Q1.

$$
\begin{tikzcd}
\text{1 Fully-functional} \arrow[d, "\lambda_1"] \\
\text{2 Degraded state} \arrow[d, "\lambda_2"] \\
\text{3 Failed state}
\end{tikzcd}
$$

$$P_1(t+\Delta t) = P_1(t) - \lambda_1 \Delta t\, P_1(t)$$

$$P_2(t+\Delta t) = P_2(t) + \lambda_1 \Delta t\, P_1(t) - \lambda_2 \Delta t\, P_2(t)$$

$$P_3(t+\Delta t) = P_3(t) + \lambda_2 \Delta t\, P_2(t)$$

$$\frac{dP_1(t)}{dt} = -\lambda_1 P_1(t)$$

$$\frac{dP_2(t)}{dt} = \lambda_1 P_1(t) - \lambda_2 P_2(t)$$

$$\frac{dP_3(t)}{dt} = \lambda_2 P_2(t)$$

$$\ln P_1(t) = -\lambda_1 t, \quad P_1(t) = e^{-\lambda_1 t}$$

$$\frac{dP_2(t)}{dt} + \lambda_2 P_2(t) = \lambda_1 e^{-\lambda_1 t}, \quad v(x) = \int \lambda_2\, dt = \lambda_2 t + C$$

$$P_2(t) = e^{-\lambda_2 t} \int e^{-\lambda_2 t} \lambda_1 e^{-\lambda_1 t}\, dt$$

$$= ce^{-\lambda_2 t} - \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1+\lambda_2)t}$$

let $P_2(0) = 1$

$$c = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

$$P_2(t) = \frac{\lambda_1}{\lambda_1+\lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}\right]$$

$$P_3(t) = 1 - P_1(t) - P_2(t)$$

$$= 1 - e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_1+\lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}\right]$$

$$R_p(t) = P_1(t) + P_2(t)$$

$$= e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_1+\lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}\right]$$

$$MTTF = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

Q2. $D(t) = \beta_1 t$

Variance

$$\beta_1 \sim \text{LogNorm}(\mu_1, \sigma_1^2)$$

Failure: $D(t) > D_f$ & $D_f \sim \text{LogNorm}(\mu_2, \sigma_2^2)$

$$P(\beta_1 t \geq D_f) = F_T(t)$$

$$\beta_1 = e^{\mu_1 + \sigma_1 Z}, \quad D_f = e^{\mu_2 + \sigma_2 Z}, \quad \text{where } Z \sim N(0,1)$$

$$\beta_1 t \sim \text{LogNorm}(\mu_1 + \ln t, \sigma_1^2)$$

$$P(\beta_1 t \geq D_f) = P(\ln(\beta_1 t) - \ln(D_f) \geq 0)$$

let $Y = \ln(\beta_1 t) - \ln(D_f) \sim N(\mu_1 + \ln t - \mu_2, \sigma_1^2 + \sigma_2^2)$

$$F_T(t) = P(Y \geq 0)$$

$$= 1 - \Phi\left[\frac{\mu_2 - \mu_1 - \ln t}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right]$$

Q3.

Jet engine — 5 modules

23A, B, C, D, E

once daily, average 5 hours

- From among the exponential, Weibull, normal, and lognormal distributions, find a best-fit failure distribution.

From histogram & best fit line plots, the best fit failure distributions:

23A: Weibull ($\beta_A = 2.51787$, $\theta_A = 1332.4332$)

23B: Exponential ($\lambda_B = 1/347.987$)

23C: Weibull ($\beta_C = 2.089338$, $\theta_C = 500.781$)

23D: Exponential ($\lambda_D = 1/1798.82$)

23E: Weibull ($\beta_E = 1.16203$, $\theta_E = 2836.11$)

Repair time: Weibull ($\beta_R = 1.24127$, $\theta_R = 17.0590$)

- Compute performance measurements

First, calculate system hazard rate, $\lambda(t)$

$$\lambda(t) = \lambda_B + \lambda_D + \sum_{i \in \{A,C,E,R\}} \frac{\beta_i}{\theta_i} \left(\frac{t}{\theta_i} \right)^{\beta - 1}$$

$$R(t) = e^{-\int \lambda(t)\, dt}$$

$$= e^{-\left[\lambda_B t + \lambda_D t + \sum_{i \in \{A,C,E,R\}} \left(\frac{t}{\theta_i} \right)^{\beta_i} \right]}$$

