Failure Distributions

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Failure Models

Constant Failure Rate Models
- Exponential Distribution
- Poisson Process

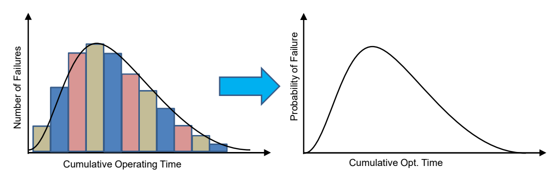
Time-Dependent Failure Models
- Weibull Distribution
- Normal Distribution
- Lognormal Distribution
- Gamma Distribution

Failures Models

- Failure models discussed in this section are theoretical distributions that are used to describe failure processes.
- They are derived mathematically and not empirically based on the data.
- An important question that we will try to answer is how adequately can a theoretical distribution describe a failure process of a component or a system.

Failures Models

- Assume that we are running a series of failure tests over 10 days.
- Each day we record the number of failed components at the end of the day.
- Let us also assume that we plot a histogram showing the total number of failures each day.



Exponential Distribution

- This function has a constant failure rate, CFR.
- In CFR models, we assume that $\lambda(t) = \lambda, t \geq 0$, and $\lambda > 0$, thus we have the following:

$f(t) = \lambda \, e^{-\lambda \, t}$

$f(t) = \frac{dF(t)}{dt}$

$F(t) = 1 - e^{-\lambda \, t}$

$R(t) = e^{-\lambda \, t}$

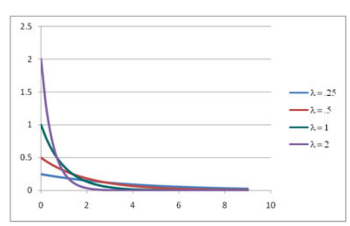
$MTTF = 1/\lambda$ and $\sigma^2 = 1/\lambda^2$

$\mu = \int_0^\infty x \, f(x) \, \mathrm{d}x$

$\sigma^2 = \int_0^\infty (x-\mu)^2 \, f(x) \, \mathrm{d}x$

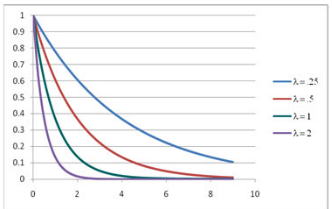
Pdf of an Exponential Distribution

- Exponential probability density function for different values of λ



Exponential Reliability Function

- Reliability function for different values of λ



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Exponential Reliability Function

- Another way to look at this is as follows:

$$R(t) = \exp\left[-\int_0^t \lambda(t')\, dt'\right]$$

$$\lambda(t) = -\frac{1}{R(t)}\frac{d\,R(t)}{dt}$$

- For the CFR case, the hazard rate is constant, λ:

$$R(t) = \exp\left[-\int_0^t \lambda(t')\, \mathrm{d}t'\right] = \exp[-\lambda t]$$

$$F(t) = 1 - e^{-\lambda t}$$

$$f(t) = -\frac{\mathrm{d}R(t)}{dt} = \lambda\, e^{-\lambda t}$$

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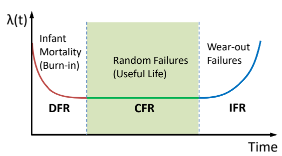
Memoryless Property

- The memoryless property is a characteristic of the CFR model
- That is, the time to failure of a component is not dependent on how long the component has been operating, i.e., there is no aging or wear-out effect.
- To see this, consider the following conditional probability:

$$\begin{aligned} R(t|T_0) &= \frac{R(t+T_0)}{R(T_0)} = \frac{e^{-\lambda(t+T_0)}}{e^{-\lambda T_0}} \\ &= \frac{e^{-\lambda t}\cdot e^{-\lambda T_0}}{e^{-\lambda T_0}} \\ &= e^{-\lambda t} \\ &= R(t) \end{aligned}$$

$R(t+T_0|T_0)$



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Example: Memoryless Property

A CFR system with $\lambda = 0.0004$ has been operating for 1000 hours.

- What is the probability that it will fail in the next 100 hrs?

