$$\frac{dP_2(t)}{dt} = \lambda_1 e^{-(\lambda_1+\lambda_2)t} - \lambda_2 P_2(t)$$

$\frac{dy}{dx} + p(x)y(x) = q(x)$ can be made integrable by defining

$v(x) = \int p(x)\,dx$, where $e^{v(x)}$ is the integrating factor

$$y(x) = e^{-v(x)} \int e^{v(x)} q(x)\,dx$$

$y(x) = P_2(t)$

$p(x) = \lambda_2$

$v(x) = \lambda_2 t$

$e^{v(x)} = e^{\lambda_2 t}$

$q(x) = \lambda_1 e^{-(\lambda_1+\lambda_2)t}$

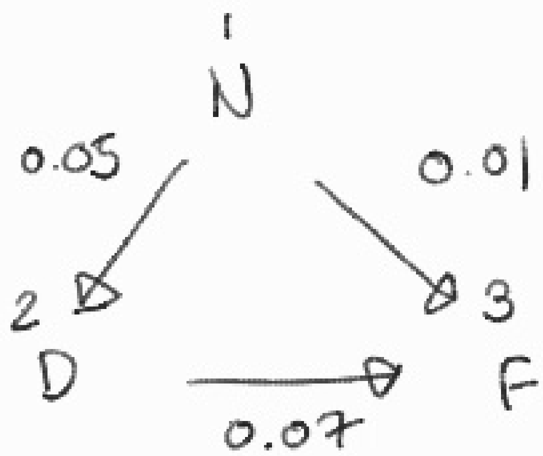
$$P_2(t) = e^{-\lambda_2 t} \lambda_1 \int e^{-(\lambda_1+\lambda_2)t} e^{\lambda_2 t}\,dt + C$$

$$P_2(t) = e^{-\lambda_2 t} \lambda_1 \int e^{-\lambda_1 t}\,dt + C$$

$$= e^{-\lambda_2 t} \lambda_1 \left(\frac{1}{-\lambda_1} e^{-\lambda_1 t} + \frac{C}{-\lambda_1} \right)$$

$$= -e^{-(\lambda_1+\lambda_2)t} - Ce^{-\lambda_2 t} \quad = e^{-\lambda_2 t} - P_1(t)$$

$$= e^{-(\lambda_1+\lambda_2)t} + Ce^{-\lambda_2 t}$$



$$P_1(t) = e^{-(0.01+0.05)t}$$

$$= e^{-0.06t}$$

$$P_2(t) = \frac{0.05}{-0.01}\left(e^{-0.07t} - e^{-0.06t}\right)$$

$$R(t) = P_1(t) + P_2(t)$$

$$MTTF = \frac{1}{\lambda_1 + \lambda_2} + \ \ldots$$

Module 1

$$R(t) = \frac{n_s(t)}{n_s(t) + n_f(t)} \quad \text{or} \quad R(t) = P(T > t)$$

$$F(t) = \text{c.d.f.} = 1 - R(t)$$

$$P(T < t)$$

$$R(0) = 1, \quad F(0) = 0, \quad \lim_{t \to \infty} R(t) = 0, \quad F(\infty) = 1$$

$$R(t) = 1 - F(t) = 1 - \int_0^t f(t')\,dt' = \int_t^\infty f(t')\,dt'$$

$$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt}$$

Eg

$$f(t) = \begin{cases} \dfrac{0.001}{(0.001t + 1)^2} & t \geq 0 \\ 0 & \text{otw.} \end{cases}$$

$$R(t) = 1 - \int_0^t \frac{0.001}{(0.001t' + 1)^2}\,dt'$$

$$= 0.001 \int_t^\infty (0.001t' + 1)^{-2}\,dt'$$

$$= -\frac{0.001}{0.001}\left[(0.001t' + 1)^{-1}\right]_{100}^{\infty}$$

$$R(100) = -\left(0 - \frac{1}{1.1}\right) = \frac{1}{1.1}$$

$$P(10 \leq t \leq 100) = R(10) - R(100)$$

$$= \int_{10}^{100} f(t')\,dt = -\left[(0.001t + 1)^{-1}\right]_{10}^{100}$$

$$= \frac{1}{1.01} - \frac{1}{1.1} = \frac{90}{1111}$$

MTTF

$$\widehat{MTTF} = \frac{1}{n}\sum_{i=1}^{n} t_i$$

$$MTTF = E[T] = \int_0^\infty t\, f(t)\, dt \quad // \quad E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$MTTF = \int_0^\infty t \times \left(-\frac{dR(t)}{dt}\right) dt \quad // \quad \int u\, dv = uv - \int v\, du$$

$$= \left[-t\, R(t)\right]_0^\infty + \int_0^\infty R(t)\, dt = \int_0^\infty R(t)\, dt$$

