Brownian Motion for Modeling Degradation Data

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Brownian Motion

- Introduction to Brownian Motion (BM)
- Properties of BM process
- Conditional Distribution of the BM
  - Condition Mean
  - Condition Variance
- Standard Brownian Bridge
- Brownian Motion with Drift
  - Mean
  - Variance
- Geometric Brownian Motion
  - Mean
  - Variance

Introduction to Standard Brownian Motion

- Markov processes with continuous state (i.e. continuous sample path) are known as diffusion processes—Brownian motion is a diffusion process.
- A standard Brownian motion, which many references designate as $\{W(t): t \geq 0\}$ is a stochastic process with the following properties
  - With probability one, $t \rightarrow W(t)$ is continuous.
  - If $t_0 < t_1 \cdots < t_n$ then $W(t_0), W(t_1) - W(t_0), \cdots W(t_n) - W(t_{n-1})$ are mutually independent.
  - If $s, t \geq 0$, then

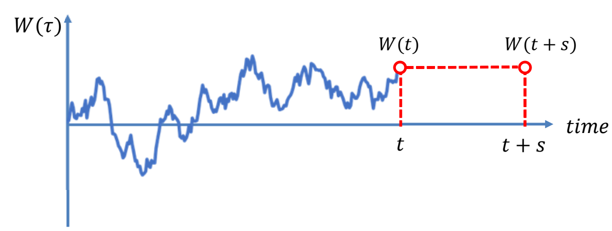
$$P(W(s+t) - W(s) \in A) = \int_A (2\pi t)^{1/2} e^{-x^2/2t} dx$$

Note: From the last property it is clear that the distribution of say, $W(t) - W(s)$ depends only on $t - s$.

Properties of a Standard Brownian Motion

- From the requirement that $W(t) \sim N(0, t)$, this implies that $W(0) = 0$ with probability 1.
- Thus, $W(t) - W(s) \sim W(t-s) - W(0)$
  - Where $W(t-s) \sim N(0, t-s)$ for $s \leq t$.
- A Brownian motion is a Martingale
  - The expected value of the process conditioned on the info at time "t" of where the process will be at time $(t+s)$ is equal to where the process currently is at time "t", i.e.,

$$\mathbb{E}[W(t+s)|\mathcal{F}_T] = W(t)$$



Properties of Brownian Motion

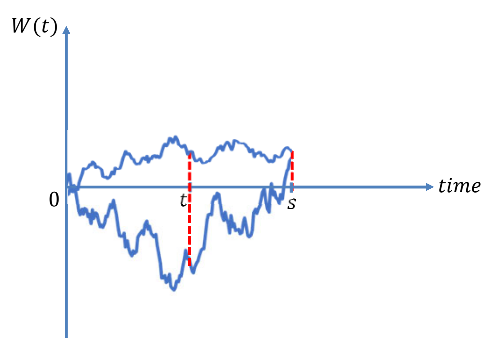
- A Brownian motion can be defined as a Gaussian process having $E[W(t)] = 0$ and for $s \leq t$, the covariance $Cov(W(s), W(t))$ is given by:
- For simplicity, we let $W(s) = W_s$

$$
\begin{aligned}
Cov(W_s, W_t) &= Cov(W_s,\ W_s + W_t - W_s) \\
&= Cov(W_s, W_s) + Cov(W_s, W_{t-s}) \\
&= \mathbb{V}(s) + 0 \\
&= s
\end{aligned}
$$

$Cov(W_s, W_{t-s})$
$= Cov(W_s - W_0, W_t - W_s)$
$= 0$ by definition of BM

Conditional Distribution of Brownian Motion

- Given the value of a Brownian motion at two points, what is the conditional distribution of the process between those two points?
- Specifically, $\mathbb{E}[W_t | W_s] = ?$ and $\mathbb{V}[W_t | W_s] = ?$
- We will start with a "claim" that we will set out to prove and use in deriving expressions for the conditional mean and variance
- Claim → $W_t - \left(\frac{t}{s}\right) W_s$ is independent of W_s



Conditional Distribution of Brownian Motion

- To show that $W_t - \left(\frac{t}{s}\right) W_s$ is independent of W_s, we need to verify that;

$cov\left(W_t - \left(\frac{t}{s}\right) W_s, W_s\right)$ is zero.

$$= cov(W_t, W_s) - \left(\frac{t}{s}\right) cov(W_s, W_s)$$

$$= t \wedge s - \frac{t}{s}(s \wedge s)$$

$Cov(W_s, W_t)$
$= \min(t, s)$
$= t \wedge s$

  - Since $t < s$:

$$= t - \frac{t}{s}(s) = 0$$

- Note: Gaussian random variables are independent if they have zero covariance.