  $R(t|T_0) = R(t)$ because of the memoryless property

  $R(100|1000) = R(100) = e^{-0.0004(100)} = 0.96$

  $P(T < 100) = F(100) = 1 - R(100) = 1 - 0.96 = 0.04$

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System Hazard Rate

- Assume a system can fail in one of multiple ways (often called "failure modes").
- Assuming independence among the failure modes, then the system reliability can be found as follows:

$$\begin{aligned} R(t) &= \prod_{i=1}^{n} \exp\left[-\int_0^t \lambda_i(t')\mathrm{d}t'\right] \\ &= \exp\left[-\int_0^t \sum_{i=1}^{n} \lambda_i(t')\mathrm{d}t'\right] \end{aligned}$$

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System Hazard Rate for CFR Model

- A similar situation occurs if a system consists of n independent components each having a constant failure rate λ_i, then,

$$\lambda(t) = \sum_{i=1}^{n} \lambda_i(t) = \sum_{i=1}^{n} \lambda_i$$

- Recall that mean time to failure of a component is

$$MTTF = 1/\lambda$$

- Thus, the mean time to failure of the system can be given by:

$$MTTF_{system} = \frac{1}{\sum_{i=1}^{n} \lambda_i}$$

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Example: Exponential and CFR

- A machine exhibits a constant failure rate with an $MTTF = 1100$ hr. Find the following:
  a) The reliability over a 200-hr period of operation.

$$R(t) = e^{-\lambda t} = e^{-t/MTTF} = e^{-t/1100} \qquad \mu = \frac{1}{\lambda}$$
$$R(200) = e^{-200/1100} = 0.834$$

  b) The design life that maintains a 0.9 reliability.

$$R(t_d) = e^{-t_d/1100} = 0.9$$
$$t_d = -1100\ln(0.9) = 115.9 \text{ hrs}$$

Poisson Process

- If a component having a constant failure rate λ and is immediately replaced upon failure, the number of failures observed over a period of time, t, follows a **Poisson distribution**.
  - Time between two consecutive failures is exponential distributed
- The probability of observing n failures in time t is given by the following probability function:

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \qquad n = 0, 1, 2, \ldots$$

- Expected number of failures over time t is λt.

Example: Poisson Process

- A specific rolling machine has a nonrepairable motor with a constant failure rate of 0.5 failures per year. The company has purchased 2 spare motors. If the design life of the roller is 3 years, what is the probability that 2 spare motors will be adequate?

First we need to find the expected number of failures over the 3 year lifetime of the roller, which is $\lambda t = 0.5 \times 3 = 1.5 \frac{\text{failures}}{3 \text{ years}}$

$$P(2 \text{ or fewer failures occuring over 3 years}) = p_0 + p_1 + p_2$$
$$= \sum_{i=0}^{2} \frac{e^{-1.5}(1.5)^i}{i!}$$
$$= e^{-1.5}\left(1 + 1.5 + \frac{(1.5)^2}{2!}\right) = 0.81$$

Poisson Process & Gamma Distribution

- The Poisson process can be used to find the time of the k^{th} failure, Y_k.

$$Y_k = \sum_{i=1}^{k} T_i$$

where T_i is the time between the failure $i-1$ and failure i, and has an exponential distribution with parameter λ.

- The sum of k independent exponential random variables has a **gamma distribution** with parameters k and λ, and has the following pdf,

$$f_Y(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{\Gamma(k)} \qquad \text{for } k, \lambda, t \geq 0$$

Poisson Process

- The cumulative distribution of Y_k, i.e., the probability that the k^{th} failure will occur by time t can be obtained as follows:

$$P(Y_k \leq t) = F_{Y_k}(t) = 1 - e^{-\lambda t}\sum_{i=0}^{k-1}\frac{(\lambda t)^i}{i!}$$

- The mean value for Y_k is k/λ, and the variance is k/λ^2.
- Note:

$$P_n(t) = P(Y_n \leq t) - P(Y_{n+1} \leq t)$$
$$= e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

Example: Time & Probability of k^th Failure

- Let Y_3 be the time of the third motor failure and follows a gamma distribution with parameters $k = 3$ and $\lambda = 0.5$, where λ is the same failure rate discussed in the previous example, i.e., $\lambda = 0.5$ failures/year.
  - What's the expected time to obtain 3 failures?

