Module 6

- Markov processes w/ continuous state $\rightarrow$ diffusion processes
- Standard Brownian motion : $\overset{B(t)}{\{W(t) : t \geq 0\}}$ is a stochastic process s.t. :
  - $t \rightarrow W(t)$ is continuous
  - if $t_0 < t_1 \dots < t_n$ then, $W(t_0), W(t_1) - W(t_0), \dots, W(t_n) - W(t_{n-1})$ are mutually independent (increments / Δ)
  - if $s, t \geq 0$, then :

$$P(W(s+t) - W(s) \in A) = \int_A (2\pi t)^{-1/2} e^{-x^2/2t} \, dx$$

Note: $W(t) - W(s)$ depends only on $t-s$

- $W(t) \sim N(0,t)$, $W(0) = 0$
- $W(t) - W(s) \sim W(t-s) - W(0)$
  - where $W(t-s) \sim N(0, t-s)$ for $s \leq t$
- $E[W(t+s) \mid F_t] = W(t)$ Martingale
- Brownian motion is Gaussian process where $E[W(t)] = 0$ & for $s \leq t$

$$\begin{aligned} Cov(W_s, W_t) &= Cov(W_s, W_s + W_t - W_s) \\ &= Cov(W_s, W_s) + Cov(W_s, W_{t-s}) \\ &= V(s) + 0 = s \quad \min(s,t) \end{aligned}$$

$Cov(W_{s-0}, W_{t-s}) = 0$ independent increments

Conditional Distribution of Brownian Motion

$E[W_t \mid W_s] = ?$, $Var[W_t \mid W_s] = ?$

- Claim $\to$ $W_t - \left(\frac{t}{s}\right) W_s$ is independent of W_s given $t < s$

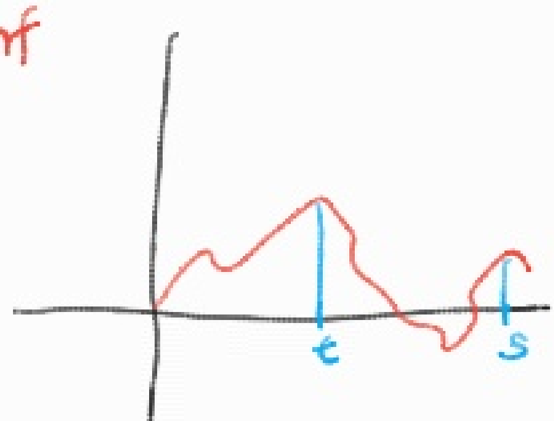
$$Cov\left(W_t - \left(\tfrac{t}{s}\right) W_s,\ W_s\right) = Cov(W_t, W_s) - \left(\tfrac{t}{s}\right) Cov(W_s, W_s)$$
$$= t \wedge s - \frac{t}{s}(s \wedge s)$$
$$= t - \frac{t}{s}(s) = 0 \quad \text{(since } t < s\text{)}$$

- Note: Gaussian r.v. are indpt if $Cov = 0$

- $E\left[W_t - \left(\tfrac{t}{s}\right) W_s\right] = E[W_t] - \left(\tfrac{t}{s}\right) E[W_s] = 0 - 0 = 0$

- $E\left[W_t - \left(\tfrac{t}{s}\right) W_s \,\middle|\, W_s\right] = 0$, since independent

$$E[W_t \mid W_s] - \left(\tfrac{t}{s}\right) W_s = 0$$
$$E[W_t \mid W_s] = \left(\tfrac{t}{s}\right) W_s$$



