# Mathematical Data Science

## Numerical Linear Algebra for Big Data

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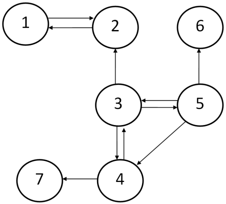
# Outline

# The PageRank and Linkspamming

- The PageRank of a webpage determines the importance of that page
- A high PageRank makes that page easy to find
- Linkspamming is the name for a collection of techniques to increase the PageRank
- In this lecture we will discuss
    - The PageRank model
    - How to manipulate it
    - How to detect that it has been manipulated

# A model of the websurfer

Webpage's are connected through hyperlinks.




Figuur: Model of part of the web

To model this mathematically we introduce

- The binary matrix $\mathbf{G}$, with $\mathbf{G}_{i,j} = 1$ if there is a link from page $j$ to page $i$. This matrix is the representation of a directed graph. Pages are nodes of this graph.
- The row-stochastic transition matrix $\mathbf{P}$, that gives the probability of going from one state to the next.

# The matrices $\mathbf{G}$ and $\mathbf{P}$

- We assume that the outlinks have equal probability to be followed. So if the number of outlinks for page $j$ is equal to $d_j$ then $\mathbf{P}_{j,i} = \mathbf{G}_{i,j}/d_j$ if $d_j \neq 0$.
- A *dangling node* is a webpage without outlinks, i.e. $d_j = 0$. We assume that these are connected to every webpage in the web with equal probability. In that case we get that $\mathbf{P}_{j,i} = \frac{1}{N}$, with $N$ the total number of pages.

# The matrices for the example

$$\mathbf{G} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

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$$\mathbf{P}^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1/7 & 1/7 \\ 1 & 0 & 1/3 & 0 & 0 & 1/7 & 1/7 \\ 0 & 0 & 0 & 1/2 & 1/3 & 1/7 & 1/7 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/7 & 1/7 \\ 0 & 0 & 1/3 & 0 & 0 & 1/7 & 1/7 \\ 0 & 0 & 0 & 0 & 1/3 & 1/7 & 1/7 \\ 0 & 0 & 0 & 1/2 & 0 & 1/7 & 1/7 \end{pmatrix}$$

# Teleportation

Teleportation is the name for jumping to a webpage without following a link.
To account for this behaviour the transition matrix is modified as follows

$$\mathbf{A} = p\mathbf{P}^T + \frac{1-p}{N}\mathbf{e}\mathbf{e}^T$$

Here the vector $\mathbf{e}$ contains all ones.
The first term says that the surfer follows an outlink with probability $p$ and the second term that the surfers teleports with probability $1-p$.

# The PageRank

- The PageRank vector is the probability vector after infinitely long surfing.
- The PageRank vector is the dominant eigenvector of the Google matrix $\mathbf{A}$.
- By the Perron-Frobenius theorem for positive matrices, $\mathbf{A}$ has a unique largest eigenvalue. Since $\mathbf{A}$ is column-stochastic this eigenvalue is $\lambda_1 = 1$.
- The corresponding eigenvector is positive and, if properly scaled, stochastic.
- The rank of a page is the index in the ordered Pagerank vector.

# Computation of the PageRank vector

- The standard way to compute the PageRank vector is by the Power method.

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- The standard way to compute the PageRank vector is by the Power method.
- Since we know that $\lambda_1 = 1$, the PageRank vector can also be computed by solving a linear system (proposed by Moler):

$$(p\mathbf{P}^T + \frac{1-p}{N}\mathbf{e}\mathbf{e}^T)\mathbf{x}^{(\mathbf{1})} = \mathbf{x}^{(\mathbf{1})} \quad \Rightarrow$$

$$(\mathbf{I} - p\mathbf{P}^T)\mathbf{x}^{(\mathbf{1})} = \frac{1-p}{N}\mathbf{e}$$

The solution $\mathbf{x}$ must be scaled to make it stochastic.

# Link spamming

Next we will discuss how to increase the PageRank by changing the link structure.

# Irreducible closed subchains

An irreducible closed subchain (subgraph):

- Once you enter a closed subchain you cannot leave;
- Every node in an irreducible subchain can be reached.
- Nodes in an irreducible closed subchain receive a high PageRank value.