$$\text{Expected time to third failure} = \frac{3 \text{ failures}}{0.5 \frac{\text{failures}}{\text{year}}} = 6 \text{ years}$$

  - What's the probability that the third failure will occur within 3 years?

$$F_{y_3}(3) = 1 - e^{-0.5\times 3}\left(1 + 0.5 \times 3 + \frac{(0.5 \times 3)^2}{2!}\right) = 0.19$$

Time-Dependent Failure Models

- Time-dependent failures models include probability distributions that model failure processes that have hazard rate functions that are **not constant** over time.
- Probability distributions that fall under this category include the Weibull, Normal, Lognormal, and Gamma distributions.

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Weibull Distribution

- The Weibull distribution is used to model both **increasing and decreasing** failure rates.
- Its hazard rate function is characterized by the following form:

$R(t) = \exp[-\int \lambda(t) dt]$

$$\lambda(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}$$

$$R(t) = \exp\left[-\int_0^t \frac{\beta}{\theta}\left(\frac{t'}{\theta}\right)^{\beta-1} dt'\right] = e^{-\left(\frac{t}{\theta}\right)^{\beta}}$$

$$f(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} e^{-\left(\frac{t}{\theta}\right)^{\beta}} \qquad \theta > 0, \beta > 0, t \geq 0$$

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Weibull Distribution

- β is referred to as the **shape parameter**.
  - It affects the shape of the distribution in the sense that for $\beta < 1$ the distribution looks similar to an exponential. In fact, for $\beta = 1$ the distribution is indeed Exponential with $\lambda = 1/\theta$
  - Whereas for $\beta > 3$, the distribution is close to symmetrical, and for $1 < \beta < 3$ it is skewed.

$$\lambda(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}$$

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Weibull Distribution

- θ is the **scale parameter** and it influences both the mean and the spread of the distribution.
- θ is also called the *characteristic life*, and usually has a unit of time.
- Reliability at the *characteristic life*, $R(\theta)$, is always 0.368. How?

$R(t) = \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right]$

$$R(\theta) = \exp\left[-\left(\frac{\theta}{\theta}\right)^{\beta}\right] = \exp(-1) = 0.368$$

- In other words, we expect that 63.2% of all the Weibull failures will occur by time $t = \theta$.

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Weibull Distribution

- Given a desired reliability R,

$$R(t) = e^{-\left(\frac{t}{\theta}\right)^{\beta}} = R$$

the design life can be estimated as follows;

$$t_R = \theta(-\ln R)^{1/\beta}$$

- When $R = 0.99$, $t_{0.99}$ is referred to as the B1 life. The time at which 1% of the population will have failed.
- When $R = 0.5$, then

$$t_{0.5} = t_{med} = \theta(-\ln 0.5)^{1/\beta} = \theta(0.69315)^{1/\beta}$$

is the **median time to failure**.

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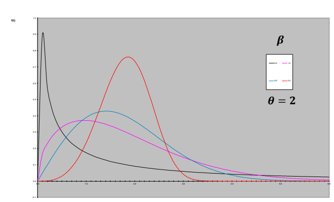
Example of Weibull plots



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Weibull Distribution

- The MTTF and the variance of the Weibull distribution are given as follows:

$$\text{MTTF} = \theta\Gamma\left(1 + \frac{1}{\beta}\right)$$

$$\sigma^2 = \theta^2\left\{\Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2\right\}$$

where;

$$\Gamma(x) = \int_0^\infty y^{x-1}e^{-y}\,\mathrm{d}y$$

$$\Gamma(x) = (x-1)\Gamma(x-1)$$

Example: Weibull Distribution

- A compressor experiences wearout that can be characterized by an linearly increasing hazard rate function:

$$\lambda(t) = \frac{2}{1000}\left(\frac{t}{1000}\right)$$

  - Calculate the MTTF and the Design Life that ensures a reliability 0.99.

