum_{i=1}^{k}\left(\frac{1}{t_i^{2(\alpha-1)}}\right)+\sigma^2\right)\left[\theta^2-2\theta\left(\frac{\sigma_0^2\sum_{i=1}^{k}\left(\frac{(S_i'/t_i)}{t_i^{2(\alpha-1)}}\right)+\mu_0\sigma^2}{\sigma_0^2\sum_{i=1}^{k}\left(\frac{1}{t_i^{2(\alpha-1)}}\right)+\sigma^2}\right)\right]\right\}\\
&\propto \exp\left\{\frac{-1}{2\sigma_p^2}(\theta-\mu_p)^2\right\}
\end{aligned}
$$

where $S_i' = S_i - \phi$

32

Linear Degradation Model

- Therefore, given the data $S_1, \ldots, S_k$, is a normal random variable with posterior mean μ_p and variance σ_p^2 where,

$$\mu_p = \frac{\sigma_0^2 \sum_{i=1}^{k} \left(\frac{(S_i'/t_i)}{t_i^{2(\alpha-1)}} \right) + \mu_0 \sigma^2}{\sigma_0^2 \sum_{i=1}^{k} \left(\frac{1}{t_i^{2(\alpha-1)}} \right) + \sigma^2} \qquad \sigma_p^2 = \frac{\sigma^2 \sigma_0^2}{\sigma_0^2 \sum_{i=1}^{k} \left(\frac{1}{t_i^{2(\alpha-1)}} \right) + \sigma^2}$$

- We can view the posterior mean of θ as a weighted average of the prior mean and a statistic calculated from the sample data, where the weights are proportional to the associated variances.

33

Linear Degradation Model

- For $\alpha = 0$, we can rewrite the posterior parameters of θ as follows:

$$\mu_p = \left(\frac{\sum_{i=1}^{k} S_i' t_i}{\sum_{i=1}^{k} t_i^2} \right) \left(\frac{\sigma_0^2}{\sigma_0^2 + \sigma^2(t_1, \ldots, t_k)} \right) + \mu_0 \left(\frac{\sigma^2(t_1, \ldots, t_k)}{\sigma_0^2 + \sigma^2(t_1, \ldots, t_k)} \right)$$

$$\sigma_p^2 = \frac{\sigma_0^2 \sigma^2(t_1, \ldots, t_k)}{\sigma_0^2 + \sigma^2(t_1, \ldots, t_k)},$$

$$\sigma^2(t_1, \ldots, t_k) = \frac{\sigma^2}{\sum_{i=1}^{k} t_i^2}.$$

34

Determining the Predictive Distribution of the Signal

- Given this posterior distribution of θ, we can find the predictive distribution for $S(t)$ for $t > t_k$.
- In this case, since $S(t) = \phi + \theta t + \epsilon(t)$, we prove that the predictive distribution of $S(t)$ given $S_1, \dots, S_k$ will be Normal with mean $\phi + \mu_p t$ and variance $t^2\sigma_p^2 + t^{2\alpha}\sigma^2$.
  - Clearly for $\alpha = 0$, the variance $t^2\sigma_p^2 + \sigma^2$
- To see this, consider the probability density function of $S(t)$ given $S_1, \dots, S_k$ can be derived through the formula

$$f(S(t)|S_1,\dots,S_k) = \int f(S(t)|\theta, S_1,\dots,S_k) f(\theta|S_1,\dots,S_k)d\theta$$

35

Determining the Predictive Distribution of the Signal

- Given this posterior distribution of θ, we can find the predictive distribution for $S(t)$ for $t > t_k$.
- In this case, since $S(t) = \phi + \theta t + \epsilon(t)$, we prove that the predictive distribution of $S(t)$ given $S_1, \dots, S_k$ will be Normal with mean $\phi + \mu_p t$ and variance $t^2\sigma_p^2 + t^{2\alpha}\sigma^2$.
  - Clearly for $\alpha = 0$, the variance $t^2\sigma_p^2 + \sigma^2$
- To see this, consider the probability density function of $S(t)$ given $S_1, \dots, S_k$ can be derived through the formula

$$f(S(t)|S_1,\dots,S_k) = \int f(S(t)|\theta, S_1,\dots,S_k) f(\theta|S_1,\dots,S_k)d\theta$$

