Real-Time Estimation of Mean Remaining Life Using Sensor-Based Degradation Models

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Advances in sensor technology have led to an increased interest in using degradation-based sensory information to predict the remaining lives of partially degraded components and systems. This paper presents a stochastic degradation modeling framework for computing and continuously updating remaining life distributions (RLDs) using in situ degradation signals acquired from individual components during their operational lives. Unfortunately, these sensory-updated RLDs cannot be characterized using parametric distributions and their moments do not exist. Such difficulties hinder the implementation of this sensor-based framework, especially from the standpoint of computational efficiency of embedded algorithms. In this paper, we identify an approximate procedure by which we can compute a conservative mean of the sensory-updated RLDs and express the mean and variance using closed-form expressions that are easy to evaluate. To accomplish this, we use the first passage time of Brownian motion with positive drift, which follows an inverse Gaussian distribution, as an approximation of the remaining life. We then show that the mean of the inverse Gaussian is a conservative lower bound of the mean remaining life using Jensen's inequality. The results are validated using real-world vibration-based degradation information. [DOI: 10.1115/1.3159045]

1 Introduction

Components and devices usually degrade during their operational lifetime. Most degradation processes are accompanied by reduced performance levels, which result in lower quality, reduced safety, and catastrophic failure. Consequently, accurate prediction of the remaining life of equipment is key to minimizing unexpected failures, eliminating unnecessary maintenance, and ensuring maximum asset utilization.

Degradation processes are typically accompanied by specific physical phenomena that evolve over time as degradation progresses. In a few cases, it is possible to observe these physical phenomena directly. However, in the majority of applications, degradation is observed indirectly by monitoring specific degradation measures, such as temperature changes and crack propagation. These measures can be observed using condition monitoring techniques [1–4], and are generally correlated with the underlying state of degradation. In most cases, these measures evolve with degradation according to characteristic patterns commonly known as degradation signals.

Degradation modeling has been widely used to predict components' remaining life. This is achieved by modeling the path followed by the degradation signal. The remaining life is defined as the time taken by the evolving degradation signal to cross a predetermined failure threshold. The challenge lies in the fact that the evolution of the degradation signal is typically stochastic in nature. Thus, it is very difficult—if at all possible—to predict the remaining life with certainty. Research efforts focus on predicting the remaining life distributions (RLDs) of operating components.

A variety of approaches have been followed in the literature on degradation modeling. One approach utilizes random coefficient models with deterministic and stochastic parameters and random error terms to model degradation signals [5–8]. The deterministic parameters capture degradation phenomena that are constant across components' populations (for example, an initial level of degradation). The random parameters, on the other hand, capture unit-to-unit variability. Error terms capture environmental and signal noise and fluctuations. The remaining life distribution is computed as the distribution of the time required for the signal trajectory to cross a given failure threshold. Another approach focuses on modeling the degradation signals using Brownian motion [9–12]. The remaining life distribution in this case is computed as the distribution of the first passage time of the signal to the failure threshold.

Very few research efforts focus on using sensory signals to revise the RLD of components as they continue to operate. Gebraeel and co-workers [4,13] developed sensor-based degradation models to predict the RLDs of degraded components and provided the capability of updating these distributions using real-time sensory signals. However, the authors were only able to determine approximate expressions for the RLDs. Given a partially observed degradation signal, the RLD was estimated as the distribution of the time it takes the trajectory of the degradation signal to cross the failure threshold. In reality, this is an approximation because the fluctuations of the signal may have already crossed the failure threshold, signifying failure, prior to predicting the remaining life. In other words, the RLDs computed in Refs. [4,13] do not represent the first passage time. In cases where the signal fluctuations are large, this approximation can be significantly imprecise. Furthermore, the obtained RLDs were similar to the Bernstein distribution [14]. Consequently, the expectation of the remaining life did not exist. To address this problem, the authors estimated the remaining life using the median of the RLD.

In this paper, we utilize recently developed sensor-based degradation models to compute approximations of the RLDs that have closed-form expressions for the mean and variance, as opposed to the RLDs in Refs. [4,13]. To do this, we model the degradation signal using a random coefficient model with error terms that follows a Brownian motion process. Using the properties of the Brownian error term, we show that the inverse Gauss-

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ian (IG) distribution provides a good approximation of the RLD. We also present a Bayesian methodology that enables revising the RLD using real-time degradation signals acquired from fielded components.

The paper is organized as follows. In Sec. 2, we survey some of the literature related to degradation modeling. In Sec. 3, we present sensor-based degradation models with linear and exponential functional forms and Brownian error terms. A Bayesian updating methodology for revising the RLDs in real time is presented and illustrated. In Sec. 4 we obtain a conservative lower bound of the mean remaining life using the properties of Brownian motion with positive drift. The proposed methodology is validated in Sec. 5 using real-world data. Section 6 compares the prediction results of the proposed models with two benchmarks from the literature. Finally, Sec. 7 provides a short discussion, conclusions, and directions for future research.

2 Literature Review

Degradation models utilize sensory signals from condition monitoring to model the path followed by degradation signals. This modeled path can be used to compute RLDs of degraded components. Lu and Meeker [5] developed a random coefficients model to estimate the failure time distribution based on degradation data from a population of components. The authors presented some special cases where closed-form expressions can be obtained for the failure time distribution. Robinson and Crowder [15] modeled fatigue crack growth using a nonlinear regression model with random coefficients. Two-stage least-squares, maximum likelihood principles, and Bayesian approaches were used to estimate the model parameters. Yang and Yang [7] developed a random coefficients approach that uses the lifetimes of failed devices plus degradation information from operating devices to estimate the parameters of lifetime distributions. Bae and Kvam [16] presented nonlinear random coefficients models to characterize the degradation of vacuum fluorescent displays (VFDs). Padgett and Tomlinson [17] used failure times and degradation measures from accelerated degradation testing to make inferences about the system's lifetime. The authors modeled the degradation of the system as a Gaussian process.