Before we start we need to identify the parameters of the Weibull distribution. Let's take a closer look at the expression of the hazard function.

$$\lambda(t) = \frac{2}{1000}\left(\frac{t}{1000}\right) \dashrightarrow \lambda(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}$$

It is clear that $\boldsymbol{\beta = 2}$ and $\boldsymbol{\theta = 1000}$ hr

Example: Weibull Distribution

- A compressor experiences wearout that can be characterized by an linearly increasing hazard rate function:

$$\lambda(t) = \frac{2}{1000}\left(\frac{t}{1000}\right)$$

  - Calculate the MTTF and the Design Life that ensures a reliability 0.99.

$$MTTF = \theta \times \Gamma\left(1 + \frac{1}{\beta}\right) = 1000 \times \Gamma\left(1 + \frac{1}{2}\right) = 886.23$$

Note that $\Gamma\left(1 + \frac{1}{2}\right)$ is evaluated as "=GAMMA(1.5)" in Excel

Example: Weibull Distribution

- A compressor experiences wearout that can be characterized by an linearly increasing hazard rate function:

$$\lambda(t) = \frac{2}{1000}\left(\frac{t}{1000}\right)$$

  - Calculate the MTTF and the Design Life that ensures a reliability 0.99.

For a reliability of 0.99, we have the following:

$$R(t) = e^{-\left(\frac{t}{\theta}\right)^\beta} = e^{-\left(\frac{t}{1000}\right)^2} = 0.99$$

Thus design life for 0.99 reliability is given by,

$$t_R = 1000\sqrt{-\ln 0.99} = 100.25 \text{ hr}$$

Failure Modes

- Complex systems may experience failures in multiple ways, i.e., resulting from different physical characteristics.
- These failure categories are referred to as **failure modes**.
- If we define $R_i(t)$ as the reliability of the i^{th} failure mode, i.e., the probability that the i^{th} failure mode does not occur before time t, and assume that these failure modes are independent, then the system reliability, which we denote by $R(t)$ can be expressed as follows:

$$R(t) = \prod_{i=1}^{n} R_i(t)$$

Note: This is the probability that none of the n failure modes occurs before time t.

Failure Modes

- Consider a system with *n* independent failure modes, (or with *n* components in series) each having an independent Weibull failure distribution with shape parameter β and scale parameter θ_i, then the **system** hazard rate function can be determined as follows:

$$\lambda(t) = \sum_{i=1}^{n}\frac{\beta}{\theta_i}\left(\frac{t}{\theta_i}\right)^{\beta-1} = \beta t^{\beta-1}\left[\sum_{i=1}^{n}\left(\frac{1}{\theta_i}\right)^\beta\right]$$

- This property is true only when each component has the same shape parameter.

Identical Weibull Components

- If a systems of n serially related and independent component have identical hazard rate functions.

$$\lambda(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}$$

- Then the hazard rate of the system can be written as follows:

$$\lambda(t) = \sum_{i=1}^{n}\frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}$$

- Therefore,

$$R(t) = \exp\left[-n\left(\frac{t}{\theta}\right)^{\beta}\right]$$

Burn-In Screening for the Weibull

- Many companies undergo what is known as Burn-in screening to reduce unexpected failures of new products.
- Recall that burn-in typically represents the first part of the bathtub curve with a decreasing hazard function.
  - If a product survives burn-in screening, it is expected to have a higher reliability.
- We can use conditional reliability to estimate reliability after burn-in screening period, T_0.

$$R(t|T_0) = \frac{\exp\left\{-\left[\frac{(t+T_0)}{\theta}\right]^{\beta}\right\}}{\exp\left[-\left(\frac{T_0}{\theta}\right)^{\beta}\right]} = \exp\left[-\left(\frac{(t+T_0)}{\theta}\right)^{\beta} + \left(\frac{T_0}{\theta}\right)^{\beta}\right]$$

Do you recall the Exponential Case?

The memoryless Property. **Doesn't apply here!**

Example: Burn-in Screening

- Givn a Weibull failure distribution with a shape parameter of $\frac{1}{3}$ and a scale parameter of 16,000.
  - Define the reliability function and calculate the design life if a 90% reliability is desired.

$$R(t) = \exp\left[-\left(\frac{t}{16{,}000}\right)^{1/3}\right]$$

$$t_R = 16{,}000(-\ln 0.9)^3 = 18.71 \text{ hr}$$

  - Calculate the design life for the same reliability level (90%) after a 10 hour burn-in screening period was conducted.

$$R(t|10) = \exp\left[-\left(\frac{(t+10)}{16{,}000}\right)^{1/3} + \left(\frac{10}{16{,}000}\right)^{1/3}\right] = 0.9$$

$$t_R = 16{,}000\left[-\ln 0.9 + \left(\frac{10}{16{,}000}\right)^{1/3}\right]^3 - 10 = 101.24 \text{ hr}$$

Normal Distribution

- One of the most popular distributions for modeling various types of phenomena .
- The normal distribution has been widely used to model fatigue and wear-out phenomena.

