Q1.

let $t_1, \dots, t_n \sim Exp(\lambda)$

(a) $L(\lambda, t_1, \dots, t_n) = \log \prod_{i=1}^{n} \lambda e^{-\lambda t_i}$

$$= \sum_{i=1}^{n} (\log \lambda + (-\lambda t_i))$$

$$= n \log \lambda - \lambda \sum_{i=1}^{n} t_i$$

let $\frac{d\, L(\lambda, t_1, \dots, t_n)}{d\lambda} = 0$

$$\frac{n}{\lambda} - \sum_{i=1}^{n} t_i = 0$$

$$\lambda = \frac{n}{\sum t_i}$$

(b) $\hat{\lambda} = \frac{50}{25} = 2$, $se\,\hat{\lambda} = \frac{\sqrt{50}}{25} = \frac{\sqrt{2}}{5}$

$$[\ \hat{\lambda} - 1.645 \times \frac{\sqrt{2}}{5}\ ,\ \hat{\lambda} + 1.645 + \frac{\sqrt{2}}{5}\ ]$$

$$\downarrow$$

$$[1.5347,\ 2.4653]$$

(c) $\lambda \sim Gamma(1, 2)$

$$\pi(\lambda|x) = \frac{f(x)\,\pi(\lambda)}{\int f(x)\,\pi(\lambda)\,d\lambda} \propto f(x)\,\pi(\lambda)$$

$$\propto \lambda^n e^{-\lambda \Sigma t_i} \cdot \frac{2}{\Gamma(1)} \lambda^{1-1} e^{-2\lambda}$$

$$\propto \lambda^n e^{-\lambda(\Sigma t_i + 2)}$$

$$\therefore \pi(\lambda|x) \sim Gamma(n+1, \sum_{i=1}^{n} t_i + 2)$$

(d) $\pi(\lambda | \boldsymbol{x}) \sim \text{Gamma}(51, 27)$

$P(\lambda \text{ lies in } [1.5347,\ 2.4653] \,|\, \boldsymbol{X})$

$$= \frac{27^{51}}{\Gamma(51)} \int_{1.5347}^{2.4653} \lambda^{26} e^{-27\lambda} \, d\lambda$$

$$= 2.08479 \times 10^{-6}$$

Q2. $S(t) = \phi + \theta t + \varepsilon(t)$, $\phi = S(0)$, $\theta \sim N(\mu_0, \sigma_0^2)$, $\varepsilon(t) \sim N(0, \sigma^2 t)$

(a)
$$\mu_p = \frac{\sigma_0^2 \sum_{i=1}^{k} (S(t_i) - \phi) t_i + \mu_0 \sigma^2}{\sigma_0^2 \sum_{i=1}^{k} t_i^2 + \sigma^2}$$

$$\sigma_p^2 = \frac{\sigma^2 \sigma_0^2}{\sigma_0^2 \sum_{i=1}^{k} t_i^2 + \sigma^2}$$

(b) let $S_i = S(t_i)$

let $F_{T|S_1,\ldots,S_k}(t) = P(T \leq t \,|\, S(t_1), \ldots, S(t_k))$

$$F_{T|S_1,\ldots,S_k}(t) = P\left\{ Z \geq \frac{D_2 - \phi - \mu_p (t + t_k)}{\sqrt{(t+t_k)^2 \sigma_p^2 + \sigma^2}} \right\}$$

$$= P\{ Z \leq g(t) \} = \Phi(g(t))$$

where $g(t) = \dfrac{\mu_p (t + t_k) + \phi - D_2}{\sqrt{(t+t_k)^2 \sigma_p^2 + \sigma^2}}$, $\Phi(\cdot)$ is the cdf for the standard normal dist.

$\because \Phi$ has negative values in its domain, we need to derive an adjusted $F^*_{T|S_1,\dots,S_k}(t) = \dfrac{\Phi(g(t)) - \Phi(g(0))}{1-\Phi(g(0))}$

(c) $S(t) - \phi = \theta t + \varepsilon(t) \sim N(\mu_0 t, \sigma_0^2 t^2 + \sigma^2 t)$

$\sim N(\theta t, \sigma^2 t)$ if deterministic $\leftrightarrow$ θ is known

$W(t) = \lambda + \omega t \sim N(\mu_\lambda + \mu_\omega t, \sigma_\lambda^2 + \sigma_\omega^2 t)$ Bernstein dist.

$\Downarrow$ when $\mu_\lambda = \lambda$, $\sigma_\lambda^2 = 0$, $\gamma = 0$

$W(t) \sim N(\mu_\omega t, \sigma_\omega^2 t)$

$$f(t) = \frac{c}{\sqrt{2\pi\alpha}} \frac{1}{t^2} \exp\left\{-\frac{1}{2\alpha}\left(1-\frac{c}{t}\right)^2\right\}$$

