插值逼近——三次样条插值

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一、三次样条插值

已知：$s(x_i) = f_i, \quad i = 1, 2, \cdots, n-1$ 。
I 型边界条件：$s(x_0) = f_0, s(x_n) = f_n \quad s'(x_0) = f_0', s'(x_n) = f_n'$
II 型边界条件：$s(x_0) = f_0, s(x_n) = f_n \quad s''(x_0) = f_0'', s''(x_n) = f_n''$
III 型边界条件：$s(x_0) = f_0, s(x_n) = s(x_0) \quad s'(x_0) = s'(x_n), s''(x_0) = s''(x_n)$
设 $h_i = x_i - x_{i-1}, e_i$ 为 $[x_{i-1}, x_i]$ 构成的小区间,$i = 1, 2, \cdots, n$

$$M_i = s_I''(x_i), i = 0, 1, \cdots, n$$

满足三次样条函数 $s(x)$

$$\begin{aligned} s(x) =& \frac{1}{6h_i}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] \\ &+ \frac{1}{h_i}\left[(x_i - x) f_{i-1} + (x - x_{i-1}) f_i\right] \\ &- \frac{h_i}{6}\left[(x_i - x) M_{i-1} + (x - x_{i-1}) M_i\right] \end{aligned}$$

其中 $x \in e_i, i = 1, 2, \cdots, n$

$$\lambda_j = \frac{h_{j+1}}{h_j + h_{j+1}}, \mu_j = 1 - \lambda_j = \frac{h_j}{h_j + h_{j+1}}$$

有

$$\mu_j M_{j-1} + 2M_j + \lambda_j M_{j+1} = d_j, \quad j = 1, 2, \cdots, n-1 \tag{1}$$

其中 $d_j = 6f\left[x_{j-1}, x_j, x_{j+1}\right]$

I 型边界条件下有：

$$\begin{cases} 2M_0 + M_1 = \dfrac{6}{h_1}\left(\dfrac{f_1 - f_0}{h_1} - f_0'\right) := d_0 \\ M_{n-1} + 2M_n = \dfrac{6}{h_n}\left(f_n' - \dfrac{f_n - f_{n-1}}{h_n}\right) := d_n \end{cases} \tag{2}$$

联立 (1) 和 (2)，得线性方程组

$$\begin{bmatrix} 2 & 1 & & & & \\ \mu_1 & 2 & \lambda_1 & & & \\ & \mu_2 & 2 & \lambda_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu_{n-1} & 2 & \lambda_{n-1} \\ & & & & 1 & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix} \tag{3}$$

II 型边界条件下有：

$$M_0 = f_0'', \quad M_n = f_n'' \tag{4}$$

联立 (1) 和 (4)，得线性方程组

$$\begin{bmatrix} 1 & & & & & \\ \mu_1 & 2 & \lambda_1 & & & \\ & \mu_2 & 2 & \lambda_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu_{n-1} & 2 & \lambda_{n-1} \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} f_0'' \\ d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ f_n'' \end{bmatrix} \tag{5}$$

进一步化简得 n-1 阶三对角方程组

$$\begin{bmatrix} 2 & \lambda_1 & & & \\ \mu_2 & 2 & \lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_{n-2} & 2 & \lambda_{n-2} \\ & & & \mu_{n-1} & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \begin{bmatrix} d_1 - \mu_1 f_0'' \\ d_2 \\ \vdots \\ d_{n-2} \\ d_{n-1} - \lambda_{n-1} f_n'' \end{bmatrix} \tag{6}$$

III 型边界条件下有：

$$M_0 = M_n, \lambda_n M_1 + \mu_n M_{n-1} + 2M_n = d_n \tag{7}$$

其中

$$\lambda_n = h_1/(h_1 + h_n), \mu_n = h_n/(h_1 + h_n)$$

$$d_n = 6(f[x_0, x_1] - f[x_{n-1}, x_n])/(h_1 + h_n)$$

联立 (1) 和 (7), 得线性方程组

$$\begin{bmatrix} 1 & & & & & -1 \\ \mu_1 & 2 & \lambda_1 & & & \\ & \mu_2 & 2 & \lambda_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu_{n-1} & 2 & \lambda_{n-1} \\ & \lambda_n & & & \mu_n & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} 0 \\ d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix} \tag{8}$$