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\frac{(t-\mu)^2}{\sigma^2}\right] \qquad -\infty < t < \infty$$

where μ and σ^2 are the mean & variance,

- The CDF of the normal distribution is,

$$F(t) = \int_{-\infty}^{t}\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\frac{(t'-\mu)^2}{\sigma^2}\right]\mathrm{d}t'$$

Normal Distribution

- The CDF of the Normal is evaluated numerically.
- The first step to evaluate cdf/pdf is to transform the normal random variable *T* as shown below,

$$z = \frac{T-\mu}{\sigma}$$

to a **Standard Normal** r.v. with mean 0 and variance 1 defined by the following PDF,

$$\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$$

and the following CDF,

$$\Pr(Z \le z) = \Phi(z) = \int_{\infty}^{z}\phi(z')\,\mathrm{d}z'$$

Example: Normal Distribution

- Wear-out in a machine follows a normal distribution with mean 10 months and standard deviation 2. Calculate the probability that a failure will occur between 9 and 11 months.

$$P(9 < X < 11) = P\left(\frac{9-10}{2} < \frac{x-10}{2} < \frac{11-10}{2}\right)$$
$$= P(-0.5 < z < 0.5)$$
$$= P(z < 0.5) - P(z < -0.5)$$
$$= 0.69146 - 0.30854 = 0.38292$$

Using Excel	
0.38292	= NORMDIST(11,10,2,TRUE) - NORMDIST(9,10,2,TRUE)

Normal Reliability Function

- The Normal reliability function is given as,

$$R(t) = P(T \geq t) = P\left\{\frac{T-\mu}{\sigma} \geq \frac{t-\mu}{\sigma}\right\}$$
$$= P\left\{z \geq \frac{t-\mu}{\sigma}\right\} = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right)$$

- The hazard function of the normal distribution can be expressed as follows:

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{\phi\big((t-\mu)/\sigma\big)}{1 - \Phi\big((t-\mu)/\sigma\big)}$$

- The hazard function for the normal distribution can be shown to be an increasing function, and thus commonly used for IFR processes.

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Example: Normal Distribution

- A new fan belt is developed from a higher grade of material. It has a time to failure distribution which is normal with a mean of 35,000 vehicle miles and a standard deviation of 7,000 vehicle miles. Find its designed life if a .97 reliability is desired.

$$R(t) = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right) = 1 - \Phi\left[\frac{(t-35{,}000)}{7{,}000}\right] = 0.97, \qquad \text{find } t?$$

From the tables we find that $1 - \Phi(-1.88) = 0.96995$

Therefore $\frac{(t-35{,}000)}{7{,}000} = -1.88$ and $t_{0.97} = 21{,}480$ miles

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Lognormal Distribution

- Lognormal model is a very popular reliability model.
- If T is lognormal then $\ln T$ is normally distributed with mean μ and variance σ^2.
- The PDF of the lognormal is

$$f(t) = \frac{1}{\sqrt{2\pi}\, st} \exp\left[-\frac{1}{2s^2} \ln\left(\frac{t}{t_{\text{med}}}\right)^2\right], t \geq 0$$

where s is the shape parameter and t_{med} is the location parameter

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Lognormal/Normal Relationship

Distribution	Lognormal	Normal
Random Variable	T	$\ln T$
Mean	$t_{\text{med}} e^{s^2/2}$	$\ln t_{\text{med}}$
Variance	$t_{\text{med}}^2 e^{s^2}\left[e^{s^2} - 1\right]$	s^2
Reliability	$R(T) = 1 - \Phi\left(\frac{1}{s} \ln \frac{t}{t_{\text{med}}}\right)$	$R(T) = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right)$

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Lognormal Distribution

- The PDF of the lognormal can be expressed in an alternative manner that uses the mean and standard deviation of $\ln T$

$$f(t) = \frac{1}{\sqrt{2\pi\, t\sigma_n}} \exp\left[-\frac{1}{2}\frac{(\ln t - \mu_n)^2}{\sigma_n^2}\right]$$

Where μ_n and σ_n are mean and standard deviation of $\ln t$.

