ON A CHARACTERIZATION OF THE NORMAL DISTRIBUTION.*

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The object of the present note is to give a simple characterization of the normal distribution. This characterization is based on an invariance principle and admits a physical interpretation.

1. Let $f(\alpha)$ be a measurable function defined in $(0,1)$. By the distribution function $\sigma_f(\omega)$ of the function $f(\alpha)$ one understands the measure of the set of those α's for which $f(\alpha) < \omega$.

THEOREM. *If for every γ the functions*

$$f_\gamma(\alpha) = \cos\gamma f(\alpha) + \sin\gamma g(\alpha), \quad g_\gamma(\alpha) = \sin\gamma f(\alpha) - \cos\gamma g(\alpha)$$

are statistically independent,[1] *then the distribution functions of $f(\alpha)$ and $g(\alpha)$ are normal and symmetric with the same precision h,*

$$\sigma_f(\omega) = \sigma_g(\omega) = h\pi^{-\frac{1}{2}} \int_{-\infty}^{\omega} \exp(-h^2u^2)\,du.$$

(*The case $h = \infty$ of the distribution function $\frac{1}{2}(1 + \operatorname{sign}\omega)$ is included.*)

Let

$$\int_0^1 \exp(i\xi f(\alpha))\,d\alpha = A(\xi), \qquad \int_0^1 \exp(i\xi g(\alpha))\,d\alpha = B(\xi)$$

and let us suppose that $A(\xi) = A(-\xi)$ and $B(\xi) = B(-\xi)$. This preliminary assumption is to the effect that the distributions are symmetric, and have, therefore real Fourier transforms $A(\xi)$, $B(\xi)$. The independence of $f_\gamma(\alpha)$ and $g_\gamma(\alpha)$ implies, for arbitrary real ξ and η, that

$$\text{(i)}\quad \int_0^1 \exp i[\xi f_\gamma(\alpha) + \eta g_\gamma(\alpha)]\,d\alpha = \int_0^1 \exp(i\xi f_\gamma(\alpha))\,d\alpha \cdot \int_0^1 \exp(i\eta g_\gamma(\alpha))\,d\alpha$$
$$= A(\xi\cos\gamma)\,B(\xi\sin\gamma)\,A(\eta\sin\gamma)\,B(-\eta\cos\gamma).$$

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[1] Cf. for instance, M. Kac, "Sur les fonctions indépendantes," I, *Studia Mathematica*, vol. 6 (1936), pp. 46-58.

On the other hand,

$$\xi f_\gamma(\alpha) + \eta g_\gamma(\alpha) = (\xi \cos\gamma + \eta \sin\gamma) f(\alpha) + (\xi \sin\gamma - \eta \cos\gamma) g(\alpha),$$

and therefore

$$\text{(ii)} \quad \int_0^1 \exp i[\xi f_\gamma(\alpha) + \eta g_\gamma(\alpha)] d\alpha = A(\xi\cos\gamma + \eta\sin\gamma) B(\xi\sin\gamma - \eta\cos\gamma).$$

Comparison of (i) with (ii) gives

$$\text{(iii)} \qquad A(\xi\cos\gamma + \eta\sin\gamma) B(\xi\sin\gamma - \eta\cos\gamma) \\ = A(\xi\cos\gamma) A(\eta\sin\gamma) B(\xi\sin\gamma) B(-\eta\cos\gamma).$$

Putting successively $\gamma = \pi/4$ and $\gamma = 3\pi/4$ and writing ξ and η instead of $\xi/\sqrt{2}$ and $\eta/\sqrt{2}$, one obtains

$$A(\xi+\eta) B(\xi-\eta) = A(\xi) A(\eta) B(\xi) B(\eta),$$
$$A(\eta-\xi) B(\xi+\eta) = A(\xi) A(\eta) B(\xi) B(\eta),$$

since $A(\xi) = A(-\xi)$, $B(\xi) = B(-\xi)$ by assumption. Consequently,

$$A(\xi+\eta) B(\xi-\eta) = A(\eta-\xi) B(\xi+\eta).$$

Placing $\eta = \xi$, it follows that

$$A(2\xi) = B(2\xi).$$

Thus, the problem is reduced to the functional equation

$$A(\xi+\eta) A(\xi-\eta) = A^2(\xi) A^2(\eta).$$

In particular, $A(2\xi) = A^4(\xi)$; so that, since $A(\xi)$ is real, $A(\xi) \geqq 0$. Hence, repeated application of $A(2\xi) = A^4(\xi)$ gives

$$A(\xi/2^k) = (A(\xi))^{1/4^k}$$

As $A(\xi/2^k) \to 1$ if $k \to \infty$, it follows from the continuity of $A(\xi)$ that $A(\xi) > 0$ for every ξ. Putting successively $\eta = \xi, 2\xi, 3\xi, \cdots$ and applying a well known method of Cauchy, it follows from $A(\xi) > 0$ in a few steps that

$$A(p\xi/q) = A^{p^2/q^2}(\xi)$$

for arbitrary integers p, q; so that, since $A(\xi)$ is continuous,

$$A(\xi) = \exp(k\xi^2) \qquad\qquad (\exp k = A(1)).$$

As $0 < A(\xi) \leq 1$, one has $k \leq 0$, and so, if $k = -h^2/4$, the proof is complete in the symmetric case under consideration.

2. According to a remark of Professor Wintner, the assumptions $A(\xi) = A(-\xi)$ and $B(\xi) = B(-\xi)$ of the symmetric case are superfluous.

In fact, let

$$A^*(\xi) = A(\xi)A(-\xi) \quad \text{and} \quad B^*(\xi) = B(\xi)B(-\xi).$$

Then the equation (iii) holds for A^* and B^*, instead of for A and B. Hence,

$$A^*(\xi) = \exp(k\xi^2).$$

Applying a well-known theorem of Cramér,[2] it follows that

$$A(\xi) = e^{a\xi^2 + b_1\xi}, \quad B(\xi) = e^{a\xi^2 + b_2\xi}.$$

Finally, an easy calculation shows that $b_1 = b_2 = 0$. This completes the proof of the theorem in the general case.

3. If instead of functions defined in $(0, 1)$ one considers independent functions defined in $(-\infty, +\infty)$ and the independence is meant in the sense of Hartman, van Kampen and Wintner,[3] one sees that the theorem remains unchanged (it is of course understood that in this case one deals with asymptotic distribution functions).

It is also clear that the theorem of § 1 holds in more than two dimensions, in the sense that instead of the two-dimensional orthogonal matrix one has to consider a more-dimensional one. The theorem in its most general form may then be stated as follows:

If a n-dimensional vector function $V(t)$ is such that its components are statistically independent in every system of Cartesian coördinates, then the density of the asymptotic distribution function of $V(t)$ is Gaussian and of radial symmetry.

If $V(t)$ is the velocity of a gas molecule $(n = 3)$ then the theorem gives the well known result of Maxwell in its "time average" formulation. It may be mentioned that Maxwell assumed not only the independence but also the radial symmetry of the distribution function. In our case the theorem follows only from the invariant character of the independence.

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[2] H. Cramér, "Random variables," *Cambridge Tracts*, 1937, p. 52.

[3] P. Hartman, E. R. van Kampen and A. Wintner, "Asymptotic distributions and statistical independence," *American Journal of Mathematics*, vol. 61 (1939), pp. 477-486.