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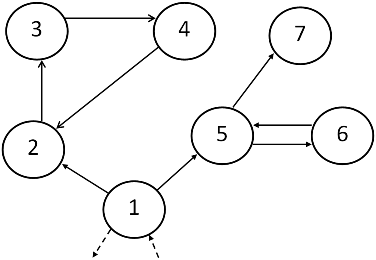
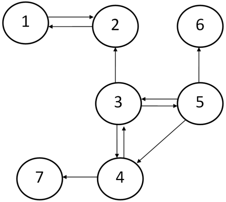



Figuur: Irreducible closed subchains?

# Linkspamming: node 4

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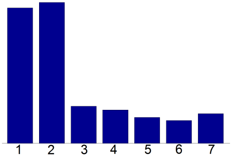

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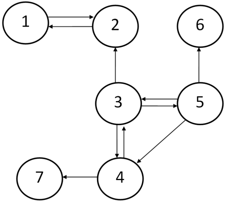


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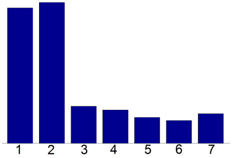


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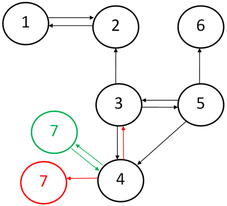

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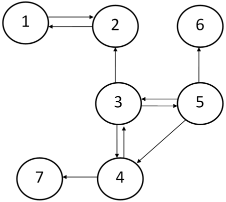
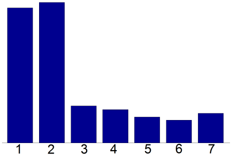



met $\mathbf{x}^{(1)} =$ 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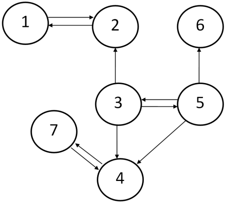

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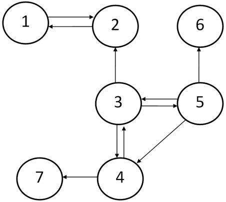


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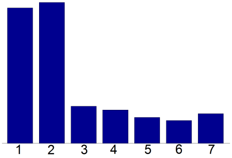


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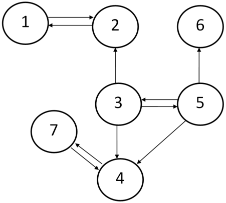


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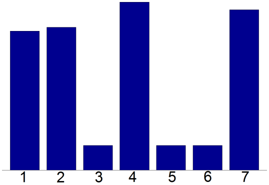

# Tarjan's algorithm

This type of link spamming can be detected by analyzing the graph of the (original) web:

- Find the strongly connected components of the graph (SCC: every node in an SSC can be reached form any of the other nodes).
- Strongly connected components that contain one node are dangling nodes.
- A group of strongly connected components without outlinks is an irreducible closed subchain.

An efficient algorithm for computing the strongly connected components of a graph is *Tarjan's algorithm*. Its complexity is linear in the number of nodes.

# The matrix P

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- The graph will in general contain many irreducible closed subchains;
- The matrix $\mathbf{P}$ can therefore be permuted to:

$$\mathbf{P} = \left(\begin{array}{cccc|cccc}
\mathbf{P_{11}} & \mathbf{P_{12}} & \cdots & \mathbf{P_{1r}} & \mathbf{P_{1,r+1}} & \mathbf{P_{1,r+2}} & \cdots & \mathbf{P_{1m}} \\
\mathbf{0} & \mathbf{P_{22}} & \cdots & \mathbf{P_{2r}} & \mathbf{P_{2,r+1}} & \mathbf{P_{2,r+2}} & \cdots & \mathbf{P_{2m}} \\
\vdots & & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{P_{rr}} & \mathbf{P_{r,r+1}} & \mathbf{P_{r,r+2}} & \cdots & \mathbf{P_{rm}} \\
\hline
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{P_{r+1,r+1}} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{P_{r+2,r+2}} & \cdots & \mathbf{0} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{P_{mm}}
\end{array}\right)$$

- The submatrices $\mathbf{P_{ii}}$ correspond to strongly connected components, $\mathbf{P_{ii}}, i = r+1, \cdots, m$ to irreducible closed subchains.