- In this form, the reliability function is,

$$R(t) = P(> t) = P\left[z > \frac{\ln t - \mu}{\sigma}\right]$$

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Example: Lognormal Distribution

- Fatigue wearout of a component has a lognormal distribution with $t_{\text{med}} = 5000$ hrs and $s = 0.2$.
  - Calculate the MTTF and the corresponding standard deviation.

    $MTTF = 5000\, e^{(0.2)^2/2} = 5101$ hr

    $\sigma^2 = 5000\, e^{(0.2)^2}\left[e^{(0.2)^2} - 1\right] = 1.0619 \times 10^6$

  - Calculate the reliability at 3000 hours

$$R(t) = P\left[z > \frac{\ln t - \mu}{\sigma}\right] = 1 - \Phi\left(\frac{1}{0.2} \ln \frac{3000}{5000}\right) = 0.995$$

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Gamma Distribution

- The gamma model covers a wide range of hazard-rate functions, decreasing, constant and increasing. It is characterized by two parameters; shape parameter ν and a scale parameter α.

$$f(t) = \frac{t^{\nu-1}e^{-t/\alpha}}{\alpha^{\nu}\Gamma(\nu)} \quad \nu, \alpha > 0 \text{ and } t \geq 0$$

  - When $0 < \nu < 1$, then failure rate decreases from infinity to $1/\alpha$ as time goes to infinity.
  - When $\nu > 1$, the failure rate increases from 1 to infinity. Furthermore, there is a single peak of the pdf which occurs at time $t = \alpha(\nu - 1)$
  - When $\nu = 1$, the failure rate is constant and equals $1/\alpha$, i.e., the gamma distribution becomes an exponential.

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Hazard Rate Function

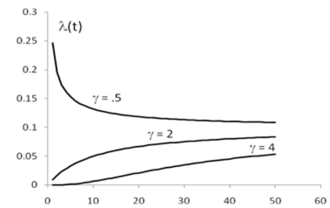
- The mean of the gamma distribution is given as,

$$MTTF = \nu\alpha$$

- Standard deviation is given as,

$$\sigma = \sqrt{\nu\alpha^2}$$

Shape Parameter	Hazard Rate Function
$0 < \nu < 1$	DFR
$\nu = 1$	CFR
$\nu > 1$	IFR



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Gamma Distribution

- The CDF can be obtained as follows;

$$F(t) = \int_0^t \frac{t'^{\nu-1}e^{-t'/\alpha}}{\alpha^{\nu}\Gamma(\nu)}\mathrm{d}t' = \frac{I\left(\frac{t}{\alpha}, \nu\right)}{\Gamma(\nu)}$$

where $I\left(\frac{t}{\alpha}, \nu\right)$ is called an incomplete gamma function which is evaluated numerically.

- The reliability function is therefore given as follows:

$$R(t) = 1 - \frac{I\left(\frac{t}{\alpha}, \nu\right)}{\Gamma(\nu)}$$

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Example: Gamma Distribution

- Failure of a critical machine part due to cyclical vibration can be characterized by gamma distribution with shape parameter 2.3 and scale parameter 2000 operating hours. Calculate the MTTF and its variance.

$$MTTF = \gamma \times \alpha = 2.3 \times 2000 = 4600 \text{ hr}$$

$$\sigma = \sqrt{\gamma\alpha^2} = \sqrt{(2.3)(2000)^2} = 3033.15 \text{ hr}$$

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Reliability Under Preventive Maintenance

- We will derive a reliability model for a system that is restored to an “as good as new state” following a preventive maintenance action.
- Let $R(t)$ be the system reliability without maintenance, T be the interval of time between PMs, and $R_m(t)$ be the reliability of the system with PM. Then

$$R_m(t) = R(T) \quad \text{for } 0 \leq t < T$$

$$R_m(t) = R(T)R(t-T) \quad \text{for } T \leq t < 2T$$

Where $R(T)$ is the reliability at the first PM, and $R(t - T)$ is the probability of surviving an additional $(t - T)$ given that the system as restored at time T.