Many researchers used Brownian motion to model degradation signals. Doksum and Hoyland [9] modeled the accumulated decay of cable insulation for units subject to accelerated testing under variable stress levels. Accumulated decay was modeled as a Brownian motion with drift. The resulting failure time followed the inverse Gaussian distribution. The authors considered cases with multiple varying stress levels, and the special case with only two varying stress levels. Whitmore and Schenkelberg [10] modeled the degradation of self-regulating heat cables subject to high stress reliability testing. The degradation process was also modeled as a Brownian motion with a constant rate of degradation. Pettit and Young [12] used a combination of failure times and degradation data from testing a sample of components to obtain the failure time distribution of component populations. Brownian motion with drift and diffusion was used to model degradation. A Bayesian approach was then used to compute the posterior densities of the drift and diffusion given data sets obtained from reliability testing for a fixed duration. The authors used Gibbs sampler to obtain the posterior distributions and the predictive distribution of the failure time.

Other approaches were also followed in the degradation modeling literature. Bae et al. [18] discussed additive and multiplicative degradation models to derive the lifetime distribution of degraded components. Park and Padgett [19] developed stochastic degradation models that incorporate several accelerating variables. Exact likelihood functions were determined for the degradation paths and used to obtain lifetime distributions. Kharoufeh and Cox [20] presented a hybrid approach for estimating the remaining life distribution of a single-unit system. The degradation rate of the system was assumed to be random and dependent on external factors in the operating environment. The evolution of these factors was modeled as a continuous-time Markov chain. Tang and Chang [21] proposed a framework to determine the reliability of components using accelerated degradation data. The authors modeled the degradation data as a collection of realizations of stochastic processes and used the Birnbaum–Saunders distribution to characterize the failure time distribution. This was motivated by the authors' claim that the Birnbaum–Saunders distribution is simpler than the generalized inverse Gaussian distribution.

We note that very few research efforts utilized condition-based sensory signals to revise the RLD of components based on their current states of degradation. The recent models developed by Gebraeel et al. [6] used a Bayesian updating methodology to predict and continuously update the RLDs of individual components using real-time sensory signals. The degradation signal was modeled using a random coefficients exponential model. The stochastic parameters were assumed to be independent. In Ref. [13], Gebraeel extended this work by assuming dependency of the stochastic model parameters. As discussed earlier, in both of these articles the authors only obtained approximate expressions for the RLD that did not have closed-form moments.

Rather than modeling the degradation signal as a Brownian motion, we model its evolution using a random coefficients model with Brownian error terms. This enables us to estimate the remaining life as the first passage time of the signal to the failure threshold. Unlike conventional degradation models, we utilize real-time degradation signals from degraded components to update the distributions of the random coefficients in our degradation models. The updated posterior parameters are then used to revise the mean and variance of the inverse Gaussian RLD. This approach is shown to result in good accuracy of failure prediction.

3 Sensor-Based Degradation Models

In this section, we present random coefficients degradation models to characterize the evolution of degradation signals. We discuss two different functional forms to model degradation and obtain expressions for the updated RLDs in both cases.

Let $\mathbf{S}=\{S(t_k),t_k>0\}$ be a continuous-time stochastic process, where $S(t_k)$ denotes the value of the degradation signal at time t_k, $k=0,1,\ldots$. This signal evolves according to the model $S(t_k)=h(t_k;\phi,\theta)+\varepsilon(t_k)$. The term $h(\cdot)$ represents the functional form characterizing the evolution of the signal $S(t_k)$. The choice of this functional form depends on the type of component being modeled and can take a variety of forms For example, in applications where preliminary degradation stages do not accelerate the degradation process, a linear functional form may be suitable. The wear of break pads on automotive wheels is a good example. Initial wear does not speed up subsequent wear of the brake pads. In contrast, the formation of spalls (pits) on the surface of bearing raceways is an initial form of bearing degradation. However, these spalls become weak points and increase the rate of subsequent degradation. The parameter ϕ is deterministic and captures constant degradation characteristics over the components' population, whereas θ is a stochastic parameter that captures random degradation characteristics of the individual components being monitored. The term $\varepsilon(t_k)$ is an error term used to model measurement noise and signal fluctuations.

Most previous research efforts assumed the error terms to be independent and identically normally distributed with mean zero and variance σ^2 across the population of components [5,6,13]. Other research efforts used Brownian motion to model the error terms [9,10,22]. Generally, Brownian error terms are more appropriate for applications where successive error fluctuations in sensor readings are correlated. In this paper, we assume that the error term follows a Brownian motion with mean zero and variance parameter σ^2. In Secs. 3.1 and 3.2, we present two base-case models, the linear and exponential models.

3.1 Linear Degradation Model. The degradation signal model can be written as follows:

$$S(t_k) = \phi + \theta t_k + \varepsilon(t_k) \tag{1}$$

where $S(t_k)$ is the value of the signal at time t_k, $k=0,1,2,\ldots$, ϕ is a constant deterministic parameter, θ is a random variable following a prior normal distribution $\pi(\theta)$ with mean μ_o and variance σ_o^2, and $\varepsilon(t_k)$ is the error term that follows a Brownian motion with mean zero and variance parameter σ^2. The next step is to update the parameters of the prior distribution using real-time sensory degradation signals as shown next.

3.1.1 Sensory Updating of the Prior Distribution Parameters. A Bayesian updating methodology is used to perform the sensory-updating process. First, we let S_i denote the increments of observed signal values at times t_{i-1} and t_i.

$$S_i = S(t_i) - S(t_{i-1}) = \theta(t_i - t_{i-1}) + \varepsilon(t_i) - \varepsilon(t_{i-1}) \tag{2}$$

We update the prior parameters by computing the conditional distribution of the stochastic parameter θ given the observed signal values up to time t_k, $P(\theta|S_1,\ldots,S_k)$. This can be computed using Bayes' theorem

$$P(\theta|S_1,\ldots,S_k) \propto f(S_1,\ldots,S_k|\theta)\pi(\theta) \tag{3}$$

The conditional probability of the observed signals given θ, $f(S_1,\ldots,S_k|\theta)$, can be computed from Eq. (4). Note that by the properties of Brownian motion, the error increments $\varepsilon(t_i) - \varepsilon(t_{i-1})$ are independent normal random variables. Thus, we have the following:

$$f(S_1,\ldots,S_k|\theta) = \frac{1}{\prod_{i=1}^{k}\sqrt{2\pi\sigma^2(t_i - t_{i-1})}} \times \exp\left(-\sum_{i=1}^{k}\frac{(S_i - \theta(t_i - t_{i-1}))^2}{2\sigma^2(t_i - t_{i-1})}\right) \tag{4}$$