Conditional Distribution of Brownian Motion

- We know that $\mathbb{E}\left[W_t - \left(\frac{t}{s}\right) W_s\right] = \mathbb{E}[W_t] - \left(\frac{t}{s}\right) \mathbb{E}[W_s] = 0$
- We can rewrite the above expression as follows:

$$\mathbb{E}\left[W_t - \left(\frac{t}{s}\right) W_s | W_s\right] = 0$$

$$\mathbb{E}[W_t | W_s] - \left(\frac{t}{s}\right) W_s = 0$$

$$\boxed{\mathbb{E}[W_t | W_s] = \left(\frac{t}{s}\right) W_s}$$

- The conditional mean of a SBM is the linear interpolation between points t and s.

Conditional Distribution of Brownian Motion

- Next, we need to find $\mathbb{V}[W_t|W_s]$
- Using the law of conditional expectation, we have:

$$\begin{aligned}
\mathbb{V}[W_t|W_s] &= \mathbb{E}\left[\left(W_t - \mathbb{E}(W_t|W_s)\right)^2 |W_s\right] \\
&= \mathbb{E}\left[\left(W_t - \left(\frac{t}{s}\right)W_s\right)^2 |W_s\right] \\
&= \mathbb{E}\left[W_t^2 - 2\left(\frac{t}{s}\right)W_s W_t + \left(\frac{t}{s}\right)^2 W_s^2\right] \\
&= t\wedge t - 2\left(\frac{t}{s}\right)s\wedge t + \left(\frac{t}{s}\right)^2 s\wedge s \\
&= t - 2\frac{t^2}{s} + \left(\frac{t}{s}\right)^2 s = \frac{t(s-t)}{s}
\end{aligned}$$

Recall: $\mathbb{E}(W_t|W_s) = \left(\frac{t}{s}\right)W_s$

Based on claim that $W_t - \left(\frac{t}{s}\right)W_s$ is independent of W_s

Assuming $t < s$

Conditional Distribution of Brownian Motion

$$\mathbb{V}[W_t|W_s] = \frac{t(s-t)}{s}$$

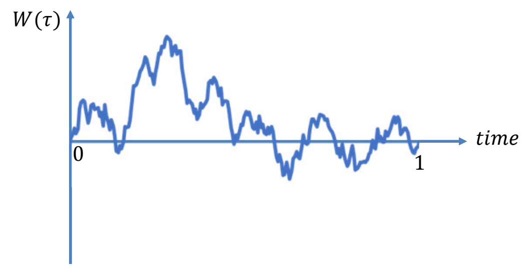
- The variance approaches "0" at $t = 0$ or $t = s$.
- Thus for a fixed "s", the conditional variance is maximized by taking $t = \frac{s}{2}$
- In summary, we have found that for $0 < t < s$, the distribution of

$$(W_t|W_s) \sim \mathcal{N}\left(\frac{t}{s}W_s, \frac{t(s-t)}{s}\right)$$

Standard Brownian Bridge

- A standard Brownian bridge over the interval $[0, 1]$ is given by the following expression $\{W(t), 0 \leq t \leq 1 | W(1) = 0\}$
- It is a standard Brownian motion $W(t)$ conditioned on the fact that $W(1) = 0$, it is tied down at time 1, therefore