Conditional mean of SBM is the linear interpolation btwn pts t & s

- Find $V[W_t \mid W_s] = E\left[(W_t - E(W_t \mid W_s))^2 \mid W_s\right]$, $Var(X) = E[(X - E[X])^2]$ law of conditional exp

$$= E\left[\left(W_t - \left(\tfrac{t}{s}\right) W_s\right)^2 \middle| W_s\right] \quad \because W_t - \left(\tfrac{t}{s}\right) W_s \perp\!\!\!\perp W_s$$
$$= E\left[W_t^2 - 2\left(\tfrac{t}{s}\right) W_s W_t + \left(\tfrac{t}{s}\right)^2 W_s^2\right]$$
$$= t \wedge t - 2\left(\tfrac{t}{s}\right) t \wedge s + \left(\tfrac{t}{s}\right)^2 s \wedge s \quad t < s$$
$$= t - 2\frac{t^2}{s} + \frac{t^2}{s} = \frac{t(s-t)}{s}$$

approaches 0 at $t = 0$, $t = s$

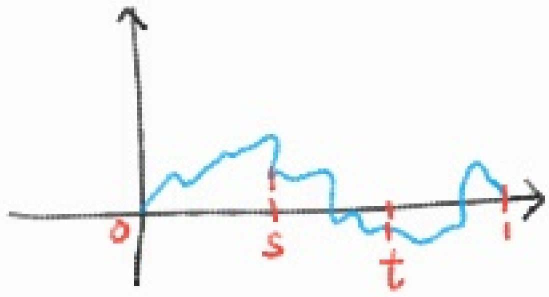
$$(W_t \mid W_s) \sim N\left(\frac{t}{s} W_s,\ \frac{t(s-t)}{s}\right) \text{ for } 0 < t < s$$

Standard Brownian Bridge

$W(0)=0$ is presumed $\forall$ SBM

- A standard Brownian bridge over the interval $[0,1]$ is given by the expression

$\{W(t), 0 \le t \le 1 \mid W(1)=0\}$



- $E[W_t \mid W_1=0] = \frac{t}{1}(0) = 0$

- Brownian bridge is Gaussian w/ Cov:

$$Cov(W_s, W_t \mid W_1=0) = E\left[(W_t - E(W_t))(W_s - E(W_s)) \mid W_1=0\right]$$

$$= E[W_t W_s \mid W_1=0]$$

$$E[W_t \mid W_s] = \left(\frac{t}{s}\right) W_s$$

$$= E\left[W_t E[W_s \mid W_t] \mid W_1=0\right]$$

$$= E\left[W_t \frac{s}{t} W_t \mid W_1=0\right]$$

$$= \frac{s}{t} E[W_t^2 \mid W_1=0] \rightarrow Var(W_t \mid W_s) = \frac{t(s-t)}{s}$$

$$= \frac{s}{t} \times \frac{t(1-t)}{1} = s(1-t)$$

First-Passage Time of Standard Brownian Motion

- Let $W(t)$ be SBM. The first passage time to a barrier defined $b>0$ is given by:

$$\tau_b = \inf\{t : W_t \ge b\}$$

- Interested in finding $P\{\tau_b \le t\}$

- After crossing, you can either be above or below b:

equally likely

$$P\{\tau_b \le t\} = P\{\tau_b \le t, W_t < b\} + P\{\tau_b \le b, W_t > b\}$$

- By path continuity, $\{\tau_b \le t, W_t > b\} = \{W_t > b\}$

$$P\{\tau_b \le t\} = P\{\tau_b \le t, W_t < b\} + P\{W_t > b\}$$

$$= \underline{P\{W_t < b \mid \tau_b \le t\}}\, P\{\tau_b \le t\} + P\{W_t > b\}$$

$\frac{1}{2}$

$$= \frac{1}{2} P\{\tau_b \le t\} + P\{W_t > b\}$$

- Path continuity ensures that $W_{\tau_b} = b$

$W_t \sim N(0,t\}$, $\therefore P\{W_t > b\} = 1 - \Phi\left(\frac{b}{\sqrt{t}}\right)$

$$P\{\tau_b \leq t\} = 2 \times P\{W_t > b\} = 2\left(1 - \Phi\left(\frac{b}{\sqrt{t}}\right)\right)$$

Brownian Motion with Drift

- $\{X(t), t > 0\}$ is a BMD μ & variance (diffusion) σ^2 if:
  - $X(0) = 0$
  - $\{X(t), t > 0\}$ has stationary & independent increments
  - $X(t) \sim N(\mu t, t\sigma^2)$

