$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

Gaussian Elimination

Operation counts:

$$
\begin{array}{c|cccc}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & & & &
\end{array}
\longrightarrow
\begin{array}{ccccc}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} & b_2^{(2)} \\
\vdots & & & \vdots &
\end{array}
$$

直接看作0.

1

$2n$

共$2n+1$步.

第一列: $(n-1)\cdot(2n+1)$

$2n-1$

第二列: $(n-2)\cdot\left(2(n-1)+1\right)$

$2n-3$

$\vdots$

第$n-1$列: $1\cdot\left(2(2)+1\right)$

求和:

$$\sum_{j=1}^{n-1}\sum_{i=1}^{j} 2(j+1)+1 = \sum_{j=1}^{n-1} 2j(j+1)+j$$

$$= \sum_{j=1}^{n-1} 2j^2+3j$$

$$= 2\sum_{j=1}^{n-1} j^2 + 3\sum_{j=1}^{n-1} j$$

$$= 2\,\frac{(n-1)n(2n-1)}{6} + 3\,\frac{n(n-1)}{2}$$

$$= (n-1)n\left[\frac{2n-1}{3}+\frac{3}{2}\right]$$

$$= \frac{n(n-1)(4n+7)}{6}$$

$$= \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n.$$

$$
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} & b_n
\end{array}\right]
\longrightarrow
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
0 & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & a_{nn} & b_n
\end{array}\right]
$$

$\frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n$ operations.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} & b_n \end{array}\right] \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

n^2 operations.

$$x_n = \frac{b_n}{a_{nn}} \qquad 1$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}} \qquad 3.$$

$$x_{n-2} = \frac{b_{n-2} - a_{n-2,n}x_n - a_{n-2,n-1}x_{n-1}}{a_{n-2,n-2}} \qquad 5.$$

$$\vdots$$

$$x_1 = \frac{b_1 - a_{12}x_2 - \cdots - a_{1n}x_n}{a_{11}} \qquad 2(n-1)+1$$

$$1+3+5+\cdots+2(n-1)+1 = \frac{(1+2(n-1)+1)\cdot n}{2} = n^2.$$

back substitution : n^2 , $O(n^2)$

Gaussian Elimination : $\frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n + n^2$, $O(n^3)$.

可以用 $\frac{2}{3}n^3$ 估计 Gaussian Elimination 的运算量.