进一步化简得 n 阶线性方程组

$$\begin{bmatrix} 2 & \lambda_1 & & & \mu_1 \\ \mu_2 & 2 & \lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_{n-1} & 2 & \lambda_{n-1} \\ \lambda_n & & & \mu_n & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix} \tag{9}$$

最终，通过解线性方程组，可以得到 $M_0, M_1, \cdots, M_n$，代入到 $s(x)$ 中，得三次样条插值函数

二、算法

对 I 型进行详细介绍, II,III 型是类似的, 不作详细说明

I 型三次样条插值

输入:

- $n+1$ 个插值节点 $(x_i, y_i), i = 0, 1, \cdots, n$ 构成向量 x_0, y_0
- I 型边界条件 f_0', f_n'
- 目标近似点 x

输出

- 近似点的值 y

实现步骤

- 步骤 1: 计算 $h_i = x_i - x_{i-1}, i = 1, 2, \cdots, n$

$$\lambda_j = \frac{h_{j+1}}{h_j + h_{j+1}}, \mu_j = 1 - \lambda_j$$

$$d_j = 6f\left[x_{j-1}, x_j, x_{j+1}\right] (\text{二阶差商}) \quad j = 1, 2 \cdots, n-1$$

$$d_0 = \frac{6}{h_1}\left(\frac{f_1 - f_0}{h_1} - f_0'\right)$$

$$d_n = \frac{6}{h_n}\left(f_n' - \frac{f_n - f_{n-1}}{h_n}\right)$$

- 步骤 2: 代入线性方程组 (3), 并用追赶法解三对角方程组，得 $M_0, M_1, \cdots, M_n$
- 步骤 3: (侧重点: 如何把分段的效果表示出来)

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for i=1:n **do**

if $x_{i-1} \le x \le x_i$ **then**

$$\begin{aligned} s(x) =& \frac{1}{6h_i}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] \\ &+ \frac{1}{h_i}\left[(x_i - x) f_{i-1} + (x - x_{i-1}) f_i\right] \\ &- \frac{h_i}{6}\left[(x_i - x) M_{i-1} + (x - x_{i-1}) M_i\right] \end{aligned}$$

end if

end for

得到 x 点处的插值近似值 $s(x)$

II 型三次样条插值

输入:

- $n+1$ 个插值节点 $(x_i, y_i), i = 0, 1, \cdots, n$ 构成向量 x_0, y_0
- II 型边界条件 f_0'', f_n''

追赶法解三对角方程组 (6)

III 型三次样条插值 前提: 判断满足 III 型条件

输入:

- $n+1$ 个插值节点 $(x_i, y_i), i = 0, 1, \cdots, n$ 构成向量 x_0, y_0

另外

$$\lambda_n = h_1/(h_1 + h_n), \mu_n = h_n/(h_1 + h_n)$$

$$d_n = 6(f[x_0, x_1] - f[x_{n-1}, x_n])/(h_1 + h_n)$$

解线性方程组 (9)