- $X(t) = X(0) + \mu t + \underbrace{\sigma W(t)}_{\text{SBM}}$

- First passage time of BMD:

$$P\{\tau \leq t\} = 1 - \Phi\left(\frac{b - \mu t}{\sqrt{t}}\right) + 2e^{2\mu b}\,\Phi\left(\frac{b - \mu t}{\sqrt{t}}\right)$$

known as the Inverse Gaussian distribution

Mean of BMD

- for a positive constant x, let $T = \inf(t : X(t) = x)$ Convert BMD → SBM

$$T = \inf\left\{t : W(t) = \frac{x - \mu t}{\sigma}\right\}$$

- If T is the stopping time of a Martingale $Y(t)$, then we know that $E[Y(T)] = E[Y(0)]$

- Similarly, $E[W(T)] = E[W(0)]$, thus $E[W(T)] = E\left[\frac{x - \mu T}{\sigma}\right] = 0$

- We can see that the mean of the FPT of BMD is:

$$E[T] = \frac{x}{\mu}$$

Variance of BMD

Martingale : $W_T^2 - T$

- If we let T be the stopping time for a BMD, then $E[W_T^2 - T] = 0$.

- Recall $W_t = \frac{X(t) - \mu t}{\sigma}$ $\quad X(0) = 0,\ X(t) = X(0) + \mu t + \sigma W_t$

$$E\left[\left(\frac{X(t) - \mu t}{\sigma}\right)^2 - T\right] = 0$$

- since $X(T) = x$

$$E\left[\left(\frac{x - \mu T}{\sigma}\right)^2 - T\right] = 0 \rightarrow E[(x - \mu \overset{\frac{1}{\mu^2}}{T})^2] = \sigma^2 E[\overset{\frac{x}{\mu}}{T}]$$

$$\boxed{V(T) = \frac{\sigma^2 x}{\mu^3}}$$

FPT BMD in terms of threshold x & params μ, σ^2

Recalling Martingales

"Markovian"

- The expected value of the process conditioned on the info at time s of where the process will be at time $(t+s)$ = where the process currently is at s

$$E[W_t | W_u, 0 \le u \le s] = E[W_s + W_t - W_s | W_u, 0 \le u \le s]$$

$$= E[W_s | W_u \ldots] + \underset{\overset{\shortparallel}{0}}{E[W_t - W_s | W_u \ldots]} \sim N(0, (t-s))$$

$$= W_s$$

- To show $W_t^2 - t$ is Martingale, we need to prove

$$E[W_t^2 - t | W_u, 0 \le u \le s] = W_s^2 - s$$

- $E[W_t^2 | W_u \ldots] = E[(W_s + W_t - W_s)^2 | W_u \ldots]$

$$= E[W_s^2 | W_s] - 2E[W_s | W_s] E[W_t - W_s | W_s] + \overset{\text{Var}}{E[(W_t - W_s)^2 | W_s]}$$

$$= W_s^2 + (t - s)$$ can be rearranged

- Since $W_t^2 - t$ is a martingale, then at $t = 0$, $E[W_t^2 - t] = E[W_0^2] = 0$

Distribution of the First Passage Time (FPT)

- T_1 is called an inverse Gaussian denoted as $IG(\nu, \lambda)$ where ν is the mean parameter, λ is the shape param, and the variance is $\frac{\nu^3}{\lambda}$

$$f(t;\nu,\lambda) = \sqrt{\frac{\lambda}{2\pi t^3}} \exp\left(-\frac{\lambda(t-\nu)^2}{2\nu^2 t}\right)$$

$$F(t;\nu,\lambda) = \Phi\left(\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\nu}-1\right)\right) + \exp\left(\frac{2\lambda}{\nu}\right)\Phi\left(-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\nu}+1\right)\right)$$