$$MTTF(T_0) = \int_{T_0}^\infty R(t|T_0)\, dt$$

$$R(t|T_0) = P\{T > t + T_0 \mid T > T_0\}$$

$$= \frac{P\{T > t + T_0\}}{P\{T > T_0\}} = \frac{R(t+T_0)}{R(T_0)}$$

$$MTTF(T_0) = \int_{T_0}^\infty \frac{R(t+T_0)}{R(T_0)}\, dt$$

$$= \int_{T_0}^\infty \frac{R(t+T_0)}{R(T_0)}\, d(t+T_0), \quad t' = t + T_0$$

$$= \frac{1}{R(T_0)} \int_0^\infty R(t')\, dt'$$

$$R(t) = e^{-0.002t}$$

$$MTTF = \int_0^\infty e^{-0.002t}\, dt = \left[\frac{1}{-0.002} e^{-0.002t}\right]_0^\infty$$

$$= \frac{1}{0.002} = 500$$

$$\int_{t_1}^{t_2} f(t)\, dt = R(t_1) - R(t_2) = F(t_2) - F(t_1)$$

$$FR = \frac{R(t_1) - R(t_2)}{(t_2 - t_1)\, R(t_1)} \rightarrow \frac{R(t) - R(t+\Delta t)}{\Delta t\, R(t)} \qquad f(t) = -\frac{dR(t)}{dt}$$

$$\lambda(t) = h(t) = \lim_{\Delta t \to 0} \frac{R(t) - R(t+\Delta t)}{\Delta t\, R(t)} = \frac{f(t)}{R(t)} = -\frac{1}{R(t)} \frac{dR(t)}{dt}$$

$$-\lambda(t)\, dt = \frac{1}{R(t)}\, dR(t)$$

$$-\int_0^t \lambda(t')\, dt' = \ln R(t)$$

$$R(t) = e^{-\int_0^t \lambda(t')\, dt'}$$

$$R(1|1) = \frac{R(2)}{R(1)} = 0.8402$$

$$AFR(t_1, t_2) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \lambda(t')\, dt'$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} -\frac{1}{R(t)}\, dR(t)$$

$$= \frac{\ln R(t_1) - \ln R(t_2)}{t_2 - t_1}$$

$$\frac{\frac{200}{(t+10)^3}}{\frac{100}{(t+10)^2}} = \frac{2}{t+10}$$

$$AFR(0, t) = \frac{-\ln R(t)}{t}$$

$$f(t) = \frac{200}{(t+10)^3} = 200\,(t+10)^{-3}$$

$$R(t) = \frac{200}{-2} \left[(t'+10)^{-2}\right]_t^{\infty} = \frac{100}{(t+10)^2}$$

$$R(1) = -100 \left(-\frac{1}{11^2}\right) = 0.826$$

$$MTTF = \int_0^\infty 100(t+10)^2\, dt = -100 \left[\frac{1}{t+10}\right]_0^\infty = 10$$

let $\frac{100}{(t+10)^2} = 0.95$, $t = 0.25978$ years

Exponential Distribution Module 2

This function has CFR.

$$R(t) = e^{-\int_0^t \lambda(t')\,dt'}$$

$$\lambda(t) = \lambda, \quad t \geq 0$$

Memoryless.

$$f(t) = \lambda e^{-\lambda t}$$

$$R(t|T_0) = \frac{R(t+T_0)}{R(T_0)} = R(t)$$

$$F(t) = 1 - e^{-\lambda t}$$

$$R(t) = e^{-\lambda t} = 1 - F(t)$$

$$MTTF = 1/\lambda, \quad \sigma^2 = 1/\lambda^2$$

System Hazard Rate

$$R(t) = \prod_{i=1}^{n} \exp\left[-\int_0^t \lambda_i(t')\,dt'\right]$$

$$= \exp\left[-\int_0^t \sum_{i=1}^{n} \lambda_i(t')\,dt'\right]$$

Hazard Function (λ)

$$\lambda(t) = \sum_{i=1}^{n} \lambda_i(t) = \sum_{i=1}^{n} \lambda_i$$

$$MTTF = 1 / \sum_{i=1}^{n} \lambda_i$$

Poisson

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, \ldots$$

Expected # of failures $= \lambda t$

time of k^{th} failure, $Y_k = \sum_{i=1}^{k} T_i$, where T_i is time btwn failure $i-1$ & i

$$f_Y(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{\Gamma(k)} \quad \text{for } k, \lambda, t \geq 0$$

(sum of k iid exp r.v.)