$$\mathbb{E}[W_t | W_1 = 0] = \frac{t}{1}(0) = 0$$



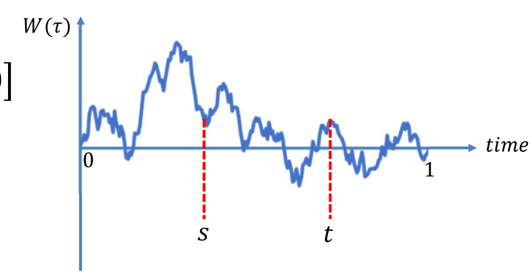
- Recall that

$$\mathbb{E}[W_t | W_s] = \left(\frac{t}{s}\right) W_s$$

Standard Brownian Bridge

- Brownian bridge is a Gaussian process with covariance:

$$\begin{aligned} &Cov(W_s, W_t | W_1 = 0) \\ &= \mathbb{E}\big[\big(W_t - \mathbb{E}(W_t)\big)\big(W_s - \mathbb{E}(W_s)\big) | W_1 = 0\big] \\ &= \mathbb{E}[W_t \, W_s | W_1 = 0] \\ &= \mathbb{E}[W_t \mathbb{E}[W_s | W_t] | W_1 = 0] \\ &= \mathbb{E}\left[W_t \frac{s}{t} W_t | W_1 = 0\right] \\ &= \left(\frac{s}{t}\right) \mathbb{E}[W_t^2 | W_1 = 0] \\ &= \left(\frac{s}{t}\right) \times \frac{t(1-t)}{1} = s(1-t) \end{aligned}$$



$\mathbb{E}(W_t | W_s) = \left(\frac{t}{s}\right) W_s$

Standard Brownian Bridge

- **Proposition:** If $\{W(t): t \geq 0\}$ is a standard Brownian motion, then $\{Z(t): t \geq 0\}$ is a Brownian bridge when $Z(t) = W(t) - t\,W(1)$:
- **Proof:** By definition, a Brownian bridge $\{Z(t): t \geq 0\}$ is a Gaussian process.
  - It can be easily shown that $\mathbb{E}[Z(t)] = 0$
  - It suffices to show that $cov(Z_s, Z_t) = s(1-t)$ for $0 < s < t < 1$
- $cov(Z_s, Z_t) = cov(W_s - s\,W_1, W_t - t\,W_1)$

$$= cov(W_s, W_t) - (t)\,cov(W_s, W_1) - (s)cov(W_1, W_t)$$
$$+(s\,t)\,cov(W_1, W_1)$$
$$= s - st - st + st = s(1-t)$$

First-Passage Time of Standard Brownian Motion

- Let $W(t)$ be a standard Brownian motion. The first passage time to a barrier defined by $b > 0$ is given by;

$$\tau_b = \inf\{t: W_t \geq b\}$$

- We are interested in finding $P\{\tau_b \leq t\}$
- Since $\tau_b \leq t$, there are two possibilities that can occur at time t, you can either be above or below the threshold barrier b, thus;

$$P\{\tau_b \leq t\} = P\{\tau_b \leq t, W_t < b\} + P\{\tau_b \leq t, W_t > b\}$$

- By path continuity and the assumption that $W_0 = 0$, the event $\{W_t > b\}$ implies that $\{\tau_b \leq t\}$, thus event $\{\tau_b \leq t, W_t > b\} = \{W_t > b\}$

$$P\{\tau_b \leq t\} = P\{\tau_b \leq t, W_t < b\} + P\{W_t > b\}$$

First-Passage Time of Standard Brownian Motion

$$P\{\tau_b \le t\} = P\{\tau_b \le t, W_t < b\} + P\{W_t > b\}$$
$$= P\{W_t < b | \tau_b \le t\} P\{\tau_b \le t,\} + P\{W_t > b\}$$
$$= \frac{1}{2} P\{\tau_b \le t\} + P\{W_t > b\}$$

- Path continuity ensures that $W_{\tau_b} = b$, hence the term "first-passage time".
- Knowing that $\tau_b \le t$, implies that the process is equally likely to continue to be above or below b at time t, therefore, $P\{W_t < b | \tau_b \le t\} = \frac{1}{2}$
- We also know that W_t is a Gaussian process, $W_t \sim \mathcal{N}(0, t)$. Therefore, $P\{W_t > b\} = 1 - \Phi\left(\frac{b}{\sqrt{t}}\right)$, and thus;

$$P\{\tau_b \le t\} = 2 \times P\{W_t > b\} = 2\left(1 - \Phi\left(\frac{b}{\sqrt{t}}\right)\right)$$