三、北太天元源程序

I 型

```
function [s,M] = spline1_interp(x0,y0,df0,dfn,x)
% I型三次样条插值
```

```
% Input: 节点向量x0,y0,两个端点的一阶导 df0,df1
%       目标点 x
% Output: 插值结果 s , M
%  子函数: divided_differences,tridiag_chase
%  Version:         1.0
%  last modified:   04/14/2024
   n = length(x0);
   h = zeros(1,n-1); h = x0(2:n)-x0(1:n-1);
   nh = length(h);
   lamda = zeros(1,nh-1);
   lamda = h(2:nh)./(h(1:nh-1)+h(2:nh));
   mu = 1-lamda;
   d= zeros(1,n);
   % 计算差商
   D = divided_differences(x0,y0,4);
   d(2:n-1) = D(3:n,4); %取二阶差商
   d(1) = 6/h(1) * ((y0(2)-y0(1))/h(1) - df0);
   d(n) = 6/h(n-1) * (dfn - (y0(n)-y0(n-1))/h(n-1));
   % 表示三对角方程组
   A = diag(2*ones(1,n),0) + diag([mu,1],-1) +diag([1,lamda],1);
   % 解三对角方程组
   [M]=tridiag_chase(A,d);
   % 分段表示
   nx = length(x);
   s = zeros(1,nx);
% 对于每个 x
for j = 1:1:nx
   % 判断在哪个小区间
   for i = 1:n-1
      if x(j) >= x0(i) && x(j) <= x0(i+1)
         hi = h(i); t2 = x0(i+1);t1 = x0(i);
         M2 = M(i+1);M1 = M(i);
         s(j) =1/(6*hi) * ((t2-x(j))^3 *M1 +(x(j) -t1)^3*M2);
         s(j) = s(j) + 1/hi *( (t2-x(j)) * y0(i) + (x(j)- t1)*y0(i+1) );
         s(j) = s(j) - hi/6 * ( (t2-x(j))*M1 + (x(j)-t1)*M2 );
         break;
      end
   end
end
end
```

将上述代码保存为 spline1_interp.m 文件。

II 型

```
function [s,M] = spline2_interp(x0,y0,dff0,dffn,x)
% II型三次样条插值
% Input: 节点向量x0,y0,两个端点的二阶导 dff0,dff1
```

```
%       目标点 x
% Output: 插值结果 s , M
%   子函数: divided_differences,tridiag_chase
%   Version:          1.0
%   last modified:    04/14/2024
    n = length(x0);
    h = zeros(1,n-1); h = x0(2:n)-x0(1:n-1);
    nh = length(h);
    lamda = zeros(1,nh-1);
    lamda = h(2:nh)./(h(1:nh-1)+h(2:nh));
    mu = 1-lamda;
    d= zeros(1,n-2);
    % 计算差商
    D = divided_differences(x0,y0,4);
    d = D(3:n,4); %取二阶差商
    d(1) = d(1) - mu(1)*dff0;
    d(n-2) = d(n-2) - lamda(n-2)*dffn;
    % 表示三对角方程组
    A = diag(2*ones(1,length(d)),0) + diag(mu(2:nh-1),-1) +diag(lamda(1:nh-2),1);
    % 解三对角方程组
    [M]=tridiag_chase(A,d); % 得到的M 是列向量
    M = [dff0;M;dffn];
    % 分段表示
    nx = length(x);
    s = zeros(1,nx);
% 对于每个 x
for j = 1:1:nx
    % 判断在哪个小区间
    for i = 1:n-1
        if x(j) >= x0(i) && x(j) <= x0(i+1)
            hi = h(i); t2 = x0(i+1);t1 = x0(i);
            M2 = M(i+1);M1 = M(i);
            s(j) =1/(6*hi) * ((t2-x(j))^3 *M1 +(x(j) -t1)^3*M2);
            s(j) = s(j) + 1/hi *( (t2-x(j)) * y0(i) + (x(j)- t1)*y0(i+1) );
            s(j) = s(j) - hi/6 * ( (t2-x(j))*M1 + (x(j)-t1)*M2 );
            break;
        end
    end
end
end
```