We can now evaluate $P(\theta|S_1,\ldots,S_k)$ to compute the posterior distribution parameters of θ as follows:

$$\begin{aligned} P(\theta|S_1,\ldots,S_k) &\propto f(S_1,\ldots,S_k|\theta)\pi(\theta) \propto \exp\left(-\sum_{i=1}^{k}\frac{(S_i - \theta(t_i - t_{i-1})^2)}{2\sigma^2(t_i - t_{i-1})}\right)\exp\left(\frac{-1}{2\sigma_o^2}(\theta - \mu_o^2)^2\right) \\ &\propto \exp\left(-\frac{1}{2\sigma^2\sigma_o^2}[\theta^2(t_k\sigma_o^2 + \sigma^2) - 2\theta(\mu_o\sigma^2 + (S(t_k) - \phi)\sigma_o^2)]\right) \\ &\propto \exp\left(-\frac{(t_k\sigma_o^2 + \sigma^2)}{2\sigma^2\sigma_o^2}\left[\theta^2 - 2\theta\left(\frac{\mu_o\sigma^2 + (S(t_k) - \phi)\sigma_o^2}{t_k\sigma_o^2 + \sigma^2}\right)\right]\right) \end{aligned} \tag{5}$$

Since the parameter θ is assumed to follow a normal distribution, its posterior distribution given the observed data is also normal taking the following form:

$$P(\theta|S_1,\ldots,S_k) = \frac{1}{\sqrt{2\pi\sigma_\theta^2}}\exp\left(-\frac{1}{2\sigma_{\theta,t_k}^2}(\theta - \mu_{\theta,t_k})^2\right) \tag{6}$$

Finally, we obtain the following expressions for the posterior parameters by comparing the last two expressions yielding

$$\mu_{\theta,t_k} = \frac{\mu_o\sigma^2 + (S(t_k) - \phi)\sigma_o^2}{t_k\sigma_o^2 + \sigma^2} \tag{7}$$

$$\sigma_{\theta,t_k}^2 = \frac{\sigma^2\sigma_o^2}{t_k\sigma_o^2 + \sigma^2} \tag{8}$$

3.1.2 Computing the Remaining Life Distribution. Given $S_1,\ldots,S_k$ are the observed increments of the degradation signal up to time t_k, we define the random variable $S(t_k+t)$ as the value of the degradation signal after t time units. The mean and variance of the random variable $S(t_k+t)$ are given as [23]:

$$\tilde{\mu}(t_k + t) = \mu_{\theta,t_k}t + S(t_k) \tag{9}$$

$$\tilde{\sigma}^2(t_k + t) = \sigma_{\theta,t_k}^2 t^2 + \sigma^2 t \tag{10}$$

The RLD is then evaluated by computing the distribution of the time until the degradation signal reaches a predetermined failure threshold δ. Let T be a random variable that denotes the remaining life of a partially degraded component given that we have observed $S_1,\ldots,S_k$. Then its distribution is given by

$$\begin{aligned} P(T \le t) &= P(S(t + t_k) \ge \delta|S_1,\ldots,S_k) \\ &= 1 - P(S(t + t_k) \le \delta|S_1,\ldots,S_k) \\ &= 1 - P\left(Z \le \frac{\delta - \tilde{\mu}(t + t_k)}{\tilde{\sigma}(t + t_k)}|S_1,\ldots,S_k\right) \\ &= \Phi\left(\frac{\tilde{\mu}(t + t_k) - \delta}{\tilde{\sigma}(t + t_k)}\right) \end{aligned} \tag{11}$$

where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of a standardized normal random variable Z.

3.2 Exponential Degradation Model. The exponential functional form is suitable for modeling applications where the degradation rate is affected by cumulative damage [8,13,24,25]. The methodology and procedure are similar to the linear degradation model. The degradation signal follows the form:

$$\begin{aligned} S(t_k) &= \phi + \theta\exp(\beta t_k + \varepsilon(t_k) - (\sigma^2 t_k/2)) \\ &= \phi + (\theta e^{\beta t_k})(e^{\varepsilon(t_k) - (\sigma^2 t_k/2)}) \end{aligned} \tag{12}$$

where $S(t_k)$ is the value of the signal at time t_k, ϕ is a constant deterministic parameter, θ and β are the stochastic model parameters, and $\varepsilon(t_k)$ is the error term following a Brownian motion. For mathematical convenience, we work with the logarithm of the degradation signal denoted by $L(t_k)$ in this model

$$\begin{aligned} L(t_k) &= \ln(S(t_k) - \phi) \\ &= \ln\,\theta + \beta t_k + \varepsilon(t_k) - (\sigma^2 t_k/2) = \theta' + \beta' t_k + \varepsilon(t_k) \end{aligned} \tag{13}$$

where the parameters $\theta' = \ln(\theta)$ and $\beta' = \beta - \sigma^2/2$.

The exponential model expressed in Eqs. (12) and (13) is similar to that presented in Ref. [6]. Although the authors assumed that the prior distributions of the stochastic parameters are independent, the updated (posterior) distribution was actually correlated. To avoid this inconsistency, we consider that case where θ' and β' are assumed to be jointly distributed random variables following a prior bivariate normal distribution, $\pi(\theta',\beta')$ with means (μ_o,μ_1'), variances (σ_o^2,σ_1^2), and correlation coefficient ρ_o. Note that $\mu_1' = \mu_1 - (\sigma^2/2)$ where μ_1 is the mean of the random variable β. We also notice the similarity between the logarithm of the degradation signal and the linear degradation model in Eq. (1).

3.2.1 Sensory Updating of the Prior Distribution Parameters. We define L_i as the difference between the logarithms of the observed signals at times t_i and t_{i-1}, where $L_i = L(t_i) - L(t_{i-1})$ for $i = 1,2,3,\ldots$ with $L_1 = L(t_1)$. By the properties of Brownian motion, L_i is normally distributed with mean $\beta'(t_i - t_{i-1})$ and variance $\sigma^2(t_i - t_{i-1})$. The prior distribution of the stochastic parameters is updated by computing the joint distribution of θ' and β' given the observed signal

$$P(\theta',\beta'|L_1,\ldots,L_k) \propto f(L_1,\ldots,L_k|\theta',\beta')\pi(\theta',\beta') \tag{14}$$