Brownian Motion with Drift

- We say that $\{X(t), t \ge 0\}$ is a Brownian motion process with drift coefficient μ and variance (or diffusion) σ^2 if:
  - $X(0) = 0$
  - $\{X(t), t \ge 0\}$ has stationary and independent increments
  - $X(t)$ is normally distributed with mean "μt" and variance "$t\sigma^2$"
- $X(t)$ can be expressed as follows:

$$X(t) = X(0) + \mu t + \sigma W(t)$$

- The first-passage time of Brownian motion with drift is given as:

$$P\{\tau \le t\} = 1 - \Phi\left(\frac{b - \mu t}{\sqrt{t}}\right) + e^{2\mu b}\, \Phi\left(\frac{b - \mu t}{\sqrt{t}}\right)$$

Known as the ***Inverse Gaussian*** distribution

Mean of a Brownian Motion with Drift

- For a positive constant x, let $T = \inf\{t: X(t) = x\}$
- Since $X(t) = X(0) + \mu t + \sigma W(t)$, we can express T as follows:

$$T = \inf \left\{t: W(t) = \frac{x - \mu t}{\sigma}\right\}$$

- If T is the stopping time of a martingale $Y(t)$, then we know that $\mathbb{E}[Y(T)] = \mathbb{E}[Y(0)]$
- Similarly, $\mathbb{E}[W(t)] = \mathbb{E}[W(0)] = 0$, thus;

$$\mathbb{E}[W(T)] = \mathbb{E}\left[\frac{x - \mu T}{\sigma}\right] = 0$$

- We can see that the mean of the first passage time of a Brownian motion with drift is;

$$\boxed{\mathbb{E}[T] = \frac{x}{\mu}}$$

Variance of a Brownian Motion with Drift

- If we let T denote the stopping time of a Brownian motion with drift, it can be shown that $\mathbb{E}[W_T^2 - T] = 0$. That is, $\mathbb{E}[W_T^2 - T]$ is a Martingale. (Prove)
- Recall that for our problem, we have $W_t = \left[\frac{X(t) - \mu t}{\sigma}\right]$, therefore;

$$\mathbb{E}\left[\left(\frac{X(t) - \mu t}{\sigma}\right)^2 - T\right] = 0$$

Recall that:
$X(t) = \mu t + \sigma W(t)$

- Since $X(T) = x$, the above expression can be expressed as follows:

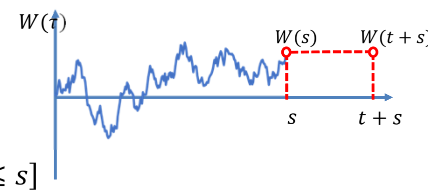
$$\mathbb{E}\left[\left(\frac{x - \mu T}{\sigma}\right)^2 - T\right] = 0 \rightarrow \mathbb{E}[(x - \mu T)^2] = \sigma^2 \mathbb{E}[T]$$

$$\boxed{\mathbb{V}(T) = \frac{\sigma^2 x}{\mu^3}}$$

Recalling Martingales

- The expected value of the process conditioned on the info at time "s" of where the process will be at time $(t+s)$ is equal to where the process currently is at time "s",
- In other words, if we have the conditional expectation of W_t given all the values of W_u, such that $0 \le u \le s$, then

$$\mathbb{E}[W_t | W_u, 0 \le u \le s] = W_s$$



$$\begin{aligned}
&\mathbb{E}[W_t | W_u, 0 \le u \le s] \\
&\qquad = \mathbb{E}[W_s + W_t - W_s | W_u, 0 \le u \le s] \\
&\qquad = \mathbb{E}[W_s | W_u, 0 \le u \le s] + \mathbb{E}[W_t - W_s | W_u, 0 \le u \le s] \\
&\qquad = W_s + \mathbb{E}[W_t - W_s] \\
&\qquad = W_s
\end{aligned}$$

Follows: $\mathcal{N}(0, (t-s))$

- This means that conditional expectation of W_t given all the values of W_u, $0 \le u \le s$ depends only on the value of W_s.