将上述代码保存为 spline2_interp.m 文件。

III 型

```
function [s,M] = spline3_interp(x0,y0,x)
% III 型三次样条插值
% Input: 节点向量x0,y0
```

```matlab
%       目标点 x
% Output: 插值结果 x , M
%   子函数: divided_differences,tridiag_chase,myJocabi
%   Version:          1.0
%   last modified:   04/14/2024
    n = length(x0);
    h = zeros(1,n-1); h = x0(2:n)-x0(1:n-1);
    nh = length(h);
    lamda = zeros(1,nh-1);
    lamda = h(2:nh)./(h(1:nh-1)+h(2:nh));
    mu = 1-lamda;
    d= zeros(1,n-1);
    % 计算差商
    D = divided_differences(x0,y0,4);
    d(1:n-2) = D(3:n,4) %取二阶差商
    % 从D中取差商时,注意位置和n的关系
    d(n-1) = 6*(D(2,3)-D(n,3))/(h(1)+h(nh));
    mu = [mu,h(nh)/(h(1)+h(nh))];
    % 表示三对角方程组
    A = diag(2*ones(1,length(d)),0) + diag(mu(2:nh),-1) +diag(lamda,1);
    A(1,n-1) = mu(1);
    A(n-1,1) = h(1)/(h(1)+h(nh));
    % 解三对角方程组
    [M]=myJacobi(A,d',zeros(1,length(d)),10e-8,100); % 得到的M 是列向量
    M = M(:,end);
    M=[M(1);M];
    % 分段表示
    nx = length(x);
    s = zeros(1,nx);
% 对于每个 x
for j = 1:1:nx
    % 判断在哪个小区间
    for i = 1:n-1
        if x(j) >= x0(i) && x(j) <= x0(i+1)
            hi = h(i); t2 = x0(i+1);t1 = x0(i);
            M2 = M(i+1);M1 = M(i);
            s(j) =1/(6*hi) * ((t2-x(j))^3 *M1 +(x(j) -t1)^3*M2);
            s(j) = s(j) + 1/hi *( (t2-x(j)) * y0(i) + (x(j)- t1)*y0(i+1) );
            s(j) = s(j) - hi/6 * ( (t2-x(j))*M1 + (x(j)-t1)*M2 );
            break;
        end
    end
end
end
```

将上述代码保存为 spline3_interp.m 文件。

四、数值算例

简单实现一下

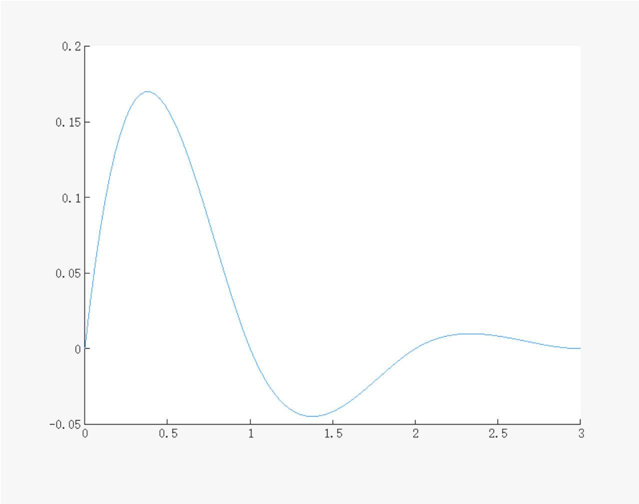
```
%% 1 I 型三次样条插值 课本P55 例2.13
clc;clear all;
x0 = [0 1 2 3];
y0 = [0 0 0 0];
df0 = 1;dfn = 0;
x = 1/2;
[s,M] = spline1_interp(x0,y0,df0,dfn,x)

x= linspace(0,3);
s = spline1_interp(x0,y0,df0,dfn,x);
plot(x,s)

%% 2 II 型三次样条插值 y = x^3
clc;clear all;
x0 = linspace(0,3,10);
y0 = x0.^3;
dff0 = 0;dffn = 18;
x= linspace(0,3,200);
y = x.^3;
[s,M] = spline2_interp(x0,y0,dff0,dffn,x);
plot(x,y,'r');
hold on
    plot(x,s,'b');
hold off

%% 3 III 型三次样条插值 y =sin(x)
clc;clear all;
x0 = linspace(0,2*pi,50);
y0 = sin(x0);
x= linspace(0,2*pi,200);
y = sin(x);
[s,M] = spline3_interp(x0,y0,x);
plot(x,y,'r');
hold on
    plot(x,s,'b');
    hold off

%% 4 Runge函数下的表现
clc;clear all;format long;
Runge = @(x)1./(x.^2+1);
x0 = linspace(-5,5,100);
y0 = Runge(x0);
df0 = 10/(25+1)^2; dfn = -10/(25+1)^2;
```