- Generalizing the above we have,

$$R_m(t) = R(T)^n R(t - nT) \quad \text{for } nT \leq t < (n+1)T$$

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Reliability Under Preventive Maintenance

- Generalizing the above we have,

$$R_m(t) = R(T)^n R(t - nT) \quad \text{for } nT \leq t < (n+1)T$$

Where $R(T)^n$ is the probability of surviving n maintenance intervals and $R(t - nT)$ is the probability of surviving $(t - nT)$ time units past the last PM.

- The MTTF under PM can be found by the following formula:

$$MTTF = \int_0^{\infty} R_m(t)\mathrm{d}t = \frac{\int_0^T R(t)\mathrm{d}t}{1 - R(T)}$$

Proof in Appendix 9A

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Example: Reliability Under Preventive Maintenance

- A compressor has a Weibull failure process with $\beta = 2$ and $\theta = 100$ days. If we assume a 20-day preventive maintenance program ($T = 20$), express the reliability of the system with PM.

$$R_m(t) = R(T)^n R(t - nT) = \exp\left[-n\left(\frac{T}{\theta}\right)^{\beta}\right]\exp\left[-\left(\frac{t-nT}{\theta}\right)^{\beta}\right]$$
$$= \exp\left[-n\left(\frac{20}{100}\right)^{2}\right]\exp\left[-\left(\frac{t-20n}{100}\right)^{2}\right]$$

for $20n \le t \le 20(n+1)$

Example: Reliability Under Preventive Maintenance

- The reliability for 90days is found by observing that $n = 4$

$$R_m(90) = \exp\left[-4\left(\frac{20}{100}\right)^{2}\right]\exp\left[-\left(\frac{90-80}{100}\right)^{2}\right] = 0.8437$$

Example: Reliability Under Preventive Maintenance

- The design life at 0.9 reliability with no PM is 32.5 days.
- Under PM, if we focus only on the compressor reliability at the end of a maintenance interval, then we have

$$\exp\left[-n\left(\frac{20}{100}\right)^{2}\right] \approx 0.9 \ \rightarrow n = 2.63$$

Thus letting $n = 2$

$$R_m(t) = \exp\left[-2\left(\frac{20}{100}\right)^{2}\right]\exp\left[-\left(\frac{t-2(20)}{100}\right)^{2}\right] = 0.9 \quad 40 \le t < 60$$

Solving for t we get $t = 55.9$.

- This is a 32% increase in the component's design life when using PM.

Section Summary

Properties of Failure distributions
- Constant Failure Rate Models
  - Exponential Distribution
  - Poisson Process
- Time-Dependent Failure Models
  - Weibull Distribution
  - Normal Distribution
  - Lognormal Distribution
  - Gamma Distribution

Knowledge Check

An aircraft engine consists of three modules having constant failure rates of $\lambda_1 = 0.002$, $\lambda_2 = 0.015$, and $\lambda_3 = 0.0025$ failures per operating hour.

- Evaluate the reliability function for the engine.
  a) $R(t) = \exp[-0.0195\, t]$
  b) $R(t) = \exp[-0.0195]$
  c) $R(t) = \exp[0.0195\, t]$
- Calculate the corresponding $MTTF$.
  a) $\text{MTTF} = 0.0195$
  b) $\text{MTTF} = \frac{1}{0.0195}$
  c) $\text{MTTF} = \frac{1}{0.015}$

Knowledge Check

In reliability testing it is of interest to know how long the test must run in order to generate a specified number of failures. A new condenser fan motor is believed to have a constant failure rate of 3.4 failures per 100 operating hours. A single test stand is to be used in which a motor is operated until failure and then replaced with a new motor from production. What is the expected test time if 10 failure are desired?

a) 1.2 hours
b) 12 days
c) 340 hours

Knowledge Check

A compressor experiences wearout that can be characterized by an linearly increasing hazard rate function:

$$\lambda(t) = \frac{2}{1000}\left(\frac{t}{1000}\right)$$

What are the values of θ and β

a) 2 and 1000
b) 3 and 1000
c) Unknown