Variance of a Brownian Motion with Drift

- To show that $W_t^2 - t$ is a martingale we need to prove,

$$\mathbb{E}[W_t^2 - t | W_u, 0 \le u \le s] = W_s^2 - s$$

  - The conditional expectation of $W_t^2 - t$ given all the values of W_u, $0 \le u \le s$ depends only on the value of W_s^2, hence $W_t^2 - t$ is a martingale.
- We will start by studying $\mathbb{E}[W_t^2]$:

$$\begin{aligned}
\mathbb{E}[W_t^2 | W_u, 0 \le u \le s] &= \mathbb{E}[(W_s + W_t - W_s)^2 | W_u, 0 \le u \le s] \\
&= \mathbb{E}[W_s^2 | \ldots] - 2\mathbb{E}[W_s | \ldots]\mathbb{E}[(W_t - W_s) | \ldots] + \mathbb{E}[(W_t - W_s)^2 | \ldots] \\
&= W_s^2 + (t-s)
\end{aligned}$$

- We can rewrite $\mathbb{E}[W_t^2 | W_u, 0 \le u \le s] = W_s^2 + (t-s)$, as follows:

$$\mathbb{E}[W_t^2 - t | W_u, 0 \le u \le s] = W_s^2 - s$$

Variance of a Brownian Motion with Drift

- Since $W_t^2 - t$ is a martingale, then at $t = 0$, $\mathbb{E}[W_t^2 - t] = \mathbb{E}[W_0^2] = 0$
- Recall $\mathbb{E}[W_T^2 - T]$ where $W_t = \left[\frac{X(t) - \mu\, t}{\sigma}\right]$, therefore;

$$\mathbb{E}\left[\left(\frac{X(t) - \mu t}{\sigma}\right)^2 - T\right] = 0$$

Recall that:
$X(t) = \mu t + \sigma W(t)$

- For a given $X(T) = x$ (threshold), the above expression can be expressed as :

$$\mathbb{E}\left[\left(\frac{x - \mu T}{\sigma}\right)^2 - T\right] = 0 \rightarrow \mathbb{E}[(x - \mu T)^2] = \sigma^2 \mathbb{E}[T]$$

$$\boxed{\mathbb{V}(T) = \frac{\sigma^2 x}{\mu^3}}$$

Distribution of the First Passage Time

- Disrtibution of the first-passage time ,T, is called an inverse Gaussian distribution and is denoted as $IG(\nu, \lambda)$ where ν is the mean parameter, λ is the shape parameter and the variance is given by $\frac{\nu^3}{\lambda}$.

$$f(t; \nu, \lambda) = \sqrt{\frac{\lambda}{2\pi t^3}} \exp\left(-\frac{\lambda(t - \nu)^2}{2\nu^2 t}\right)$$

$$F(t; \nu, \lambda) = \Phi\left(\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\nu} - 1\right)\right) + \exp\left(\frac{2\lambda}{\nu}\right)\Phi\left(-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\nu} + 1\right)\right)$$

Properties of the IG Distribution

- Some popular properties of the IG distribution
  - If $X \sim IG(\nu, \lambda)$ then $kX \sim IG(k\nu, k\lambda)$ for any $k > 0$ → Scaling
  - If $X_i \sim IG(\nu, \lambda)$ then $\sum_i^n X_i \sim IG(n\nu, n^2\lambda)$ → Summation
  - If $X_i \sim IG(\nu, \lambda)$ for $i = 1, \dots, n$ then $\bar{X} \sim IG(\nu, n\lambda)$
  - If $X \sim IG(\nu, \lambda)$ then $\lambda(X - \mu)^2/\mu^2 X \sim \chi^2$

Likelihood Function of the IG Distribution

- Let $X_1, \dots, X_n \sim \mathrm{IID}\ IG(\nu, \lambda)$, with the following sampling density

$$f(\boldsymbol{x}|\nu, \lambda) = \prod_{i=1}^{n} \left(\frac{\lambda}{2\pi x_i^3}\right)^{1/2} \exp\left(-\sum_{i=1}^{n} \frac{\lambda(x_i - \nu)^2}{2\nu^2 x_i}\right), \qquad \text{for all } \boldsymbol{x} \geq 0$$