Suppose that we have observed the increments $L_1,\ldots,L_k$ at times $t_1,\ldots,t_k$. Since the error increments $\varepsilon(t_i)-\varepsilon(t_{i-1})$, $i = 1,2,\ldots,k$, are independent normal random variables, the joint density function of $L_1,\ldots,L_k$ given the stochastic parameters θ' and β' can be expressed as

$$f(L_1,\ldots,L_k|\theta',\beta') = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^k \times \exp - \left(\frac{(L_1-\theta'-\beta' t_1)^2}{2\sigma^2 t_1} + \sum_{i=2}^{k}\left(\frac{(L_i-\beta'(t_i-t_{i-1}))^2}{2\sigma^2(t_i-t_{i-1})}\right)\right) \tag{15}$$

The posterior (updated) distribution of θ' and β' can be found by evaluating $P(\theta',\beta'|L_1,\ldots,L_k)$ given the observed signals as follows:

$$\begin{aligned} P(\theta',\beta'|L_1,\ldots,L_k) &\propto f(L_1,\ldots,L_k|\theta',\beta')\pi(\theta',\beta') \\ &\propto \exp\left(-\frac{L_1-\theta'-\beta' t_i}{2\sigma^2 t_i} - \sum_{i=2}^{k}\frac{(L_i-\beta'(t_i-t_{i-1}))^2}{2\sigma^2(t_i-t_{i-1})}\right) \times \left(-\frac{1}{2(1-\rho_o^2)}\left[\frac{(\theta'-\mu_o)^2}{\sigma_o^2} - \frac{2\rho(\theta'-\mu_o)(\beta'-\mu_1')}{\sigma_o\sigma_1} + \frac{(\beta'-\mu_1')^2}{\sigma_1^2}\right]\right) \\ &\propto \exp\left(-\frac{1}{2}\left[\left(\frac{1}{\sigma^2 t_1}+A\right)\theta'^2 + \left(\frac{t_k}{\sigma^2}+B\right)\beta'^2 - 2\theta'\left(\frac{L_1}{\sigma^2 t_1}+C\right) - 2\beta'\left(\frac{\sum_{i=1}^{k} L_i}{\sigma^2}+D\right) + 2\theta'\beta'\left(\frac{1}{\sigma^2}+E\right)\right]\right) \end{aligned} \tag{16}$$

where

$$A = \frac{1}{\sigma_o^2(1-\rho_o^2)}, \quad B = \frac{1}{\sigma_1^2(1-\rho_o^2)}, \quad C = \frac{\mu_o}{\sigma_o^2(1-\rho_o^2)} - \frac{\mu_1'\rho_o}{\sigma_o\sigma_1(1-\rho_o^2)}, \quad D = \frac{\mu_1'}{\sigma_1^2(1-\rho_o^2)} - \frac{\mu_o\rho_o}{\sigma_o\sigma_1(1-\rho_o^2)}$$

and

$$E = -\frac{\rho_o}{\sigma_o\sigma_1(1-\rho_o^2)}$$

The parameters θ' and β' are assumed to follow a prior bivariate normal distribution. Thus, their posterior distribution given the observed data is also bivariate normal with means $(\mu_{\theta'},\mu_{\beta'})$, variances $(\sigma_{\theta'}^2,\sigma_{\beta'}^2)$, and correlation coefficient ρ, and given by the form

$$P(\theta',\beta'|L_1,\ldots,L_k) = \frac{1}{2\pi\sigma_{\theta'}^2\sigma_{\beta'}^2\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(\theta'-\mu_{\theta'})^2}{\sigma_{\theta'}^2} - \frac{2\rho(\theta'-\mu_{\theta'})(\beta'-\mu_{\beta'})}{\sigma_{\theta'}\sigma_{\beta'}} + \frac{(\beta'-\mu_{\beta'})^2}{\sigma_{\beta'}^2}\right]\right) \tag{17}$$

By comparing expressions (16) and (17), we get the following updated posterior parameters:

$$\mu_{\theta',t_k} = \frac{(t_k+B\sigma^2)\left(\frac{L_1}{t_1}+C\sigma^2\right) - \left(\sum_{i=1}^{k} L_i + D\sigma^2\right)(1+E\sigma^2)}{\left(\frac{1}{t_1}+A\sigma^2\right)(t_k+B\sigma^2) - \left(\frac{1}{\sigma^2}+E(E\sigma^2+2)\right)} \tag{18}$$

$$\mu_{\beta',t_k} = \frac{\left(\sum_{i=1}^{k} L_i + D\sigma^2\right)\left[\left(\frac{1}{t_1}+A\sigma^2\right) - (1+E\sigma^2)\right]}{\left(\frac{1}{t_1}+A\sigma^2\right)(t_k+B\sigma^2) - \left(\frac{1}{\sigma^2}+E(E\sigma^2+2)\right)} \tag{19}$$

$$\sigma_{\theta',t_k}^2 = \frac{[t_k\sigma_1^2(1-\rho_o^2)+\sigma^2](\sigma^2\sigma_o^2 t_1)}{\sigma^2(\sigma_o^2+\sigma_1^2 t_k t_1+\sigma^2 t_1+2\sigma_o\sigma_1\rho_o t_1) - [\sigma_o^2\sigma_1^2(1-\rho_o^2)(t_1-t_k)]} \tag{20}$$

$$\sigma_{\beta',t_k}^2 = \frac{\sigma_1^2\sigma^2[\sigma_o^2(1-\rho_o^2)+\sigma^2 t_1]}{\sigma^2(\sigma_o^2+\sigma_1^2 t_k t_1+\sigma^2 t_1+2\sigma_o\sigma_1\rho_o t_1) - [\sigma_o^2\sigma_1^2(1-\rho_o^2)(t_1-t_k)]} \tag{21}$$

$$\rho_{t_k} = \frac{\sigma^2\rho_1 - \sigma_0\sigma_1(1-\rho_o^2)\sqrt{t_1}}{\sqrt{[(\sigma_0^2(1-\rho_o^2)+\sigma^2 t_1)(t_k\sigma_1^2(1-\rho_o^2)+\sigma^2)]}} \tag{22}$$

3.2.2 Computing the Remaining Life Distribution. The remaining life distribution is computed in a procedure similar to that presented in the case of the linear degradation model presented above. Given $L_1,\ldots,L_k$ are the observed increments of the degradation signal logarithms up to time t_k, we define the random variable $L(t_k+t)$ as the logarithm of the degradation signal after t time units. The mean and variance of the random variable $L(t_k+t)$ are given as [6]

$$\tilde{\mu}(t+t_k)=\sum_{i=1}^{k}L_i+\mu_{\beta'}t=L(t_k)+\mu_{\beta'}t \tag{23}$$

$$\tilde{\sigma}^2(t+t_k)=\sigma_{\beta'}^2t^2+\sigma^2t \tag{24}$$

Let T be the random variable that denotes the remaining life. Then its distribution is given as follows:

$$P(T\le t)=P(L(t+t_k)\ge \ln(\delta-\phi)|L_1,\ldots,L_k)$$
$$=\Phi\left(\frac{\tilde{\mu}(t+t_k)-\ln(\delta-\phi)}{\tilde{\sigma}(t+t_k)}\right) \tag{25}$$

where $\Phi(\cdot)$ is the CDF of a standardized normal random variable Z.

