Exercises for the Stochastic part of Mathematical Data Science WI4231
Tan Rei Yun, 4763777

1a. $E((U-U')^2) = E\left(U^2 + U'^2 - 2UU'\right)$

$$= E(U^2) + E\left(U'^2\right) - E(2UU') \quad \text{(by linearity of expectation)}$$

$$= \frac{1}{12} + \left(\frac{1}{2}\right)^2 + \frac{1}{12} + \left(\frac{1}{2}\right)^2 - E(2UU') \quad \text{(using variance of standard uniform)}$$

$$= \frac{1}{6} + \frac{1}{2} - 2E(U)E(U') \quad \text{(as U and U' are independent)}$$

$$= \frac{1}{6} + \frac{1}{2} - 2 * \frac{1}{2} * \frac{1}{2} \quad \text{(as U and U' are independent)}$$

$$= \frac{1}{6}$$

1b. The moments of U and U' are given by:

$$E(U) = E(U') = \frac{1}{2}(0+1) = \frac{1}{2}$$

$$E(U^2) = E\left(U'^2\right) = \frac{1}{3}(0^2 + 0 * 1 + 1) = \frac{1}{3}$$

$$E(U^3) = E\left(U'^3\right) = \frac{1}{4}(0+1) * (0^2 + 1^2) = \frac{1}{4}$$

$$E(U^4) = E\left(U'^4\right) = \frac{1}{5}(0^4 \pm 0^3 * 1 + 0^2 * 1^2 + 0 * 1^3 + 1^4) = \frac{1}{5}$$

$$Var((U-U')^2) = E((U-U')^4) - \left(E((U-U')^2)\right)^2$$

$$= E((U-U')^4) - \left(\frac{1}{6}\right)^2$$

$$= E(U^4 - 4U^3U' + 6U^2U'^2 - 4UU'^3 + U'^4) - \frac{1}{36}$$

$$= E(U^4) - 4E(U^3)E(U') + 6E(U^2)E(U'^2) - 4E(U)E(U'^3) + E(U'^4) - \frac{1}{36}$$

$$= E(U^4) - 4E(U^3)E(U') + 6E(U^2)E(U'^2) - 4E(U)E(U'^3) + E(U'^4) - \frac{1}{36}$$

$$= \frac{1}{5} - 4 * \frac{1}{4} * \frac{1}{2} + 6 * \frac{1}{3} * \frac{1}{3} - 4 * \frac{1}{2} * \frac{1}{4} + \frac{1}{5} - \frac{1}{36}$$

$$= 0.038888888$$

$$\approx 0.04$$

3. Let Z be a random variable with a standard normal N(0,1)-distribution. It can be seen that P(|Z|<= z) is twice of the normal probability from positive z and 0 otherwise as distribution is folded over. For z>0,

$$P(|Z| \geq z) = 2\int_z^{\infty} e^{-\frac{x^2}{2}} dx$$

$$= 2\int_z^{\infty}(-x)e^{-\frac{x^2}{2}}(-\frac{1}{x})dx$$
$$= \left[-\frac{2}{(2\pi)^{0.5}x}\ e^{-\frac{x^2}{2}}\right]_z^{\infty} - \frac{2}{(2\pi)^{0.5}}\int_z^{\infty} x^{-2}e^{-\frac{x^2}{2}}dx$$
$$= \left[-\sqrt{\frac{2}{\pi}}\frac{e^{-\frac{x^2}{2}}}{x}\right]_z^{\infty} - \sqrt{\frac{2}{\pi}}\int_z^{\infty} x^{-2}e^{-\frac{x^2}{2}}dx$$
$$= -0 + \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{z^2}{2}}}{z} - \sqrt{\frac{2}{\pi}}\int_z^{\infty} x^{-2}e^{-\frac{x^2}{2}}dx$$
$$= \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{z^2}{2}}}{z} - \sqrt{\frac{2}{\pi}}\int_z^{\infty} x^{-2}e^{-\frac{x^2}{2}}dx$$
$$= \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{z^2}{2}}}{z} + \sqrt{\frac{2}{\pi}}\int_z^{\infty}(-x)x^{-3}e^{-\frac{x^2}{2}}dx$$
$$= \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{z^2}{2}}}{z} + \left[\frac{\sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}}}{x^3}\right]_z^{\infty} + 3\int_z^{\infty} x^{-4}e^{-\frac{x^2}{2}}dx$$
$$= \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{z^2}{2}}}{z} + 0 - \frac{\sqrt{\frac{2}{\pi}}e^{-\frac{z^2}{2}}}{z^3} + 3\int_z^{\infty} x^{-4}e^{-\frac{x^2}{2}}dx$$
$$= \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{z^2}{2}}}{z} - \frac{\sqrt{\frac{2}{\pi}}e^{-\frac{z^2}{2}}}{z^3} + 3\int_z^{\infty} x^{-4}e^{-\frac{x^2}{2}}dx$$
$$= \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{z^2}{2}}}{z}(1 - \frac{1}{z^2}) + 3\int_z^{\infty} x^{-4}e^{-\frac{x^2}{2}}dx$$