$c = \dfrac{D-\lambda}{\mu_\omega}$, $\alpha = \dfrac{\sigma_\omega^2}{\mu_\omega^2}$, $\gamma = 0$

$$\frac{n}{c} + \frac{1}{\alpha}\sum_i \left[\frac{1}{t_i}\left(1-\frac{c}{t_i}\right)\right] = 0 \quad \left(\text{set } \frac{\partial L}{\partial c} = 0\right)$$

$$-\frac{n}{2\alpha} + \frac{1}{2\alpha^2}\sum_i \left(1-\frac{c}{t_i}\right)^2 = 0 \quad \left(\text{set } \frac{\partial L}{\partial \alpha} = 0\right),$$

where t_i is failure time of component i

solving: $c = 170.70$, $\alpha = 0.20218$

$\mu_0 = \hat{\mu}_\omega = \dfrac{D-\lambda}{c}$, $\lambda = \phi$, $D = D2 = 21$

$\sigma_0^2 = \hat{\sigma}_\omega^2 = \hat{\mu}_\omega^2 \alpha$

the parameters of model are:

$\phi = 0.20115$

$\sigma^2 = 0.28904$

$\mu_0 = 0.12184$

$\sigma_0^2 = 0.0030016$

(d) let D, D2 = 21 and using

$$F^{*}_{T|s_1,\dots,s_k}(t) = \frac{\Phi(g(t)) - \Phi(g(0))}{1 - \Phi(g(0))},$$

$E[F^{*}_{T|s_1,\dots,s_k}(t)]$ for each component is as follows:

21: 521.579

22: 61.517

23: 151.074

24: 172.073

25: 93.042

26: 97.789

27: 135.320

28: 109.289

29: 14.105

30: 39.555

(e) instead of using historically calculated ϕ and μ_0, we use the value from the new partially observed data

21: 525.126

22: 61.690

23: 150.820

24: 171.966

25: 92.866

26: 97.465

27: 135.756

28: 108.944

29: 14.097

30: 39.546

Q8.

reformulate $F^{*}_{T|S_1,\dots,S_k}(t) = \dfrac{\Phi(g(t)) - \Phi(g(0))}{1 - \Phi(g(0))}$

using the new $S_1, \dots, S_k$

t_r = optimal planned replacement time

t_o = optimal spare part ordering time

L = lead time $= 7$

$c_p = 50, \; c_f = 100, \; k_h = 0.5, \; k_s = 200, \; L = 7$

$$C_r = \frac{c_p \bar{F}(t_r) + c_f F(t_r)}{\int_0^{t_r} \bar{F}(t)\, dt}$$

$$C_o = \frac{k_s \int_{t_o}^{t_o+L} F(t)\, dt + k_h \int_{t_o+L}^{t_r} \bar{F}(t)\, dt}{\int_{t_o}^{t_o+L} F(t)\, dt + \int_0^{t_r} \bar{F}(t)\, dt)}$$

$$t_o + L \leq t_r$$

After plugging all the values, functions, and constraints, and using scipy solver to minimize $C_r + C_o$ over all values of t_r and t_o s.t. $t_o + L \leq t_r$, we get

$t_o = 153.08899$

$t_r = 160.08899$

Q4. $S(t) = \phi + \theta t + \varepsilon(t)$, $\phi = S(0)$, $\theta \sim N(\mu_0, \sigma_0^2)$, $\varepsilon(t) \sim N(0, \sigma^2 t)$

$A[t_i, t_j]$: event that degradation does not cross D in $[t_i, t_j]$

$A[t_i, t_j]$ implies $A[t_{i+x}, t_{j-y}]$ is true $\forall\ 0 \le x, y \le j-i$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad S_j = s_j - s_{j-1}, \quad s_i = S(t_i)$$

$$f(S_1, \ldots, S_k \mid A[0, t_k], \theta)$$

$$= \frac{f(S_1, \ldots, S_k, A[0, t_k] \mid \theta)}{P(A[0, t_k] \mid \theta)}$$

S_i is independent of other S_j where $j \neq i$, since $S_i = s_i - s_{i-1}$ depends only on the period of change $(i-(i-1)=1)$

$$f(S_i \mid S_1, \ldots, S_k \theta) = f(S_i \mid \theta)$$

$$\therefore f(S_1, \ldots, S_k \mid \theta) = \prod_{i=1}^{k} f(S_i \mid \theta)$$

$$f(A[t_{j-1}, t_j] \mid S_1, \ldots, S_k, A[0, t_k] \mid \theta) = f(A[t_{j-1}, t_j] \mid S_1, \ldots, S_k, s_1, \ldots, s_k, A[0, t_k], \theta)$$

(since s_0 is known, and $s_1 = S_1 + s_0$, $s_2 = S_2 + s_1$, $\ldots$, $s_k = S_k + s_{k-1}$)

$$= f(A[t_{j-1}, t_j] \mid s_{j-1}, S_j, \theta)$$

($s_j = S_j + s_{j-1}$, $\therefore$ given s_{j-1}, S_j, $A[t_{j-1}, t_j]$ is independent of everything else)

$$f(S_1, \ldots, S_k \mid A[0, t_k], \theta)$$

$$= \frac{1}{P(A[0,t_k] \mid \theta)} \prod_{i=1}^{k} f(S_i) \prod_{j=1}^{k} f(A[t_{j-1}, t_j] \mid S_j, S_{j-1}, \theta)$$

(multiply both terms: $P(A|B)P(B) = P(A, B)$)

$$= \frac{1}{P(A[0,t_k] \mid \theta)} \prod_{j=1}^{k} f(S_j, A[t_{j-1}, t_j] \mid S_{j-1}, \theta)$$