4 Conservative Estimation of the Mean Remaining Life

In most engineering applications, the RLD is utilized in making critical decisions such as equipment replacement and spare parts ordering policies. Consequently, it is crucial to have expressions for the RLD that are less computationally intensive. To evaluate the RLD, we need to evaluate the expressions in Eqs. (11) and (25) (for the linear and exponential models, respectively) at each time epoch t_k. Furthermore, note the similarity of expressions (11) and (25) to the expression of the Bernstein distribution. The Bernstein distribution belongs to a bimodal family of distributions for which the moments do not exist [14]. To overcome this problem, we utilize the properties of Brownian motion to obtain a conservative estimate of the mean of the sensory-updated RLD.

We first start by reviewing the first passage time distribution of Brownian motion with positive drift, the inverse Gaussian distribution. We then utilize Jensen's inequality to show that the mean of the inverse Gaussian distribution can be used as a conservative estimate of the mean remaining life.

4.1 First Passage Time Distribution of Brownian Motion With Positive Drift. Let $\{X(t),t>0\}$ be a Brownian motion process with drift $\lambda>0$ and variance parameter σ^2 having the following form:

$$X(t)=\lambda t+W(t) \tag{26}$$

where $W(t)$ is a Brownian motion with mean zero and variance parameter σ^2. Then $X(t)$ has the following properties:

1. $X(0)=0$.
2. The increments $X(t_j)-X(t_i)$, $t_j>t_i$, are independent and identically normally distributed with mean $\lambda(t_j-t_i)$ and variance $\sigma^2(t_j-t_i)$.

Given some critical level ψ, let $\nu=\psi/\lambda$ and $\gamma=\psi^2/\sigma^2$. Furthermore, we define the random variable T to be the first passage time of $X(t)$ to ψ. Then T has an inverse Gaussian distribution with mean ν and shape parameter γ, $IG(\nu,\gamma)$, with the following probability density function (PDF) [26]:

$$f_T(t;\nu,\gamma)=\sqrt{\frac{\gamma}{2\pi t^3}}\exp\left\{-\frac{\gamma}{2\nu^2}\frac{(t-\nu)^2}{t}\right\},\quad t,\nu,\gamma>0 \tag{27}$$

The IG distribution has been used to model failure times in many research efforts [9,27–30]. It has useful properties such as the existence of closed-form expressions for the PDF, CDF, moment generating function (MGF), and well defined statistics such as the population mean, mode, and variance. Table 1 summarizes these properties of the inverse Gaussian distribution [26,31].

Table 1 Summary of the IG distribution properties

Parameters	$\nu>0$ location parameter $\gamma>0$ shape parameter
Support	$t\in(0,\infty)$
PDF	$\sqrt{\frac{\gamma}{2\pi t^3}}\exp\left\{-\frac{\gamma}{2\nu^2}\frac{(t-\nu)^2}{t}\right\}$
CDF	$\Phi\left(\sqrt{\frac{\gamma}{t}}\left(\frac{t}{\nu}-1\right)\right)$ $+\exp\left\{\frac{2\gamma}{\nu}\right\}\Phi\left(-\sqrt{\frac{\gamma}{t}}\left(\frac{t}{\nu}+1\right)\right)$, where $\Phi(\cdot)$ is the CDF of a standard normal random variable
Mean	ν
Variance	$\frac{\nu^3}{\gamma}$
Mode	$\nu\left[\left(1+\frac{9\nu^2}{4\gamma^2}\right)^{1/2}-\frac{3\nu}{2\gamma}\right]$
MGF	$\exp\left\{\left(\frac{\gamma}{\nu}\right)\left[1-\sqrt{1-\frac{2\nu^2t}{\gamma}}\right]\right\}$

4.2 Conservative Mean of the Sensory-Updated RLDs. Consider the linear degradation model presented in Sec. 3.1. We can rewrite the signal as follows:

$$S(t_k)-\phi=\theta t_k+\varepsilon(t_k) \tag{28}$$

We notice the similarity between expression (28) and Brownian motion with positive drift in Eq. (26). Hence, the mean of the first passage time distribution, which is inverse Gaussian, can be expressed as $(\delta-S(t_k))/\theta$, where $(\delta-S(t_k))$ represents the critical level at time t_k and θ is the drift of the Brownian motion at time t_k. Since θ is a random variable, we use its expected value μ_{θ,t_k} to estimate the updated drift at each updating epoch t_k. Next, we show, using Jensen's inequality, that this results in a conservative lower bound of the mean remaining life.

We know that for any positive random variable X, the following inequality holds (Jensen's inequality [32]);

$$E\left[\frac{1}{X}\right]\ge\frac{1}{E[X]} \tag{29}$$

Therefore, we can say that

$$E\left[\frac{\delta-S(t_k)}{\theta}\right]\ge\frac{\delta-S(t_k)}{E[\theta]}=\frac{\delta-S(t_k)}{\mu_{\theta,t_k}}=\frac{\zeta(k)}{\lambda(k)} \tag{30}$$

where

$$\zeta(k)=\begin{cases}\delta-\phi, & k=0\\ \delta-S(t_k), & k=1,2,\ldots\end{cases} \tag{31}$$

$$\lambda(k)=\begin{cases}\mu_o, & k=0\\ \mu_{\theta,t_k}, & k=1,2,\ldots\end{cases} \tag{32}$$

Equations (31) and (32) state that each time a signal is observed, the updated mean μ_{θ,t_k} is used to estimate the drift of the Brownian motion, and thus the mean of the remaining life is $\zeta(k)/\mu_{\theta,t_k}$. In other words, the updated RLD at updating epoch t_k

can be approximated by an inverse Gaussian distribution with mean $\zeta(k)/\lambda(k)$ and shape parameter $(\zeta(k)/\sigma)^2$, where $\lambda(k)$ and $\zeta(k)$ are the updated drift and updated threshold at time t_k, respectively.