- The log-likelihood function can be expressed as follows:

$$\ln\big(\mathcal{L}(\nu, \lambda)\big) = C + \frac{n}{2}\ln(\lambda) - \sum_{i=1}^{n} \frac{\lambda(x_i - \nu)^2}{2\nu^2 x_i}$$

Likelihood Function of the IG Distribution

- Let $\ell_{\boldsymbol{x}}(\nu, \lambda) = \ln\big(\mathcal{L}(\nu, \lambda)\big)$, the partial derivatives are:

$$\frac{\partial \ell_{\boldsymbol{x}}(\nu, \lambda)}{\partial \nu} = \frac{n\lambda}{\nu^3}(\bar{x} - \nu)$$

$$\frac{\partial \ell_{\boldsymbol{x}}(\nu, \lambda)}{\partial \lambda} = \frac{n}{2\lambda} - \frac{1}{2\nu^2}\sum_{i=1}^{n}\frac{(x_i - \nu)^2}{x_i}$$

- Solving yields the following parameter estimates $\hat{\nu} = \bar{x}$ and $\frac{1}{\hat{\lambda}} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{x_i} - \frac{1}{\hat{\nu}}\right)$
- $\hat{\lambda}$ can also be expressed as follows: $\frac{1}{\hat{\lambda}} = \frac{1}{\tilde{x}} - \frac{1}{\bar{x}}$ where $\tilde{x}$ is the know as the harmonic mean $\tilde{x} = \left(\frac{\sum_{i=1}^{n} x_i^{-1}}{n}\right)^{-1}$

IG in Terms of Parameters of a Brownian Motion with Drift

- For a Brownian motion with drift :

$$X(t) = X(0) + \mu t + \sigma W(t),$$

- For a given threshold value x, we have the following relationships with the parameters of $IG(\nu, \lambda)$.

$$\nu = \frac{x}{\mu}$$

$$\lambda = \frac{x^2}{\sigma^2}$$

- The variance of $IG(\nu, \lambda)$ is $\frac{\nu^3}{\lambda}$ and can also be expressed as $\frac{\sigma^2 x}{\mu^3}$.

Geometric Brownian Motion

- If $\{X(t), t \geq 0\}$ is a Brownian motion process with drift coefficient μ and variance (or diffusion) σ^2, then the process $\{Z(t), t \geq 0\}$ defined by the following expression is a Geometric Brownian motion.

$$Z(t) = z_0 \exp\{X(t)\}$$

- In contrast to a conventional Brownian motion process and even the Brownian motion with drift, the Geometric Brownian motion process is everywhere positive (with a probability 1).

Geometric Brownian Motion

- A random variable X is said to follow a lognormal distribution if $\log(X)$ is normally distributed, i.e., $\log(X) \sim \mathcal{N}(\mu, \sigma^2)$.
- At fixed time t, **the Geometric Brownian motion $\boldsymbol{Z(t) = z_0\, exp\{X(t)\}}$ has a lognormal distribution** with parameters $(\ln(z_0) + \mu t)$ and $\sigma\sqrt{t}$.
- To see this, consider the following cdf of a lognormal r.v. X:

$$F_Z(z) = P\{Z \leq z\} = P\{z_0 \exp\{\mu t + \sigma W(t)\} \leq z\}$$

$$= P\left\{\mu t + \sigma W(t) \leq \ln\left(\frac{z}{z_0}\right)\right\}$$

$$= P\left\{W(t) \leq \frac{(\ln(z/z_0) - \mu t)}{\sigma}\right\}$$

Geometric Brownian Motion

$$F_Z(z) = P\left\{W(t) \le \frac{(\ln(z/z_0) - \mu t)}{\sigma}\right\}$$

$$= P\left\{\frac{W(t)}{\sqrt{t}} \le \frac{(\ln(z/z_0) - \mu t)}{(\sigma\sqrt{t})}\right\}$$

$$= \int_{-\infty}^{\frac{(\ln(z/z_0)-\mu t)}{(\sigma\sqrt{t})}} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)\ \mathrm{d}y$$

Dividing by $\sqrt{t}$ generates a Standard Normal

➢ Differentiating with respect to z, we obtain that:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\,\sigma z\sqrt{t}} \exp\left\{-\frac{1}{2}\left[\frac{(\ln(z) - \ln(z_0) - \mu t)}{(\sigma\sqrt{t})}\right]^2\right\}$$

Geometric Brownian Motion

➢ We can calculate the mean of a Geometric Brownian motion by using the moment generating function of the normal distribution.