36

Determining the Predictive Distribution of the Signal

- Note that $f(S(t)|\theta, S_1, \dots, S_k) = f(S(t)|\theta)$ because $S(t)$ conditional on θ is independent of $S_1, \dots, S_k$.
- Therefore, we can obtain the pdf of $S(t)$ given $S_1, \dots, S_k$ through the updated formula

$$
\begin{aligned}
& f(S(t)|S_1, \dots, S_k) \\
&= \int f(S(t)|\theta) f(\theta|S_1, \dots, S_k) d\theta \\
&= \int \frac{1}{\sqrt{2\pi\sigma^2 t^{2\alpha}}} \exp\left\{-\frac{(S(t)-\phi-\theta t)^2}{2\sigma^2 t^{2\alpha}}\right\} \frac{1}{\sqrt{2\pi\sigma_p^2}} \exp\left\{-\frac{(\theta-\mu_p)^2}{2\sigma_p^2}\right\} d\theta
\end{aligned}
$$

37

Determining the Predictive Distribution of the Signal

$$
\begin{aligned}
&= \int \frac{1}{\sqrt{2\pi\sigma^2 t^{2\alpha}}} \exp\left\{-\frac{(S(t)-\phi-\theta t)^2}{2\sigma^2 t^{2\alpha}}\right\} \frac{1}{\sqrt{2\pi\sigma_p^2}} \exp\left\{-\frac{(\theta-\mu_p)^2}{2\sigma_p^2}\right\} d\theta \\
&\propto \exp\left\{-\left(\frac{(S(t)-\phi)^2}{2\sigma^2 t^{2\alpha}} + \frac{\mu_p^2}{2\sigma_p^2} + \frac{[(S(t)-\phi)\sigma_p^2 + \mu_p \sigma^2 t^{2\alpha-1}]^2}{2t^{2\alpha}\sigma^2\sigma_p^2(t^{2\alpha-2}\sigma^2+\sigma_p^2)}\right)\right\} \times \\
&\qquad \int \exp\left\{-\frac{t\sigma_p^2+\sigma^2}{2\sigma_p^2\sigma^2}\left(\theta - \frac{(S(t)-\phi)\sigma_p^2 + \mu_p\sigma^2 t^{2\alpha-1}}{t\sigma_p^2 + t^{2\alpha-1}\sigma^2}\right)^2\right\} d\theta \\
&\propto \exp\left\{-\frac{1}{2(t^2\sigma_p^2 + t^{2\alpha}\sigma^2)}\left(S(t)-\phi-\mu_p t\right)^2\right\}.
\end{aligned}
$$

- Note that $S(t)$ given $S_1, \dots, S_k$ is still Normally distributed and it is now easy to know that its mean and variance are $\phi + \mu_p t$ and $t^2\sigma_p^2 + t^{2\alpha}\sigma^2$.

38

Determining the Distribution of Remaining Life

- Once the predictive distribution of $S(t)$ given $S_1, \ldots, S_k$ has been determined, we can estimate the time to failure for the component.
- We will assume that failure occurs when the degradation signal hits some given failure threshold, D_2, and thus our objective is to estimate the distribution of the time until the signal reaches D_2.
- We assume that the threshold value, D_2, is fixed and known.
- We let the r.v., T, denote the distribution of the component's remaining life, where T satisfies $S(T + t_k) = D_2$. Thus,

$$S(T + t_k) = \phi + \theta(T + t_k) + \epsilon(T + t_k) = D_2$$

  - where $\epsilon(T + t_k)$ is the random error term at time $T + t_k$

39

Determining the Distribution of Remaining Life

- We need to find the conditional CDF of T given $S_1, \ldots, S_k$;

$$F_{T|S_1,\ldots,S_k}(t) = P\{T \leq t | S_1, \ldots, S_k\}$$

- We can write the following expression:

$$F_{T|S_1,\ldots,S_k}(t)$$
$$= P\{T \leq t|S_1, \ldots, S_k\} = 1 - P\{\theta(t + t_k) + \epsilon(t + t_k) < D_2 - \phi|S_1, \ldots, S_k\}$$
$$= P\left\{Z \geq \frac{D_2 - \phi - \mu_p(t + t_k)}{\sqrt{(t + t_k)^2\sigma_p^2 + \sigma^2}}\right\} = P(Z \leq g(t)) = \Phi(g(t)),$$

40