From Eq. (30), we see that the mean of the IG distribution used to approximate the RLD represents a conservative lower bound of the expected remaining life. It is worth mentioning that for Jensen's inequality to hold in the form shown in Eq. (29), we need to have $\theta > 0$. We are assuming here that the variance of the random variable θ is small such that the probability of attaining negative values is negligibly small. In Sec. 5, we will demonstrate an example where this assumption is valid for an application including real-world data. It is worth mentioning, however, that if the variance of θ is not small such that its probability of being negative cannot be neglected, the presented results are not precise.

A similar argument can be made to obtain a conservative estimate of the mean remaining life for the exponential degradation model. We start by rewriting the logarithm of the degradation signal model in Eq. (13) as

$$L(t_k) - \theta' = \beta' t_k + \varepsilon(t_k) \tag{33}$$

By comparing expressions (26) and (33), we can approximate the RLD at each time t_k by an inverse Gaussian distribution with mean $\zeta(k)/\lambda(k)$ and shape parameter $(\zeta(k)/\sigma)^2$ where

$$\zeta(k) = \begin{cases} \ln(\delta) - \mu_o, & k = 0 \\ \ln(\delta) - L(t_k), & k = 1,2,\ldots \end{cases} \tag{34}$$

$$\lambda(k) = \begin{cases} \mu_1', & k = 0 \\ \mu_{\beta',t_k}, & k = 1,2,\ldots \end{cases} \tag{35}$$

Again, the mean of this inverse Gaussian distribution $\zeta(k)/\lambda(k)$ represents a conservative lower bound of the expected remaining life. In Sec. 5, we discuss the practical application of the proposed stochastic degradation models to real-world degradation data associated with rolling element bearings.

5 Implementation and Validation

We choose to model bearing degradation using the exponential functional form. This assumption is supported by existing literature on bearing condition monitoring and degradation modeling, such as Harris in Ref. [24], Shao and Nezu in Ref. [33] and Gebraeel et al. in Ref. [6].

5.1 Experimental Test Bed. Although there are several physical phenomena that characterize bearing degradation, we focus our attention on bearing vibration. The experimental setup, shown in Fig. 1, is used to perform accelerated degradation testing on thrust ball bearings. In accelerated testing, components are subjected to loading conditions beyond their design specifications in order to accelerate their failure, thus reducing the duration of the tests [34]. A test bearing is placed in the testing chamber where its lower race is fixed to a stationary housing and its upper race is fastened to a rotating shaft. The degradation data used in this study are obtained from bearings that were tested under constant operating conditions, a load of 200 lbs and a rotational speed of 2200 rpm.

Accelerometers—attached to the testing chamber—are used to capture vibration signals during the test. Time varying vibration signals are acquired using a data acquisition program designed in LABVIEW. The time domain signals are transformed into frequency domain using standard fast Fourier transformation (FFT). The evolution of the vibration spectra at different points in time is shown in Fig. 2.

Given the rotational speed and the specifications of the bearing, the amplitudes of the bearing's defective frequencies are extracted and used to develop a vibration-based degradation signal. Several possible degradation signals can be developed. The degradation signal used in this research represents the evolution of the average amplitude of the defective frequency and its first six harmonics over time (Fig. 3). The evolution of the degradation signal is correlated with the degradation state of the bearing and indirectly represents the physical transitions associated with the degradation process. The failure threshold is defined using the root mean square (rms) of the overall vibration acceleration. According to industrial standards for machinery vibration, ISO 2372, 2.0–2.2 Gs represents a "vibration-based danger level" for applications involving general-purpose machinery. For the degradation signal developed in this paper, we define the failure threshold as the

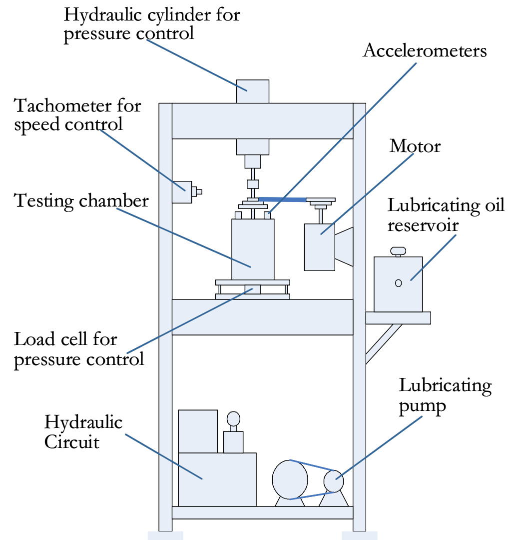

Fig. 1 Setup for accelerated degradation testing

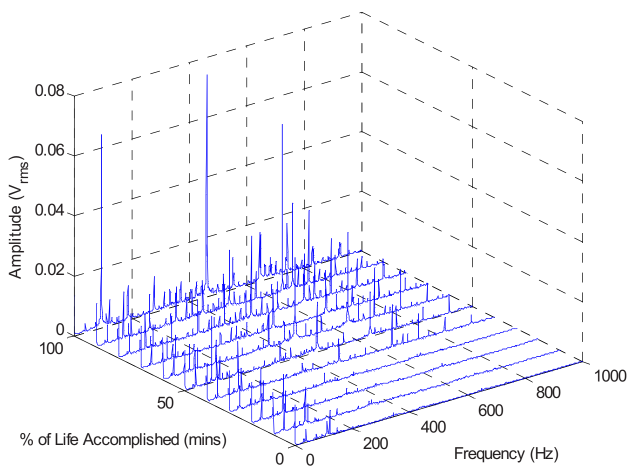

Fig. 2 Evolution of bearing vibration spectra

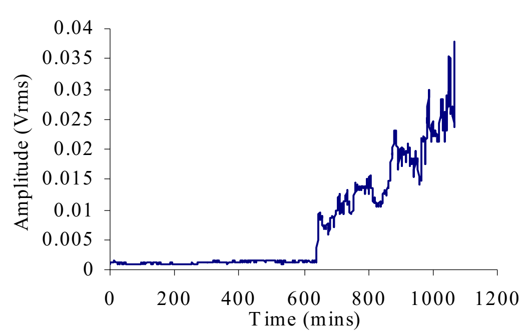

Fig. 3 Vibration-based degradation signal

amplitude of the degradation signal corresponding to 2.2 Gs of overall vibration. After observing several degradation signals, the failure threshold of the degradation signals was identified as 0.025 V_{rms}.