➢ Specifically, let $X \sim \mathcal{N}(\mu, \sigma^2)$, then the moment generating function of X is,

$$M_X(s) = \mathbb{E}[e^{sX}] = e^{\mu s + \frac{\sigma^2 s^2}{2}}$$

Verify as an exercise

➢ The mean of a Geometric Brownian motion is

$$\mathbb{E}[Z(t)] = \mathbb{E}[z_0 \exp\{\mu t + \sigma W(t)\}] = z_0 \exp\left[\mu t + \frac{1}{2}\sigma^2 t\right]$$

Geometric Brownian Motion

- To show this, we start with the following;

$$\mathbb{E}[Z(t)] = \mathbb{E}[z_0 \exp\{\mu t + \sigma W(t)\}]$$
$$= z_0 \exp(\mu t)\, \mathbb{E}[\exp\{\sigma W(t)\}]$$

$\sigma W(t) \sim \mathcal{N}(0, \sigma^2 t)$

- If we let $Y = \sigma W(t)$, then using the moment generating function of a normal random variable we get $\mathbb{E}[e^{sY}] = \exp\left(\frac{\sigma^2 t s^2}{2}\right)$.

- Thus, for $s = 1$,

$$\boxed{\mathbb{E}[Z(t)] = z_0 \exp\left(\mu t + \frac{1}{2}(\sigma^2 t)\right)}$$

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Geometric Brownian Motion

- Nex we show that the variance of the geometric Brownian motion is given as:

$$\mathbb{V}[Z(t)] = z_0 \exp(2\mu t + \sigma^2 t)[\exp(\sigma^2 t) - 1]$$

- We know that $\mathbb{V}[Z(t)] = \mathbb{E}[Z(t)^2] - \mathbb{E}[Z(t)]^2$, we start by computing

$$\mathbb{E}[Z(t)^2] = \mathbb{E}\left[z_0^2 \exp\left(\mu t + \sigma W(t)\right)^2\right]$$
$$= \mathbb{E}\left[z_0^2 \exp\left(2\mu t + 2\sigma W(t)\right)\right]$$
$$= z_0^2 \exp(2\mu t)\, \mathbb{E}\left[\exp\left(2\sigma W(t)\right)\right]$$
$$= z_0^2 \exp(2\mu t) \exp\left(\frac{4\sigma^2 t s^2}{2}\right)$$

$\sigma W(t) \sim \mathcal{N}(0, 4\sigma^2 t)$
Using the MGF

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Geometric Brownian Motion

- For $s = 1$, we get $\mathbb{E}[Z(t)^2] = z_0^2 \exp(2\mu t + 2\sigma^2 t)$, thus

$$\mathbb{V}[Z(t)] = \mathbb{E}[Z(t)^2] - \mathbb{E}[Z(t)]^2$$

$$= z_0^2 \exp(2\mu t + 2\sigma^2 t) - z_0^2 \exp\left(\mu t + \frac{1}{2}(\sigma^2 t)\right)^2$$

$$= z_0^2 \exp(2\mu t + 2\sigma^2 t) - z_0^2 \exp(2\mu t + \sigma^2 t)$$

$$= z_0^2 \exp(2\mu t + \sigma^2 t)[\exp(\sigma^2 t) - 1]$$

- Thus,

$$\boxed{\mathbb{V}[Z(t)] = z_0^2 \exp(2\mu t + \sigma^2 t)[\exp(\sigma^2 t) - 1]}$$

Section Summary

Understood properties of the Brownian motion process

Derived Conditional Distribution of the standard Brownian motion

Standard Brownian Bridge

Brownian Motion with Drift

    Derivations of Mean and Variance

Geometric Brownian Motion

    Derivations of Mean and Variance