We see we can continuously integrate the remaining the integral by parts to get smaller order of z. Hence, we can conclude the following as needed.

$$P(|Z| \geq z) = \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{z^2}{2}}}{z}(1 + O\left(\frac{1}{z^2}\right))$$

b. For $Z_1, \dots, Z_p\ i.i.d$ with N(0,1) standard Gaussian distribution and α>0,
probability one of the random variable, Z_i, is greater than $\sqrt{\propto \log(p)}$ is

$$P\left(|Z_i| \geq \sqrt{\propto \log(p)}\right)$$

probability that one of the random variable, Z_i, is not greater than $\sqrt{\propto \log(p)}$ is

$$1 - P\left(|Z_i| \geq \sqrt{\propto \log(p)}\right)$$

probability that none of the independent random variables is not greater than $\sqrt{\propto \log(p)}$ is

$$\left(1 - P\left(|Z_i| \geq \sqrt{\propto \log(p)}\right)\right)^p$$

Hence, when p → ∞

$$P\left(\max_{j=1,\dots,p}|Z_j| \geq \sqrt{\propto \log(p)}\right) = 1 - \left(1 - P\left(|Z_1| \geq \sqrt{\propto \log(p)}\right)\right)^p$$
$$= 1 - \left(1 - \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{\sqrt{\propto\log(p)}^2}{2}}}{\sqrt{\propto\log(p)}}(1 + O\left(\frac{1}{\sqrt{\propto\log(p)}^2}\right)\right)^p$$

$$= 1 - \left(1 - \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{\propto \log(p)}{2}}}{\sqrt{\propto \log(p)}} \left(1 + O\left(\frac{1}{\propto \log(p)}\right)\right)^{p}$$

$$= 1 - \left(1 - \sqrt{\frac{2}{\propto \pi}} \frac{e^{\log(p^{-\frac{\propto}{2}})}}{\sqrt{\log(p)}} \left(1 + O\left(\frac{1}{\log(p)}\right)\right)^{p}$$

$$= 1 - \left(1 - \sqrt{\frac{2}{\propto \pi}} \frac{p^{-\frac{\propto}{2}}}{\sqrt{\log(p)}} + O\left(\frac{p^{-\frac{\propto}{2}}}{(\log(p))^{\frac{3}{2}}}\right)\right)^{p}$$

$$= 1 - \left(1 - \left(\sqrt{\frac{2}{\propto \pi}} \frac{p^{1-\frac{\propto}{2}}}{\sqrt{\log(p)}} + O\left(\frac{p^{1-\frac{\propto}{2}}}{(\log(p))^{\frac{3}{2}}}\right)\right) * \frac{1}{p}\ \right)^{p}$$

$$= 1 - \exp\left(-\sqrt{\frac{2}{\propto \pi}} \frac{p^{1-\frac{\propto}{2}}}{\sqrt{log(p)}} + O\left(\frac{p^{1-\frac{\propto}{2}}}{(log(p))^{\frac{3}{2}}}\right)\right)$$