Relationship w/ BM

- For BMD: $X(t) = X(0) + \mu t + \sigma W(t)$

$$\nu = x/\mu, \quad \lambda = x^2/\sigma^2, \quad \frac{\nu^3}{\lambda} = \frac{\sigma^2 x}{\mu^3}$$

Geometric Brownian Motion

- if $\{X(t), t \geq 0\}$ is a BMD μ, σ^2, then $\{Z(t), t \geq 0\}$ is a Geometric BM s.t.

$$Z(t) = z_0 \exp\{X(t)\}$$ strictly positive

- lognormal X : $\log(x) \sim N(\mu, \sigma^2)$
- at fixed time t, $Z(t) = z_0 \exp\{X(t)\}$ is lognormal w/ param $(\ln(z_0)+\mu t)$ & σt
- Consider c.d.f. of a lognormal r.v. X:

$$\begin{aligned}
F_Z(z) = P\{Z < z\} &= P\{z_0 \exp\{\mu t + \sigma W(t)\} \leq z\} \\
&= P\{\mu t + \sigma W(t) \leq \ln\left(\tfrac{z}{z_0}\right)\} \\
&= P\left\{W(t) \leq \frac{(\ln(z/z_0) - \mu t)}{\sigma}\right\} \\
&= P\left\{\frac{W(t)}{\sqrt{t}} \leq \frac{(\ln(z/z_0) - \mu t)}{\sigma\sqrt{t}}\right\} \\
&= \int_{-\infty}^{\frac{(\ln(z/z_0)-\mu t)}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)\, dy
\end{aligned}$$

dividing by $\sqrt{t}$ generates a standard normal

$W(t) \sim N(0, t)$

Differentiating wrt z,

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\,\sigma x\sqrt{t}} \exp\left\{-\frac{1}{2}\left[\frac{\ln(z)-\ln(z_0)-\mu t}{(\sigma\sqrt{t})}\right]^2\right\}$$ lognormal

- let $X \sim N(\mu, \sigma^2)$

$$M_x(s) = E[e^{sX}] = e^{\mu s + \frac{\sigma^2 s^2}{2}}$$ Moment generating fn

- Mean of GBM: $E[Z(t)] = E[z_0 \exp\{\mu t + \sigma W(t)\}] = z_0 \exp[\mu t + \frac{1}{2}\sigma^2 t]$

$$E[Z(t)] = E[z_0 \exp\{\mu t + \sigma W(t)\}]$$
$$= z_0 \exp(\mu t) E[\exp\{\sigma W(t)\}]$$ $\sigma W(t) \sim N(0, \sigma^2 t)$

- if we let $Y = \sigma W(t)$, then using MGF of N, $E[e^{sY}] = \exp(\frac{\sigma^2 t s^2}{2})$
- Thus, for $s=1$, $\boxed{E[Z(t)] = z_0 \exp(\mu t + \frac{1}{2}(\sigma^2 t))}$

<u>Variance</u> $V[Z(t)] = z_0 \exp(2\mu t + \sigma^2 t)[\exp(\sigma^2 t) - 1]$

$$V[Z(t)] = E[Z(t)^2] - E[Z(t)]^2$$

$$E[Z(t)^2] = E[z_0^2 \exp(\mu t + \sigma W(t))^2]$$ $\exp(X(t)) = \exp(0) = 1$ 0

$$= E[z_0^2 \exp(2\mu t + 2\sigma W(t))]$$ $\sim N(0, 4\sigma^2 t)$

$$= z_0^2 \exp(2\mu t) E[\exp(2\sigma W(t))]$$

$$= z_0^2 \exp(2\mu t) \exp\left(\frac{4\sigma^2 t s^2}{2}\right)$$

sub $s=1$, $E[Z(t)^2] = z_0^2 \exp(2\mu t + 2\sigma^2 t)$

$$V[Z(t)] = z_0^2 \exp(2\mu t + 2\sigma^2 t) - z_0^2 \exp(2\mu t + \sigma^2 t)$$
$$= z_0^2 \exp(2\mu t + \sigma^2 t)[\exp(\sigma^2 t) - 1]$$