$$P(Y_k \leq t) = F_{Y_k}(t) = 1 - e^{-\lambda t} \sum_{i=0}^{k-1} \frac{(\lambda t)^i}{i!}$$

$$E(Y_k) = k/\lambda, \quad Var(Y_k) = k/\lambda^2$$

$$p_n(t) = P(Y_n \leq t) - P(Y_{n+1} \leq t)$$

$$= e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

Weibull Dist

$$\lambda(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}$$

$$R(t) = \exp\left[-\int_0^t \frac{\beta}{\theta}\left(\frac{t'}{\theta}\right)^{\beta-1} dt'\right] = e^{-\left(\frac{t}{\theta}\right)^{\beta}}$$

$$f(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} e^{-\left(\frac{t}{\theta}\right)^{\beta}}$$

scale, shape

$\theta > 0, \beta > 0, t \geq 0$

$$R(\theta) = e^{-1} = 0.368$$

$$t_R = \theta(-\ln R)^{1/\beta}$$

$$MTTF = \theta\,\Gamma(1 + 1/\beta), \quad \sigma^2 = \theta^2\left\{\Gamma\left(1+\frac{2}{\beta}\right) - \left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right\}$$

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y}\, dy, \quad \Gamma(x) = (x-1)\,\Gamma(x-1) = (x-1)!$$

Failure Modes

$$R(t) = \prod_{i=1}^{n} R_i(t)$$

$$\lambda(t) = \sum_{i=1}^{n} \frac{\beta}{\theta_i}\left(\frac{t}{\theta_i}\right)^{\beta-1} = \beta t^{\beta-1}\left[\sum_{i=1}^{n}\left(\frac{1}{\theta_i}\right)^{\beta}\right]$$

if iid: $\lambda(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}$

$$\lambda(t) = \sum_{i=1}^{n} \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}, \quad R(t) = e^{-n\left(\frac{t}{\theta}\right)^{\beta}}$$

$$R(t|T_0) = \frac{\exp\left\{-\left[\frac{(t+T_0)}{\theta}\right]^{\beta}\right\}}{\exp\left[-\left(\frac{T_0}{\theta}\right)^{\beta}\right]} = \exp\left[-\left(\frac{(t+T_0)}{\theta}\right)^{\beta} + \left(\frac{T_0}{\theta}\right)^{\beta}\right]$$

Normal Distribution

$$f(t) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\frac{(t-\mu)^2}{\sigma^2}\right], \quad F(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\frac{(t'-\mu)^2}{\sigma^2}\right] dt'$$

$$Z = \frac{T-\mu}{\sigma}, \quad \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$R(t) = P(T \geqslant t) = P\left\{\frac{T-t}{\sigma} \geqslant \frac{t-\mu}{\sigma}\right\}$$

$$= P\left\{Z \geqslant \frac{t-\mu}{\sigma}\right\} = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right)$$

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{\phi((t-\mu)/\sigma)}{1-\Phi((t-\mu)/\sigma)}$$

Lognormal

if T is lognormal, $\log T$ is normal

$$f(t) = \frac{1}{\sqrt{2\pi}\,st} \exp\left[-\frac{1}{2s^2} \ln\left(\frac{t}{t_{med}}\right)^2\right], \quad t \geqslant 0$$

	Lognormal	Normal
R.V.	T	$\ln T$
Mean	$t_{med}\, e^{s^2/2}$	$\ln t_{med}$
Var	$t_{med}^2\, e^{s^2/2}\left[e^{s^2} - 1\right]$	s^2
$R(t)$	$1 - \Phi\left(\frac{1}{s}\ln\frac{t}{t_{med}}\right)$	$1 - \Phi\left(\frac{t-\mu}{s}\right)$

$$f(t) = \frac{1}{\sqrt{2\pi}\,t\sigma_n} \exp\left[-\frac{1}{2}\frac{(\ln t - \mu)^2}{\sigma^2}\right]$$

$$R(t) = P(>t) = P\left[z > \frac{\ln t - \mu}{\sigma}\right]$$

Gamma Distribution

$$f(t) = \frac{t^{\nu-1} e^{-t/\alpha}}{\alpha^{\nu} \Gamma(\nu)} \qquad MTTF = \nu\alpha$$

$$\sigma = \sqrt{\nu\alpha^2}$$

$$F(t) = \int_0^t \frac{t'^{\,\nu-1} e^{-t'/\alpha}}{\alpha^{\nu} \Gamma(\nu)} \, dt' = \frac{I\left(\frac{t}{\alpha}, \nu\right)}{\Gamma(\nu)}$$

$$R(t) = 1 - \frac{I\left(\frac{t}{\alpha}, \nu\right)}{\Gamma(\nu)}$$

Identifying Failure Distributions

Module 3

1. Identify candidate distributions (hypothesis)
2. Estimate its parameters
3. Perform a goodness-of-fit test

$\downarrow$

1. Construct histogram of failure times
2. Compute descriptive statistics
3. Analyse failure rate
4. Use prior knowledge of failure process
5. Use properties of theoretical distributions
6. Construct a probability plot

MLE

$$f(t_1, t_2, \ldots, t_n) = f(t_1) f(t_2) \ldots f(t_n)$$

$L(\theta_1, \theta_2, \ldots, \theta_k) = \prod_{i=1}^{n} f(t_i)$, where $\theta_1, \ldots, \theta_k$ are distribution parameters