Now, we can plug the values into this formula in Python to calculate the measurements:

- $R(1^{st}$ mission$) = R(5) = 0.982312$
- R (25^{th} mission $= R(125) = 0.598757$
- Median # of missions to failure = 162
- Mean time to repair $= \theta_R \Gamma\left(1 + \frac{1}{\beta_R}\right) = 15.9141$
- 90^{th} percentile of the repair time $= t_{90}$

$$F_R(t_{90}) = 0.9$$

$$t_{90} = 33.40136$$

- Which subsystem displays the worst reliability?

  Clearly, the 23B module has the worst reliability out of all the modules, based on the MTTF and the distribution/histogram of failure times.

- What should the reliability of that subsystem be to achieve a system reliability of R(10 missions) = 0.9?

  Current R(10 missions) = 0.827774

  Need to solve for λ_B^* in the equation below:

$$0.9 = e^{-\left[\lambda_B^* t + \lambda_D t + \sum_{i \in \{A,C,E,R\}} \left(\frac{t}{\theta_i}\right)^{\beta_i}\right]}$$

  Use the Python solver, we get a $\lambda_B^* = 1/832.92989$

  Reliability of 23B $= R_B(t) = e^{-\frac{t}{832.92989}}$

$$R_B(10 \text{ missions}) = 0.941737$$

Q4. a. Using TTF info (only):

Solve using MLE on pdf of 15 iid Weibull r.v.s

$$L(\beta, \theta) = \prod_{i \in F} f(t_i) \prod_{i \in C} R(120\,000),$$

where $F = \{1, 2, 3, 5, 6, 7, 8, 9, 10\}$, (specimens with complete failure data)

$C = \{4, 11, 12, 13, 14, 15\}$, (specimens w/ censored failure data)

$$f(t_i) = \frac{\beta}{\theta}\left(\frac{t_i}{\theta}\right)^{\beta-1} e^{-\left(\frac{t_i}{\theta}\right)^{\beta}}, \theta > 0, \beta > 0,$$

$$R(120\,000) = 1 - F(120\,000) = e^{-(120\,000/\theta)^{\beta}}$$

$$\ln\left[L(\beta, \theta)\right] = 6 \ln\left[1 - F(120\,000)\right] + \sum_{i \in F} \ln f(t_i)$$

Using MLE solver on R:

$\theta = 166420.226$

$\beta = 1.07888043$

$$F(t) = 1 - R(t), \quad R(t) = e^{-\left(\frac{t}{\theta}\right)^{\beta}}$$

let $F(t) = \alpha$

$$e^{-(t/\theta)^{\beta}} = 1 - \alpha$$

$$(t/\theta)^{\beta} = -\ln(1-\alpha)$$

$$\beta\left(\ln t - \ln \theta\right) = \ln\left(-\ln(1-\alpha)\right)$$

$$t = e^{\frac{\ln(-\ln(1-\alpha))}{\beta} + \ln \theta}$$

$F(20669.94) = 0.1$

$F(52442.04) = 0.25$

$F(118486.6) = 0.5$

$F(225263.1) = 0.75$

$F(360528) = 0.9$

b. Assume that the degradation signals follow a linear degradation path:

$D_f = \beta_1 + \beta_2 t$

$\beta_1 \sim \mu_{\beta_1}$, $\beta_2 \sim N(\mu_{\beta_2}, \sigma_{\beta_2}^2)$

$D_f \sim N(\mu_{\beta_1} + \mu_{\beta_2} t, \sigma_{\beta_2}^2 t^2)$

$F_T(t) = P(\beta_1 + \beta_2 t \geq D_f)$

$P(\beta_2 \leq x) = \Phi\left[\frac{x - \mu_{\beta_2}}{\sigma_{\beta_2}}\right]$

$$F_T(t) = P\{T \leq t\} \approx \Phi\left[\frac{t - \frac{D_f - \mu_{\beta_1}}{\mu_{\beta_2}}}{\frac{\sigma_{\beta_2} t}{\mu_{\beta_2}}}\right]$$