5.2 Model Implementation. Each testing experiment was conducted and vibration signals were acquired every 2 min for the entire duration of the test until failure was observed. The Bayesian updating methodology for computing and updating remaining life distributions was implemented in MATLAB. This sensory-updating procedure is outlined below.

5.2.1 Calculate the Values of the Prior Distribution Parameters. Bearings 1 to 25 were used to obtain the values of the parameters of the prior distribution $\pi(\theta', \beta')$. The assumption that the error terms follow a Brownian motion requires that $\varepsilon(0)=0$, thus the value of the initial degradation is given by $L(0)=\theta'$.

In order to estimate β', we utilize the first property of Brownian motion, which states that the error increments are independent. We use Eq. (36) to define the random variables R_k as follows:

$$R_k = \frac{L(t_k) - L(t_{k-1})}{t_k - t_{k-1}}, \quad k = 1,2, \ldots, \frac{t_f}{2} \tag{36}$$

where $t_k - t_{k-1}$ is 2 min and t_f is the observed failure time of the bearing. The random variables R_k are independent and identically distributed with mean β'. We use $\bar{R}$ to estimate β' for each bearing.

The θ' and β' values for these 25 bearings are used to estimate the prior means, variances, and correlation coefficient. The computed values of these parameters are given as $\mu_o = -6.031$, $\mu_1' = 0.00806$, $\sigma_o^2 = 0.3464 \times 10^{-5}$, $\sigma_1^2 = 1.0347 \times 10^{-5}$, and $\rho_o = -0.3464$. The variance parameter of the error terms σ^2 was computed and found to be equal to 0.007348.

5.2.2 Test the Model Assumptions. The deterministic parameter ϕ was observed to be ≈ 0.002 V_{rms}. As a simplifying assumption, the value of ϕ was set to zero. Now we recall that θ' and β' are assumed to follow a bivariate normal distribution. There is no specific goodness-of-fit test for the bivariate normal distribution. However, there are consequences that must be satisfied in order not to reject the assumption [35].

1. The marginal distribution of each random variable has to follow a univariate normal distribution.
2. Given that the data consist of pairs of observations (θ_i', β_j'), where $i = 1, \ldots, 25$ and $j = 1, \ldots, 25$, roughly 50% of the sample observations $\mathbf{x} = (\theta_i', \beta_j')$ must lie within the contour of an ellipse defined by the following expression:

$$(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \le \chi_2^2(0.5) \tag{37}$$

   where μ is estimated by $\bar{x}$ and Σ^{-1} is estimated by S^{-1}, with S being the sample covariance matrix.
3. A plot of the squared generalized distances given by Eq. (38) versus the corresponding chi-square percentiles has to follow a straight line.

$$d_j^2 = (\mathbf{x}_j - \bar{\mathbf{x}})^T S^{-1} (\mathbf{x}_j - \bar{\mathbf{x}}), \quad j = 1,2, \ldots, n \tag{38}$$

The three consequences were evaluated and found to hold true for the values of θ' and β'. Consequently, the assumption that θ' and β' follow a bivariate normal distribution was not rejected.

The error term $\varepsilon(t_k)$ is assumed to follow a Brownian motion with mean zero and variance parameter σ^2. By properties of Brownian motion, this means that the increments $\varepsilon(t_i) - \varepsilon(t_{i-1})$ are independent and normally distributed with mean zero and variance $\sigma^2 t$. Figure 4 shows a plot of the computed error increments. The mean of these increments was also computed and shown to be equal to $9.48 \times 10^{-6} \approx 0$, justifying the assumption.

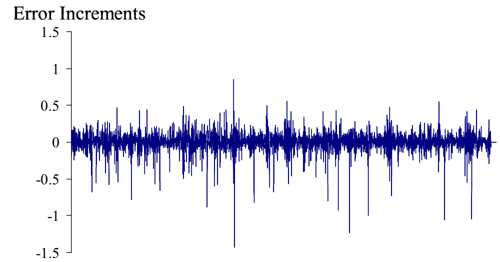

Fig. 4 Plot of the computed error increments

5.2.3 Updating of the RLD. A second group of 25 validation bearings, 26–50, was used to evaluate the performance of the model. Figures 5 and 6 show updated CDFs and PDFs, respectively, at different degradation percentiles for one of the validation bearings. The bearings were run-to-failure under the same loading and operating conditions. Every 2 min a vibration acquisition was performed, and the degradation signal evaluated and used to update the remaining life distribution.

We evaluated the mean remaining life using our proposed conservative approximation approach. At each updating epoch, the posterior mean of the degradation signal's trajectory μ_{β', t_k} was computed. Next, we calculated the conservative mean of the component's remaining life using $(\ln(\delta) - L(t_k)) / \mu_{\beta', t_k}$. After the failure time was observed for a given bearing, we computed a pre-