# The largest eigenvalue of P

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- Therefore, by the theorem of Perron each of these submatrices has an eigenvalue $1$, with a corresponding positive eigenvector.
- This means that $\mathbf{P}$ has $m-r$ eigenvalues 1. Since $\mathbf{P}$ is row stochastic, this is also the in modulus largest eigenvalue (complex eigenvalues with modulus 1 may exist).

# A property of the second eigenvector of $\mathbf{A}$

Because $\mathbf{A}$ is column stochastic, we have

$$\mathbf{e}^T \mathbf{A} = \mathbf{e}^T$$

this is, $\mathbf{e}$ is the left-eigenvector corresponding to the eigenvalue $\lambda_1 = 1$.

Since left and right eigenvectors are bi-orthogonal we have that

$$\mathbf{e}^T \mathbf{x}^{(\mathbf{2})} = 0$$

so the coefficients of the second eigenvector(s) add up to zero.

# The second eigenvectors of $\mathbf{A}$

The (second) eigenvectors of $\mathbf{A}$ satisfy

$$(p\mathbf{P}^T + \frac{(1-p)}{N}\mathbf{e}\mathbf{e}^T)\mathbf{x}^{(\mathbf{2})} = \lambda \mathbf{x}^{(\mathbf{2})}$$

and

$$\mathbf{e}^T \mathbf{x}^{(\mathbf{2})} = 0 \quad .$$

This means that

$$p\mathbf{P}^T \mathbf{x}^{(\mathbf{2})} = \lambda \mathbf{x}^{(\mathbf{2})}$$

If $\mathbf{P}^T$ contains at least two irreducible submatrices $\mathbf{P_{ii}}$ we can construct an eigenvector $\mathbf{x}$ for the eigenvalue 1 of $\mathbf{P}^T$ that satisfies $\mathbf{e}^T\mathbf{x}$.

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If $\mathbf{P}^T$ contains at least two irreducible submatrices $\mathbf{P_{ii}}$ we can construct an eigenvector $\mathbf{x}$ for the eigenvalue 1 of $\mathbf{P}^T$ that satisfies $\mathbf{e}^T\mathbf{x}$.
This is an eigenvector of $\mathbf{A}$ for the eigenvalue $\lambda = p$.

# Computation of all the second eigenvectors

To compute a second eigenvector of $\mathbf{A}$ we can solve the homogeneous equation

$$(\mathbf{I} - \mathbf{P}^T)\mathbf{x} = 0 \quad .$$

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From the indices of the nodes in the same subchain we can form the submatrix $\mathbf{P_{ii}}$. The eigenvectors of two such submatrices can be combined to one second eigenvector of $\mathbf{A}$.

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In total there are $m - r$ second eigenvectors. Since these eigenvectors are pairwise combinations of nonoverlapping vectors they have in total not more than $2N$ nonzero entries.

# Detection of linkspamming

Now there are two obvious algorithms to detect linkspamming:

1. Search the web for irreducible subchains (Tarjan's algorithm)
2. Compute a first eigenvector of $\mathbf{P}^{\mathbf{T}}$, determine the nonzero entries and use Tarjan's algorithm only on the nodes corresponding to nonzero entries.

# Results

| Test problem | Size | Closed subchains | CPU-time Tarjan | CPU-time Eigvec |
|---|---|---|---|---|
| wb-cs-stanford | 9914 | 113 | 0.3 | 1.4 |
| flickr | 820878 | 5394 | 399.3 | 160.8 |
| wikipedia-20051105 | 1634989 | 68 | 1515.3 | 140.2 |
| wikipedia-20060925 | 2983494 | 63 | 5077.1 | 166.6 |
| wikipedia-20061104 | 3148440 | 59 | 5696.9 | 155.1 |
| wikipedia-20070206 | 3566907 | 58 | 7462.7 | 313.6 |
| wb-edu | 9845725 | 49573 | 75703.2 | 2825.6* |

Computing time for web crawls by Gleich

Note: For `wb-edu` the eigenvector algorithm found 41606 subchains.

# Concluding remarks

- We have discussed the PageRank algorithm and the relation between linkspamming and the second eigenvector of the Google matrix:
- The PageRank algorithm is one example of a mathematical method to order importance of nodes in a graph
- Many other applications exist.
- We also discussed some mathematical properties of the transition matrix and developed ideas that can be used for 'Big Data'.