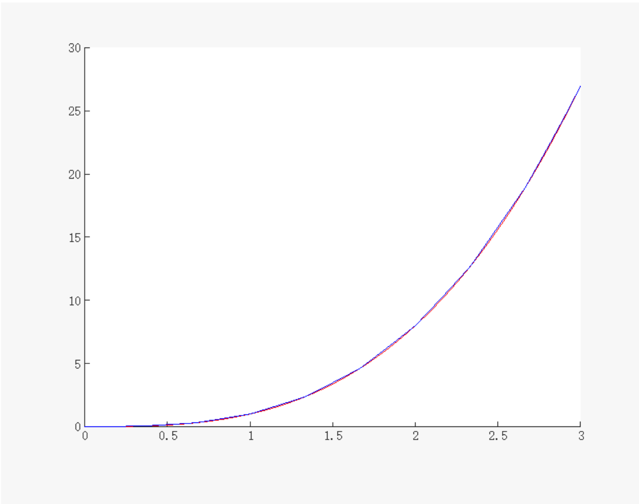
```
x=linspace(-5,5,200);
y = Runge(x);
s = spline1_interp(x0,y0,df0,dfn,x);
plot(x,y,'r');
hold on
    plot(x,s,'b');
    hold off
```

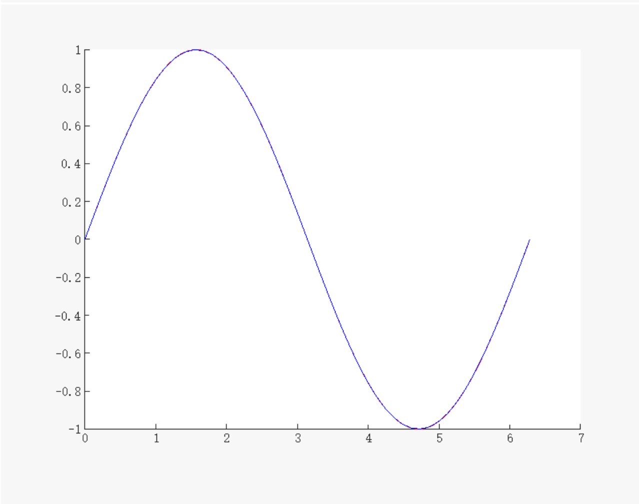
将上述代码保存为 `test_1.m` 文件。

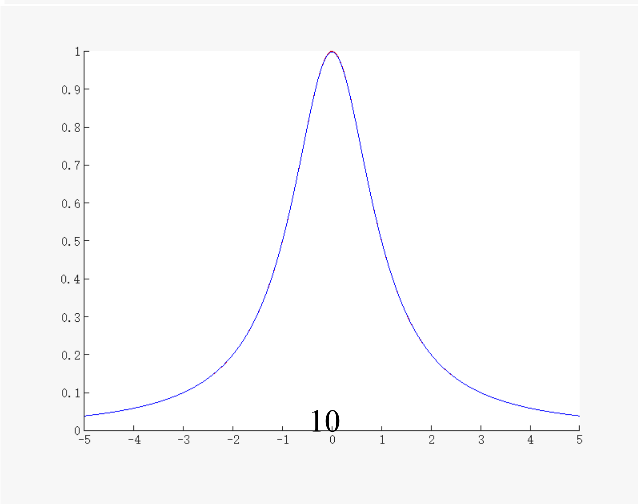
下图中红色为原函数, 蓝色为插值函数可以发现, 在插值节点个数足够的情况下, 近似效果还是非常好的.

另外在对 Runge 函数进行样条插值时, 随着节点个数的增加, 逼近效果也是越来越好的, 不会出现误差很大的现象.


0.2
0.15
0.1
0.05
0
-0.05
0
0.5
1
1.5
2
2.5
3


30
25
20
15
10
5
0
0
0.5
1
1.5
2
2.5
3


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
1
2
3
4
5
6
7


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-5
-4
-3
-2
-1
0
1
2
3
4
5