From numerical estimation using the data,

$\beta_1 \sim \mu_{\beta_1} = 8.39793572$

$\beta_2 \sim N(5.2731428 \times 10^{-5}, (8.9685798 \times 10^{-6})^2)$

$$\frac{t - \frac{D_f - \mu_{\beta_1}}{\mu_{\beta_2}}}{\frac{\sigma_{\beta_2} t}{\mu_{\beta_2}}} = N_\alpha, \quad t - \frac{D_f - \mu_{\beta_1}}{\mu_{\beta_2}} = \frac{N_\alpha \sigma_{\beta_2} t}{\mu_{\beta_2}}$$

$$t = \frac{\frac{D_f - \mu_{\beta_1}}{\mu_{\beta_2}}}{1 - \frac{N_\alpha \sigma_{\beta_2}}{\mu_{\beta_2}}}$$

$F(102\ 795.66) = 0.1$

$F(112\ 316.98) = 0.25$

$F(125\ 201.70) = 0.5$

$F(141\ 425.69) = 0.75$

$F(160\ 097.66) = 0.9$

c. Assume the same linear degradation path, use approximate degradation path to estimate pseudo failure times.

$$S_{ij} = D_{ij} + \varepsilon_{ij}$$

Pseudo random failure times = 108 266.44, ..., 146 556.57

After estimating the 15 pseudo random failure times, the estimated Weibull distribution has parameters:

$\theta = 138\ 468.62$

$\beta = 5.7392593$

$F(93\ 554.40) = 0.1$

$F(111\ 448.22) = 0.25$

$F(129\ 902.36) = 0.5$

$F(146\ 577.75) = 0.75$

$F(160\ 126.53) = 0.9$

d. Bootstrapping:

$\beta_1 \sim$ constant, μ_{β_1}

$\beta_2 \sim N(\mu_{\beta_2}, \sigma_{\beta_2}^2)$

From numerical estimation of the 15 paths

$$\left.\begin{array}{l} \beta_1 \sim \mu_{\beta_1} = 8.39793572 \\ \beta_2 \sim N(5.2731428 \times 10^{-5}, (8.9685798 \times 10^{-6})^2) \end{array}\right\} \hat{\theta}_\beta$$

$$F_T(t) = p\{T \leq t\} \approx \Phi\left[\frac{t - \frac{D_f - \mu_{\beta_1}}{\mu_{\beta_2}}}{\frac{\sigma_{\beta_2} t}{\mu_{\beta_2}}}\right]$$

$\hat{F}(102\,795.66) = 0.1$

$\hat{F}(112\,316.98) = 0.25$

$\hat{F}(125\,201.70) = 0.5$

$\hat{F}(141\,425.69) = 0.75$

$\hat{F}(160\,097.66) = 0.9$

$S_{ij}^* = D(t_{ij}; \beta_1^*, \beta_2^*) + \varepsilon_{ij}^*$, where for each i, there will be 35 equally spaced t_{ij} points from 0 to 120 000, and $\varepsilon_{ij}^* \sim N(0, 0.63382^2)$ which is the standard dev. of the residual data points.

Set $B = 50{,}000$

After running algorithm to calculate Bootstrap Confidence Intervals at $B = 50{,}000$ & $\alpha = 0.1$, the confidence intervals are as follows:

$$\hat{F}(102\,795.66) = 0.1$$
$$[\underline{F}(102\,795), \overline{F}(102\,795)] = [7.7561 \times 10^{-5}, 0.27012]$$

$$\hat{F}(112\,316.98) = 0.25$$
$$[\underline{F}(112\,316), \overline{F}(112\,316)] = [0.016758, 0.49062]$$

$$\hat{F}(125\,201.70) = 0.5$$
$$[\underline{F}(125\,201), \overline{F}(125\,201)] = [0.14228, 0.85985]$$

$$\hat{F}(141\,425.69) = 0.75$$
$$[\underline{F}(141\,425), \overline{F}(141\,425)] = [0.49426, 0.98548]$$

$$\hat{F}(160\,097.66) = 0.9$$
$$[\underline{F}(160\,097), \overline{F}(160\,097)] = [0.86178, 0.99901]$$