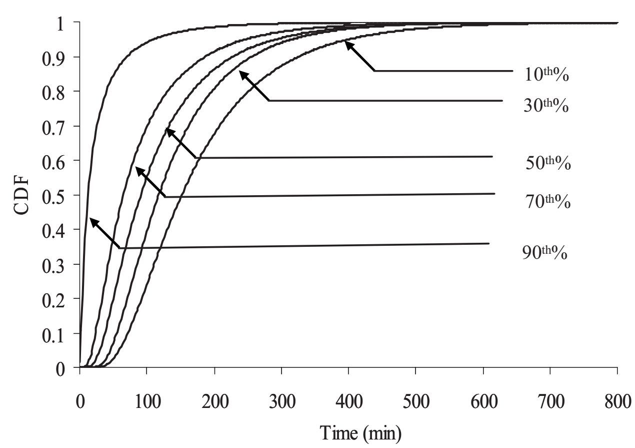

Fig. 5 Updated CDFs at different degradation percentiles for bearing 50

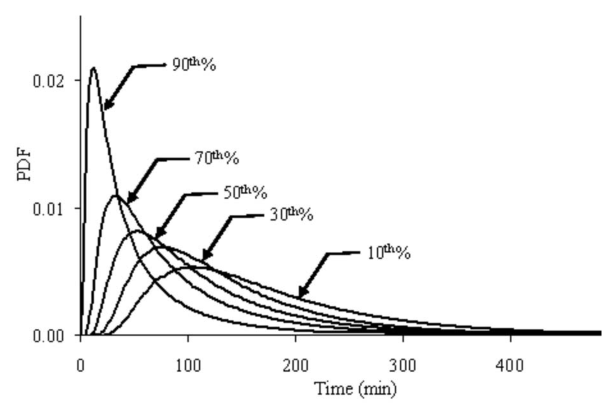

Fig. 6 Updated PDFs at different degradation percentiles for bearing 50

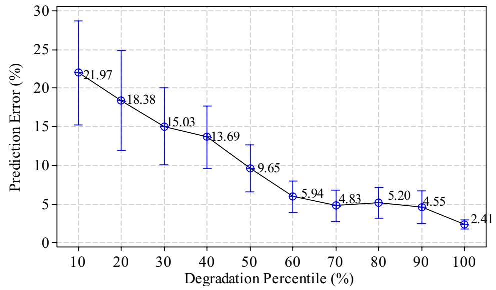

Fig. 7 Proposed methodology—95% CI for prediction errors based on the mean

diction error for each estimated mean remaining life that was evaluated at each updating epoch. The errors were computed as follows:

$$D_k^j = \frac{|(p_k^j + \tilde{t}_k^j) - t_N^j|}{t_N^j} \times 100 \tag{39}$$

where D_k^j is the percentage prediction error associated with bearing j, computed at sampling epoch (time) k; t_N^j is the actual observed failure time of bearing j; and p_k^j is the current total operating time of bearing j at sampling epoch k. This is obtained by summing the predicted remaining life and the total time the bearing has been in operation, where $\tilde{t}_k^j$ is the remaining life estimator at the k^{th} updating epoch (in this case $\tilde{t}_j^k = (\ln(\delta) - L(t_k)) / \mu_{\beta',t_k}$).

The prediction errors are then grouped by degradation percentile. We define a degradation percentile as the percentage duration in the degradation phase, given that we have observed the bearing's failure time. Figure 7 summarizes the confidence interval of the prediction errors for the 25 validation bearings (bearings 26–50).

6 Benchmarking

Our prediction results are benchmarked against three other procedures for predicting the failure. The first procedure focuses on using the median of the remaining life distribution as the estimator of the component's remaining life. The results of the same 25 validation bearings are outlined in Fig. 8. The median results demonstrate that the median estimator has relatively more accurate prediction accuracy. Nevertheless, if the RLD did not have a closed-form expression as shown in Eq. (25), it will be computationally difficult to estimate the mean, let alone the median. In summary, having a conservative estimate of the mean of the remaining life distribution eases computations and is relatively accurate compared with the median.

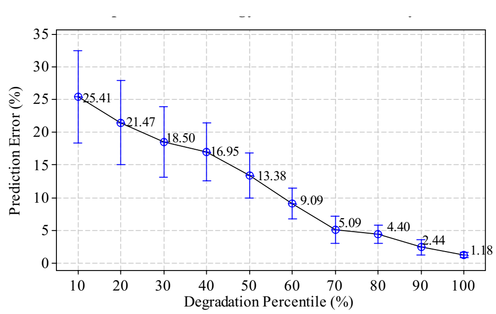

Fig. 8 Proposed methodology—95% CI for prediction errors based on the median

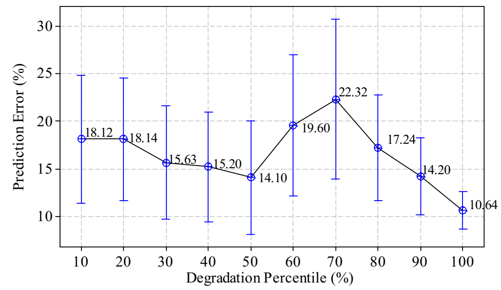

Fig. 9 Policy 1—95% CI of prediction errors based on the median

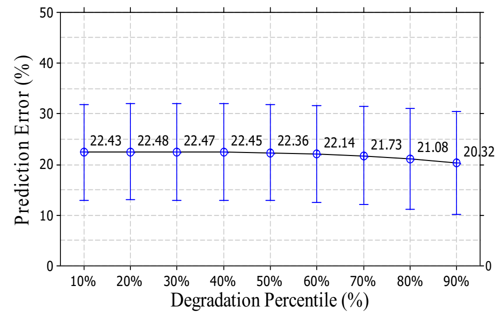

Fig. 10 Policy 2—95% CI for prediction errors based on the median

The results of the degradation model developed in this paper are benchmarked against two other policies. Policy 1 is based on the work done by Gebraeel [13], and policy 2 is based on the model developed by Lu and Meeker [5].

In policy 1 [13], the degradation model assumes that the stochastic model parameters are jointly distributed and follow a bivariate normal distribution. However, the error terms are assumed to be independent and identically normally distributed. A 95% confidence interval plot of the prediction errors using this policy is shown in Fig. 9.

Policy 2 uses the stochastic degradation model proposed by Lu and Meeker [5] and it demonstrates the importance of sensory-updating using real-time degradation information. In Ref. [5], the path of the degradation signal is still assumed to follow an exponential functional form. However, no updating of the remaining life distribution is performed. Thus, the degradation model is based entirely on the degradation characteristics of the sample bearings (bearings 1–25) used to estimate the prior parameters. A 95% confidence interval plot of the prediction errors using this policy is shown in Fig. 10.

7 Conclusions

This paper builds on previous work related to the development of sensor-based degradation models for computing and updating the remaining life distributions of partially degraded components. One of the major shortcomings of these sensory-updated degradation models was the inability to compute moments for the revised distributions, which are typically necessary for real-world implementation and decision making. Consequently, in all the previous work, either the median was used as an estimate of the remaining life or the distributions had to be evaluated numerically.

In contrast, this paper utilizes the appealing properties of the inverse Gaussian distribution as a distribution of the first passage time for a Brownian motion with positive drift. We investigate the similarities between our proposed degradation models and a Brownian motion model with positive drift. Next, we show that we can express the mean of the remaining life distribution as the mean of the corresponding inverse Gaussian distribution. Finally, we use Jensen's inequality to show that this approximation is a conservative estimate for the component's mean remaining life.

The approximation results of a validation case study using real-world vibration-based degradation signals from 25 bearings are summarized and show that the proposed methodology performs favorably to other benchmark procedures. Current extensions of this work are geared toward developing a similar methodology to account for degradation processes that occur under time-varying operating and environmental conditions.

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